ON COMMUTATOR SUBGROUPS OF ARTIN GROUPS

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E. A. Gorin and V. Ya. Lin [1; §2] found a finite presentation for the commutator subgroups of braid groups. Using partially computations from [1], V. M. Zinde [2] found presentations (not all of them are finite) for the commutator subgroups of other Artin groups (Artin-Tits groups of spherical type according to the modern terminology). Following [2], we denote Artin groups of types A_n, $B_n, \dots$ just by A_n, $B_n, \dots$ and we denote the commutator subgroup of G by G'. In Sect. 1, we give a finite presentation for H_3' which is missing in [2]. It is obtained as a partial case of a finite presentation (see Proposition 1) for $\ker(e : G \to \mathbb{Z})$ where G is a homogeneous Garside group (see [3, 4]) and e is the homomorphism that takes each atom to 1. In Sect. 2, we correct two mistakes in [2] for groups of series B and we give sketches of proofs missing in [2].

After corrections and completions to [2] done in this article, the groups in question that are (are not) finitely generated/presented are as follows. The groups $I_2(2k)'$, $k \ge 2$ (including B_2' and G_2') are free groups on a countable set of generators; the groups B_3' and F_4' are finitely generated but the question of their finite presentedness is still open; the commutator subgroups of other irreducible Artin groups (B_n' for $n \ge 4$, $I_2(2k+1)'$, A_n', D_n', E_n', H_n') are finitely presented.

In Sect. 3, we discuss when epimorphisms of commutator subgroups of Artin groups onto nontrivial free groups exist. The groups $I_2(p)'$ for $p \ge 3$ (including A_2', B_2', H_2', and G_2') are free themselves. Each of the groups A_3', B_3', B_4', D_4' can be mapped onto a free group with two generators; for other irreducible Artin groups G, we have $G'' = G'$, i. e., G' cannot be mapped onto any non-trivial abelian group, and hence onto any non-trivial free group.

1. Let G be a Garside group of finite type, i. e., the group of fractions of a Garside monoid P with a Garside element Δ and a (finite) set of atoms A (see the definitions in [3]). Then $\tau(P) = P$ and $\tau(A) = A$ where $\tau(x) = \Delta^{-1}x\Delta$. We suppose that there exists a homomorphism $e : G \to \mathbb{Z}$ such that $e(A) = \{1\}$ — in this case G is called a *homogeneous* Garside group (e. g., Artin groups have this property). For $p \in P$, we denote $e(p)$ by $|p|$. Let $K = \ker e$. If G is an Artin group such that $G/G' = \mathbb{Z}$ (i.e. A_n, D_n, E_n, H_n, or $I_2(2k+1)$), then $K = G'$.

The fact that K is finitely presented is obvious. Indeed, let $m = |\Delta|$. Then K is generated by $s_p = \Delta^{-1}p$ where $p \in P$, $|p| = m$, subject to relations $s_\Delta = 1$ and $s_p s_q = s_{p'} s_{q'}$ for $p\tau(q) = p'\tau(q')$. This presentation is huge. For example, for $G = H_3$ it has more than a thousand generators and more than a million relations. However, combining Garside approach with Reidemeister-Schreier method, one can obtain a more compact presentation. Let $\langle a, b, \dots \mid R = R', S = S', \dots \rangle$ be a presentation for P such that $\{a, b, \dots\} = A$ (then, the homogeneity implies

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$|R| = |R'|$, $|S| = |S'|, \dots$). We choose $\{a^n\}_{n\in\mathbb{Z}}$ as Schreier representatives (everything below can be easily adapted for any other choice). Then K is generated by $\{a_k, b_k, \dots\}_{k\in\mathbb{Z}}$ subject to relations $a_k = 1$, $R_k = R'_k$, $S_k = S'_k, \dots$, $k \in \mathbb{Z}$, where, for a word $T = uvw\dots$, we denote the word $u_k v_{k+1} w_{k+2}\dots$ by T_k (this is the Reidemeister-Schreier presentation).

Preposition 1. *The group K is generated by $\{a_k, b_k, \dots\}_{0\le k\le m+l-2}$ where $l = \max(|R|, |S|, \dots)$ subject to relations $a_k = 1$ $(0 \le k \le m + l - 2)$, $U_k = U'_k$ $(0 \le k \le m + l - |U| - 1$, $U = R, S, \dots)$.*

Proof. We fix a positive word representing Δ (we shall denote it also by Δ). Since Δ is a Garside element, we may assume that the chosen word is of the form Γa. We add new relations $\mathfrak{R}_k : \Delta_k \tau(x)_{k+m} = x_k \Delta_{k+1}$ $(x = a, b, \dots; k \in \mathbb{Z})$ to the Reidemeister-Schreier relations. Using them, we can reduce any relation $U_{k+m} = U'_{k+m}$, $U \in \{R, S, \dots\}$, to relations $W_j = W'_j$ where $W \in \{R, S, \dots\}$ and $k \le j \le k+|U|-|W|$. Indeed, by replacing each letter x_{j+m} in U_{k+m} and in U'_{k+m} by $(\Delta_j)^{-1}\tau^{-1}(x)_j\Delta_{j+1}$ using $\mathfrak{R}_j$, we obtain $(\Delta_k)^{-1}V_k\Delta_{k+|U|} = (\Delta_k)^{-1}V'_k\Delta_{k+|U|}$, with $\tau(V) = U$ and $\tau(V') = U'$. Since the identity $V = V'$ holds in P, the word V' is obtained from V by subword replacements $W \leftrightarrow W'$, $W \in \{R, S, \dots\}$, hence V'_k is obtained from V_k by the replacements $W_j \leftrightarrow W'_j$ with $k \le j \le k + |U| - |W|$. Proceeding in this manner, we exclude all the relations $U_k = U'_k$ with indices exceeding the required limits. Using the relation $a_{k+m} = 1$, we replace all $\mathfrak{R}_k$ by the relation $\Delta_k\tau(x)_{k+m} = x_k\Gamma_{k+1}$ which express $\tau(x)_{k+m}$ via generators with smaller indices.

Similarly, if we choose a word for Δ of the form $a\Gamma$, then we exclude all the generators and relations with negative indices. □

In particular, H'_3 is generated by $a = \sigma_1\sigma_3^{-1}$, $p_k = \sigma_3^k\sigma_2\sigma_3^{-(k+1)}$ $(0 \le k \le 18)$ subject to relations $p_kp_{k+2}p_{k+4} = p_{k+1}p_{k+3}$ $(0 \le k \le 14)$, $p_kap_{k+2} = ap_{k+1}a$ $(0 \le k \le 16)$.

2. *The group B'_3* (it is erroneously claimed in [2] that this group is free). Choosing $\{\sigma_1^k\sigma_3^j\}_{j,k\in\mathbb{Z}}$ as Schreier representatives, the method of Reidemeister-Schreier yields the generators $p_{j,k} = \sigma_3^j\sigma_1^k\sigma_2\sigma_1^{-(k+1)}\sigma_3^{-j}$ and the relations (1) $p_{j,k}p_{j,k+2} = p_{j,k+1}$ and (2) $p_{j,k}p_{j+1,k+1} = p_{j+1,k}p_{j+2,k+1}$ $(j, k \in \mathbb{Z})$. We introduce new relations (3) $[p_{j,k}, p_{j+1,k}^{-1}] = [p_{j,k+1}, p_{j+1,k+1}^{-1}]$. Assuming that $(1)_{j,k-2}$, $(1)_{j,k-1}$, $(1)_{j+1,k}$, $(1)_{j+1,k+1}$, and (2) hold true, it is easy to derive the equivalencies $(1)_{j-1,k-2} \Leftrightarrow (3)_{j,k} \Leftrightarrow (1)_{j+2,k+1}$. Hence B'_3 is generated by $p_{j,k}$ subject to relations $(1)_{j=0,1}$, (2) and (3). Using $(1)_{j=0,1}$ and (2), we can express all generators via $p_k = p_{0,k}$, $q_k = p_{1,k}$ $(k = 0, 1)$ and then only the generators p_0, p_1, q_0, q_1 and the relations obtained from (3) for all $j, k \in \mathbb{Z}$ remain.

The groups B'_n, $n \ge 5$. It seems that the following relations are forgotten in [2] by misprint: $(*)$ $p_0x = xp_1$, $p_1x = xp_0^{-1}p_1$ for $x = q_4, \dots, q_{n-2}, t_0, t_1$. If we add them, then the fact that the obtained presentation defines B'_n can be proved as follows. By Tietze transformations we replace d and the relations containing it by t_2 and the relations $t_0t_1 = t_1t_2$, $t_2q_{n-2}t_2 = q_{n-2}t_2q_{n-2}$. Applying [4; Lemma 2.9], we may add generators t_i $(i \in \mathbb{Z} \setminus \{0, 1, 2\})$ and relations $t_{t-1}t_i = t_it_{i+1}$, $t_iq_{n-2}t_i = q_{n-2}t_iq_{n-2}$, $t_iq_j = q_jt_i$, and $(*)$ for $x = t_i$ $(i \in \mathbb{Z}, 3 \le j \le n-2)$. The obtained presentation can be also obtained from the Reidemeister-Schreier presentation for B'_n (with respect to the Schreier representative system $\{\sigma_1^k\sigma_n^j\}$) by the method described in [1; §2].

The group B_4'. The proof of the fact that B_4' is presented as written in [2] is almost the same as for B_n', $n \geq 5$. The only difference is that first we apply Lemma 2.9 from [4] (modified in a suitable way), and then we apply the transformation from [1].

The group D_4'. Under the choice of $\{\sigma_3^k\}$ as the system of Schreier representatives (we assume that σ_2 corresponds to the central vertex of the Coxeter graph), the method of Gorin-Lin [1; §2] yields a presentation with generators $p_0 = \sigma_2\sigma_3^{-1}$, $p_1 = \sigma_3\sigma_2\sigma_3^{-2}$, $q_i = \sigma_i\sigma_3^{-1}$, $d_i = \sigma_2\sigma_i\sigma_3^{-1}\sigma_2^{-1}$ $(i = 1, 4)$ and relations $q_1q_4 = q_4q_1$ and

$$p_0q_ip_0^{-1} = d_i,\ p_0d_ip_0^{-1} = d_i^2q_i^{-1}d_i,\ p_1q_ip_1^{-1} = q_i^{-1}d_i,\ p_1d_ip_1^{-1} = (q_i^{-1}d_i)^3q_i^{-2}d_i \quad (i = 1, 4).$$

These generators and those from [2] are expressed in terms of each other as follows: $a_0 = q_1$, $a_1 = d_1$, $b_0 = q_4$, $b_1 = d_4$, $c_0 = p_0^{-1}$, $c_1 = p_0p_1^{-1}p_0^{-1}$, $p_0 = c_0^{-1}$, $p_1 = c_0c_1^{-1}c_0^{-1}$. Using these formulas, it is easy to check the equivalence of the two presentations.

Other presentations in [2] are obtained either directly by Reidemeister-Schreier method, or by an easy modification of the presentation for A_n' from [1].

Remark. After the identification $a_0 = b_0$, $a_1 = b_1$ (which corresponds to the standard epimorphism $D_4 \to A_3$), the presentation for D_4' found in [2] yields the following presentation for A_3'. Generators: $a_0 = \sigma_3\sigma_1^{-1}$, $a_1 = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2$, $c_0 = \sigma_1\sigma_2^{-1}$, $c_1 = \sigma_2\sigma_1\sigma_2^{-2}$. Relations: $c_0^{-1}a_0c_0 = a_1$, $c_0^{-1}a_1c_0 = a_1^2a_0^{-1}a_1$, $c_1^{-1}a_0c_1 = a_0a_1^{-1}$, $c_1^{-1}a_1c_1 = a_1a_0^{-1}a_1$. Perhaps, for certain problems, this presentation would be better than Gorin-Lin's one (anyway, it is shorter).

3. According to [1], the group A_3' has a normal series with free quotients. Namely, let T be the subgroup of A_3' generated by $\sigma_3\sigma_1^{-1}$ and $\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}$. Then T is freely generated by these two elements and the quotient group A_3'/T is a free group with two generators $\sigma_2\sigma_1^{-1}T$ and $\sigma_1\sigma_2\sigma_1^{-2}T$.

It is claimed in [2; Theorem 2] that the groups B_4' and D_4' have normal series with free quotients. V. A. Zinde communicated to me that it is a mistake. She meant only that there exist epimorphisms onto a non-trivial free group. An epimorphism of B_4' and D_4' onto a free group with two generators can be obtained by adding the relations $a_0 = a_1 = b_0 = b_1 = 1$ to the presentation from [2].

The group B_3' also admits an epimorphism onto a free group with two generators. Indeed, by adding the relations $p_{j,k} = p_{j+1,k}$ to the presentation from Sect. 2, we obtain the group freely generated by $p_0 = q_0$ and $p_1 = q_1$.

It can be easily seen that the kernels of the above epimorphisms of B_3', B_4', and D_4' onto the free group are not free.

References

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