To A.T. Fomenko on his 50^{th} birthday

A topological classification of integrable Hamiltonian systems

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Abstract: This paper is an introduction to a new theory of topological classification of finite-dimensional integrable Hamiltonian systems.

1 Introduction

This paper arises from 2 talks that I gave in Montpellier in February 1995 (one in the seminar Gaston Darboux and the other one in the conference "Nonlinearity and Integrability: From Mathematics to Physics"). It is intended to serve as an introduction to a new theory of topological classification of integrable Hamiltonian systems. The material presented here is based on two more technical papers [19, 20].

The topological (qualitative) study of integrable Hamiltonian systems has attracted many mathematicians around the world. It is an interesting problem in its own right, and it also has direct applications in symplectic geometry, KAM and Poincaré-Melnikov theories, celestial mechanics, etc. (We know that our solar system is more or less integrable). Integrable systems are also a special case of systems with symmetries, whose topological study was initiated by Smale [17], using the moment map. Since then, may authors have computed the bifurcation diagram of the moment map, for a large number of classical integrable systems.

The first and most fundamental result about the qualitative behavior is the celebrated Arnold-Liouville theorem, which classifies integrable systems locally near nonsingular tori. The globalization of this theorem for systems without singularities was achieved by Nekhoroshev [16], Duistermaat [9] and Dazord and Delzant [7]. However, realistic systems always have singularities. Fomenko and his collaborators (e.g., [4, 10, 11]) developed a Morse-type topological theory for integrable systems, to take into account singularities. However, he studied only codimension 1 singularities, and hence his theory works best only for systems with two degrees of freedom restricted to isoenergy 3-manifolds. Recently, we have obtained a topological classification of nondegenerate singularities of any codimension, which makes possible a generalization of Fomenko's theory to higher dimensions.

Our theory of topological classification of nondegenerate integrable systems, which may be regarded as a nontrivial generalization of Fomenko's theory, is based on the following 3 kinds of topological invariants: structure of nondegenerate singularities (in terms of normal forms), global monodromy, and singular Chern classes. Each integrable Hamiltonian system, under some mild assumptions, has an associated singular Lagrangian torus foliation, and the orbit (base) space is a stratified integral affine manifold. Our main result is the following (cf. [20]):

Theorem 1.1 *Two nondegenerate integrable Hamiltonian systems are topologically equivalent (as singular torus foliations) if and only if there is a homeomorphism between the orbit spaces, which also maps singularities to singularities, global monodromy to global monodromy and singular Chern classes to singular Chern classes isomorphically.*

The notion of global monodromy is a nontrivial generalization of monodromy as defined by Duistermaat [9] to the general case. The notion of singular Chern class is also a generalization of Duistermaat's Chern class to the case with singularities. In case there are only elliptic singularities, our definition of singular Chern classes coincides with that given earlier by Boucetta and Molino [5].

The organization of this paper is as follows. In the first half of the paper we recall the well-known classification of integrable Hamiltonian systems without singularities (§3), which is based on Arnold-Liouville theorem (§2), affine monodromy (§4), and Duistermaat-Chern class (§5). In §6 we discuss the structure of nondegenerate singularities. In §7 and §8 we give the notions of global monodromy and singular Chern classes, respectively. As a conclusion, in §9 we give some remarks on some related results and problems.

Acknowledgments. This work is done mainly during the author's visit to Montpelllier in February 1995. He would like to thank Jean-Paul Dufour and Pierre Molino for the warm hospitality and for many stimulating discussions.

2 Arnold-Liouville theorem

Let (M^{2n}, ω) be a smooth symplectic manifold, and $H : M^{2n} \to \mathbb{R}$ a smooth Hamiltonian function. The Hamiltonian system $\dot{x} = X_H(x)$ on M^{2n} (defined by $i_{X_H}\omega = -dH$) is called *integrable* in the sense of Liouville if there are n commuting first integrals $F_1, ..., F_n$ which are functionally independent almost everywhere: $\{F_i, H\} = \{F_i, F_j\} = 0; dF_1 \wedge \cdots \wedge dF_n \neq 0$ *a.e.*. One can put $F_1 = H$. The map $\mathbf{F} = (F_1, ..., F_n) : M^{2n} \to \mathbb{R}^{2n}$ is called the *moment map.* Of course, for a given integrable Hamiltonian H, this moment map is not unique. However, under the nonresonance condition (see below), the regular level sets of this moment map are uniquely determined by the system, so we can also fix a moment map.

We will always assume that *the level sets of the moment map are compact* (without this assumption the Arnold-Liouville theorem may fail). Let $\Sigma = \{y \in \mathbb{R}^n | \ \exists x \in \mathbf{F}^{-1}(y), \operatorname{rank} d\mathbf{F}(x) < n\}$ be the *bifurcation diagram.* Put $M_0 = M \setminus \mathbf{F}^{-1}(\Sigma)$, and denote by O_0 the space of connected components of the regular level sets (i.e. each component is considered as one point). Then we have a natural projection $\pi : M_0 \to O_0$, and the map $\mathbf{F} : M_0 \to \mathbb{R}^n$ can be factored through this projection to a map $\tilde{\mathbf{F}} : O_0 \to \mathbb{R}^n$.

The following *Arnold-Liouville theorem* is an analog of Darboux's theorem in symplectic geometry, and it gives the normal form for an integrable system near a regular level set of the moment map.

Theorem 2.1 *$\pi : M_0 \to O_0$ is a regular Lagrangian T^n-torus fibration. Moreover, for each $y \in O_0$ there is a small neighborhood $D^n = D(y)$ of y in O_0 such that $(\pi^{-1}(D^n), \omega)$ can be written as $(D^n \times T^n, \sum_1^n dp_i \wedge dq_i)$ via a fibration preserving symplectomorphism, where (p_i) is a system of coordinates in D^n, $(q_i \bmod 1)$ is a system of periodic coordinates in T^n, and the fibration of $(D^n \times T^n, \sum_1^n dp_i \wedge dq_i)$ into Lagrangian tori is the projection $D^n \times T^n \to D^n$.*

p_i and q_i are called *action* and *angle* coordinates. Each torus of the fibration $\pi : M_0 \to O_0$ is called a *Liouville torus.*

Consequently, in $\pi^{-1}(D^n)$, H (and F_i) is a function of the action coordinates p_i only, $H = H(p_i)$, and the Hamiltonian system has the form

$$\begin{cases} \dot{p}_i = 0 \\ \dot{q}_i = \partial H / \partial p_i \end{cases}$$

In particular, the motion of the system is quasi-periodic in each Liouville torus.

The fact that $M_0 \to O_0$ is a torus fibration and the system is quasi-periodic in each torus was known to Liouville. Action-angle coordinates were found by Arnold (e.g., [1]), under some additional assumptions. The theorem was then reproved by several people, including Jost, Markus, Meyer, Nekhoroshev. For a modern treatment see e.g. Dazord-Delzant [7].

It follows from Arnold-Liouville theorem that each Liouville torus has a natural *flat structure.* But a more important consequence of Arnold-Liouville theorem is that the (regular part of the) orbit space O_0 is an *integral affine manifold.* That is, it admits an atlas whose transformation maps are affine with the integral linear part. In our case, the charts are given by small open sets with local action coordinates (p_i).

If for some point $y \in O_0$ the numbers $\partial H/\partial p_i(y)$ are non-commensurable (i.e. their $\mathbb{Z}$-span is a subgroup in $\mathbb{R}$ which is isomorphic to $\mathbb{Z}^n$), then each orbit of the Hamiltonian flow in the torus $T^n_y = \pi^{-1}(y)$ is dense. Thus this torus is uniquely determined by X_H (and a point on it), regardless of the moment map $\mathbf{F}$. The Hamiltonian system X_H is called *non-resonant* if for almost every point $y \in O_0$ the numbers $\partial H/\partial p_i(y)$ are non-commensurable. For example, if H satisfies the *Kolmogorov condition*: $\det(\partial^2 H/\partial p_i \partial p_j) \neq 0$ *a.e.*, then H is nonresonant. Clearly, nonresonant condition is a kind of general position condition. However, its verification for concrete systems may be not so simple (see e.g. [12]), and it may even fail for some integrable systems (e.g. the so called non-commutatively integrable systems (cf. [15]) are resonant).

From now on we will assume all integrable systems under consideration to be nonresonant, so that the singular foliation given by the connected components of the level sets of the moment map does not depend on the moment map itself, but only on the initial Hamiltonian.

3 Global action-angle coordinates

Suppose now that we have a regular Lagrangian torus fibration $\pi : (M_0^{2n}, \omega) \to O_0^n$. It is a natural geometrical setting of integrable systems without singularities, because any two functions $f_1, f_2 : O_0^n \to \mathbb{R}$ Poisson-commute if considered as function on M^{2n} (i.e. $\{f_1 \circ \pi, f_2 \circ \pi\} = 0$), and any Hamiltonian function $H = h \circ \pi, h : O_0^n \to \mathbb{R}$, is integrable.

We can ask if there are global action-angle coordinates. That is, can $(M_0, \omega) \to O_0^n$ be written in the form

$$(O_0^n \times T^n, \sum_1^n dp_i \wedge dq_i) \to O_0^n$$

where $(p_i) : O_0^n \to \mathbb{R}^n$ is an immersion, $q_i \bmod 1$ are periodic coordinates on T^n.

More generally, we can ask for a classification of such fibrations $\pi : (M_0^{2n}, \omega) \to O_0^n$, assuming that O_0^n is known.

A natural way to solve the above problem is via *obstruction theory.* If $(M_0, \omega) \to O_0^n$ admits a global action-angle coordinate system, then it has the following properties:

a) $M_0 \to O_0$ is a principal $\mathbb{T}^n$-bundle (where $\mathbb{T}^n$ is considered as an Abelian Lie group).

b) $M_0 \to O_0$ has a global section.

c) Moreover, it has a *Lagrangian* global section.

Conversely, if the above conditions are satisfied then one can show easily that $\pi : (M_0^{2n}, \omega) \to O_0^n$ admits global action-angle coordinates.

The obstruction for condition a) to be fulfilled is called the *(affine) monodromy* and is considered in §4. It will be shown that the monodromy, besides of being a topological invariant, can also be determined from the affine structure of the base space O^n_0 alone. Hence the adjective affine. The obstruction to b) is called the *Duistermaat-Chern class*. The obstruction to c) is called the *Lagrangian Duistermaat-Chern class*. They will be considered in §5

Even without knowing explicitly what are the affine monodromy and Duistermaat-Chern class, we can write down the following natural theorems (due to Duistermaat [9] and Dazord and Delzant [7]):

Theorem 3.1 *If the topological structure of O^n_0 is known, then $\pi : (M^{2n}_0, \omega) \to O^n_0$ is classified topologically (as a topological fibration) by the affine monodromy and the Duistermaat-Chern class.*

Theorem 3.2 *If the affine structure of O^n_0 is known, then $\pi : (M^{2n}_0, \omega) \to O^n_0$ is classified geometrically (with the symplectic form) by the Lagrangian Duistermaat-Chern class.*

If O^n_0 is 2-connected, then there is no room for the monodromy and Lagrangian Duistermaat-Chern class, and one obtains the following result due to Nekhoroshev [16]:

Corollary 3.3 *If $\pi_1(O^n_0) = \pi_2(O^n_0) = 0$, then there is a unique Lagrangian torus fibration over O^n_0, and it admits global action-angle coordinates.*

4 Affine monodromy

Affine monodromy was first defined by Duistermaat [9]. As in §3, consider a Lagrangian torus fibration $\pi : (M_0, \omega) \xrightarrow{T^n} O^n_0$. One has an associated vector bundle

$$E \xrightarrow{H_1(T^n,k)} O_0$$

where k can be $\mathbb{Z}$ or $\mathbb{R}$ (or something else). On this vector bundle there is a unique natural locally flat connection, called the *Gauss-Manin connection* (e.g., [2]). Indeed, each (first) homology class in a fiber of $\pi : M_0 \xrightarrow{T^n} O^n_0$ can be moved in a unique way homologically to a homology class in any nearby fiber, and that moving defines the flat connection. The *monodromy* is defined as the holonomy of this connection, and is an element of $\mathrm{Hom}(\pi_1(O_0), GL(n, \mathbb{Z}))$.

From the definition it is clear that the monodromy is a topological invariant. We will now show that it is also an invariant of O_0 as an integral affine manifold. Hence the adjective *affine*.

Indeed, the vector bundle $E_{\mathbb{R}} \xrightarrow{H_1(T^n,\mathbb{R})} O_0$ can be identified with the bundle of constant vector fields on the fibers of $M_0 \xrightarrow{T^n} O_0^n$. If X is a constant vector field on $T_y^n, y \in O_0$, then $\alpha(X) = -\omega(X,.)$ can be identified with an element of $T_y^* O_0$, and the map $X \mapsto \alpha(X)$ is an isomorphism. Hence $E_{\mathbb{R}} \xrightarrow{H_1(T^n,\mathbb{R})} O_0$ is isomorphic to the cotangent bundle T^*O_0 of O_0, and we have a natural flat connection on it. On the other hand, since O_0 is an affine manifold, the tangent bundle TO_0 has a natural flat connection (defined by the local trivializations given by the local affine charts). The dual connection on the cotangent bundle T^*O_0 is therefore also flat. This connection should coincide with the connection defined before, because of naturality. (The proof is left to the reader).

Let us notice also that $E_{\mathbb{Z}} \xrightarrow{H_1(T^n,\mathbb{Z})} O_0$ is a discrete subbundle of $E_{\mathbb{R}} \xrightarrow{H_1(T^n,\mathbb{R})} O_0$. Under the natural isomorphism between $E_{\mathbb{R}}$ and T^*O_0, $E_{\mathbb{Z}}$ maps to a subbundle of T^*O_0 (consisting of "integral" covectors). We will denote this subbundle, or the discrete sheaf associated to it, by $\mathcal{R}$. It will be used in §5.

It is an interesting problem to find integrable systems with nontrivial affine monodromy. First examples, namely the spherical pendulum and the Lagrange top, were found by Cushman and others, cf. [9, 6]. It is now known that the existence of nontrivial affine monodromy in integrable systems is mainly due to the existence of *focus-focus singularities* [18, 19, 20]. Using this result, one can add to the above list of examples many other systems: Garnier systems, Euler equations in so(4), Clebsch case of the motion of a rigid body in ideal incompressible fluid, etc.

5 Duistermaat-Chern class

The Duistermaat-Chern class is defined as the obstruction for the torus fibration $M_0^{2n} \to O_0^n$ to admit a global section. This fibration is locally trivial. Let (U_i) be a trivializing open covering of O_0^n. Over each U_i there is a section, denoted by s_i. The difference between two local sections, s_i and s_j, over $U_i \cap U_j$, can be written as

$$\mu_{ij} = s_j - s_i \in C^\infty(E_{\mathbb{R}}/E_{\mathbb{Z}})(U_i \cap U_j) = C^\infty(T^*O_0/\mathcal{R})(U_i \cap U_j)$$

Here $C^\infty(.)$ denotes the sheaf of smooth sections. It is immediate that (μ_{ij}) is a 1-cocycle, and it defines a first cohomology class, not depending on the choice of local sections:

$$\mu_{DC} \in H^1(O_0, C^\infty(T^*O_0/\mathcal{R}))$$

μ_{DC} is called the *Duistermaat-Chern class.*

One notices the short exact sequence $\mathcal{R} \to C^\infty(T^*O_0) \to C^\infty(T^*O_0/\mathcal{R}) \to 0$, with $C^\infty(T^*O_0)$ being a fine sheaf. It follows from the associated long exact sequence that

$H^1(O_0, C^\infty(T^*O_0/\mathcal{R}))$ is isomorphic to $H^2(O_0, \mathcal{R})$. Thus the Duistermaat-Chern class may be considered as a second cohomology element. In case the monodromy is trivial, i.e. $M_0^{2n} \to O_0^n$ is a principal bundle, the Duistermaat-Chern class coincides with the usual Chern class (cf. [7]).

If one requires local sections s_i to be Lagrangian, then one has that

$$\mu_{ij} \in Z(T^*O_0/\mathcal{R})(U_i \cap U_j)$$

(Z means closed 1-forms), and it will define the *Lagrangian Duistermaat-Chern class*:

$$\mu_{LDC} \in H^1(O_0, Z(T^*O_0/\mathcal{R}))$$

From the short exact sequence $\mathcal{R} \to Z(T^*O_0) \to Z(T^*O_0/\mathcal{R}) \to 0$ follows the long exact sequence

$$\cdots \to H^1(O_0, Z(T^*O_0/\mathcal{R})) \xrightarrow{i} H^2(O_0, \mathcal{R}) \xrightarrow{\hat{d}} H^2(O_0, Z(T^*O_0)) = H^3(O_0, \mathbb{R}) \to \cdots$$

Under the maps i and $\hat{d}$ we have

$$\mu_{LDC} \overset{i}{\mapsto} \mu_{DC} \overset{\hat{d}}{\mapsto} 0$$

Thus, if the integral affine manifold O_0 is given, then any element of $H^1(O_0, Z(T^*O_0/\mathcal{R}))$ will be the Lagrangian Duistermaat-Chern class of some torus Lagrangian fibration over O_0, and the necessary and sufficient condition for an element μ in $H^2(O_0, \mathcal{R})$ to be the Duistermaat-Chern class of some Lagrangian torus fibration is that $\hat{d}\mu = 0$ (cf. [7]).

There is no known example of a physically meaningful integrable system with nontrivial Duistermaat-Chern class. However, it is not difficult to construct artificial examples: Take O_0 to be the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then $H^1(O_0, Z(T^*O_0/\mathcal{R})) = H^2(O_0, \mathcal{R}) = \mathbb{Z}^2$, and we have a discrete 2-dimensional family of different integrable systems with the orbit space $\mathbb{T}^2$. The corresponding symplectic manifolds are closed 4-dimensional, and many of these manifolds have $\dim H^1 = 3$, hence are non-Kähler.

6 Nondegenerate singularities

A point $x \in (M^{2n}, \omega)$ is called *singular* with respect to the moment map $\mathbf{F} : (M^{2n}, \omega) \to \mathbb{R}^n$ if $\operatorname{rank} d\mathbf{F}(x) < n$. x is a fixed point of the Poisson $\mathbb{R}^n$-action generated by $\mathbf{F}$ iff $d\mathbf{F}(x) = 0$. A fixed point x is called *nondegenerate* if the linear parts of the Hamiltonian vector fields $X_{F_1}, ..., X_{F_n}$ of the components of the moment map span a Cartan subalgebra of the algebra of all Hamiltonian linear vector fields (which is isomorphic to $sp(2n, \mathbb{R})$).

In general, a singular point of the moment map is called nondegenerate, if it becomes a nondegenerate fixed point after a local Marsden-Weinstein reduction.

Let $y \in \Sigma$ be a singular value of the moment map $\mathbf{F} : (M^{2n}, \omega) \to \mathbb{R}^n$, and denote by $N = \mathbf{F}^{-1}(y)$ a connected component of the singular level set corresponding to y. We will assume that N is a *nondegenerate singular leaf*, that is each of its points is either nonsingular or nondegenerate singular. The number $k = max_{x \in N}(n - \operatorname{rank} d\mathbf{F}(x))$ is called the *codimension* of N. Put $r = n - k$. Under an additional mild condition of *topological stability*, which is satisfied for all known systems and is conjectured to be satisfied for all systems with analytic coefficients, we have the following classification theorem (see [19] for details).

Theorem 6.1 *There is a neighborhood* $(U(N), \omega, \mathcal{L})$ *of* N*, which is saturated with respect to the singular Lagrangian foliation (denoted by* $\mathcal{L}$*) given by the moment map, and a natural (unique minimal) normal finite covering* $(\overline{U(N)}, \overline{\omega}, \overline{\mathcal{L}})$ *of* $(U(N), \omega, \mathcal{L})$ *such that:*
a) There is a free foliation-preserving Hamiltonian $\mathbb{T}^r$*-action in* $(\overline{U(N)}, \overline{\omega}, \overline{\mathcal{L}})$*.*
b) $(\overline{U(N)}, \overline{\omega}, \overline{\mathcal{L}}) = (D^n \times T^n \times P^{2k}, \sum_1^r dp_i \wedge dq_i + \omega_0, T^n_{(p_i)} \times \mathcal{L}_{0,(p_i)})$*, where* p_i *are coordinates on* D^r*,* $q_i \bmod 1$ *are periodic coordinates on* T^r*,* $T^r_{(p_i)}$ *denotes the trivial Lagrangian foliation* $D^n \times T^n \to D^r$*,* ω_0 *a symplectic form on* P^{2k}*, and* $\mathcal{L}_{0,(p_i)}$ *a singular Lagrangian foliation on* (P^{2k}, ω_0) *corresponding to a moment map which depends smoothly on the parameter* (p_i)*.*
c) $(P^{2k}, \mathcal{L}_{0,(p_i)})$ *as a topological singular foliation does not depend on the parameter* (p_i)*, and it is decomposed diffeomorphically into a direct product*

$$(P^{2k}, \mathcal{L}_{0,(p_i)}) \stackrel{diff}{=} (P_1, \mathcal{L}_1) \times \cdots \times (P_s, \mathcal{L}_s)$$

where $(P_i, \mathcal{L}_i)$ *are primitive nondegenerate singular Lagrangian foliations (*$\dim P_i = 2$ *if it is an elliptic or hyperbolic singularity,* $\dim P_i = 4$ *if it is a focus-focus singularity)*

In a sense, the above result completes the work done over a long period by several authors, including A.V. Bolsinov, J.-P. Dufour, L.H. Eliasson, A.T. Fomenko, L. Lerman, P. Molino, H. Rüssmann, Ya. Umanskii, J. Vey, about the structure of nondegenerate singularities of integrable systems. See [19] for a history of the question. Using spectral theory, N. Ercolani, M.G. Forest, D.W. McLaughlin and others proved some similar but weaker statements about singularities of infinite-dimensional integrable (soliton) systems (see e.g. [14] and references therein).

An integrable Hamiltonian system is called *strongly nondegenerate* if all singular leaves of the associated Lagrangian foliation are nondegenerate and topologically stable. In the rest of this paper we will consider only strongly nondegenerate integrable systems, and for brevity we will omit the word "strong".

It follows from Theorem 6.1 that for a nondegenerate integrable Hamiltonian system on a symplectic manifold (M^{2n}, ω), the orbit space, that is the space of leaves of the associated singular Lagrangian foliation (given by the connected components of the level set of a moment map), with the induced topology, is a Hausdorff space. Moreover, it has the structure of a stratified integral affine manifold, each stratum being an integral affine manifold (cf. [19, 20]).

7 Global monodromy

Let O denotes the orbit space of a nondegenerate integrable Hamiltonian system on a symplectic manifold (M^{2n}, ω), and $\pi : M^{2n} \to O$ the natural projection. For each open subset $U \subset O$, we associate to it an Abelian group, namely $H^1(\pi^{-1}(U), \mathbb{Z})$. Such an assignment gives a presheaf of Abelian groups over O. Consider the associated sheaf and call it the *monodromy sheaf* of the system. The stalk of this sheaf at each point $y \in O$ is $H^1(\pi^{-1}(y), \mathbb{Z})$.

For example, if $O = O_0$ does not contain singular points, then the monodromy sheaf is nothing but the dual of the sheaf $\mathcal{R}$ defined in §3. In this case, the structure of this sheaf is determined by the affine monodromy.

By analogy, we will call the structure of the monodromy sheaf the *global monodromy* of the system. Thus, two integrable systems with the same orbit space O have the same global monodromy iff their monodromy sheaves are isomorphic.

The above definition of global monodromy is rather abstract. However, in principle, using Abelian group theory (things like amalgrams), one can characterize the global monodromy in terms of invariants of combinatorial type. For example, if O is diffeomorphic to a graph times an interval, then the global monodromy is equivalent to the set of rational and integral marks in the so called Fomenko-Zieschang invariant, which (according to a theorem by Bolsinov-Fomenko-Matveev [4]) classifies topologically nondegenerate integrable Hamiltonian systems with two degrees of freedom restricted to a nondegenerate isoenergy 3-manifold.

It is still an open problem how to give a good characterization of global monodromy, even for the case O is an arbitrary 2-dimensional stratified integral affine manifold.

8 Singular Chern class

Recall that the Duistermaat-Chern class is defined as the obstruction for a regular Lagrangian torus fibration $M_0 \to O_0$ to admit a global section. Since in general we have a singular Lagrangian foliation $M \to O$, our idea is to define the singular Chern class as the obstruction for this singular foliation to admit a singular global section.

First we have to describe what is a local singular section. The formal definition (cf. [20]) is rather long. Here we will only outline some main points.

a) A singular local section of $\pi : M^{2n} \to O^n$ over a conected open subset $U \subset O$ is a multivalued map ϕ from U to M such that $\pi \circ \phi = id$.

b) The image of ϕ is a stratified space of dimension n.

c) If the rank of $y \in U$ is r ($r = n$ if y is a regular point), then $\phi(y)$ is a $(n-r)$-dimensional submanifold of $N_y = \pi^{-1}(y)$, which is transversal to a $\mathbb{T}^r$ torus action given by Theorem 6.1.

d) To each $y \in U$ there is associated a finite group $G_y \subset \mathbb{T}^r$ such that $\phi(y)$ is invariant under the action of G_y.

Then we say that ϕ is a *G-equivariant singular section over U* of the singular Lagrangian foliation $M \to O$, where $G = \{G_x\}$ is a family of finite groups satisfying some compatibility condition. The reason we need G is that the Hamiltonian torus actions in singular leaves are generally not free but only locally free.

For example, incase of one degree of freedom, there always exists a global singular section, with G being trivial.

Analogously, one can define local *Lagrangian G-equivariant singular sections*, by requiring the image of ϕ to be (singular) Lagrangian.

Like in §5, we can form the difference between two local singular sections (assuming that G is fixed). It will be an element

$$\mu_{ij} = \phi_j - \phi_i \in \mathcal{F}_G(U_i \cap U_j)$$

of some Abelian group $\mathcal{F}_G(U_i \cap U_j)$ which acts freely and transitively on the set of all G-equivariant singular sections over $U_i \cap U_j$.

In this way, we define a sheaf $\mathcal{F}_G$ and a cohomology element

$$\mu^G_{sC} \in H^1(O, \mathcal{F}_G)$$

We can replace O by any open subset $O' \subset O$ and get

$$\mu^G_{sC}(O') \in H^1(O', \mathcal{F}_G)$$

For each O' there is a unique canonical choice of G with some minimality property. If we take this minimal G, then we can omit it and write

$$\mu_{sC}(O') \in H^1(O', \mathcal{F}_{O'})$$

This element is called the *singular Chern class over O'* of the singular Lagrangian torus foliation $M \to O$.

Analogously, we can define the *Lagrangian singuar Chern class* over O':

$$\mu_{LsC} \in H^1(O', \mathcal{Z}_{O'})$$

(with some appropriate sheaf $\mathcal{Z}_{O'}$).

In case O is regular, the singular Chern class coincides with the Duistermaat-Chern class. In case O contains only elliptic singularities, it coincides with a generalization by Boucetta and Molino [5] of the Duistermaat-Chern class.

One should notice that, in general, unlike the case of Duistermaat-Chern class, there are no short exact sequences as presented in §5. (There are longer exact sequences).This circumstance makes the study of singular Chern classes much more difficult than the Duistermaat-Chern class.

It is easy to construct artificial integrbale systems with singularities and non-trivial singular Chern class. A possible way to do it is via *integrable surgery* (cf. [20]). For example, one can "twist" the ruled symplectic 4-manifolds (cf. [13]) to obtain non-Kähler closed symplectic 4-manifolds admitting nondegenerate integrable Hamiltonian systems.

9 Concluding remarks

Above we have outlined a classification theory of nondegenerate integrable hamiltonian systems. There are 3 main ingredients in the theory: singularities, global monodromy and singular Chern classes. Much work is still required to get a better feeling of global monodromy and singular Chern class, and to explore their relations with the ambient symplectic manifolds. On the other hand, it is interesting to develope methods to compute our new toplogical invariants for well-known integrable systems. It is also desirable to extend our theory to infinite-dimensional case.

One may wish to classify integrable systems ont only topologically, but also geometrically, or up to orbital equivalence. Recently, an orbital classification for systems with two degrees of freedom was obtained by Bolsinov and Fomenko [3]. Some geometric invariants (which allow to classify also the symplectic form) in the simplest cases were obtained by Dufour, Molino, Toulet [8] and the author.

References

[1] V.I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag, 1978.

[2] V.I. Arnold, S.M. Givental, A.N. Gussein-Zade, Singularities of differentiable maps, Birkhauser, 1988.

[3] A.V. Bolsinov, A.T. Fomenko, *Orbital equivalence of integrable Hamiltonian systems with two degrees of freedom. Classification theorm*, I and II, Russ. Math. Sb., 185 (1994), N. 4, 27-80 and N. 5, 27-78 (Russian).

[4] A.V. Bolsinov, S.V. Matveev, A.T. Fomenko, *Topological classification of integrable Hamiltonian systems with two degrees of freedom. List of all systems of small complexity* , Russ. Math. Surv., 45 (1990), No. 2, 59-99.

[5] M. Boucetta, P. Molino, *Géométrie globale des systèmes hamiltoniens complètement integrables*, CRAS Paris, Ser. I, 308 (1989), 421-424.

[6] R. Cushman, H. Knörrer, *The energy momentum mapping of the Lagrange top*, Lecture Notes in Math., 1139 (1985), 12-24.

[7] P. Dazord, T. Delzant, *Le problème général des variables action-angles*, J. Diff. Geom., 26 (1987), No. 2, 223-251.

[8] J.-P. Dufour, P. Molino, A. Toulet, *Classification des systems integrables en dimension 2 et invariants des modeles de Fomenko*, C.R.A.S. Paris, Ser. I, 1994.

[9] J.J. Duistermaat, *On global action-angle variables*, Comm. Pure Appl., 33 (1980), 687-706.

[10] A.T. Fomenko, *Topological classification of all integrable Hamiltonian systems of general types with two degrees of freedom*, MSRI Publ., Vol. 22 (1991), 131-340.

[11] A.T. Fomenko, ed., *Topological classification of integrable Hamiltonian systems*, Advances in Soviet Mathematics, Vol. 6, 1991.

[12] H. Knörrer, *Singular fibres of the momentum mapping for integrable hamiltonains systems*, J. Reine Angew. Math., 355 (1985), 67-107.

[13] D. McDuff, *The structure of rational and ruled symplectic manifolds*, Journ. AMS, 3 (1990), 679-712, and *Notes on ruled symplectic manifolds*, preprint 1992.

[14] D.W. McLaughlin, *Whiskered tori for NLS equations*, in Important Developments in Soliton Theory, A. Fokas and V. Zakharov eds., Springer series in Nonlinear Dynamics, 1993.

[15] A.S. Mishchenko, A.T. Fomenko, *Generalized Liuoville method for integrating hamiltonian systems*, Funct. Anal. Appl., 12 (1978), N. 2, 46-56.

[16] N.N. Nekhoroshev, *Action-angle variables and their generalizations*, Trans. Moscow Math. Soc., 26 (1972), 180-198.

[17] S. Smale, *Topology and Dynamics I*, Invent. Math., 10 (1970), 305-331.

[18] Nguyen Tien Zung, *A note on focus-focus singularities*, Diff. Geom. Appl. (to appear).

[19] Nguyen Tien Zung, *Decomposition of nondegenerate singularities of integrable Hamiltonian systems*, Lett. Math. Phys., 33 (1995), 187-193, and *Symplectic topology of integrable Hamiltonian systems, I: Arnold-Liouville with singularities*, Compositio Math. (to appear).

[20] Nguyen Tien Zung, *Symplectic topology of integrable Hamiltonian systems II: Characteristic classes and integrable surgery*, preprint Jan/1995.

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