COMPATIBLE CONTACT STRUCTURES FOR INTEGRABLE HAMILTONIAN SYSTEMS

NGUYEN TIEN ZUNG

Abstract. This note contains some old unpublished simple results on compatible contact structures for integrable Hamiltonian systems

1. Introduction

This note is a slightly revised version of some unpublished notes/results that I have written around 1992-1994, concerning compatible contact structures for integrable Hamiltonian systems.[1]

The question is : given an integrable Hamiltonian system with two degrees of freedom, with a moment map $(H, F) : (M^4, \omega) \to \mathbb{R}^2$, restricted to a compact regular isoenergy 3-manifold $Q = \{H = const\}$, does there exist a contact structure on Q which is transversal to the Hamiltonian vector field $v = X_H$ of H ?

Of course, one can ask the above question without the system being integrable. The question will then be related to the convexity of the system. For example, if $H : T^*N \to \mathbb{R}$ is a Hamiltonian function on the cotangent bundle of a compact manifold N, which is convex on each cotangent space, then the Liouville form $\sum p_j dq_j$ will be a compatible contact form on $Q = \{H = const\}$ provided that the constant is large enough.

In the integrable case, we have some additional structures : the singular fibration by Liouville tori, and local symplectic $\mathbb{S}^1$-action. We will introduce the notion of almost-convexity for integrable systems, and show that if an integrable system (Q, ω, v) (restricted on an isoenergy 3-manifold Q) admits a contact structure which is invariant under some natural local $\mathbb{S}^1$-action if and only if it is almost-convex. This note also contains a result on necessary conditions for the existence of a non-locally-$\mathbb{S}^1$-invariant compatible contact structure, which may probably be enhanced to give a kind of "if and only if" theorem.

The space of fibers of the singular fibration of (Q, ω, v) by Liouville tori is a graph, which we denote by $Base(v)$ or $Base(Q, v)$ (the base space). To each edge of this base space, we will associate an *angle function*, and a number called the index (or winding number) of this edge. Almost-convexity is mainly a property of the angle functions, and the winding number is a kind of total angle. We will show that in the case when there is a compatible contact structure, if some index is large enough then this contact structure must be *overtwisted*.

Date: Old notes dated from 1992-1993.

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[1] The results of this note are also included in my PhD thesis, Strasbourg 1994. I'll not try to publish this note, but simply put it on the web for archival purposes.

2. Angle functions and indices

For simplicity, in this note we will assume that all singularities of the system (Q, ω, v) are topologically stable nondegenerate (in the sense of [6]), and all hyperbolic singular orbits are assumed to be normally-orientable.

Consider now an edge W of the base space $Base(v)$. Recall that, by Arnold-Liouville theorem, every point t of the edge W corresponds to a Lagrangian torus T^2_t, and there is a natural flat structure on this torus. Furthermore, v restricted to the torus T^2_t is a constant vector field.

Let us make some conventions about the orientations. On $M^4 = Q^3 \times (-\epsilon, +\epsilon)$ the positive orientation is given by $\omega \wedge \omega$. On Q^3 the positive orientation is the orientation for which the gradient of the Hamiltonian generating v is pointing outside (with respect to the orientation on M^4). Suppose that we fix on every edge of $Base(v)$ an orientation. Then on T^2_t the positive orientation with respect to the orientation on W is the orientation for which the product orientation on $T^2 \times W$, considered as an open subset of Q, is inverse to that induced from Q. For example, $M^4 = \mathbb{R}^2 \times \mathbf{T}^2 = \{(p_1, p_2, q_1, q_2), q_1, q_2 - \mathrm{mod}\, 1\}$ with the natural foliation by tori, $\omega = \Sigma dp_i \wedge dq_i, H = p_1, Q = \{H = 0\} \subset M$. The orientation on Q is given by the basis (q_2, q_1, p_2). Q has one edge W parametrized by p_2. Let p_2 give the positive orientation on W, then (q_1, q_2) gives the positive orientation on $\mathbf{T}^2$ with respect to it, since (q_1, q_2, p_2) gives the negative orientation on Q.

Take now some basis (γ_1, γ_2) of $\pi_1(T_t)$, which is presented by two simple closed curves of the same names, which are linear with respect to the flat structure of T_t. We can assume that the homotopy classes of $\gamma_1(t), \gamma_2(t)$ (in $\{\cup T_t, t \in W\}$) do not depend on t. Since every T_t is a flat torus, (γ_1, γ_2) gives us a linear basis in T_t. Let this basis be orthogonal, i.e. provide T_x with a conformal metric. Then we can talk about the algebraic angle $\angle(\gamma_1, v)$. Note that for every $t \in W$, since v is a constant vector field with respect to the flat structure of T_t, this angle is constant on T_t, hence we obtain an **angle function** $Ang := Ang_{(W,\gamma_1,\gamma_2)} : W \to \mathbb{R}/2\pi\mathbf{Z}, Ang(t) = \angle(\gamma_1(t), v(t))$. (The terminology may be not very good; in any case don't confuse it with angle functions in systems of action-angle coordinates). This function is obviously continuous, at least outside the singularities of the system v. Clearly, invariants of these angle functions (for example, their local min-maxs) are orbital invariants of the system v. We want to derive from these angle functions some numerical invariants, which will be called indices. The idea is very simple: these indices measure 'total geometric torsions' of systems on their edges (i.e. one-parameter families of Liouville tori).

An important observation is that angle functions as defined above behave well near singularities, hence on every edge W they can be considered as continous functions on a closed interval. More precisely, we have:

Proposition 2.1. *The angle function Ang(t) (for any fixed pair of cycles (γ_1, γ_2)) converges when $x \in W$ tends to an end of W (i.e. to a singularity). More precisely, this angle function converges to $\angle(\gamma_1, h)$ (mod 2π) in the hyperbolic case, and to some value different from $\angle(\gamma_1, e)$(mod π) in the elliptic case.*

Remark. This proposition does not hold if the system is not topologically stable.

The proof of the above proposition rests on the following canonical form of of nondegenerate singularities:

Proposition 2.2 ([6]). *If N is a connected singular level set of the additional integral F in (Q, ω, v), then a tubular neighborhood $\mathcal{U}(N)$ of N in $(M^4, \omega) = (Q \times (-\epsilon, +\epsilon), \omega)$ is symplectomorphic to $D^1 \times S^1 \times P^2$ with the symplectic form*

$$\omega = dx_1 \wedge dy_1 + \pi^*(\omega_1) \qquad (\star)$$

where x_1 is the coordinate on an interval D^1, y_1 (mod 1) *is the coordinate on the circle S^1, ω_1 is a symplectic form on a 2-surface P^2 with boundary, and π means the projection. Under this symplectomorphism, H and F become functions independent of y_1. A form $(\star)$ as above is called a* **canonical form**. $\diamondsuit$

Proof of Proposition 2.1. The existence of the limit does not depend on the choice of (γ_1, γ_2), so we will choose the basis as we like. According to the above proposition, we have the following canonical form for the symplectic form in a tubular neighborhood of a singularity in the thickened Q, i.e. $Q \times (\epsilon, -\epsilon)$:

$$\omega = dx_1 \wedge dy_1 + \pi^*(\omega_1)$$

with respect to decomposition of $\mathcal{U}_M(N)$ into $D^1 \times S^1 \times P^2$. Furthermore, the Hamiltonian H and first integral F are invariant with respect to y_1. The vector field $\partial/\partial y_1$ is Hamiltonian and periodic with period 1. In the elliptic case, P^2 is a disk with the usual area form on it, and H has the type $H = H(x_1, (x_2^2 + y_2^2))$ with $\partial H/\partial x_1 \neq 0$. The proof in this case is trivial, since the flows of x_1 and $\pi(x_2^2 + y_2^2)$ are periodic of period 1 and give rise to a basis in Liouville tori. Consider now the hyperbolic case. Let $\{O_i\}$ be the set singular points of F restricted to P^2. Because of the topological stability, we can assume that $(\partial F/\partial x_1)(O_i) = 0$ (cf. [6]). Let us look at one of the orbits of X_F in an annulus A in the singular level in $P^2 \times S^1$. Every such orbit is homoclinic with respect to the hyperbolic curve. Since $(\partial F/\partial x_1)(O) = 0$, i.e. the $\partial/\partial y_1$ component of X_F at O is 0, and F is nondegenerate in the transversal direction to closed singular orbits, this orbit does not wind around any of the hyperbolic orbits. In other words, the limit set of it in positive time is just one point, say E_1, in the hyperbolic curve, and in negative time is another point, say E_2. Now suppose $T_t^2, t \simeq 0$, is a Liouville torus in Q^3, which is near to the annulus A in the sense of Hausdorff. let γ_1 be the hyperbolic cycle in T_t^2, and γ_2 be the cycle given by $T_2 \cap \{y_1 = const\}$. Then by continuity, on T_t^2 the flow of X_F going from a point $E_1(t)$ near to E_1 will meet the circle $S_{E_1(t)} = \{(x_2, y_2) = (x_2(E_1(t)), y_2(E_1(t)))\} \subset T_t^2$ in the first time at a point $E_2(t)$ near to E_2. In the flat structure of T_t^2, homotopically the orbit of X_F from $E_1(t)$ to $E_2(t)$ can be written as $c\gamma_1 + \gamma_2$, where c tends to a finite number as $t \to 0$. The time for X_F to go from $E_1(t)$ to $E_2(t)$ will be denoted by $d(t)$. Note that $d(t) \to \infty$ as $t \to 0$. Now recall that $\partial/\partial y_1 = a\,dH + b\,dF$, and $a(O) \neq 0$ because $(\partial F/\partial x_1)(O) = 0$. It means that, homotopically, the orbits of the time 1 flow of X_H on T_t^2 are equivalent to

$$\frac{1}{a}\gamma_1 - \frac{b}{a}\frac{1}{d(t)}(c\gamma_1 + \gamma_2) = (\frac{1}{a} - \frac{b}{ad(t)})\gamma_1 - \frac{b}{ad(t)}\gamma_2$$

When $t \to 0$, it tends to $(1/a)\gamma_1$. It means that angle $\angle(X_H, \gamma_1)$ tends to 0 with respect to the basis (γ_1, γ_2). Thus we have proved our statement for one family of tori. For the other families of tori the proof is the same. $\diamondsuit$

To define the index on the edge W, we will need some canonical choice of the basis (γ_1, γ_2). Recall that there are 2 types of nondegenerate critical periodic orbits : elliptic and hyperbolic. For shortness, we will denote elliptic things by e and hyperbolic by h. Every vertex of $Base(Q, v)$ must be of one of the two above types. Thus there are 3 cases for the vertices of the edge W:

I. e —– e

II. e —– h

III. h —– h

Consider the first case: e —– e. (In this case Q, if it is connected, is a lens space and $Base(v)$ has only one edge). Then near every one of these two elliptic critical cycles there is a uniquely defined simple vanishing cycle in $\pi_1(T)$. Denote these vanishing cycles by e_1 and e_2 respectively. There are two subcases:

Ia. $e_1 = e_2$ up to the orientation. In this case we do as follows: Fix an orientation on W (It is easy to show that the definition of index does not depend on the choice of orientation on W). Take $\gamma_1 = e_1 = e_2$, and fix an arbitrary cycle γ_2 such that (γ_1, γ_2) is a positive basis on T^2_t (with respect to the orientation of W). In the cases below we will also require that the orientation given by (γ_1, γ_2) be positive with respect to the orientation of W.

Ib. $e_1 \wedge e_2 \neq 0$.

In the second case e —– h near the first end of W there is a vanishing cycle e_1, and near the second end of W there is a uniquely defined "hyperbolic" cycle h_2, which is homotopic to the critical hyperbolic periodic cycles of the second vertex of W. There are also two subcases:

IIa. $e_1 = h_2$ up to the orientation.

IIb. $e_1 \wedge h_2 \neq 0$.

In the subcases Ib, IIa, IIb, also take γ_1 to be e_1. Similarly, in the third case there are two subcases:

IIIa. $h_1 = h_2$ (up to the orientation).

IIIb. $h_1 \wedge h_2 \neq 0$.

In both subcases take γ_1 to be h_1.

γ_2 can be taken arbitrarily, since the definition below will not depend on it.

Instead of the angle function $Ang(t)$, it is more convenient for us to consider the "direction map" $Dir(t) := \exp(iAng(t))$. According to the above proposition, this map gives a continuous path in $S^1 \subset \mathbb{R}^2$, which is parametrized by [0,1] and smooth outside the end points. In the case Ib we have to add to this direction path a (smallest) arc in the circle from $Dir(1)$ to the point (-1,0), if $Dir(1)$ and (-1, 0) lie in the same side of the line generated by e_2, or from $Dir(1)$ to (1,0) if otherwise. We will call this new path $Dir(t)$ and forget about the old path in the case Ib.

Now define the index of W as twice a Maslov index of the direction curve in the plane with respect to the straight line given by γ_1. More precisely, let $\zeta(t)$ be a continuous piece-wise smooth curve $\zeta : [0, 1] \to S^1 = \{x^2+y^2 = 1\} \subset \mathbb{R}^2 = \{(x, y)\}$. Then the Maslov index of the path ζ with respect to the line $\{x = 0\}$ is given as follows: in case both $\zeta(0)$ and $\zeta(1)$ lie outside the line $\{x = 0\}$ the Maslov index is the algebraic number of times the curve crosses the line $\{x = 0\}$, i.e. points where it crosses $\{x = 0\}$ in anti-clockwise direction are counted as positive. Note that by homotopy theory this number is well-defined. If at least one of the ends of $\zeta(t)$ lies in $\{x = 0\}$, then perturb ζ a little bit in the anti-clockwise direction: $\zeta' = exp(i\epsilon)\zeta$, $\epsilon > 0$ small. Then the Maslov index is the algebraic number of times ζ' crosses the line $\{x = 0\}$, if $\zeta(0), \zeta(1) \in \{x = 0\}$, this number plus 1/2 if only $\zeta(0) \in \{x = 0\}$, or this number minus 1/2 if only $\zeta(1) \in \{x = 0\}$. With the above definition of Maslov index, we have:

Definition 2.3. *Twice the Maslov index of the path $Dir(t), t \in W$, with respect to the line given by γ_1, is called* index *of the system v on the edge W.*

Remarks. 1) It follows from the definition that in the cases Ia ,IIb, IIIa, index is an even number. In the other cases index is an odd number. Aside from this fact, any choice of indices is admissible, i.e. there will be a system with the given base space and given indices. 2) Indices depend essentially on the orientation of the system. If we turn a system upside down (replace v by $-v$), then all the indices will get the opposite sign. 3) Though we said from the beginning of the paper that all hyperbolic orbits are assumed to be normally-orientable, Proposition 2.1 and Definition 2.3 hold without modification also for the case when there are normally-nonorientable hyperbolic orbits.

Examples. 1) The base space of the harmonic oscillator with the moment map $(H = 1/2(x_1^2 + y_1^2 + x_2^2 + y_2^2), F = x_1^2 + y_1^2)$, on $Q = S^3 = \{H = 1\}$ in $\mathbb{R}^4$, has only one edge, and this edge has index 1. 2) Integrable geodesic flows of Liouville metrics on a 2-dimensional Riemannian torus have all indices equal to zero (conjecture).

3. Compatible contact structures

A **compatible contact structure** on (Q, ω, v) is a contact 1-form α, defined up to multiplication with positive functions on Q, such that $\alpha(v) > 0$ and $\alpha \wedge d\alpha$ gives the positive orientation on Q. Recall that in a tubular neighborhood of each singularity of (Q, ω, v) there is a natural free Hamiltonian S^1-action which preserves the Liouville tori, cf. Proposition 2.2 or [5, 6]. Outside singularities we have a natural $S^1 \times S^1 = \mathbf{T}^2$-action, which can be thought of as a large set of S^1-actions. So every point of Q lies under at least one natural S^1-action. We call the set of all that S^1-actions a **local S^1-action on** Q (with respect to a given choice of small neighborhoods of singularities). Thus for every triple (Q, ω, v)there is a natural local S^1 action associated with it. Therefore we are interested in 2 questions: how to construct compatible contact structures, and how to construct compatible contact structures which are invariant under a local S^1-action.

In $\mathbb{R}^{2n}$, star-shaped hypersurfaces are contact. An analogous situation holds in our case. We will define the notion of **almost-convex** systems and show that such systems admit compatible contact structures.

Fix an orientation on every edge of $Base(Q, v)$, and fix a positive basis $(\gamma_1(t), \gamma_2(t))$ on every Liouville torus, so that tori over the same edge have the same basis up to homotopy (as in Section 2). Associate to first end and second end (with respect to the orientation) of any edge W the real numbers $A_1(W)$ and $A_2(W)$, called angles, respectively, so that $A_1(W) < A_2(W)$. We say that these angles satisfy Property (P) if the following conditions hold:

> 1) Near every elliptic singularity, the angle of the only one end-of-edge contiguous to it, say $A(W)$, is equal to $\angle(\gamma_1(W), e)$ modulo 2π, where e is a positive vanishing cycle, positive means that (e, X_H) gives rise to a positive basis.
> 2) Near every hyperbolic singularity, say N, suppose $A(W_1), \dots, A(W_k)$ are angles of end-of-edges contiguous to it, and $s_1, \dots, s_k$ are their "sign": s_i equals +1 if $A(W)$ is $A_1(W)$ and -1 if $A(W)$ is $A_2(W)$. Denote by e_i the positive boundary cycles of 2-surface P^2 in the canonical form $(\star)$ (as in Proposition 2.2) associated to this singularity: $\partial P^2 = \Sigma e_i$ (recall that P^2 has a positive orientation given by the symplectic form).

Every angle $A(W_i)$ is given (mod. 2π) by a direction in the flat structure on a torus $T_t, t \in W$. Denote this direction by g_i in case $s_i = 1$, or the inverse of this direction by g_i in case $s_i = -1$. We require that g_i can be given by in the flat structure by $e_i + a_i\, h$ for some numbers a_i, where h means a common positive hyperbolic cycle for all the contiguous edges. Now property (P) says that $\Sigma a_i = 0$.

The meaning of the above property will be clear in the construction of locally S^1-invariant contact structures. With the notations as in the previous section, we have the following definition.

Definition 3.1. *A system (Q, ω, v) is called* **almost-convex** *if, for some choice of orientation of edges in $Base(v)$ and positive bases $(\gamma_1(W), \gamma_2(W))$, there are angles $A_1(W), A_2(W)$ satisfying the property (P), so that for every edge W, identified with the interval (0,1) via a homeomorphism, the following conditions hold:*
1. $Ang(t) - Ang(t') > -\pi$ if $t > t'$
2. $min_{[0,1]} Ang(t) > A_1(W) > A(0) - \pi$
3. $max_{[0,1]} Ang(t) - \pi < A_2(W) < A(1)$

Definition 3.2. *A system (Q, ω, v) is called* **convex** *if all the indices are positive, and for some choice of orientation of edges in $Base(v)$ and positive bases $(\gamma_1(W), \gamma_2(W))$, angle functions are monotone strictly increasing on every edge.*

It is clear that Definition 3.1 does not depend on the choice of bases, i.e. if angle functions are increasing with respect to some choice of bases, then they are so with respect to any other choice. The same is true for Definition 3.2. Just note that if we change orientation on some edge W and change the basis (γ_1, γ_2) over it to (γ_2, γ_1), then $Ang(t)$ changes to $\pi/2 - Ang(t)$, and $A_1(W), A_2(W)$ can be changed to $-A_1(W) - \pi/2, -A_2(W) - \pi/2$ respectively. It is also easy to see that if some system is convex, then it is almost-convex.

Recall that we can write $Q = \bigcup U_j$, where each U_j is a connected saturated open subset of Q, such that U_1 does not contains any hyperbolic singular fiber, and each U_j with $j > 1$ contains exactly one hyperbolic singular fiber of the singular fibration by Liouville tori. On U_1 we have a local torus action (near each fiber) which preserves the system, and on each U_j we have an S^1-action [6]. We will say that a contact structure on (Q, ω, v) is *locally $\mathbb{S}^1$-invariant*, if it is invariant under the above local $\mathbb{S}^1$-actions and torus actions for some covering $Q = \bigcup U_j$ of Q.

Theorem 3.3. *Each almost-convex system (Q, ω, v) admits a compatible locally $\mathbb{S}^1$-invariant contact structure.*

Proof. We will construct a contact 1-form at singularities first and then extend it along families of nonsingular tori to get a global contact structure.

Near every elliptic singularity N, simply take a canonical contact 1-form in a neighborhood of a transversal simple closed curve, which is in our case an elliptic periodic orbit: $\alpha = dy_1 + \frac{1}{2}(xdy - ydx)$, where y_1 is the periodic coordinate (mod 1) in a canonical form $(\star)$ (cf. Proposition 2.2), x, y- Euclidean coordinates in a disk $D^2 = \mathcal{U}_Q(N) \cap \{y_1 = const\}$. Note that α in this case can be made $S^1 \times S^1 = \mathbf{T}^2$-invariant. Also, it is clear that $\alpha(X_H) > 0$ if $\mathcal{U}_Q(N)$ is small enough, since at every (fixed) elliptic orbit X_H is equal to ∂y_1 times a positive number

For every hyperbolic singularity N, suppose it has k contiguous end-of-edges. Recall that we have a set of numbers $a_1, \dots, a_k$, which corresponds to the set

of boundary components $\zeta_1, \dots, \zeta_k$ of $P^2 = \mathcal{U}_Q(N) \cap \{y_1 = const\}$ under some canonical form ($\star$), and $\Sigma a_i = 0$. Set $\alpha = dy_1 + \beta$, where β is some one-form on P^2 with the following properties:
a) $d\beta$ is a positive nondegenerate (area) form on P^2. Recall that P^2 is a symplectic submanifold, thus it has a predefined positive orientation.
b) For every $i = 1, \dots, k$, $\beta(\zeta_i) - a_i$ is positive and small (as small as we want).
c) Near the boundary of $\mathcal{U}_Q(N)$ (for some $\mathcal{U}_Q(N)$), α is $\mathbf{T}^2$-invariant.
d) α is compatible: $\alpha(X_H) > 0$.

Properties a), b), c) can be achieved simply by Poincaré's Lemma. Property d) can be achieved as follows: Take any $\alpha_1 = dy_1 + \beta_1$ which satisfies a), b), c). Denote by $g^s_{X_F}$ the flow with time s of vector field X_F of an additional integral F chosen as in Proposition 2.1. Then set

$$\alpha = dy_1 + \int_{-S}^{S} (g^s_{X_F})^* \beta_1 \, ds.$$

One can check that, for S big enough, the above form α will satisfy also the property d).

Now we have to extend the above constructed contact compatible S^1-invariant 1-form α near the singularities to a contact 1-form on the whole Q, making it $\mathbf{T}^2$-invariant outside singularities.

For every edge W, identify it with the interval (0,1) and write a $\mathbf{T}^2$-invariant 1-form α which we are looking for in the following way:

$$\alpha = a_1 dq_1 + a_2 dq_2 + a_3 dt,$$

where q_1, q_2 are coordinates with respect to the chosen basis (γ_1, γ_2), $t \in (0+\epsilon, 1-\epsilon)$ and a_1, a_2, a_3 - functions, depending only on t (ϵ positive small, since we consider only outside singularities).

The compatibility yields that α does not vanish when restricted to the tori $T^2 \times \{point\}$. In other words, at every point at least one of two functions a_1, a_2 does not vanish. We can assume that $a_1 \neq 0$ (locally). Then

$$\alpha \wedge d\alpha = dq_2 \wedge dq_1 \wedge dt(\frac{da_2}{dt} a_1 - \frac{da_1}{dt} a_2),$$

and positive contactness of α means that

$$(\frac{da_2}{dt} a_1 - \frac{da_1}{dt} a_2) > 0$$

It is equivalent to the fact that the angle $\mathrm{arctg}(a_2/a_1)$ increases when t increases. Denote by $e_\alpha(t) \propto a_2(t)\partial/\partial q_1 - a_1(t)\partial/\partial q_2$ the characteristic direction field given by contact the form α on $T^2(t)$. Then the positive contactness of α is also equivalent to the fact that the angle $\angle(\gamma_1, e_\alpha(t))$ is increasing with t.

The compatibility of α now means that $\angle(\gamma_1, e_\alpha(t))$ can be chosen so that

$$\angle(\gamma_1, e_\alpha(t)) < Ang(t) < \angle(\gamma_1, e_\alpha(t)) + \pi.$$

In other words, it is sufficient and necessary to chose an angle function $B(t) = \angle(\gamma_1, e_\alpha(t))$ so that $B(t) < Ang(t) < B(t) + \pi$ and $B(t)$ satisfy some "boundary conditions": it can be made to be equal to the value given by a constructed before 1-form α near the singularities. These boundary conditions mean exactly that $B(1-\epsilon) - A_2(W)$ is positive and near to zero, $B(0+\epsilon) - A_1(W)$ is negative and near to zero. In other words, we should be able to extend B to a monotone increasing angle function on [0,1] such that $B(0) = A_1(W), B(1) = A_2(W)$, and

$B(t) < Ang(t) < B(t) + \pi$ for all $t \in [0, 1]$. Such an angle function B can be constructed as follows: for $t \in [0, \epsilon)$, $B(t)$ is a linear function going from $A_1(W)$ to $\min\{\min_{[0,1]} Ang(s), A_2(W)\} - \epsilon_1$ for some small positive ϵ_1. For $t \in [\epsilon, 1]$, $B(t)$ has the form $B(t) = \min\{\min_{[t,1]} Ang(s), A_2(W)\} - \epsilon_2(t)$, with some small correcting function ϵ_2, $\epsilon_2(1) = 0$, such that $B(t)$ is continuous strictly increasing. $\diamondsuit$

Remark. Lutz [4] constructed all the contact structures invariant with respect to some group action of S^1. One can easily show that every contact structure of this type is compatible for some system. What we have constructed above may be viewed a generalization of [4].

Analogously, one can show that the converse statement to the above theorem is also true:

Proposition 3.4. *If a system (Q, ω, v) admits a locally S^1 invariant compatible contact structure, then it is almost convex.* $\diamondsuit$

If contact compatible structures are not required to be locally S^1- invariant, then their behavior can be rather complicated, especially at singularities of the systems. Anyway, some conditions near singularities and along the Lagrangian tori, not much weaker than almost-convexity, are still necessary. Here we will give a result only about conditions along Lagrangian tori.

Proposition 3.5. *If a system (Q, ω, v) admits some compatible contact structure, then for any angle function $Ang(t)$ on any edge of $Base(Q, v)$ we have that $Ang(t) - Ang(t') > -\pi$ if $t > t'$.*

Proof. Suppose we have a compatible contact 1-form α with contact distribution $\xi = \{\alpha = 0\}$ over the edge W of (Q, ω, v). Identify W with the interval (0,1). Then for every $t \in (0, 1)$ the tangent planes of the torus $T^2(t)$ intersects with ξ transversally, and give rise to a characteristic direction distribution $\zeta(x) \in T_x T^2(t), x \in T^2(t)$. This direction is, generally speaking, non-constant with respect to the torus action on $T^2(t)$, unlike the locally S^1-invariant case. Still we can define an "average direction" of $\zeta(x)$ over $T^2(t)$ as follows. Note that $\zeta(x)$ is transversal to $X_H(x)$. So it is also transversal to some constant vector field $X(x)$, which is near to $X_H(x)$, and which has periodic orbits. Define a basis (γ_1, γ_2) on $T^2(t)$, such that γ_2 is parallel to the above periodic orbits, and denote (q_1, q_2) the coordinates system corresponding to this basis. (q_1, q_2) will be an Euclidean system of coordinates for $\mathbb{R}^2$, which is $\mathbf{Z}^2$-covering of $T^2(t)$. In these coordinates, $\zeta(x)$ can be given by the vector field of the form $\partial/\partial q_1 + a(q_1, q_2)\partial/\partial q_2$, where $a(q_1, q_2)$ is some real function which is periodic with period 1 in both variables. Suppose $g^s(q_1, q_2) = (q_1 + s, g_s(q_1, q_2))$ is a time s map of the flow of the above vector field. Then take the limit

$$\lim_{s \to \infty} g_s(q_1, q_2)/s.$$

One can check that this limit exists and does not depend on the initial value (q_1, q_2). Denote it by $a(t)$ and denote by $\zeta(t)$ the direction given by $\partial/\partial q_1 + a(t)\partial/\partial q_2$. $\zeta(x)$ changes smoothly with respect to t, so $\zeta(t)$ is a continuous function on t. From the construction it is also clear that $\zeta(t)$ is transversal to X_H. Denote by $B(t)$ the angle $\angle(\gamma_1, \zeta(t))$ with respect to some basis (γ_1, γ_2) of $\pi_1(T)$ over the edge W. Then Proposition 3.5 is a consequence of the following lemma:

Lemma 3.6. *$B(t)$ is a monotone increasing function.*

Proof. As before, we can assume that $\alpha = a_1 dq_1 + a_2 dq_2 + a_3 dt$. Then

$$\alpha \wedge d\alpha = [(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1})a_3 + (\frac{\partial a_3}{\partial q_1}a_2 - \frac{\partial a_3}{\partial q_2}a_1) + (\frac{\partial a_2}{\partial t}a_1 - \frac{\partial a_1}{\partial t}a_2)]dt \wedge dq_2 \wedge dq_1.$$

Positively contactness of α means that

$$(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1})a_3 + (\frac{\partial a_3}{\partial q_1}a_2 - \frac{\partial a_3}{\partial q_2}a_1) + (\frac{\partial a_2}{\partial t}a_1 - \frac{\partial a_1}{\partial t}a_2) > 0$$

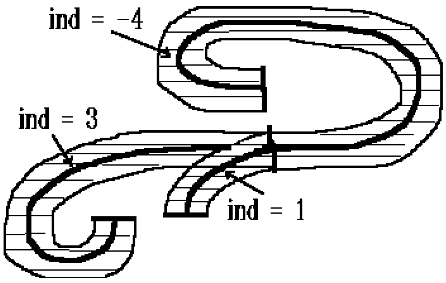
Fix any $t_0 \in (0,1)$. Near $T^2(t_0)$, define new form $\alpha' = a'_1 dq_1 + a'_2 dq_2 + a'_3 dt$ such that $\alpha' \wedge d\alpha' = 0$ and restrictions of α and α' on $T^2(t_0)$ coincide. It implies that (near to $T^2(t_0)$),

$$(\frac{\partial a_2}{\partial t}a_1 - \frac{\partial a_1}{\partial t}a_2) > (\frac{\partial a'_2}{\partial t}a'_1 - \frac{\partial a'_1}{\partial t}a'_2).$$

Consequently, at $T^2(t_0)$ we have $\frac{\partial}{\partial t}\frac{a_2}{a_1} > \frac{\partial}{\partial t}\frac{a'_2}{a'_1}$. It follows that, for $t > t_0$ and near to t_0, the average angle given by α is greater than that one given by α'. But since the plane distribution given by α' is integrable (by Frobenius's theorem), the average angle given by it does not depend on t (because of the conjugacy). Thus we have $B(t) > B_{\alpha'}(t) = B_{\alpha'}(t_0) = B(t_0)$ for $t > t_0$, t near to t_0. Analogously, $B(t) < B(t_0)$ for $t < t_0$, t near to t_0. $\diamondsuit$

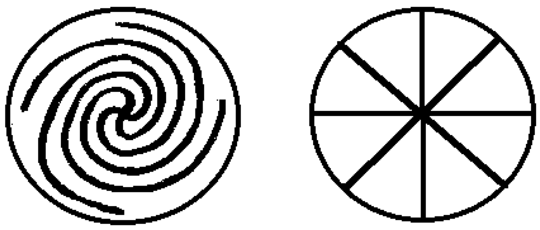
Corollary 3.7. *If a system (Q, ω, v) has a edge with index smaller or equal to -2, then it does not admit any compatible contact structure.* $\diamondsuit$

Example. The following figure shows an integrable system (on $Q^3 \times (-\epsilon, \epsilon)$), given in the form of orbit space, with an edge with highly negative index. (The lines there represent the level sets of a action function). Thus this system cannot admit any compatible contact structure.



Remarks. 1) From Theorem 3.3 it is easy to construct an example of a stable system, which admits a compatible contact structure, and which has an edge of index -1. 2) From the results of Eliashberg [3] and Lutz [4], one sees that any contact structure on S^3 is invariant under some S^1-action. It is interesting to know wether every contact structure on a graph 3-manifold is invariant under some local S^1-action or not. 3) The notion of convexity can be generalized to the case of higher dimensional integrable systems. In the same vein, one can show that convex integrable systems with many degrees of freedom admit a natural compatible locally S^1-invariant contact structure. For example, in this way one can recover the standard contact structure on the sphere S^{2n-1} by using a multi-component harmonic oscillator.

The following definition of overtwisted contact structure appeared first in the work of Bennequin [1]. Suppose (Q^3, ξ) is a contact 3-manifold, and $P^2 \subset Q$ is an arbitrary embedded surface (may be with boundary) in it. Then the intersection of ξ with the tangent planes of P^2 gives rise to a singular line field on P^2. The foliation given by this line field is called a characteristic foliation on P^2. It is an unique foliation by Legendrian curves on P^2. A contact 3-manifold (Q^3, ξ) is called **overtwisted** if it contains an embedded disk with the characteristic foliation as shown in the left hand side of the following figure, or equivalently, in the right hand side. (notice that the boundary itself is a Legendrian curve in both cases). It is not difficult to construct explicit examples of overtwisted contact structure on any orientable 3-manifold. Bennequin [1] showed that if a contact structure in S^3 is overtwisted, then it is non-standard.



a) **b)**

Following Eliashberg [2], we say that a contact 3-manifold **bounds symplectically** if it is a contact type boundary for some symplectic compact 4-manifold, and **bounds in complex terms** if it is a quasiconvex boundary for some almost complex compact 4-manifold, and the contact structure is given by complex lines. Of course, if something bounds symplectically, then it also bounds in complex terms. Using techniques of filling by holomorphic curves, Eliashberg [2] showed that, f a contact 3-manifold is overtwisted, then it does not bound (in complex terms). Using Eliashberg's result, we can prove the following :

Proposition 3.8. *If a system (Q, ω, v) is almost convex, and there is a final edge of $Base(v)$, i.e. an edge which is contiguous to at least one elliptic singularity, with index greater than 2, then (Q, ω, v) admits an overtwisted compatible contact structure. In particular, (Q, ω) does not bound.*

Proof. We will show that the contact structure constructed in Theorem 3.3 in this case is overtwisted, by detecting an overtwisted disk.

Suppose v is convex, edge W of $Base(v)$ has index greater than 2, and the first end of W is elliptic, i.e. $B(0) = A_1(W) = \angle(\gamma_1, e)$. From our assumptions, it follows that there exists a (unique) point $t_0 \in (0,1)$ such that $B(t_0) = B(0) + \pi$. Consider the solid torus $\Pi = \cup_{t=0}^{t_0} T^2(t)$. By changing coordinates in a $\mathbf{T}^2$- equivariant way, we can assume that $t_0 = \pi$ and the contact structure can be given as follows:

$$\alpha = \cos t \, dq_2 + t \sin t \, dq_1$$

Here (q_1, q_2) are coordinates on tori, $\{q_2 = const\} \cap T^2(t)$ are vanishing cycles, and $t \in [0, \pi]$. In these coordinates, the disk $D^2 = \{q_2 = 0\} \subset \Pi$ is an overtwisted disk, i.e. it has a characteristic foliation as in the previous figure. $\diamondsuit$

Example. Let $(M, \omega) = (\mathbb{R}^4, \omega_0)/\mathbf{Z}$ where the group action of $\mathbf{Z}$ is generated by $y_2 \to y_2 + 1$. Consider the Hamiltonian $H = (x_1^2 + y_1^2)^2 + x_2^2$. Then one can easily see that the restriction of X_H on every positive energy surface is a up-cobordant-to-0 system on $S^1 \times S^2$, whose base space has only one edge, and with index 2. This example shows that the assumption *greater than 2* in Theorem 3.8 is necessary.

Corollary 3.9. *There are non-degenerate integrable systems* (Q, ω, v)*, which are not a boundary of any integrable system on a compact symplectic 4-manifold.* $\diamondsuit$

Actually, there are plenty of such systems, since we can choose our angle functions rather arbitrarily. In particular, we can chose them to be monotone, with indices as high as we want.

Similarly, we have :

Proposition 3.10. *If a system* (Q, ω, v) *is convex, and there is an edge of* $Base(v)$ *with index greater than 6, then it admits an overtwisted compatible contact structure.*

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GTA, UMR 5030 CNRS, Département de Mathématiques, Université Montpellier II
E-mail address: `tienzung@math.univ-montp2.fr`