# Singularities of integrable and near integrable Hamiltonian systems

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**Abstract**

The aim of this Letter is to show that singularities of integrable Hamiltonian systems, besides being important for such systems themselves, also have many applications in the study of near integrable systems. In particular, we will show how they are related to Kolmogorov's nondegeneracy condition (in the famous KAM theorem), the Poincaré-Melnikov function and its generalizations, topological entropy, and nonintegrability criteria.

**Key words**: *KAM theory, integrable system, homo/heteroclinic orbits, Poincaré-Melnikov function, nonintegrability*

**AMS subject classification**: 58F14, 58F07, 58F05, 70H05

## 1 Introduction

Integrable Hamiltonian systems form a rather special, but very important class of dynamical systems. Over the last decades, they have attracted a great number of mathematicians, because of their appearance in many problems of classical dynamics, physics, control theory, etc., and because of the discovery of new methods (inverse scattering, Lie group-theoretic, algebro-geometric,etc.) for finding and dealing with them. It is also of great interest to study their topological, qualitative properties. As a first step, one would like to study the bifurcations and singularities (of their moment maps). This study was influenced in part by a paper of Smale [18] on topology of dynamical systems with symmetries. Notice that singularities also play a central role in [18].

Recently, it became clear to us that singularities of integrable classical Hamiltonian systems also have many applications in symplectic geometry (e.g. construction of nonstandard symplectic spaces $\mathbb{R}^{2n}$), quantum dynamical systems, and especially near integrable systems.

In this Letter, we will concentrate on near integrable systems. Poincaré [16] considered such systems as the *basic problem of mechanics*. Following Poincaré, let us write a Hamiltonian function $H$ as the perturbation of an integrable Hamiltonian function $H_0$:

$$H = H_0 + \epsilon H_1 + ...$$

Here $\epsilon$ is a small perturbation parameter. It is natural to expect that the knowledge about the properties of the integrable Hamiltonian $H_0$ (and its singularities in particular) will give us also some information about the dynamical properties of the perturbed Hamiltonian $H$. As one will see, it is really the case.

There are various approaches to Poincaré's basic problem, the most famous one being the KAM (Kolmogorov-Arnold-Moser) theory (e.g., [1, 10, 14]). The classical KAM theorem says that when the perturbation is small, most of the invariant tori of the unperturbed integrable system will not be destroyed but only slightly deformed. This theorem is stated under the following nondegeneracy condition, also known as Kolmogorov's condition: $\det(\partial^2 H_0/\partial I_i \partial I_j) \neq 0$, where $(I_i)$ is a local system of action coordinates. In practice, this condition is not easy at all to verify directly, because the computation of the determinant often involves transcendental functions. However, inspired by an earlier work of Knörrer [8], we will show that one can most often verify this condition easily using singularities.

While KAM theory shows the "almost stability" of near integrable systems, Poincaré-Melnikov function (or method) is an effective tool to show their chaotic behavior, things like stochastic webs, Smale horseshoes, Arnold diffusions, and phase space transport. This method was developed by many authors (see e.g. [1, 9, 11, 12, 13, 17, 20] and references therein). While the classical Poincaré-Melnikov function makes use of simplest hyperbolic singularities in its definition, its generalizations ask for higher dimension and codimension singularities (e.g., [9, 17, 20]). The importance of a topological study of singularities of integrable systems for this purpose was already noticed by Koiller [9] and others.

Topological entropy is another measure of chaotic behavior. Since integrability is the opposite of chaoticity, it is natural that integrable systems, under some mild assumptions on their singularities, must have zero topological entropy. Surprisingly enough, this fact was proved only quite recently by G. Paternain [15] (under some strong nondegeneracy conditions of singularities) and I. Taimanov [19]. As a consequence, one obtains that near integrable systems have small entropy (if the perturbation is small). Thus near integrable systems are also not very chaotic in the sense of topological entropy.

The Poincaré-Melnikov function, the topological entropy, and the topological structure of singularities of integrable systems give rise to various criteria for nonintegrability of near integrable systems, some of which we will consider in this Letter.

We also suspect that higher-codimension singularities have a lot to do with overlapping resonances, and hence with the regions where Arnold diffusion is relatively fast, but so far we don't have any result about that.

This Letter is organized as follows: In §2 we present two basic results about the structure of singularities of integrable systems. In §3 a method for checking Kolmogorov's nondegeneracy condition, based on singularities, is derived. In §4 various nonintegrability criteria, based on the Poincaré-Melnikov function and the structure of singularities, are presented.

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# 2 Singularities of integrable systems

In this section let $(M^{2n}, \omega)$ be a symplectic manifold and $H : (M^{2n}, \omega) \to \mathbb{R}$ an integrable Hamiltonian with a moment map $\mathbf{F} = (F_1, ..., F_n) : M^{2n} \to \mathbb{R}^n$. That is, $\{F_i, F_j\} = \{F_i, H\} = 0$ and $\mathrm{rank} d\mathbf{F} = n$ almost everywhere. We will assume everything to be sufficiently smooth, and sometimes even real analytic. The moment map $\mathbf{F}$ will always be assumed to be a proper map.

For each $x \in \mathbb{R}^n$ in the image of $\mathbf{F}$, let $N_x$ denotes a connected component of the level set $\mathbf{F}^{-1}(x)$ of the moment map. We define the *rank* of $N_x$ to be $\mathrm{rank} N_x = \min_{z \in N_x} \mathrm{rank} d\mathbf{F}(z)$. $N_x$ is called singular if its rank is less than $n$. In this case $n - \mathrm{rank} N_x$ is called the *codimension* of $N_x$. When $N_x$ is nonsingular, it consists of an $n$-dimensional torus orbit of the Poisson action of $\mathbf{F}$, by Liouville theorem. When $N_x$ is singular it may contain many orbits of the Poisson action.

**Theorem 2.1** *If all data are real analytic, or if $N_x$ is not too degenerate, then there is a locally free Hamiltonian $\mathbb{T}^{\mathrm{rank} N_x}$-action in a neighborhood $\mathcal{U}(N_x) \subset M^{2n}$ of $N_x$ which preserves the moment map (and the Hamiltonian). This action is analytic in the real analytic case.*

In the above theorem, *not too degenerate* means that $N_x$ has a good topological stratification similar to Whitney stratification for analytic spaces. Of course, one can cook up other sufficient conditions for being not too degenerate as well. The proof of the above theorem is absolutely analogous to the proof of theorems about torus actions in [21].

When $\mathrm{rank} N_x = n$, the above theorem is nothing but the classical Arnold-Liouville theorem. Thus we have an extension of Arnold-Liouville theorem to the singular case. We notice that an analogous but weaker result (near a singular torus orbit, under a nondegeneracy condition...) was obtained earlier by H. Ito [7].

For nondegenerate singularities, a much stronger result was obtained in [21]. Here a singularity $N_x$ is called nondegenerate if each of its points is nonsingular or nondegenerate singular, and an additional mild condition of topological stability is satisfied (see [21] for details). It is an experimental fact that most singularities of known integrable Hamiltonian systems are nondegenerate. Denote by $\mathcal{L}$ the singular Lagrangian foliation given by the connected components of the moment map (generally $\mathcal{L}$ does not depend on the choice of the moment map but only on the original integrable Hamiltonian function). A nondegenerate singularity is called *simplest* (or *irreducible*) if it is an elliptic or hyperbolic singularity of a system with 1 degree of freedom (on a 2-surface), or a focus-focus singularity of a system with 2 degrees of freedom on a symplectic 4-manifold (cf. [21] for details). We have:

**Theorem 2.2 ([21])** *Under the nondegeneracy condition, an appropriate neighborhood $\mathcal{U}(N_x)$ of $N_x$ can be decomposed topologically together with the singular foliation into a direct product of simplest singularities, up to a finite covering:*

$$(\mathcal{U}(N_x), \mathcal{L}) \overset{\text{diff}}{\simeq} \mathbb{T}^r \times D^r \times (\mathcal{U}_1, \mathcal{L}_1) \times ... \times (\mathcal{U}_k, \mathcal{L}_k) \;/\; \Gamma$$

*Here $r = \mathrm{rank} N_x$, $\mathbb{T}^r$ a torus like in the previous theorem, $D^r$ is a ball, $(\mathcal{U}_i, \mathcal{L}_i)$ are singular foliations of some simplest (irreducible) singularities, $\sum \dim \mathcal{U}_i = 2n - 2r$. $\Gamma$ is a finite group acting freely on the product, and component-wise (i.e. this action commutes with the projection to each component). It acts trivially on $D^n$ and on elliptic components. The above decomposition is unique if $r = 0$ and $\Gamma$ is required to be minimal.*

For example, in case of the Kovalevskaya's top there is a codimension 2 hyperbolic singularity, which can be described by the following picture:

This picture means the quotient by $\mathbb{Z}_2$ of the product of two simplest hyperbolic singularities, with the singular sets as given in the picture. The group $\mathbb{Z}_2$ acts on these singularities by rotation by $180^o$. The Kovalevskaya's top also contains some degenerate singularities (for which Theorem 2.1 holds - that is, there is an $S^1$-action). For more details and other examples see [21].

The above theorem is a kind of topological classification of nondegenerate singularities of integrable systems. Notice that it says nothing about the symplectic structure. Indeed, the symplectic form cannot be decomposed in general. This theorem has a long story. It grew out from the attempts to do surgery with integrable systems, from local results by Rüssmann, J. Vey, Eliasson, and from some partial results and ideas by J.-P. Dufour, P. Molino, L. Lerman, Ya. Umanskii, A. Bolsinov, A. Fomenko and others (cf. [21]).

We should notice here some dynamical consequences of the above theorem. First, if $\mathrm{rank} N_x = 0$ (i.e. $N_x$ contains a fixed point) and all the irreducible components are hyperbolic, then there are transversal homoclinic or heteroclinic orbits. The existence of such orbits was first observed by R. Devaney [4] in a concrete example. Now we know that it is rather a rule than an exception. Second, if there is no elliptic component, then there are stable and unstable manifolds of dimension $n$ which nearly coincide. This fact was already used by several authors (without proof) in dealing with generalizations of the Poincaré-Melnikov function (e.g., [9, 11]). More generally, $N_x$ has a natural stratification, and a Poincaré-Melnikov function can be defined for each stratum (of dimension $> r$) of this stratification. (Such strata consist of homoclinic or heteroclinic orbits, and the Poincaré-Melnikov function is $\mathbb{R}^s$-valued on each $s$-dimensional stratum).

# 3 Kolmogorov's nondegeneracy condition

As we said in the introduction, the Kolmogorov's condition $\det(\partial^2 H/\partial I_i \partial I_j) \neq 0$ *a.e.* is not easy to verify directly in general. Indeed, direct computations (using Abelian integrals), as done by Arnold himself, E. Horozov [6], and other people, are usually very long and complicated. A method to check this condition was found in 1985 by H. Knörrer [8], who used codimension 2 singularities. (S. Bolotin told us that before Knörrer a similar method was applied to the Kovalevskaya top by some student of Stëpin, but he didn't remember the reference). Knörrer applied his method successfully to the geodesic flow on multi-dimensional ellipsoids and to the Newmann problem. Using Theorem 2.2, we will construct a generalization of Knörrer result to include focus-focus components and any codimension singularities. First results in this direction were obtained in [22].

**Theorem 3.1** *If the following conditions are satisfied:*
*a) $N_x$ is a nondegenerate singularity with only hyperbolic and focus-focus irreducible components (i.e. no elliptic components),*
*b) $H$ on a center manifold in $\mathcal{U}(N_x)$ of the Poisson action (which is symplectomorphic to $\mathbb{T}^r \times D^r$, $r = \mathrm{rank} N_x$) satisfies Kolmogorov's condition if considered as an integrable Hamiltonian on that manifold,*
*c) the linear part of the Hamiltonian vector field $X_H$ on the transversal subspace to the above center manifold have all eigenvalues different from pure imaginary or 0,*
*then $H$ satisfies Kolmogorov's condition, at least near $N_x$.*

The proof of the above theorem is by induction on the number of irreducible components, using Knörrer arguments for hyperbolic components, and an additional argument about monodromy as in [22] for focus-focus components.

Conditions a) and c) in the above theorem are relatively easy to check in practice because they usually involve only polynomials. Condition b) is empty or almost empty if we consider rank 0 or rank 1 singularities. We notice also that if all data are real analytic then if $H$ satisfies Kolmogorov condition near some Liouville torus, it will satisfy this condition near any other Liouville torus which can be connected to the former torus by a path of Liouville tori. Thus it is enough to prove Kolmogorov's condition near singularities.

The above theorem is applicable to all well-known integrable tops, spherical pendulum, Garnier systems, ... and perhaps to many other systems.

A similar result holds for the isoenergetic nondegeneracy condition.

# 4 Poincaré-Melnikov function and nonintegrability

Consider a perturbation of an integrable Hamiltonian: $H(z) = H_0(z) + \epsilon H_1(z,t,\epsilon)$. In this section everything is assumed to be real analytic, and $H_1$ periodic in time $t$.

Suppose that $z_1, z_2$ are two nondegenerate fixed points of the Hamiltonian flow of $H_0$ (which may coincide), such that the unstable manifold $W^u_{z_1}$ of $z_1$ and the stable manifold $W^s_{z_2}$ of $z_2$ have dimension $n$ and contain a common domain of dimension $n$. Let $\phi(z,t)$ be a heteroclinic orbit of the Hamiltonian vector field of $H_0$, $z$ belonging to the above domain. Fixing $\phi(z,t)$, one can define the Poincaré-Melnikov function:

$$PM(t) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(\phi(z, t+s), s, 0) ds$$

Using Theorem 2.1 and some arguments of Bolotin [2] (which in turn may go back to Poincaré, Kozlov,...), one can prove the following theorem:

**Theorem 4.1** *If $PM(t)$ is not a constant function in $t$ (or equivalently, $\int_{-\infty}^{+\infty}\{H_0,\{H_0,H_1\}\}(\phi(z,t+s),s,0)ds \neq 0$), then $H$ is not integrable for $\epsilon \neq 0$ small (i.e. it does not have a complete set of analytic first integrals).*

The above theorem improves in a significant way an earlier result by Bolotin (see e.g. [2, 11]). Bolotin proved the same statement under an additional condition that some Birkhoff transformation converges. Unfortunately, this additional condition is both difficult to verify and restrictive. (For most non-integrable systems the formal Birkhoff transformation does not converge). Another sufficient condition of Bolotin, namely the existence of a heteroclinic orbit, is also restrictive and difficult to verify.

The above theorem also generalizes some earlier results by Poincaré, Cushman, Ziglin and others on nonintegrability. Another generalization of the results by Poincaré and Cushman from the 2-degree-of-freedom case to the many-degree-of-freedom case is the following:

**Theorem 4.2** *If a periodically perturbed system in Poincaré's basic problem of mechanics has a topologically transversal homoclinic or heteroclinic orbit connecting fixed points, then it is nonintegrable.*

*Proof.* Time-periodic Hamiltonians with $n$ degrees of freedom may be considered as systems with $n+1$ degrees of freedom. Under this transformation, fixed points go to periodic orbits. If the perturbed system were integrable, it would follow from Theorem 2.1 that there is locally free $S^1$-action on the singular level set containing the homo/hetero-clinic orbit in question. But it is impossible because of the transversality.

One can use Theorem 4.1 and Theorem 4.2 to show, for example, that most perturbations of the metrics on the multi-dimensional ellipsoids give rise to nonintegrable geodesic flows.

We also suspect that the group $\Gamma$ in Theorem 2.2 enjoys some rigidity under integrable perturbations, and therefore it may give rise to some (topological) obstructions to integrability. For example, we have:

**Theorem 4.3** *Let an integrable system with 2 degrees of freedom have a hyperbolic codimension 2 singularity, which contains exactly two fixed points* $z_1, z_2$ *and whose decomposition as in Theorem 2.2 has* $\Gamma = \mathbb{Z}_2$. *Under a small autonomous perturbation the system still has 2 hyperbolic fixed points* $z_1'$ *and* $z_2'$ *near to* $z_1$ *and* $z_2$. *Then if the perturbed Hamiltonian function has different values at* $z_1'$ *and* $z_2'$, *it will not admit an additional nondegenerate first integral (which would make it integrable with nondegenerate singularities).*

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