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KOLMOGOROV CONDITION FOR INTEGRABLE SYSTEMS WITH FOCUS-FOCUS SINGULARITIES

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Abstract. We construct a method of proving the Kolmogorov's nondegeneracy condition, which guaranties the validity of KAM theorem, for two degree of freedom integrable Hamiltonian systems possessing focus-focus singularities. This method is then applied to various classical integrable systems.

1. Introduction

The celebrated KAM theorem says that under a small Hamiltonian perturbation, most of the invariant tori of an integrable system will not be destroyed but only perturbed (e.g., [8]). This theorem is stated under the following nondegeneracy condition, also called the *Kolmogorov condition*:

$$\det(\partial^2 H/\partial p_i \partial p_j) \neq 0 \text{ almost everywhere}$$

Here H is the integrable Hamiltonian, (p_i) is a local system of action coordinates.

Kolmogorov condition, though clearly a general position condition, is not easy to verify, and a rigorous proof of this condition exists for very few systems (cf. [6, 5, 11]). Moreover, this condition fails for some classical integrable systems (e.g., the 3-vortex equation in ideal incompressible fluid is resonant and hence does not satisfy the Kolmogorov condition). In [6], Knörrer developed a method for checking the Kolmogorov condition which uses codimension 2 hyperbolic singularities. This is the first general method available, and it was applied successfully in [6] to the Neumann equation and the geodesic flow on multi-dimensional ellipsoids.

In this Letter, we will use another type of singularities, namely the so called *focus-focus singularities*, for checking the Kolmogorov condition. Our main result is in Section 2. It says that, for integrable systems with two degrees of freedom, under a small additional assumption which can be verified easily in practice, focus-focus singularities imply the Kolmogorov condition.

Section 3 contains several applications of our main result. The first application is the spherical pendulum under a quadratic potential field. This application is inspired by two earlier papers by Horozov [5] and Zou [11]. The second application is the Lagrange top, and the third application is the "champagne bottle" (a special case of the Garnier system with two degrees of freedom, cf. [1]). There are some other systems possessing focus-focus singularities, e.g. the Clebsch case of motion of a rigid body in ideal fluid. The reader is invited to check our additional assumption for these other systems.

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Using our recent results about the topological structure of singularities of integrable systems [13], one can generalize Knörrer's and our methods and consider other types of singularities. This will be done in a subsequent paper.

2. The main result

Let $H : (M^4, \omega) \to \mathbb{R}$ be an integrable Hamiltonian with two degrees of freedom, and with an additional first integral $F : M^4 \to \mathbb{R}$. We will assume everything to be *real analytic*, and the moment map $(H, F) : M^4 \to \mathbb{R}^2$ a proper map. Recall that a fixed point $x \in M^4$ (i.e. $dH(x) = dF(x) = 0$) is called a *focus-focus point* if there are symplectic coordinates (x_i, y_i) near x (i.e. $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$) such that

$$\begin{aligned} H = &\quad a(x_1y_1 + x_2y_2) + b(x_1y_2 - x_2y_1) + \text{higher order terms} \\ F = &\quad c(x_1y_1 + x_2y_2) + d(x_1y_2 - x_2y_1) + \text{higher order terms} \end{aligned}$$

where a, b, c, d are constants with $ad - bc \neq 0$. It is well-known that if $a \neq 0$ then the Hamiltonian vector field X_H has two local invariant (2-dimensional Lagrangian) submanifolds at x, the stable and unstable ones with respect to X_H (e.g. [4, 7]). On each of these submanifolds the system behaves like a focus, and this property explains the term "focus-focus".

According to Arnold-Liouville theorem, the moment map $(H, F) : M^4 \to \mathbb{R}^2$ gives rise to a *singular Lagrangian torus foliation* of (M^4, ω), with leaves being connected components of the level sets of this map. A regular leaf of this foliation is nothing but a Liouville torus. A singular leaf is called a *focus-focus leaf* if it contains a focus-focus fixed point. A focus-focus singular leaf will be called *topologically stable* if any singular point y (i.e. $dH(y) \wedge dF(y) = 0$) in it is a focus-focus fixed point. That is, we do not allow other types of singular points in focus-focus leaves. (Of course, by a small perturbation which leaves the system integrable, one can achieve that on each focus-focus leaf there is only one singular point, cf. [12]. However, since the systems may have some discrete symmetries which will be destroyed by such a perturbation, we want to keep here the possibility of several focus-focus singular points in the same leaf). Our topological stability is not a stability in the sense of Lyapunov, but rather a structural stability. It can be easily verified for systems in practice, and is satisfied for most systems. From now on we will assume all focus-focus singularities under consideration to be topologically stable.

Recall that a homoclinic or heteroclinic orbit is an orbit going from one fixed point to the same point or another fixed point. (In more general contexts, fixed points may be replaced by other invariant sets). We have the following theorem.

Theorem 2.1. *Let the integrable Hamiltonian system with the moment map $(H, F) : M^4 \to \mathbb{R}^2$ contain a focus-focus singular leaf. Then if the Hamiltonian vector field X_H has a homoclinic or heteroclinic orbit connecting focus-focus points, H will satisfy the Kolmogorov condition (at least near the focus-focus singular leaf).*

Remark. Since everything is real analytic, if $\det(\partial^2 H/\partial p_i \partial p_j) \neq 0$ somewhere it will be different from zero on a semi-algebraic open subset of the orbit space, which in many cases will be dense in the orbit space.

Proof. Denote the singular focus-focus leaf by N, the singular Lagrangian foliation associated to the system by $\mathcal{L}$, the number of focus-focus points in N by n ($n \geq 1$), a small tubular saturated neighborhood of N by $U(N)$. Restrict our attention to $U(N)$. It is known that $(U(N), \mathcal{L})$ has the following remarkable properties (see [12, 13] for details):

1. N is a finite union of orbits of the $\mathbb{R}^2$-Poisson action (of the moment map). In fact, it is an easy exercise to see that N consists of n zero-dimensional orbits (i.e. points) and n orbits of of the type $\mathbb{R}^2/\mathbb{Z}$ (i.e. cylinders). N is homeomorphic to a cyclic chain of n 2-spheres, with one point of intersection for each pair of consecutive spheres. When $n = 1$, N is simply an immersed sphere with one point of self-intersection.

2. The focus-focus points of N are the only singular points in $U(N)$. The base space of the singular Lagrangian foliation $\mathcal{L}$ (restricted to $U(N)$) is a 2-disk $D^2 \ni z_0$, z_0 being the image of N. D^2 has a real analytic structure with (H, F) being a system of coordinates. $D^2 \setminus \{z_0\}$ corresponds to the regular part of the Lagrangian foliation. Homeomorphically, this foliation is determined uniquely by the number n. (The situation here is a little bit similar to that of elliptic fibrations in algebraic geometry, except for the fact that here we have Lagrangian instead of holomorphic fibers).

3. Recall (e.g., [3]) that the regular part of the base space of the associated Lagrangian foliation of an integrable Hamiltonian system is an *integral affine manifold* (where the affine chart is given by small open sets with local systems of action coordinates). This integral affine structure gives rise to a natural flat connection on the (co)tangent bundle of the base space, and the holonomy of this connection is called the *monodromy*. In particular, the punctured disk $D^2 \setminus \{z_0\}$ above has the structure of an integral affine manifold (different from the usual Euclidean structure). The monodromy of this affine manifold is nontrivial, and is generated by the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

4. In $U(N)$ there is a unique (up to the orientation) natural Hamiltonian S^1-action which preserves the moment map. In fact, using arguments similar to that used in the proof of the Arnold-Liouville theorem, one can extend the natural S^1-action from an orbit of the type $\mathbb{R}^2/\mathbb{Z}$ in N to the whole $U(N)$. We will denote by p_1 the Hamiltonian function which generates this S^1-action.

Now we will use the above properties (mainly 3 and 4) to prove our theorem. We will not use the existence of homo/heteroclinic orbits in the assumptions of the theorem directly, but only the following consequence of it:

H and p_1 are functionally independent. (Indeed, if $dH \wedge dp_1 = 0$, all orbits of H would be periodic, and there would be no homo/heteroclinic orbit). p_1 may be considered as a single-valued analytic function on D^2. Near a point in D^2 different from z_0, p_1 can be completed to a system of action coordinates (p_1, p_2). When one extends p_2 to $D^2 \setminus \{z_0\}$, it becomes a multi-valued function, due to the nontriviality of the monodromy. More explicitly, when one goes around z_0 once (in an appropriate direction), p_2 will change to $p_2 + np_1$, where n is the number of focus-focus fixed points in the singular leaf. Correspondingly, the vector field $\partial/\partial p_2$ is single-valued, but $\partial/\partial p_1$ will change to $\partial/\partial p_1 - n\partial/\partial p_2$ after going around z_0 once. The homotopy type of the (periodic) orbits of the Hamiltonian vector field X_{p_1} (of the Hamiltonian p_1) is trivial. On the other hand, the homotopy type of the orbits of X_{p_2} is nontrivial and is the generator of $\pi_1(U(N)) = \mathbb{Z}$.

Consider H as a function on D^2 and put $f_1 = \partial H/\partial p_1, f_2 = \partial H/\partial p_2$ (in some local system of action coordinates). Then f_2 is a single-valued analytic function on $D^2 \setminus \{z_0\}$, and f_1 changes to $f_1 - nf_2$ after going around z_0 once. We notice that f_2 can be extended continuously to z_0 with $f_2(z_0) = 0$. Indeed, let x_0 be a focus-focus point of N, and $U(x_0)$ a small ball in $U(N)$ containing x_0. Then if an orbit of the

Hamiltonian vector field X_H passes nearby x_0, it will take a long time to get out of $U(x_0)$ (the time is of order at least $O(-\ln\epsilon)$ where ϵ is the distance from the orbit to x_0). Thus it will take a long time for a point near x_0 to go away by X_H and return to another point near x_0, making a nontrivial homotopy element of $U(N)$. The time it takes for a point to go by the vector field X_{p_2} and make a nontrivial cycle (generator of $\pi_1(U(N))$) is always 1. Since $X_H = f_1 X_{p_1} + f_2 X_{p_2}$, we get that the solution of X_H modulo X_{p_1} projects to a cycle in time equal to $1/f_2$. It follows that $1/f_2$ is big, and f_2 is small near z_0 (of order $1/(-\ln\epsilon)$ or smaller). In the limit we have $f_2(z_0) = 0$. On the other hand, by the assumption of the theorem, H and p_1 are functionally independent. Hence $f_2 \neq 0$ (almost everywhere). It follows that f_2 is not a constant function on D^2, and $df_2 \neq 0$ almost everywhere.

Suppose now by contradiction that $\det(\partial^2 H/\partial p_i \partial p_j) \equiv 0$. It means that $df_1 \wedge df_2 = 0$. We will show that the local level sets of f_2 are straight lines in D^2 (with respect to the affine structure), which do not intersect, and f_1 is constant on these lines. Indeed, if $z \in D^2 \setminus \{z_0\}$ such that $df_2(z) \neq 0$, then $df_1 \wedge df_2 = 0$ implies that near z, f_1 is constant on the level sets of f_2, the quotient df_1/df_2 makes sense, and $df_1/df_2.\partial f_2/\partial p_2 = \partial f_1/\partial p_2 = \partial^2 H/\partial p_1 \partial p_2 = \partial f_2/\partial p_1$. Hence the ratio between $\partial f_2/\partial p_2$ and $\partial f_2/\partial p_1$ is constant on the level sets of f_2 near z, and therefore the local level sets of f_2 near z are straight lines. By the uniqueness of analytic continuation, it follows that the level sets of f_2 in D^2 are (pieces of) straight lines, except maybe for singular points. One notices that no two lines can intersect, otherwise it can be shown easily that f_2 is constant. The density of regular points implies that singular level sets of f_2 are also straight lines. It follows easily that D^2 (including the point z_0) has a regular topological foliation by the connected components of the level sets of f_2. All leaves of this foliation are straight lines, except for one leaf which goes through z_0. This special leaf is a curve consisting of z_0 and two straight lines, because we don't have the affine structure at z_0. Denote this leaf by l, and choose any two points z_1, z_2 of l lying on different sides with respect to z_0. Let l_1 and l_2 be two (pieces of) straight lines going through z_1 and z_2 respectively, and transversal to l. Each point of l_1 is mapped to a unique point of l_2 via the above foliation on D^2. Denote this map by ϕ. We will show that ϕ is a local *analytic* isomorphism, with respect to the natural analytic structure on l_1 and l_2 induced from D^2. Indeed, using the fact that no two lines (level sets of f_2) can intersect, one can see easily that the degree of degeneracy of f_2 at z_1 and z_2 must be the same. That is, if the Taylor series of f_2 on l_1 at z_1 with respect to some analytic coordinate starts with the power of order k ($k \geq 1$ because $f_2(z_1) = f_2(z_0) = 0$), then the same number k will play the same role for f_2 on l_2 at z_2. The function $f_2^{1/k}$ will be well-defined to be an analytic coordinate on both l_1 and l_2 (if these lines are short enough). Since the map ϕ preserves this function, it is an analytic isomorphism. What we have proved in fact is that the base space of the foliation by connected components of the level sets of f_2 is an interval with a well-defined analytic structure induced from $D^2 \setminus z_0$. Recall that f_1 is functionally dependent on f_2 and is hence constant on each leaf of the above foliation. In other words, f_1 can be factored (at least locally) to an analytic function on the base space of this foliation. Since this base space is just an interval, it follows easily from the uniqueness of analytic continuation that f_1 must be a single-valued function on $D^2 \setminus \{z_0\}$. But we know that after going around z_0 once, f_1 changes to $f_1 - nf_2$, and remember that n is a positive integer, and f_2 is a non-zero function. Thus we came to contradiction. $\square$

Remark. Another proof, which is also valid for the C^∞ case, goes as follows: Proceed as in the above proof until we obtain the curve $l \ni z_0$, on which $f_2 = 0$ and $f_1 = c$ (some non-zero constant). It follows that on the preimage of l in $U(N)$ all the orbits of X_H are periodic of period $1/c$. In particular, all orbits of X_H in N are periodic, so there is no homo/heteroclinic orbit. Notice that this second proof uses the existence of homo/heteroclinic orbits in an essential way, but not only the fact that H and p_1 are functionally independent.

3. Examples

3.1. **Spherical pendulum under a quadratic potential.** Consider a spherical pendulum, that is motion of a particle on the sphere $S^2 = \{\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3, |\mathbf{q}|^2 = \sum q_i^2 = 1\}$, under a potential field $V = V(\mathbf{q})$, which depends only on q_3 and is a quadratic function of q_3: $V = aq_3^2 + bq_3 + c$. It is a constrained Hamiltonian system on $T^*\mathbb{R}^3 = \{(\mathbf{p}, \mathbf{q})\} = \{(p_i, q_i)\}$, and can be written as a faithful Hamiltonian system on a symplectic manifold diffeomorphic to TS^2. This Hamiltonian system is integrable: it has the additional first integral $F = p_1q_2 - p_2q_1$, corresponding to the obvious axial symmetry.

When $a > 0$ or $a = 0, b \neq 0$ (where a, b are coefficients of the potential), the system has a focus-focus point. Indeed, in these cases, the north pole, or the south pole, or both of them (but none other), will be the position of maximal potential on the sphere, and it follows that the point $(\mathbf{p} = 0, \mathbf{q} = (0, 0, 1))$, or the point $(\mathbf{p} = 0, \mathbf{q} = (0, 0, -1))$, or both of them, will be focus-focus in the phase space. It can also be checked easily that these points, when they are focus-focus, lie in topologically stable focus-focus leaves. Moreover, the regular part of the orbit space is connected (it is a disk with one hole in case there is only one focus-focus point and in case $b = 0$ where 2 focus-focus points have the same value of the moment map, and a disk with two holes otherwise).

The additional integral $F = p_1q_2 - p_2q_1$ is also the function which generates S^1-symmetry near focus-focus singularities, and the Hamiltonian is obviously functionally independent of this function. Thus, (the proof of) our main result implies that Kolmogorov condition is satisfied for the spherical pendulum under the quadratic potential field $V = aq_3^2 + bq_3 + c$ (with $a > 0$ or $a = 0, b \neq 0$). (It is also easy to show that there are homo/heteroclinic orbits in the focus-focus leaf).

We remark that Kolmogorov condition for the case $a = 0, b \neq 0$ and the case $a > 0, b = 0$ were proved by Horozov [5] and Zou [11], by the use of Abelian integrals. The advantage of their method is that it allows them not only to prove the Kolmogorov condition, but also to compute the set where the determinant in question vanishes. (To compute this set, perhaps one cannot avoid dealing with explicit formulae). The disadvantage of their method is that the computations involved may be very complicated, and that is why they could not handle the general quadratic case (cf. [11]).

3.2. **Lagrange top.** Lagrange top is the motion of a heavy axially symmetric rigid body with a fixed point (which lies on the symmetry axis), under a constant gravitational force. It can be written as a Hamiltonian system on the coadjoint orbits of the Lie algebra $e(3) = so(3) \ltimes \mathbb{R}^3$ of rigid motions in $\mathbb{R}^3$ (e.g., [9]). General coadjoint orbits of $e(3)$ are symplectic manifolds isomorphic to TS^2. The system is integrable: it has an additional integral, which is the angular momentum corresponding to the symmetry axis. It is known that this system (for some coadjoint orbits) has a topologically stable focus-focus singularity (e.g., [2, 9]). The energy function and the

angular momentum are functionally independent, and hence again we can apply our main result. Notice that, by analytic continuation, Kolmogorov condition on coadjoint orbits which contain focus-focus fixed points imply Kolmogorov condition on most regular coadjoint orbits which do not contain focus-focus points.

3.3. **Champagne bottle.** The Garnier system with two degrees of freedom is the Hamiltonian system in $(\mathbb{R}^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$ with the following Hamiltonian:

$$H = 1/2(p_1^2 + p_2^2) - (a_1 q_1^2 + a_2 q_2^2) + b(q_1^2 + q_2^2)^2$$

This system is integrable by the method of separation of variables (e.g., [10]). Assume that $a_1, a_2, b > 0$. Then the energy level sets are compact, and the point $(q_1, q_2) = (0, 0)$ is the local maximal point of the potential $V = -(a_1 q_1^2 + a_2 q_2^2) + b(q_1^2 + q_2^2)^2$.

If $a_1 = a_2$ then there is an obvious S^1-symmetry, the origin of $\mathbb{R}^4$ is a focus-focus singular point (cf. [1]), and we can proceed just like in the previous examples.

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