RUDOLPH DIAGRAMS AND ANALYTIC REALIZATION OF VITUSHKIN'S COVERING

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Introduction

In his paper [1] (see also [2]) A.G.Vitushkin has constructed an example of a real 4-manifold X with its two-dimensional submanifold M and a branching covering $f : X \to \mathbf{R}^4$ branched only along M, such that $X - M$ is homeomorphic to $\mathbf{R}^4$, M is homeomorphic to $\mathbf{R}^2$ and $f|_M$ is an embedding. This example is very important for understanding the topological nature of the well-known Jacobian Conjecture (see [2,3]), and A.G.Vitushkin asked if there exist analytic mappings with similar topological properties.

In this paper we give in some sense an affirmative answer to this question. We show that it is possible to realize the Vitushkin's covering as an analytic mapping of a Stein manifold onto a ball in $\mathbf{C}^2$. Thus, one has

Theorem 1. *There exists a complex analytic 2-manifold X with the boundary ∂X, a smooth analytic disk $M \subset X$, transversal to ∂X, with $\partial M \subset \partial X$, and a holomorphic three-sheeted branching covering $f : X \to B^4$, where B^4 is the unit ball in $\mathbf{C}^2$, such that* $\operatorname{Int} X - M$ *is homeomorphic to $\mathbf{R}^4$, the restriction of f onto M is an embedding, and f has branching of order two along M, being an immersion (i.e. local homeomorphism) everywhere on $X - M$.*

This implies immediately that ∂X is strictly pseudoconvex outside ∂M. It is not difficult to deduce also that the both $\operatorname{Int} X$ and $\operatorname{Int} X - M$ are Stein spaces, in particular, there exists an exhaustion of $\operatorname{Int} X - M$ by strictly pseudoconvex domains with smooth boundaries, which are homeomorphic to the 4-ball. According to Eliashberg's result [10, Theorem 5.1] there exists a proper plurisubharmonic function on $\operatorname{Int} X - M$ with a single isolated minimum and without other critical points.

Essentially, the proof of Theorem 1 is nothing more than an interpretation of the Vitushkin's construction in terms of Rudolph diagrams of multi-valued complex functions (see below the definition) and direct applying a Rudolph's construction from [4]. As A.G.Vitushkin informed me, while constructing the example, he used pictures similar to (and maybe even the same as!) Rudolph diagrams. However, writing the paper [1] he reformulated everything in terms of explicit cuttings and gluings. According to my opinion, this made the paper more difficult to read. One of the purposes of this paper is to give an exposition of the Vitushkin's example which seems to be more clear than the original one given in [1]. (Note, that topologically the example is not modified here.)

It is easy to write down necessary and sufficient conditions for a given Rudolph diagram in general position to be the diagram of a multi-valued continuous function (for example, all smooth diagrams satisfy them). Moreover, the graph of the function is uniquely determined by the diagram up to an ambient isotopy. Of course, the Vitushkin's construction

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of $f(M)$ is the same as the application of this general construction in a particular case, as well as his gluing the covering is the same as a defining of the covering by a homomorphism of the fundamental group into a symmetric group S_n.

I did not succeed to find a more easy than in [1] proof of the fact that $\operatorname{Int} X - M$ is homeomorphic to $\mathbf{R}^4$ (in [1] it is proved by explicit writing down successive deformation retractions which are simple in the sense of Whitehead). Nevertheless, the contractedness of $\operatorname{Int} X - M$ and its simple-connectedness at infinity can be easily obtained by standard topological arguments using the presentation of the fundamental group of the complement to the graph of a multi-valued function via its Rudolph diagram, presented in [5]. This presentation is just an interpretation of the classical Zariski—van Kampen presentation and it is a word-by-word generalization for a higher dimension of the Wirtinger presentation for a knot group.[1] Thus, we prove here the homeomorphism $\operatorname{Int} X - M \simeq \mathbf{R}^4$ only modulo Poincaré Conjecture. However, with this single (but essential!) exception, we give a self-contained alternative exposition of all the results (with proofs) of [1].

The fact that $\operatorname{Int} X - M$ is diffeomorphic to an open ball, can be reduced, also by standard topological methods, to the Eliashberg's theorem [10, Theorem 5.1] which states that if the boundary of a complex manifold is strictly pseudoconvex and diffeomorphic to S^3 then the manifold is diffeomorphic either to B^4 or to B^4 blown up at several points. (Clearly, the latter is not the case, since a covering over a ball can not contain a compact curve). To prove this reduction, it is enough to write down a Heegaard splitting for $\partial(X - (\text{tubular neighbourhood of } M))$ and to show its equivalence to S^3 (see Remark 4.3 below). Since this manifold is obtained by a covering of S^3 branched along an explicitly defined knot, followed by a one-step surgery along a handle, the required equivalence of Heegaard splittings can be obtained by a standard technique (described, for instance, in [11]). However, these computations do not seam to be much easier than those, given in [1] and we do not present them.

In §1 we give the definition of the Rudolph diagram of a multi-valued complex function. §2 is devoted to the theorem on realizability of a smooth Rudolph diagram by an algebraic function. In fact, this theorem was proved in [4] though it was not explicitly formulated there. In §3 we construct the covering $f : X \to B^4$ as a realization of some Rudolph diagram. In §4 the simple-connectedness of $\operatorname{Int} X - M$ and its simple-connectedness at infinity are proved. In §5 we prove that $X - M$ is contractible. In §6 we show that the Rudolph diagram of the covering from [1], is equivalent to the one constructed in §3. The §§1–5 are independent of [1] and contain a proof of Theorem 1 modulo Poincaré Conjecture; §6 is independent of §§3–4 and contains the reduction of Theorem 1 to [1] and [4].

In Appendix we summarize the results from [5,6] on existence of non-degenerate Rudolph diagrams for algebraic functions. These results are not used in the other sections of the paper.

I am grateful to A.G.Vitushkin for setting the problem and a lot of useful discussions. I am grateful to Lee Rudolph for a useful discussion which caused the appearance of this paper, and to I.A. Taimanov for pointing out to the paper [11]. The work was partially supported by Grant 93-011-225 of Russian Foundation for Fundamental Researches. This paper was written when I was invited at the "Université Bordeaux I" in the framework of the program "PAST", and I am grateful to these institutions for the hospitality and financial support.

[1]However, it is shown in [5] that this presentation gives a direct proof of Fulton—Deligne theorem (Zariski's Conjecture) that the group of the complement to a nodal curve in $\mathbf{P}^2$ is abelian.

§1. Rudolph diagrams

By words multi-valued complex function we shall mean in this paper a multi-valued function with values in $\mathbf{C}$, not necessary analytic, but which looks like an algebraic function in the sense, that it has isolated branching points and outside them can be uniquely continued along any path in the domain of its definition. Now, we give the formal definition.

Let $\mathcal{C}$ be some category of manifolds (smooth, piecewise smooth, real or complex analytic etc.). Let $D \in \mathcal{C}$ and $\mathrm{pr}_D : D \times \mathbf{C} \to D$ be the projection onto the first factor (further D will be usually a domain in $\mathbf{C}$). An *n-valued complex $\mathcal{C}$-function on D* is said to be an equivalence class $F = [K, j]$ of pairs (K, j) where $K \in \mathcal{C}$ has the same dimension as D and $j : K \to D \times \mathbf{C}$ is a $\mathcal{C}$-mapping such that $\mathrm{pr}_D \circ j : K \to D$ is a branched covering of degree n. Pairs (K, j) and (K_1, j_1) are equivalent if there exists a $\mathcal{C}$-isomorphism $\varphi : K \to K_1$ such that $j = j_1 \circ \varphi$. The image $j(K)$ is called *the graph* of the function F and the set of values of F at $z_0 \in D$ is defined as $F(z_0) = \mathrm{graph}(F) \cap (z_0 \times \mathbf{C})$. In any simply connected subset of D which does not intersect the image of branching locus, F splits into n single-valued branches.

Clearly, that if D is a circle then the notion of multi-valued complex function is the same as the notion of braid. So, maybe it would be reasonable to call a multi-valued function on D a *(multi-dimensional) braid over D*.

Denote by $B(F)$ the set of all $z \in D$ such that $F(z)$ contains less than n elements. It consists of the branching points of F and the points where two single-valued branches coincide. Let

$$B_+ = B_+(F) := B(\mathrm{Re}\, F) = B(F) \cup \{z \in D \mid \exists w_1 \neq w_2 \in F(z), \text{ with } \mathrm{Re}\, w_1 = \mathrm{Re}\, w_2\}.$$

It is defined by one real equation, so, generically it is a real hypersurface in D, maybe with boundary.

Let us assume now that D is orientable and F is piecewise smooth. Then we define the *Rudolph diagram of the function* F as the set B_+ equipped with the following additional structure (see [4]). A point $z \in B_+ - B$ is said to be *regular* if $\mathrm{Re}\, F(z)$ contains exactly $n-1$ elements and the intersection of the graphs of $\mathrm{Re}\, f_1$, and $\mathrm{Re}\, f_2$ is transversal where f_1 and f_2 are those single-valued branches of F in some neighborhood of z for which $\mathrm{Re}\, f_1(z) = \mathrm{Re}\, f_2(z)$. Otherwise z is called *singular*. We denote the set of singular points of B_+ by $\mathrm{Sing}(B_+)$. With each component L of $B_+ - \mathrm{Sing}(B_+)$ we associate an integer k such that $\mathrm{Re}\, w_1 < ... < \mathrm{Re}\, w_k = \mathrm{Re}\, w_{k+1} < ... < \mathrm{Re}\, w_n$ for $\{w_1, ..., w_n\} = F(z)$, $z \in L$. In other words, we define a locally constant function $N : B_+ - \mathrm{Sing}(B_+) \to \mathbf{Z}_+$, $N(z) = k$. Define the orientation O on $B_+ - \mathrm{Sing}(B_+)$ as follows. If we cross a segment L of $B_+ - \mathrm{Sing}(B_+)$ marked by k by some transversal path $\alpha(t)$ then we obtain a braid which is equivalent either to σ_k or to σ_k^{-1}. We say that α has positive intersection with L in the former case, and negative in the latter case.

A stratification $B \subset \mathrm{Sing}(B_+) \subset B_+ \subset D$ of D together with the data (O, N, n) is called a *Rudolph diagram* on D, the number n being called its *degree*. If a Rudolph diagram was constructed as above starting with a multi-valued function F, it is called the Rudolph diagram of F. We shall denote it also just by $B_+(F)$ when this does not abuse the notation.

Picturing Rudolph diagrams for two-dimensional D, we use the following convention for orientations. On $B_+(\sqrt{z})$ the arrow points from ∞ to 0, and on $B_+(\sqrt{\bar{z}})$ from 0 to ∞ (in the both cases B_+ is the half-line of non-positive real numbers).

Now let us restrict ourselves by the case when $\dim_{\mathbf{R}} D = 2$. Say that $B(F)$ is *non-degenerate* if it is discrete and for any $z \in B$ in some its neighborhood U all the connected components of $\operatorname{graph}(F|_U)$ but one are graphs of single-valued functions. Say that $B_+(F)$ is *non-degenerate* if $\operatorname{Sing}(B_+)$ is discrete, $B(F)$ is non-degenerate and for any $z \in B$ the real parts of all the values of F at z are distinct. Say that B_+ is *strictly non-degenerate* if it is non-degenerate and at any point of $\operatorname{Sing}(B_+) - B$ it is either a transversal crossing of two curves marked by integers k, j with $|k - j| \geq 2$ (see Fig. 1a) or a transversal crossing of three curves oriented and marked as in Fig. 1b with $|k - j| = 1$.

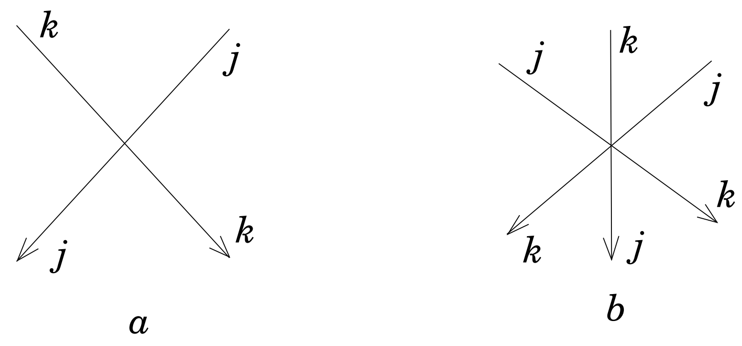

Fig.1

(Such points correspond to standard relations of the braid group $\sigma_k\sigma_j = \sigma_j\sigma_k$, $|k-j| \geq 2$ and $\sigma_j\sigma_k\sigma_j = \sigma_k\sigma_j\sigma_k$, $|j-k| = 1$ respectively.)

A Rudolph diagram is said to be *smooth* if B_+ is a smooth manifold with boundary, $\partial B_+ = B \cup (\partial D \cap B_+)$ and $\operatorname{Sing}(B_+) = B$. Smooth diagrams are evidently strictly non-degenerate. A Rudolph diagram is called *quasipositive* if near $z \in B$ all its segments are oriented to the direction of z (like for $\sqrt{z - z_0}$). It is *positive* if near ∂D all its segments are oriented inwards D.

We give without proof the following evident statement (neither this statement, nor the notion of strict non-degeneracity will be used in the rest of this paper)

Proposition 1.1. *Let D be a (real) two-dimensional piecewise smooth manifold and F a piecewise smooth complex multi-valued function on D. By an arbitrary small smooth perturbation of the graph of F one can make $B_+(F)$ to became strictly non-degenerate. Any strictly non-degenerate Rudolph diagram on D can be realized as $B_+(F)$ for some piecewise smooth function F which is then uniquely defined up to an isotopy of $D \times \mathbf{C}$ preserving the projection onto D.*

Remarks. 1. Rudolph diagrams were introduced in [4] for studying braids arising as multi-valued functions of the form $F \circ \gamma$, where F is an algebraic function without poles on $D \subset \mathbf{C}$ and $\gamma : S^1 \to D$ is a path. If $D = S^1$ then a non-degenerate B_+ of a braid is the same as a decomposition of the braid into a product of standard generators of the braid group.

2. Non-degenerate Rudolph diagrams of complex analytic functions are always quasi-positive (see [4]) and the distance between those two values of F whose real parts coincide, decreases while moving in the positive direction along the regular part of $B_+(F)$ (see [5]). In particular, B_+ can not contain closed components. Lee Rudolph informed me that B. Moishezon has proved some less evident restrictions when the graph of F is a cuspidal curve.

3. In Appendix we discuss the existence of non-degenerate Rudolph diagrams for algebraic and analytic functions.

§2. Realizability of smooth Rudolph diagrams by algebraic functions

Let D be a disk and B_+ a smooth Rudolph diagram in D. We shall say that B_+ is *reduced* if it does not contain cycles and the points of $B_+ \cap \partial D$ can be divided into successive (with respect to circuit along ∂D) groups with odd number of points in each one, so that the central point of a group is incident to a segment of B_+ whose the other end is inside D, and the k-th point from the right is connected by a segment of B_+ with the k-th point from the left. (Example: the diagram in Fig.2 below is reduced.)

In fact, in [4] it was proven (though it was not explicitly formulated) that any quasipositive reduced Rudolph diagram is diffeomorphic to a diagram of an algebraic function without poles on the unit disk. One just have to replace the "universal function" defined by $(w-z)P(w)+\varepsilon$ (see [4, Example 3.3]), with the function defined by

$$(w-z)(1-w)(2-w)...(n-w)+\varepsilon p(z)=0, \tag{2.1}$$

where $\varepsilon > 0$ is a sufficiently small real number and $p(z)$ is a polynomial of degree n whose values at odd integers belonging to $[1,n]$ are positive, and the values at the even ones are negative.

Actually, all we need for the further purposes of this paper, is the realizability of the diagram in Fig.2 below, which is reduced. Nevertheless, we present here a slightly more general statement whose proof is almost a reproduction of the proof of the main theorem in [4] adopted to this case.

Proposition 2.1. *Any smooth quasipositive Rudolph diagram B_+ on the unit disk $D \subset$ $\mathbf{C}$ without cycles (components, diffeomorphic to S^1) is diffeomorphic to $B_+(F)$ for some algebraic function F on D without poles, whose graph is a smooth analytic subvariety of $D \times \mathbf{C}$.*

Proof. Let $U_1,...,U_k$ be the connected components of $D - B_+$. Chose a point a_i in each U_i, and let $B=\{b_1,...,b_m\}$. Join every a_i with all points of $B\cap U_i$ by distinct paths $p_{i1},...,p_{ik_i}$ inside U_i, and then connect every couple of points a_i, a_j for which $L_{ij}=\partial U_i \cap \partial U_j \neq \varnothing$, by a path q_{ij} meeting L_{ij} transversally in a single point. It is possible to do this in such a way that all the q_{ij}'s intersect neither each other, nor the p_{ij}'s. Chose sufficiently small disks A_i and B_i around all the a_i and b_i respectively. Chose also thin ribbons P_{ij}, Q_{ij} along the paths p_{ij}, q_{ij} (the widths of the ribbons being much smaller then the radii of the disks). Let D_0 be the union of all A_i, B_i, P_{ij} and Q_{ij}. Clearly, that the pairs (D, B_+) and $(D_0, B_+\cap D_0)$ are diffeomorphic.

Denote by n the degree of the given Rudolph diagram B_+. Let $w=F_\varepsilon(z)$ be the multivalued algebraic function defined as the implicit function from the equation (2.1), and let D_u be a disk containing the points $\{0,...,n\}$. Then for the non-perturbed function F_0 (which is just $F_0(z)=\{z,1,2,...,n\}$) we have $B(F_0)=\{1,...,n\}$ and $B_+(F_0)$ is the union of vertical lines $L_k=\{z \mid \operatorname{Re} z = k\}$, $k=1,...,n$, where L_k being marked by k. Hence, for the perturbed function, due to the properly chosen signs of the perturbation near B, for a sufficiently small $\varepsilon>0$, we have that $B_+(F_\varepsilon|_{D_u})$, up to an ambient diffeomorphism, can be obtained from the $B_+(F_0)$ by removing from every line a segment of length $O(\varepsilon^{1/2})$ around the intersection with the real axis. Denote the ends of the upper half-lines of $B_+(F_\varepsilon)$ by $z_1,...,z_n$ respectively.

Now construct an immersion of the D_0 into D_u, I quote [4], "handle by handle" (where handles are, of course, the above disks and ribbons). First, we map every B_i onto a small disk around z_k, where k is the mark on B_+ near b_i, so that $B_+\cap B_i$ is mapped into $B_+(F_\varepsilon)$. Next, we map all the A_i's onto a disk outside of $B_+(F_\varepsilon)$, and immerse the ribbons P_{ij}, also avoiding $B_+(F_\varepsilon)$. And finally, immerse each ribbon Q_{ij} as follows. Let k be the integer which marks L_{ij}. Immerse q_{ij} so that it makes one turn around z_k (clockwise or counterclockwise depending on the orientation of L_{ij}) and has no other intersections with $B_+(F_\varepsilon)$, and continue this immersion onto Q_{ij}. Denote the obtained immersion of D_0 by q. It "transports" the analytic structure to D_0. In this structure $F_\varepsilon \circ q$ is an analytic multi-valued function on D_0 with the given Rudolph diagram. According to the Riemann uniformization theorem D_0 is isomorphic to the unit disk. To obtain an algebraic function with the given B_+, we approximate $q \circ \varphi$ where φ is the isomorphism of the unit disk onto D_0, by a polynomial. The graph of the constructed function is smooth because the curve (2.1) is smooth.

§3. Construction of the covering

Let $w = F(z)$ be a complex continuous n-valued function on a simply connected domain $D \subset \mathbf{C}$ with non-degenerate Rudolph diagram B_+, and let K be the graph of F. Then one can write down a presentation for $\pi_1(D \times \mathbf{C} - K)$ with the positive infinite point of the axis Im w as the base point (see [5]).

Namely, let us introduce the generators $a_1(U)$,..., $a_n(U)$ for each connected component U of $D - B_+$ as follows. Chose any $z_0 \in U$ and let $\{w_1, ..., w_n\} = F(z_0)$ being $\operatorname{Re} w_1 < ... < \operatorname{Re} w_n$. Then $a_j(U)$ is represented by a path in the fiber $z = z_0$ which arrives from the infinity along the real half-line $\operatorname{Re} w = \operatorname{Re} w_j$, $\operatorname{Im} w > \operatorname{Im} w_j$, then turns around w_j in the positive direction, and returns back to the infinity.

With each connected component L of $B_+ - \operatorname{Sing}(B_+)$ we associate the following n relations. Let U and V be the connected components of $D - B_+$, to the left and to the right of L respectively (recall that B_+ is oriented). Let i be the integer marking L on B_+, i.e. w_i and w_{i+1} are those values of F whose real parts coincide on L. Then the relations are:

$$a_j(U) = a_j(V), \quad j \neq i, i+1, \tag{3.1}$$

$$a_{i+1}(U) = a_i(V), \quad \text{(denote this element by } b) \tag{3.2}$$

$$a_i(U) = b\, a_{i+1}(V)\, b^{-1}. \tag{3.3}$$

Proposition 3.1. *If K is a graph of a piesewise smooth multi-valued function with a non-degenerate (not necessarily strictly) Rudolph diagram B_+, then the generators $a_i(U)$, $i = 1, ..., n$, $U \in \pi_0(D - B_+)$ and the relations (3.1)–(3.3) written for all $L \in \pi_0(B_+ - \operatorname{Sing}(B_+))$ form a presentation of the group $\pi_1(D \times \mathbf{C} - K)$.*

We omit the proof, because it repeates word by word the proof of the Wirtinger presentation of a knot group (see, for example, [9]). Our presentation is also evidentely equivalent to the classical Zariski—van Kampen presentation for the fundamental group of the complement to a plane algebraic curve.

Now, let $w = F(z)$ be a 4-valued analytic function in the unit disk D with the Rudolph diagram, homeomorphic to the diagram in Fig.2 (existence of such a function was proved in §2). Let K be the graph of F.

Proposition 3.2. *The manifold K obtained from the diagram shown in Fig.2, is homeomorphic to the disk.*

Proof. The restriction onto K of the projection $D \times \mathbf{C} \to D$ is a branched 4-covering over D with three branch points of the order two. The boundary of K is the link defined

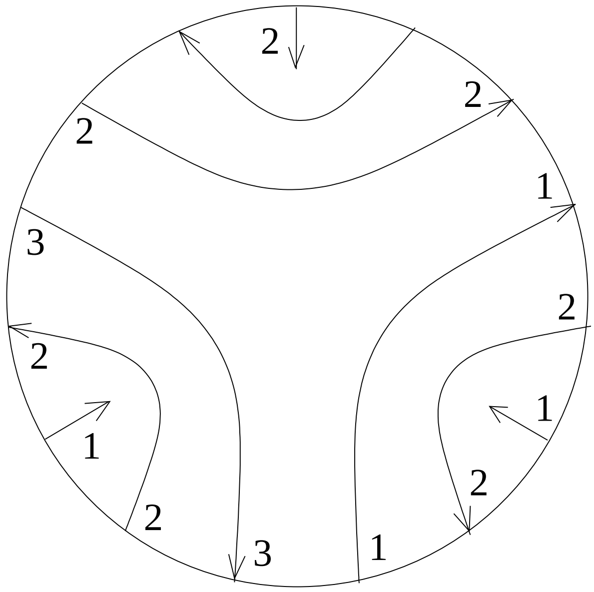

FIG.2

by the braid

$$(\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})(\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2)(\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1}). \tag{3.4}$$

Easy to see that it is a knot which covers ∂D with degree 4. Collapsing ∂K and ∂D, we obtain one more branching point of order 4. The required statement follows from Riemann—Hurwitz formula. Q.E.D.

Denote by U_0 the central component of $D - B_+$, and let

$$a = a_1(U_0),\ b = a_2(U_0),\ c = a_3(U_0),\ d = a_4(U_0).$$

The relations (3.1)–(3.3) corresponding to the 6 arcs with the both ends on ∂D, allows us to express all the $a_i(U)$ via a, b, c, d. Substituting them into the relations corresponding to the rest 3 arcs, we get: $b = b^{-1}abcb^{-1}a^{-1}b,\ \ a = bdb^{-1},\ \ d = bcb^{-1}$. Thus,

$$\pi_1(D \times \mathbf{C} - K) = \langle a, b, c, d\ :\ abc = bab,\ ab = bd,\ bc = db\rangle \tag{3.5}$$

As known, equivalence classes of unbranched m-coverings $p : \widetilde{Y} \to Y$ for a given space Y, are in one-to-one correspondence with homomorphisms of $\pi_1(Y, y_0)$ into the symmetric group S_m in the following way: if an element g is presented by a path α and $p^{-1}(y_0) = \{y_1, ..., y_m\}$ then $\varphi(g)$ takes i to j where y_j is the end of that lifting of α which starts at y_i. We use here a convention that S_m acts on $\{1, ..., m\}$ from the right, i.e. to apply a product of permutations $s_1 s_2$, we first apply s_1 and then s_2.

Consider the homomorphism $\varphi : \pi_1(D \times \mathbf{C} - K) \to S_m$ (for the K constructed above), given by

$$\varphi(a) = (12),\ \varphi(b) = (23),\ \varphi(c) = (12),\ \varphi(d) = (13). \tag{3.6}$$

This definition obeys the relations in (3.5), hence, φ is well-defined. It defines an unbranched covering over $D \times \mathbf{C} - K$ which can be uniquely extended to a branched covering over the cylinder $D \times \mathbf{C}$. To obtain an equivalent covering over the unit ball

$B^4 = \{(z,w) \mid |z|^2+|w|^2=1\}$, it is enough to note that for a sufficiently small $\varepsilon>0$ the pair $(B^4, \operatorname{graph}(\varepsilon F))$ is diffeomorphic to $(D\times\mathbf{C}, K)$.

Denote this covering by $f: X\to B^4$.

Since the permutation corresponding to a small loop around K consists of two cycles of lengths one and two, the preimage of K decomposes into a disjoint union of two components $f^{-1}(K) = M\cup M_1$ where f is a local homeomorphism along M_1 and is branched with order two along M.

Proposition 3.3. *The restrictions of f onto M and M_1 are diffeomorphisms.*

Proof. Otherwise K would be singular.

§4. Simple-connectedness of $X-M$ and its simple-connectedness at infinity

The content of this section is just a direct application of Reidemeister—Schreier theorem on the fundamental group of a finite covering (see [8, Theorem 2.9], [9]).

Given an unbranched m-covering $p:\widetilde{Y}\to Y$ defined by a homomorphism $\varphi: G\to S_m$, $G=\pi_1(Y,y_0)$ (see above), p_* isomorphically maps $\pi_1(\widetilde{Y},y_1)$ onto the subgroup $H\subset G$ of all $h\in G$ such that the permutation $\varphi(h)$ does not move 1. The homomorphism φ can be interpreted as the action of G by shifts on the right cosets of G modulo H. The right cosets are $H_1=H, H_2, ..., H_m$ where $H_j=\{g\in G\mid \varphi(g) \text{ sends } 1 \text{ to } j\}$.

Let $\langle a_1,...,a_n \mid R_1=1,...,R_r=1\rangle$ be any finite presentation of G and let $1=K_1,...,K_m$ be a system of words in a_ν which define representatives of all right cosets. Suppose that any initial segment of any K_μ is again some $K_{\mu'}$ (such systems are called *Schreier systems*). Then the Reidemeister—Schreier theorem provides us a finite presentation of H with generators $s_{\nu\mu}$, $\nu=1,...,n$, $\mu=1,...,m$ and relations

$$s_{\nu\mu}=1 \qquad \text{if } K_\mu a_\nu = K_{\mu'} \text{ for some } \mu', \tag{4.1}$$

$$\tau(K_\mu R_\rho K_\mu^{-1}) = 1, \qquad \mu=1,...,m;\ \rho=1,...,r. \tag{4.2}$$

where τ is the *Reidemeister rewriting process* which rewrites any word W in generators a_ν into a word in generators $s_{\nu\mu}$ as follows. If $W=\prod_i a_{\nu_i}^{\varepsilon_i}$, $\varepsilon_i=\pm1$ then $\tau(W)=\prod_i s_{\nu_i\mu_i}^{\varepsilon_i}$ where K_{μ_i} is the Schreier representative of the coset of the initial segment of W up to $a_{\nu_i}^{\varepsilon_i}$ excluding $a_{\nu_i}^{\varepsilon_i}$ if $\varepsilon_i=1$ and including it if $\varepsilon_i=-1$. The inclusion $H\to G$ is then defined by $s_{\nu\mu}\mapsto K_\mu a_\nu K_{\mu'}^{-1}$ where $K_{\mu'}$ is the Schreier representative for the right coset $HK_\mu a_\nu$.

Now, let $f: X\to B^4\subset\mathbf{C}^2$ be the covering constructed in §3. Denote: $G=\pi_1(B^4-K)$, $G_\partial=\pi_1(S^3-\partial K)$ where $S^3=\partial B^4$, $\partial K = K\cap S^3$. Then G is defined by (3.5) and f is defined by (3.6) and by the homomorphism $\varphi: G\to S_3$. Chose $1, a, d$ as a Schreier system of representatives. Then Reidemeister—Schreier theorem provides us a presentation of $\pi_1(X-f^{-1}(K))$ with generators

$$a_\mu, b_\mu, c_\mu, d_\mu, \qquad \mu=1,2,3 \tag{4.3}$$

and relations

$$a_1=d_1=1, \tag{4.4}$$

$$\begin{array}{lll} a_1b_2c_3b_2^{-1}a_1^{-1}b_1^{-1}, & a_1b_2d_1^{-1}b_1^{-1}, & b_1c_1b_3^{-1}d_1^{-1}, \\ a_2b_1c_1b_3^{-1}a_3^{-1}b_2^{-1}, & a_2b_1d_3^{-1}b_2^{-1}, & b_2c_3b_2^{-1}d_2^{-1}, \\ a_3b_3c_2b_1^{-1}a_2^{-1}b_3^{-1}, & a_3b_3d_2^{-1}b_3^{-1}, & b_3c_2b_1^{-1}d_3^{-1}, \end{array} \tag{4.5}$$

where (4.4) corresponds to (4.1) and (4.5) to (4.2), the rows of (4.5) corresponding to the Schreier representatives $1, a, d$ respectively, and the columns of (4.5) corresponding to the three relations in (3.5) respectively.

To obtain a presentation for $\pi_1(X - M)$, one has to equate to 1 a word, representing a small loop around M_1. Such a word is just b_1. Indeed, by definition b_1 is the lifting of b onto the covering, starting from the base point y_1, followed by the lifting of the Schreier representative $K^{-1}_{\varphi(b)(1)} = K^{-1}_1 = 1$. Hence, b_1 is conjugated to a small loop around $f^{-1}(K)$ whose image b is a simple (not doubled) small loop around K. It means that b_1 is a loop around that component of $f^{-1}(K)$, on which f is not ramified.

Thus, $\pi_1(X - M)$ is generated by (4.3) with relations (4.4), (4.5) and $b_1 = 1$. Substituting (4.4) and $b_1 = 1$ into (4.5), we get $\pi_1(X - M) = \langle c_1, a_2, ..., d_2, a_3, ..., d_3 \mid\ c_3 = 1,$ $b_2 = 1,\ c_1 = b_3,\ a_2c_1 = b_2a_3b_3,\ a_2 = b_2d_3,\ b_2c_3 = d_2b_2,\ a_3b_3c_2 = b_3a_2,\ a_3b_3 = b_3d_2,$ $b_3c_2 = d_3\rangle = ...$ (eliminate c_3, b_2, c_1 by rels. 1,2,3) ... $= \langle a_2, c_2, d_2, a_3, b_3, d_3 \mid\ a_2 = a_3,$ $a_2 = d_3,\ 1 = d_2,\ a_3b_3c_2 = b_3a_2,\ a_3b_3 = b_3d_2,\ b_3c_2 = d_3\rangle = ...$ (eliminate a_3, d_3, d_2 by rels. 1,2,3) ... $= \langle a_2, c_2, b_3 \mid\ a_2b_3c_2 = b_3a_2,\ a_2 = 1,\ b_3c_2 = a_2\rangle = \langle 1\rangle$.

Thus, we have proven

Proposition 4.1. $\pi_1(X - M) = 1$.

To prove that $X - M$ is simply connected at infinity, we use similar calculations for the group of the knot $\partial K \subset S^3$ which is presented by "the boundary braid of the Rudolph diagram" (3.4).

Proposition 4.2. $\pi_1(\partial(X - N)) = 1$ *where* N *is a tubular neighbourhood of* M *in* X.

Proof. Clearly, that Wirtinger presentation for ∂K is equivalent to the presentation (3.1)—(3.3) written for a thin annulus A along the border of D. Denote the components of $A - B_+$ successively by $U_1, ..., U_{15}$ moving counterclockwise starting with the lower (according to Fig.2) component of $A \cap U_0$ where U_0 is the central component of $D - B_+$ (the order, used in the braid (3.4)). Let us denote by a, b, c, d the standard generators $a_i(U_1)$ and keep the rest of notation without changing.

Till the end of this section for group elements a, b we shall denote a^{-1} by $\bar{a}$ and $b^{-1}ab$ by a^b. Let us express all the rest $a_i(U_j)$'s via a, b, c, d, using 7 crossings through B_+ clockwise and 7 ones counterclockwise:

$j:$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a_1(U_j):$	a	b	b	$c^{\bar{b}\bar{a}b}$	$c^{\bar{b}\bar{a}b}$	a^{W_1}	a^{W_1}	a^{W_1}	$d^{\bar{b}\bar{a}}$	$d^{\bar{b}\bar{a}}$	$d^{\bar{b}\bar{a}}$	$d^{\bar{b}\bar{a}}$	$d^{\bar{b}\bar{a}}$	a	a
$a_2(U_j):$	b	a^b	$c^{\bar{b}\bar{a}b}$	$b^{abc\bar{a}^b}$	a^b	$c^{\bar{b}\bar{a}b}$	b^{W_2}	b^{W_2}	$c^{d\bar{b}\bar{a}}$	$c^{d\bar{b}\bar{a}}$	b	b	a	$d^{\bar{b}}$	b
$a_3(U_j):$	c	c	a^b	a^b	b^{abc}	b^{abc}	$c^{\bar{b}\bar{a}b}$	d	a^b	b	$c^{d\bar{b}\bar{a}b}$	a^b	b	b	d
$a_4(U_j):$	d	d	d	d	d	d	d	$c^{\bar{b}\bar{a}bd}$	b^{ab}	a^b	a^b	c^d	c^d	c^d	c^d

where $W_1 = b\bar{c}\bar{b}\bar{a}b$, $W_2 = abc\bar{b}abc\bar{c}\bar{b}\bar{a}b$. The last crossing (between U_8 and U_9) gives us the relations: $d^{\bar{b}\bar{a}}\bar{a}^{W_1} = c^{d\bar{b}\bar{a}}\bar{d} = a^b\bar{b}^{W_2} = b^{ab}\bar{c}^{\bar{b}\bar{a}bd} = 1$.

Since the old a, b, c, d are the images of the new ones, $\varphi : G_\partial \to S_3$ is still defined by (3.6). Adding, like above, the relation $b_1 = 1$ to the Reidemeister—Schreier presentation for $\pi_1(f^{-1}(S^3 - \partial K))$ (with the same Schreier system $1, a, d$), we obtain a presentation for $G_1 = \pi_1(\partial X - (\partial X \cap N))$ whose generators are $a_1, a_2, a_3, b_1, ..., d_3$. Eliminating successively $a_1 = 1$, $b_1 = 1$, $d_1 = 1$, then $a_3 = 1$, $c_3 = 1$, and then $d_2 = 1$, we obtain $G_1 =$

$\langle c_1, a_2, b_2, c_2, b_3, d_3 \mid \ \ d_3\bar a_2 c_2 \bar a_2 b_2,\ a_2\bar b_2\bar c_2,\ b_2 c_1 \bar b_3,\ b_3 c_2 \bar b_2 \bar d_3,\ \bar b_2 a_2 \bar d_3 \bar b_2 c_2,\ \bar a_2 b_2 b_3 \bar c_1 \bar a_2 b_2 d_3,$ $a_2 \bar d_3 \bar b_2 a_2 \bar c_2\rangle$ (denote these relations $R_1, ..., R_7$ respectively). Successively eliminating $c_1 = \bar b_2 b_3$ from R_3, $c_2 = a_2 \bar b_2$ from R_2, $d_3 = \bar b_2 a_2 b_2$ from R_1, $a_2 = b_2^3$ from R_5 and $b_3 = b_2^2$ from R_4, we obtain that G_1 is an infinite cyclic group generated by b_2 and the other generators are expressed via b_2:

$$a_1 = b_1 = d_1 = d_2 = a_3 = c_3 = 1, \quad c_1 = b_2, \quad c_2 = b_3 = b_2^2, \quad a_2 = d_3 = b_2^3. \tag{4.6}$$

By definition of a tubular neighborhood there exists a diffeomorphism $t : M \times D^2 \to N$. The required manifold $\partial(X - N)$ is obtained from $\partial X - (\partial X \cap N)$ by attaching a handle $M \times S^1$ along the torus $\partial M \times S^1$ which is identified by t with $\partial(\partial X \cap N)$. Due to Propositions 3.2 and 3.3 M is isomorphic to a disk, so, to prove that $\pi_1(X - N) = 1$, we have to show that the image in G_1 of the homotopy class of $\partial M \times p$ for $p \in S^1$ coincides with b_2.

Denote by K_ε the graph of $F + \varepsilon i$ (i.e. K, moved a little higher). Then for sufficiently small ε the intersection index $K \cdot K_\varepsilon$ is equal to 3, because there are exactly 3 intersection points, one near each branching point of F (these points evidently are over B_+). It is known that the intersection index of two surfaces in a 4-ball is equal to the linking number of their boundaries. Thus, $\mathrm{link}(\partial K, \partial K_\varepsilon) = 3$.

Denote by α a closed path parameterizing ∂K_ε and by β a small positive loop around K. Suppose that they begin at the same point. Then $\mathrm{link}(\alpha\beta^{-3}, \partial K) = 0 = \mathrm{link}(f(\partial M \times p), \partial K)$, hence these paths are homotopic. Put $\beta = a$ and connect the base point of the fundamental group with the start point of α by the same path, which connects the base point with a (vertical imaginary half-line).

Let us express α via the generators $a_i(U_j)$. To do it, one has to start with the empty word, to chose a single-valued branch of F and to continue it along ∂D till it takes again the initial value. Every time when some other value of F passes over the value, which we are continuing (any such event corresponds to a crossing through B_+, but not conversely), we write to the end of our word the generator $a_i(U_j)$ corresponding to the upper value of F if the intersection with B_+ is positive, and $a_i(U_j)^{-1}$ if it is negative. Clearly, that there is no difference (and this was the reason for (3.2)), either we take U_j before or after the intersection with B_+.

Due to our choice of β, we must start from a point over U_1 near the first (i.e. with minimal real part) single-valued branch of F. Execute this procedure. We obtain

$$\begin{aligned}\alpha = \big(a_2(U_1)\, a_1(U_5)^{-1}\, a_1(U_{14})\, a_2(U_{15})\big) \cdot \big(a_2(U_{11})\, a_3(U_{12})\, a_4(U_1)\big)\cdot \\ \cdot \big(a_2(U_2)^{-1}\, a_4(U_7)\, a_4(U_{10})^{-1}\big) \cdot \big(a_2(U_3)\, a_3(U_4)\, a_2(U_6)^{-1}\, a_3(U_8)\, a_3(U_{13})^{-1}\big),\end{aligned}$$

and in terms of a, b, c, d, this is $\alpha = ab\bar c\bar b\bar a babab\bar d\bar b\bar a bdc\bar b ab\bar c\bar b\bar a bd\bar b$ (we used here the above table of $a_i(U_j)$). Since the path $\alpha\beta^{-3}$ is trivial in $\pi_1(D \times \mathbf{C} - K)$, all its liftings are closed, one of them going along M_1, and the two others along M. As we explained above, $\partial M \times p$ is homotopic to any of those two, which go along M. Since $\alpha\beta^{-3}$ is connected with the base point by the same path as a, and since $\varphi(a) = (12)$ involves the symbol "1", the two liftings which go along M can be defined by $\tau(\alpha\beta^{-3})$ and $\tau(a\alpha\beta^{-3}a^{-1})$ where τ is the Reidemeister rewriting process described above. Computing $\tau(\alpha\beta^{-3})$ and applying (4.6), we see that this element is equal to b_2. Q.E.D.

Remark 4.3. As we mentioned in Introduction, by standard technique described, for instance, in [11], one can write down a Heegaard splitting for $\partial(X - N)$ and to show that it is diffeomorphic to S^3. In our case this can be done even in a more easy way, because the branching has order two. Indeed, presenting the knot ∂K by a knot diagram with over- and underpasses, one can cut the $\mathbf{R}^3$ by a plane H, so that all the overpasses are higher than H and all the underpasses are lower than H. This gives a Heegaard splitting of the covering manifold where the disks cutting the upper (resp. lower) handlebody, are the coverings of the half-disks spanned between the over- (resp. under-) -passes and H.

§5. Contractibility of $X - M$

Now, contractibility of the $X - M$ immediately follows from the fact that the Euler characteristic $\chi(X - M)$ is equal to 1 (Lemma 5.3 below), using the following simple lemma.

Lemma 5.1. *Let Y be a simply connected oriented 4-manifold with connected boundary $\partial Y \neq \varnothing$, such that $\chi(Y) = 1$. Then Y is contractible.*

Proof. According to Hurewitcz theorem, it suffices to prove that $H_q(Y; \mathbf{Z}) = 0$ for $q > 0$ what, by universal coefficients formula, follows from the fact that $H_q(Y; k) = 0\,,\ q > 0$ for any coefficient field k of arbitrary characteristic. By hypothesis, we already have this for $q = 1$ and for $q \geq 4$. By Poincaré—Lefschetz duality we have $H_3(Y; k) = H^1(Y, \partial Y; k)$, and from the exact sequence of the pair

$$... \to H^0(Y; k) \xrightarrow{epi} H^0(\partial Y; k) \to H^1(Y, \partial Y; k) \to H^1(Y; k) = 0 \to ...$$

we see that $H^1(Y, \partial Y; k) = 0$. It remains to prove that $H_2(Y, k) = 0$. This follows from $1 = \chi(Y) = \sum_q (-1)^q \dim_k H_q(Y; k)$. Q.E.D.

A standard method of calculating Euler characteristic of a branched covering is convenient to be formulated in the form of the following lemma, which we reproduce (in a little modified form) together with the proof from [7]. For Riemann surfaces this lemma turns out to be Riemann—Hurwitz formula.

Given a continuous mapping of topological spaces $g : A \to B$, call the *local multiplicity of g at $a \in A$* (notation: $\mu_a g$) the maximal number k such that in any neighborhood of a there exist distinct points $a_1, ..., a_k$ for which $g(a_1) = ... = g(a_k)$. The mapping g is said to be *of constant degree* if there exists a number (denoted by $\deg g$) such that for any $b \in B$ one has

$$\sum_{a \in g^{-1}(b)} \mu_a g = \deg g \tag{5.1}$$

It is clear that all branched coverings are mappings of constant degree.

Lemma 5.2. (cf. [7, lemma 4.1]) *Let $g : A \to B$ be a simplicial mapping of constant degree of finite simplicial complexes. Let $A_k = \{a \in A \mid \mu_a g \geq k\}$ (Clearly, each A_k is a closed subcomplex of A). Then*

$$\chi(A) = \chi(B) \deg g - \sum_{k>1} \chi(A_k).$$

Proof. The word "simplex" will mean here "open simplex". For a simplex τ in B denote by $N(\tau)$ the number of the simplices in $g^{-1}(\tau)$. If C is a union of open simplices σ, denote

$\chi_0(C)=\sum_{\sigma\subset C}(-1)^{\dim\sigma}$. Clearly, that if C is closed a subcomplex then $\chi_0(C)=\chi(C)$. Let $A'_k=\{a\in A\mid \mu_a g=k\}$. From (5.1) we have

$$N(\tau)=\deg g-\sum_{\sigma\subset g^{-1}(\tau)}(\mu_\sigma g-1).$$

Hence,

$$\begin{aligned}\chi(A)&=\sum_{\tau\subset B}(-1)^{\dim\tau}N(\tau)=\sum_{\tau\subset B}(-1)^{\dim\tau}\Big[\deg g-\sum_{\sigma\subset g^{-1}(\tau)}(\mu_\sigma g-1)\Big]\\&=\sum_{\tau\subset B}(-1)^{\dim\tau}\deg g-\sum_{\sigma\subset A}(-1)^{\dim\sigma}(\mu_\sigma g-1)=\chi(B)\deg g-\sum_k(k-1)\chi_0(A'_k)\\&=\chi(B)\deg g-\sum_k(k-1)(\chi(A_k)-\chi(A_{k+1}))=\chi(B)\deg g-\sum_{k>1}\chi(A_k).\end{aligned}$$

Q.E.D.

Lemma 5.3. *$\chi(X)=2$; $\chi(X-M)=1$.*

Proof. By Propositions 3.2 and 3.3, M is a disk. Hence, applying Lemma 5.2 to the covering $f:X\to B^4$, we have $\chi(X)=\chi(B^4)\deg f-\chi(M)=3-1=2$. Let N be an open tubular neighbourhood of M. Since $M\simeq D^2$, N is homeomorphic to a 4-cell, and one can chose a cell complex subdivision of X which contains N as an open cell. Therefore, $\chi(X-N)=\chi(X)-1=1$, but $X-N$ is homeomorphic to $X-M$. Q.E.D.

§6. Rudolph diagram of the Vitushkin's covering

In this section we use the notation from [1, §1] and refere to the pictures [1, Fig.1 — Fig.4].

First, note that if F is a complex multi-valued function in a disk D, $D_0\subset D$ is a domain bounded by a Jordan (i.e. simple closed) curve $\partial D_0\subset D$, and F has neither branching points nor intersections of different single-valued branches in $D-D_0$, (i.e. $B(F)\cap(D-D_0)=\varnothing$) then the pairs $(D\times\mathbf{C},\mathrm{graph}(F))$ and $(D_0\times\mathbf{C},\mathrm{graph}(F|_{D_0}))$ are homeomorphic.

Therefore, it is enough to calculate the Rudolph diagram for $f|_{D_0}$ where f is the 4-valued function constructed in [1,§1], and D_0 is an ε_1-neighborhood of $K_1\cup K_2\cup K_3\cup I_1\cup I_2\cup I_3$ (see [1, Fig.1] and Fig.3 below). Unfortunately, the diagram for the f is degenerate. However, if we perturb f, replacing it by $e^{i\varepsilon_2}f$ and replacing the terms "$2i$" and "$-2i$" respectively with the terms "$2i+\varepsilon_3$" and "$-2i+\varepsilon_3$" everywhere in the definition of f in [1,§1], then for $0<\varepsilon_1\ll\varepsilon_2\ll\varepsilon_3\ll 1$ we obtain the diagram shown in Fig.3. Evidently, the diagrams in Fig.2 and Fig.3 are diffeomorphic.

To show that the Rudolph diagram B_+ really looks like in Fig.3, we just write down explicit formulas for real polynomial parameterizations of all the segments of B_+. We display them only for $B_+\cap K_3$ (other computations are either the same or easier).

Denote: $\alpha=e^{i\varepsilon_2}$, $a_1=2i+\varepsilon_3$, $a_2=-2i+\varepsilon_3$. By definition we have $B_+\cap K_3=L_0\cup L_1\cup L_2\cup L_3$ where $L_0=\{x\mid \mathrm{Re}(\alpha\sqrt{x+2})=-\mathrm{Re}(\alpha\sqrt{x+2})\}$, $L_j=\{x\mid \mathrm{Re}(\alpha\sqrt{x+2})=\mathrm{Re}(\alpha a_j)\}$, $j=1,2$, and $L_3=\{x\mid \mathrm{Re}(\alpha a_1)=\mathrm{Re}(\alpha a_2)\}=\varnothing$. Clearly,

$$\begin{aligned}L_0&=\{x\in K_3\mid \mathrm{Arg}(\alpha\sqrt{x+2})=\pi/2\}=\{x\in K_3\mid \mathrm{Arg}(x+2)=\pi-2\,\mathrm{Arg}\,\alpha\}\\&=\{-2+te^{i(\pi-2\varepsilon_2)}\mid t\in\mathbf{R}_+\}\cap K_3,\\L_j&=\{x\in K_3|\exists t\in\mathbf{R};\ \alpha(\sqrt{x+2}-a_j)=it\}=\{-2+(it/\alpha+a_j)^2\mid t\in\mathbf{R}\}\cap K_3.\end{aligned}$$

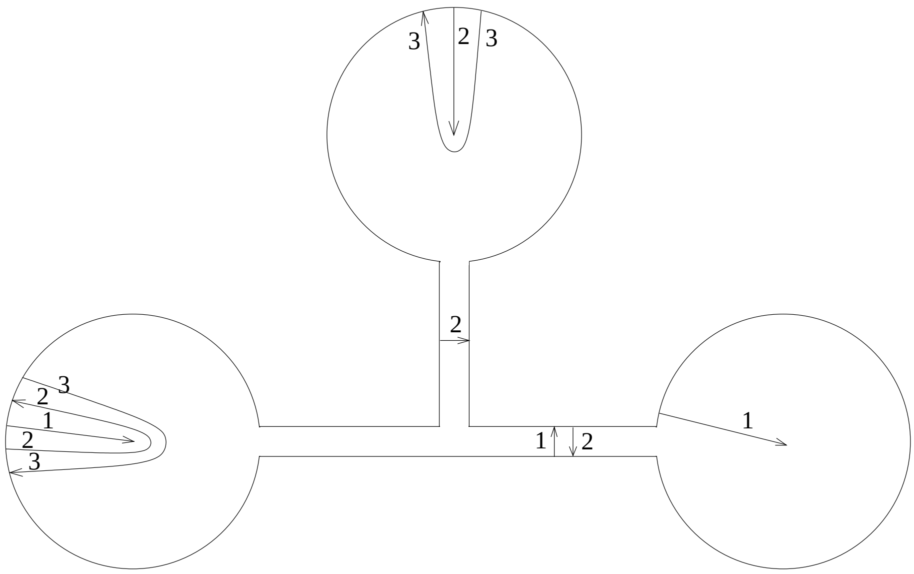

FIG.3

To find the integer numbers marking on the segments of B_+, it is enough to trace, in what order the values of f pass one over another (see [1,Fig.4] for the considered case of K_3, and see [1, Fig.2,3] for the cases K_1 and K_2).

APPENDIX. EXISTENCE OF NON-DEGENERATE RUDOLPH DIAGRAMS FOR ALGEBRAIC CURVES

In this appendix we summarize the results from [5,6] on possibility to present a plane algebraic curve by a multi-valued function with non-degenerate Rudolph diagram. Combining together lemmas 2.3, 2.4 of [5] and lemmas 1, 2 of [6], one obtains

Proposition. *Let K be a reduced (i.e. without multiple components) algebraic curve in $\mathbf{C}^2$ which contains no pair of components of form*

$$(Q(z,w)-a)(Q(z,w)-b)=0, \qquad Q \in \mathbf{C}[z,w], \deg Q = 2, a \neq b \in \mathbf{C}, \tag{A.1}$$

other than a union of 4 parallel lines. Then there exist coordinates (z,w) obtained by an arbitrary small linear transformation of $\mathbf{C}^2$, such that the multi-valued function $F(z) = \{w \mid (z,w) \in K\}$ has a non-degenerate Rudolph diagram $B_+(F)$ which satisfies the following additional condition. $B_+(F) - B(F)$ is a union of immersed real lines with only transversal intersections and self-intersections, and if $j : \mathbf{R} \to \mathbf{C}$ is one of these immersions then for any $t \in \mathbf{R}$ there exists a unique pair f_1, f_2 of germs of single-valued branches of F at $j(t)$, such that $\operatorname{Re} f_1 - \operatorname{Re} f_2$ locally defines $j([t-\varepsilon, t+\varepsilon])$ as a smooth real-analytic curve. All such pairs of germs corresponding to the same immersion j, are obtained one from another by a simultaneous analytic continuation along j.

Clearly, that the Rudolph diagram of a curve containing components of the form (A.1) (different from 4 lines) is degenerate for any choise of coordinates. Indeed, the multi-valued function always has two pairs of branches f_1, f_2 and f_3, f_4 whose differences $f_2 - f_1$ and $f_4 - f_3$ identically coincide, and hence, vanishing of $\operatorname{Re} f_2 - \operatorname{Re} f_1$ implies vanishing of $\operatorname{Re} f_4 - \operatorname{Re} f_3$. There are two ways to avoid this problem. On one hand, such pairs disappear after changing the infinite line (if we are interested in the projective case), or after an affine but non-linear change of variables (if we are interested in the affine case). On the other hand, one can consider Rudolph diagrams with more than one integer marking the segments of the curves. If the difference between any two marks on the same segment is grater than 2, what means that real parts of no three values of F coincide (and the lemmas in [5,6] provide existence of such a projection for any curve) then such diagrams possess all the essential properties of Rudolph diagrams, including the unique reconstruction of the graph and the existence of the presentation for the fundamental group of the complement given by Proposition 3.1.

REFERENCES

1. A.G.Vitushkin, *Certain examples in connection with the problems about polynomial transformations of* $\mathbf{C}^n$, Izvestiya AN SSSR, Ser. Math. **35** (1971), 269–279 (Russian); English transl. in Math. USSR-Izv. **5** (1971), 278–288.
2. A.G.Vitushkin, *On polynomial transformation of* $\mathbf{C}^n$, in: Manifolds, Tokyo, 1973, Tokyo Univ. Press, Tokyo, 1975, pp. 415–417.
3. H.Bass, E.H.Connell, D.Wright, *The Jacobian Conjecture: reduction of degree and formal expansion of the inverse*, Bull. AMS (N.S.) **7** (1982), 287–330.
4. L.Rudolph, *Algebraic functions and closed braids*, Topology **22** (1983), 191-202.
5. S.Yu.Orevkov, *The fundamental group of the complement of a plane algebraic curve*, Mat. Sbornik **137(179)** (1988), no. 2, 260–270 (Russian); English transl. in Math. USSR-Sbornik **65** (1990), 267–277.
6. S.Yu.Orevkov, *The commutant of the fundamental group of the complement of a plane algebraic curve*, Uspekhi Mat.Nauk **45** (1990), no. 1, 183–184 (Russian); English transl. in Russ. Math. Surv. **45** (1990), 221–222.
7. S.Yu.Orevkov, *On three-sheeted polynomial mappings of* $\mathbf{C}^2$, Izvestiya AN SSSR, Ser. Math. **50** (1986), 1231–1240 (Russian); English transl. in Math. USSR-Izvestija **29** (1987), 587–596.
8. W.Magnus, A.Karrass, D.Solitar, *Combinatorial Group Theory,* Pure and applied Math., Vol. XIII, Interscience Publishers, N.Y., 1966.
9. R.H.Fox, *A quick trip through knot theory*, in: Topology of 3-manifolds and related topics. Proc. Univ. Georgia Inst. 1961, Prentice Hall, 1962, pp. 120–167.
10. Ya.Eliashberg, *Filling by holomorphic discs and its applications*, in: Geometry of low dimensional manifolds. London Math. Soc. Lect. Notes, vol. 151, 1991, pp. 45–67.
11. J.S.Birman, H.M.Hilden, *Heegaard splittings of branched covering of* S^3, Trans. A.M.S **213** (1975), 315–352.

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