Decomposition of Nondegenerate Singularities of Integrable Hamiltonian Systems

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Abstract

The main purpose of this note is to give a topological classification of stable nondegenerate singularities of smooth integrable Hamiltonian systems. Namely, we show that all such singularities can be decomposed diffeomorphically, after a finite covering, to the direct product of simplest (codimension 1 and codimension 2 focus-focus) singularities.
Key words: Hamiltonian system, integrable, singularity.

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1 Introduction

Integrable Hamiltonian systems met in classical mechanics and physics always have singularities, and most often these singularities are nondegenerate in a natural sense. Thus the study of nondegenerate singularities is one of the main step toward understanding topological structure of general integrable Hamiltonian systems.

The local structure of nondegenerate singularities has been known for some time (cf. [8, 4, 3] and references therein), although the semi-local structure of them has to date been unknown, except for the case of codimension 1 (cf. [5, 11]) and some particular cases of codimension 2 (cf. [6, 7, 2, 11]).

This Letter is devoted to nondegenerate singularities of integrable Hamiltonian systems of any codimension. It is based on a simple observation, which turns out to be a theorem. It can be stated as follows:

THEOREM. *All stable*[1] *nondegenerate singularities of integrable Hamiltonian systems can be, after a finite covering, decomposed topologically to a direct product of simplest codimension 1 and/or codimension 2 singularities.*

In a sense, the above theorem gives a topological classification of all stable nondegenerate singularities. It also gives a way to compute singularities in concrete cases. Without the stability condition, the theorem does not hold but a similar statement is true for the the singular leaf.

2 Preliminaries

In this work by an integrable Hamiltonian system we will always mean a Poisson action on a symplectic manifold (M^{2n}, ω), generated by n commuting Hamiltonian vector fields $X_1, ..., X_n$. Let $\mathbf{F} = (F_1, ..., F_n) : M^{2n} \to \mathbf{R}^n$ be a corresponding moment map, $X_i = X_{F_i}$. We will always assume that the level sets of $\mathbf{F}$ are compact.

Let x_0 be a singular point of the above Poisson action. The number $n - rank D\mathbf{F}(x_0)$ is called the *corank* of x_0. Let $\mathcal{K}$ be the kernel of $D\mathbf{F}(x_0)$ and let $\mathcal{I}$ be the space generated by $X_i(x_0)(1 \le i \le n)$. $\mathcal{I}$ is a maximal isotropic subspace of $\mathcal{K}$ with respect to the symplectic structure $\omega_0 = \omega(x_0)$. Hence the quotient space $\mathcal{K}/\mathcal{I} = \overline{R}$ carries a natural symplectic structure $\overline{\omega_0}$. $\overline{R}$ is symplectically isomorphic to a subspace R of $T_{x_0}M$ of dimension $2k$, k being the corank of the singular point. The quadratic parts of $F_1, ..., F_n$ at x_0 generate a subspace $\mathcal{F}^{(2)}_R(x_0)$ of the space of quadratic forms on R. This subspace is a commutative subalgebra under the Poisson bracket, and is often called in the literature *transversal linearization of* $\mathbf{F}$. The singular point x_0 is called *nondegenerate* of corank k if $\mathcal{F}^{(2)}_R(x_0)$ is a Cartan subalgebra of the algebra of quadratic forms on R, i.e. if it has dimension k. Denote by $Q(2n)$ the algebra of quadratic forms in $\mathbf{R}^{2n}$ with respect to a standard Poisson bracket.For the local structure of nondegenerate singularities, we have (cf. [9, 1, 8, 4, 3]):

Theorem 2.1 (Williamson) *For any Cartan subalgebra* $\mathcal{C}$ *of Q(2n) there is a symplectic system of coordinates* $(x_1, ..., x_n, y_1, ...y_n)$ *in* $\mathbf{R}^{2n}$ *and a basis* $f_1, ..., f_n$ *of* $\mathcal{C}$ *such that each* f_i *is one of the following*

$f_i = x_i^2 + y_i^2$ *(elliptic type)*

$f_i = x_i y_i$ *(hyperbolic type)*

$$\begin{cases} f_i &= x_i y_{i+1} - x_{i+1} y_i \\ f_{i+1} &= x_i y_i + x_{i+1} y_{i+1} \end{cases} \quad \text{(focus-focus type)}$$

[1]addendum: "stable" in this Letter means "topologically stable", a topological property which has nothing to do with dynamical stability a la Lyapunov. Hyperbolic orbits can be topologically stable

Theorem 2.2 (Vey-Eliasson) *Locally near a nondegenerate singular point, the Lagrangian foliation associated to the Poisson action is diffeomorphic (and even symplectomorphic) to that one given by the linearized action*

The above theorems are still not enough in order to understand the topology of integrable Hamiltonian systems. We need the following (cf [11]):

Definition 2.3 *Let* $\mathbf{F} : (M^{2n}, \omega) \to \mathbf{R}^n$ *be the moment map of a given Poisson action, and assume that the preimage of every point in* $\mathbf{R}^n$ *under* $\mathbf{F}$ *is compact and the differential* $D\mathbf{F}$ *is nondegenerate almost everywhere.*
a) The leaf through a point $x_0 \in M$ associated to the Poisson action *is the minimal closed invariant subset of* M *which contains* x_0 *and which does not intersect the closer of any orbit of the action, except the orbits contained in it.*
b) From a) it is clear that every point is contained in exactly one leaf. The (singular) foliation given by these leaves is called the Lagrangian foliation associated to the Poisson action. *The orbit space of this foliation is called the* orbit space *of the Poisson action*
c) A leaf is called singular *if it contains a singular point. An* nondegenerate singular leaf *is a singular leaf whose singular points are all nondegenerate. A singular leaf is called* of codimension k *if* k *is the maximal corank of its singular points.*

If a leaf is nonsingular or nondegenerate singular, then it is a connected component of the preimage of a point under the moment map, and a tubular neighborhood of it can be made saturated with respect to the foliation (i.e. it consists of the whole leaves only). Later on, a *tubular neighborhood of a nondegenerate leaf* will always mean a *saturated* tubular neighborhood. The Lagrangian foliation in a tubular neighborhood of a nondegenerate singular leaf N will be denoted by $(\mathcal{U}(N), \mathcal{L})$. By a *singularity* of the Poisson action we will mean either a singular leaf N or (a germ of) a singular foliation $(\mathcal{U}(N), \mathcal{L})$ (which object is referred to will be clear in the context). In this paper all singularities are assumed to be nondegenerate.

Definition 2.4 *A nondegenerate singular point* x *of corank* k *is said to have* Williamson type (m_1, m_2, m_3) *if it has* m_1 *elliptic,* m_2 *hyperbolic and* m_3 *focus-focus components in the Williamson's classification* $(k = m_1 + m_2 + 2m_3)$.

Lemma 2.5 *If a point* x *in a singular leaf* N *has corank* k *equal to the codimension of* N *(maximal possible) and has Williamson type* (m_1, m_2, m_3)*, then any other point* x' *in* N *with the same corank will have the same Williamson type.*

The proof of the above lemma only uses some standard arguments similar to that of [LU1,LU2]. If $x \in N$ has maximal corank and has Williamson type (m_1, m_2, m_3) then we will call (m_1, m_2, m_3) the *Williamson type* of N (or of the singularity $(\mathcal{U}(N), \mathcal{L})$). *Box*

3 Torus action and reduction

Theorem 3.1 *Let $(\mathcal{U}(N), \mathcal{L})$ be a nondegenerate singularity of codimension k of an integrable system with n degrees of freedom. Then we have:*
a) There is a locally free Hamiltonian $\mathbf{T}^{n-k}$ action in $(\mathcal{U}(N), \mathcal{L})$ which preserves the singular Lagrangian foliation.
b)[2] There is a finite covering of $(\mathcal{U}(N), \mathcal{L})$, denoted by $(\overline{\mathcal{U}}(N), \mathcal{L})$, of order at most 2^{m_2} where (m_1, m_2, m_3) is the Williamson type, such that the Hamiltonian torus $\mathbf{T}^{n-k}$ action in $(\overline{\mathcal{U}}(N), \mathcal{L})$) as in a) is free. So we one can use Marsden-Weinstein reduction to obtain a $(n-k)$-dimensional family of codimension k singularities in systems with k degrees of freedom.

The above theorem generalizes a result in [11] about codimension 1 singularities. In view of the above theorem, to prove our main result it is enough to consider singularities of maximal codimension, i.e. singularities which contain a fixed point.

For the proof, one first shows that near every closed singular $\mathbf{T}^{n-k}$ orbit in $(\mathcal{U}(N), \mathcal{L})$ (i.e. the set of singular points of maximal corank) there exists a required action. Then one shows that these actions can be extended to the whole $(\mathcal{U}(N), \mathcal{L})$ (and they will agree on the extension). *Box*

Lemma 3.2 *There is a local function $\mathbf{H} = \mathbf{H}(\mathbf{F}) : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, which is nondegenerate at $0 = \mathbf{F}(N)$, such that for any point $y \in N$, if y has corank r then there exist exactly r components $H_{i_1}, ..., H_{i_r}$ of $\mathbf{H}$ $i_1, ..., i_r \leq k$, such that $dH_{i_1}(y) = ... = dH_{i_r}(y) = 0$.*

From the topological point of view, the moment maps $\mathbf{F}$ and $\mathbf{H}$ give the same thing. So later on instead of considering the Poisson action generated by $\mathbf{F}$ we will consider the one generated by $\mathbf{H}$. For shortness, we will call $\mathbf{H}$ again by $\mathbf{F}$.

Take a point $x \in N$ of maximal corank k. Then we can also reorder (F_i) to assume that $F_1, ..., F_{m_1}$ correspond to elliptic components, $F_{m_1+1}, ...F_{m_1+m_2}$ correspond to hyperbolic components and $F_{m_1+m_2+1}, ..., F_k$ correspond to focus-focus components in the local

[2]erratum: this part b) requires some additional condition to be correct (topological stability will suffice). The number 2^{m_2} is not correct and must be replaced by 2^{n-k}

classification. (We say that F_i corresponds to a component, if in the local classification, the differential of F_i, restricted to N, does not vanish only on the intersection of N with that component) Then it is also true for any other singular point of maximal corank in N (without reordering again).

4 Stability

Assume now that $k = n$ and $\mathbf{F}$ as in Lemma 3.2 Set $V_i = \{x \in (\mathcal{U}(N), \mathcal{L}) \mid dF_j(x) = 0 \mathrm{if} j \neq i\}$ (i-elliptic or hyperbolic component) or $V_{i,i+1} = \{x \in (\mathcal{U}(N), \mathcal{L}) \mid dF_j(x) = 0 \mathrm{if} j \neq i, i+1\}$ ($(i, i+1)$- focus-focus component) Then V_i are 2-dimensional symplectic surfaces and $V_{i,i+1}$ are 4-dimensional symplectic submanifolds, and they have induced singular Lagrangian foliations.

By generalizing a notion in [2], we will call the set of all V_i and $V_{i,i+1}$ the *l-type* of $(\mathcal{U}(N), \mathcal{L})$. It is an important invariant of the singularity, which will be used in the proof of the main result.

Definition 4.1 *A nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ of codimension n (maximal possible) is called stable if the following two conditions are satisfied:*
1) There is a diffeomorphism $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ such that by taking the composition $F' = \Phi \circ F$ of the moment map with this diffeomorphism, we have that, under the new moment map F' restricted to $\mathcal{U}(N)$, $F' : \mathcal{U}(N) \to \mathbf{R}^n = \{(x_1, ..., x_n)\}$, the singular value set of this map in $\mathbf{R}^n$ is contained in the union of the hyperplanes $\cup_{i=1}^n \{x_i = 0\}$ (and N is mapped to the origin).
2) In N there is no closed orbit except fixed points.

Comments The first condition means that singularities near N (of codimensions less than $k = n$) do not "break up" when going from N. For example, all the codimension 1 singularities near N are just finite-sheet coverings of V_i where V_i are elements of the $l - type$ of N. The second condition implies that focus-focus elements in the l-type don't contain closed singular orbits. It is justified by the fact that in view of the existence of a Hamiltonian S^1 action [11], in 4-dimensional case we can split closed singular orbits from focus-focus singularities by a small perturbation.

Definition 4.2 *A nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ of codimension k will be called* stable *if in the reduction given by Theorem 3.1 we obtain a $(n-k)$ family of topologically equivalent stable codimension k singularities (of systems with k degrees of freedom)*

In case $k = 1$, Definition 4.2 coincides with a definition of stable codimension 1 singularities given in [11]. The following proposition can also be taken as a definition of stable singularities.

Proposition 4.3 *A nondegenerate codimension k singularity $(\mathcal{U}(N), \mathcal{L})$ is stable if and only if the following two conditions are satisfied:*
1) There is a diffeomorphism $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ such that by taking the composition $F' = \Phi \circ F$ of the moment map with this diffeomorphism, we have that, under the new moment map F' restricted to $\mathcal{U}(N)$, $F' : \mathcal{U}(N) \to \mathbf{R}^n = \{(x_1, ..., x_n)\}$, the singular value set of this map in $\mathbf{R}^n$ is contained in the union of the hyperplanes $\cup_{i=1}^k \{x_i = 0\}$ (and N is mapped to the origin).
2) All closed orbits in N have the same dimension k. $\diamond$

5 Decomposition

If N_1 and N_2 are two nondegenerate singularities, $N_i \subset M^{2n_i}$ of codimension k_i, with the corresponding Lagrangian foliation $(\mathcal{U}(N_1), \mathcal{L}_1)$ and $(\mathcal{U}(N_2), \mathcal{L}_2)$, then the *direct product* of these singularities is the singularity $N = N_1 \times N_2$ of codimension $k_1 + k_2$ with the associated Lagrangian foliation equal to the direct product of the given Lagrangian foliations:

$$(\mathcal{U}(N), \mathcal{L}) = (\mathcal{U}(N_1), \mathcal{L}_1) \times (\mathcal{U}(N_2), \mathcal{L}_2).$$

Definition 5.1 *A nondegenerate singularity N of codimension k in a symplectic $2n$-manifold is called* of direct-product type topologically *(or a* direct-product singularity *if the Lagrangian foliation $(\mathcal{U}(N), \mathcal{L})$ associated to it is homeomorphic to the following direct product of Lagrangian foliations:*
$(\mathcal{U}(N), \mathcal{L}) \overset{homeo}{=}$
$(\mathcal{U}(\mathbf{T}^{n-k}), \mathcal{L}_r) \times (P_1^2(N_{s1}), \mathcal{L}_{s1}) \times \ldots \times (P_i^2(N_{si}), \mathcal{L}_{si}) \times (P_1^4(N_{f1}), \mathcal{L}_{f1}) \times \ldots \times (P^4(N_{fj}), \mathcal{L}_{fj})$,
where $(\mathcal{U}(\mathbf{T}^{n-k}), \mathcal{L}_r)$ denotes the Lagrangian foliation in a tubular neighborhood of a regular Lagrangian $(n-k)$-torus in a symplectic $2(n-k)$-manifold, $(P_t^2(N_{st}), \mathcal{L}_{st})$ for $1 \leq t \leq i$ denotes a Lagrangian foliation associated to a codimension 1 nondegenerate surface singularity (= singularity on a symplectic 2-manifold), $(P^4(N_{ft}), \mathcal{L}_{ft})$ for $1 \leq t \leq j$ denotes a Lagrangian foliation associated to a focus-focus singularity on a symplectic 4-manifold, $i, j \geq 0, i + 2j = k$.

Definition 5.2 *A nondegenerate singularity N is called* of almost-direct-product type *topologically (or simply an* almost-direct-product singularity*) if the associated Lagrangian* *foliation* $(\mathcal{U}(N),\mathcal{L})$ *has the property that a finite covering of it is homeomorphic to a Lagrangian associated to a direct-product singularity.*

Notice if (m_1, m_2, m_3) is the Williamson type of N then in the above definition we have $i = m_1 + m_2, j = m_3$.

Theorem 5.3 *Any stable nondegenerate singularity* $(\mathcal{U}(N),\mathcal{L})$ *is of almost-direct product type. More precisely, it can be written (diffeomorphically) in the form of a quotient of a direct product singularity*
$(\mathcal{U}(\mathbf{T}^{n-k}),\mathcal{L}_r)\times(P_1^2(N_{s1}),\mathcal{L}_{s1})\times\ldots\times(P_i^2(N_{si}),\mathcal{L}_{si})\times(P_1^4(N_{f1}),\mathcal{L}_{f1})\times\ldots\times(P^4(N_{fj}),\mathcal{L}_{fj})$, *by a free action of a finite group* G. G *acts on the above product component-wise, i.e. it commutes with the projections onto the components.*

Proof (Sketch). We consider only the case when $k = n$. Consider the $l - type$ of N. Then for each pair of surfaces and/or 4-dimensional spaces U, V in this $l - type$, there is a natural action of the fundamental group of U on V. By taking covering, we can trivialize these actions step by step. Once these actions are trivial, the singularity is also of direct-product type. $\diamond$

A direct product with an action group in the above theorem will be called a *model* of a stable nondegenerate singularity $(\mathcal{U}(N),\mathcal{L})$. A model is called *minimal* if there does not exist a nontrivial element of G which acts trivially on all the components except one.

Proposition 5.4 *Suppose that* $k = n$. *Then there exists an unique minimal model for each stable nondegenerate singularity* $(\mathcal{U}(N),\mathcal{L})$. $\diamond$

Open question. If two stable singularities are such that their near-by singularities (of less codimension) are equivalent in a natural way, can we deduce that these two singularities are equivalent?

The answer seems to be YES, at least in the case when the components in their decompositions are rather simple. The meaning is that in concrete problems we can compute codimension 1 and codimension 2 singularities - a relatively easy task - and then deduce from that the structure of higher codimension singularities. Some computations were done in [10, 11] in this direction.

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