

COUNTING LATTICE TRIANGULATIONS: FREDHOLM EQUATIONS IN COMBINATORICS

S. YU. OREVKOV

ABSTRACT. Let $f(m, n)$ be the number of primitive lattice triangulations of $m \times n$ rectangle. We compute the limits $\lim_n f(m, n)^{1/n}$ for $m = 2$ and 3 . For $m = 2$ we obtain the exact value of the limit which is equal to $(611 + \sqrt{73})/36$. For $m = 3$, we express the limit in terms of certain Fredholm's integral equation on generating functions. This provides a polynomial time algorithm for computation of the limit with any given precision (polynomial with respect to the number of computed digits).

1. INTRODUCTION

A *lattice triangulation* of a (lattice) polygon in \mathbb{R}^2 is a triangulation with all vertices in \mathbb{Z}^2 . As it was discovered in [3], lattice triangulations are important in algebraic geometry (see also [9]). A lattice triangulation is called *primitive* (or *unimodular*) if each triangle is primitive, i.e., has the minimal possible area $1/2$ (a translate of $[(0, 0), (x_1, y_1), (x_2, y_2)]$ with $x_1 y_2 - x_2 y_1 = 1$). We denote the number of primitive lattice triangulations of the rectangle $m \times n$ by $f(m, n)$. Let

$$c(m, n) = \frac{\log_2 f(m, n)}{mn}, \quad c_m = \sup_n c(m, n) = \lim_{n \rightarrow \infty} c(m, n),$$

$$c = \sup_m c_m = \lim_{m \rightarrow \infty} c_m = \sup_n c(n, n) = \lim_{n \rightarrow \infty} c(n, n).$$

The existence of the limits is proven in [4; Proposition 3.6]. The number $c(m, n)$ is called in [4] the *capacity* of the rectangle $m \times n$. In [8] I gave an upper bound $c < 6$ (which can be easily improved by the same arguments up to $c < \log_2 27 = 4.755$: it is enough just not to distinguish the cases $v_j = 1$ and $v_j = 2$ in the notation of [8]). Later on, a much better estimate $c < 3$ was obtained by Anclin [1] as well as $c_m < 3 - 1/m$. A yet better upper bound $c < 4 \log_2 \frac{1+\sqrt{5}}{2} = \log_2 6.854 = 2.777$ is obtained in [7] and announced in [12] (I have not seen the manuscript [7] but Professor Welzl kindly sent me the slides of his talk [13] where the proof of this bound is clearly exposed).

Easy to see that

$$f(1, n) = \binom{2n}{n} \quad \text{whence} \quad c_1 = 2 \tag{1}$$

which yields a lower bound $c \geq 2$. It is also computed in [4] that $c \geq c_4 \geq c(4, 32) = 2.055702$. It is written in [4; §2.1]: “For $f(2, n)$ we have no explicit formula, and we cannot evaluate the asymptotics precisely”. We still have no explicit formula for $f(2, n)$ but we give here the principal term of the asymptotics:

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Theorem 1. $\lim_{n \rightarrow \infty} f(2, n)^{\frac{1}{2n}} = \sqrt{\alpha} = 4.148440\dots$ where

$$\alpha = \frac{611 + \sqrt{73}}{36}, \quad \text{hence} \quad c_2 = \frac{1}{2} \log_2 \alpha = 2.05256897$$

An exact value of c_3 , in a sense, is given in Proposition 4.5 where we express c_3 in terms of Fredholm's integral equations on certain generating functions. In particular, Proposition 4.5 provides an algorithm to compute c_3 up to n digits in a polynomial time in n . A Mathematica code implementing the main step of this algorithm is presented in Figure 7 at the end of §4.

Theorem 2. $\lim_{n \rightarrow \infty} f(3, n)^{\frac{1}{3n}}$, up to 360 digits, is equal to

4.239369481548025671877625742045235772100695711251795499830801
687833358238276728987837054831763341276708855553395893005289
580195934799338289257489707990192054275721787374165246347114
466096241741151814326914780021501337938335813142441896953051
597942032082556780952912032761797534112146994900056374798271
988378451540168358202181556482461979420039542105330977266751

and hence $c_3 = 2.0838497\dots$

We computed c_3 with this high precision hoping to find an algebraic equation for it, or to relate it with some known constants, but we did not succeed so far.

In §2.2 we present the results of computations of exact values of the numbers $f(m, n)$ for some m and n . These computations show in particular that $c \geq c(5, 115) = 2.10449551\dots$

In §6 we give an asymptotic upper bound for the number of all (not necessarily primitive) lattice triangulations. However it seems to be far from optimal.

The rest of the paper (besides §2.2 and §6) is devoted to the proof of Theorems 1 and 2. Namely, in §2.1 recurrent formulas for the number of primitive lattice triangulations of polygons contained in strips of fixed width are presented. They are similar to those in [4] but, in our opinion, simpler and more efficient. Theorems 1 and 2 are proven in §3 and §4 respectively using the relations between generating functions coming from the recurrent relations from §2.1. As it is mentioned already, c_3 is expressed in terms of a Fredholm integral equation on the Cauchy integrals of the generating functions. In §5 we estimate the approximation error of (the most straightforward) numerical solution of a Fredholm equation with analytic kernel.

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2. RECURRENT RELATIONS FOR STRIPS OF FIXED WIDTH

2.1. Recurrent relations. Given a polygon $P \subset \mathbb{R}^2$, the *upper part* of its boundary is the set $\{(x, y) \in P \mid y' > y \Rightarrow (x, y') \notin P\}$. A *vertical side* of P is a side of P contained in a line $\{x = x_0\}$. Let \mathcal{T} be a triangulation of a polygon P in \mathbb{R}^2 . We say that Q is a *tile of \mathcal{T}* in the following three cases:

- (1) Q is a triangle of \mathcal{T} without vertical sides;
- (2) Q is a triangle of \mathcal{T} whose vertical side lies on the boundary of P ;
- (3) Q is a union of two triangles of \mathcal{T} which share a common vertical side.

A polygon is called *y-convex* if its intersection with any line $x = \text{const}$ is either the empty set, or a point, or a segment.

Lemma 2.1. *Let \mathcal{T} be a triangulation of a y -convex polygon P in \mathbb{R}^2 . Then there exists a tile Q of \mathcal{T} such that the upper part of the boundary of Q is contained in the upper part of the boundary of P .*

Proof. Let Γ_P be the upper part of the boundary of P . Let Q_1, \dots, Q_n be all the tiles of \mathcal{T} which have at least one side lying on Γ_P . Let Γ_i be the union of the sides of Q_i lying on Γ_P . It is clear that each Γ_i is either a side of Q_i or a union of two sides with a common vertex. It is also clear that the projections of the Γ_i onto the x -axis have pairwise disjoint interiors, hence we may assume that $\Gamma_1, \dots, \Gamma_n$ are numbered from the left to the right. We say that a tile Q_i is *shadowed on the left* (resp. *shadowed on the right*) if the upper part of the boundary of Q_i contains a segment I such that $I \not\subset \Gamma_P$ and I is on the left (resp. on the right) of Γ_i ; see Figure 1. It is clear that none of the tiles Q_1, \dots, Q_n can be shadowed on the left and on the right simultaneously. Hence without loss of generality we may assume that at least one of these tiles is not shadowed on the right. Let i_0 be the minimal number such that Q_{i_0} is not shadowed on the right. Then Q_{i_0} is the desired tile with the upper part contained in Γ_P . Indeed, it is not shadowed on the right by its definition. It cannot be shadowed on the left neither because otherwise Q_{i_0-1} would not be shadowed on the right which contradicts the minimality of i_0 . \square

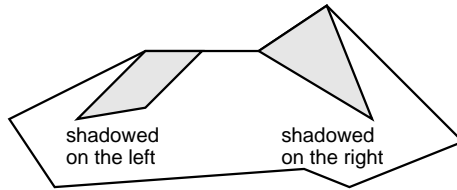


FIGURE 1

Now we fix an integer $m > 0$ and consider primitive lattice triangulations of polygons contained in the vertical strip $\{0 \leq x \leq m\}$ bounded by two graphs of continuous piecewise linear functions.

By analogy with the terminology introduced in [4, §2.2], we say that $\varphi : [0, m] \rightarrow \mathbb{R}$ is an *admissible function* if it is a continuous piecewise linear function whose graph is a union of segments with endpoints at \mathbb{Z}^2 . Let us fix an admissible function φ_0 and say that a function $\varphi : [0, m] \rightarrow \mathbb{R}$ is φ_0 -*admissible* if it is admissible and $\varphi(x) \geq \varphi_0(x)$ for any $x \in [0, m]$. A φ_0 -*admissible shape* is a polygon S of the form $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq m, \varphi_0(x) \leq y \leq \varphi(x)\}$ for some φ_0 -admissible function φ .

As in the above definition of a tile of a triangulation, we say that Q is a *primitive lattice tile* in the following three cases:

- (1) Q is a primitive lattice triangle without vertical sides;
- (2) Q is a primitive lattice triangle whose vertical side is contained in the boundary of the strip $0 \leq x \leq m$;
- (3) $Q = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are primitive lattice triangles such that $\Delta_1 \cap \Delta_2$ is a common vertical side of Δ_1 and Δ_2 .

A primitive lattice tile Q is P -*maximal* for a polygon P if $Q \subset P$ and the upper part of the boundary of Q is contained in the upper part of the boundary of P . We say that S' is a φ_0 -*admissible subshape* of a φ_0 -admissible shape S , if S' is the

closure of $S \setminus (Q_1 \cup \dots \cup Q_n)$, where Q_1, \dots, Q_n are S -maximal primitive lattice tiles with pairwise disjoint interiors. Following [4], in this case we set $\#(S', S) = n$. Note that the relation “to be an admissible subshape” is not transitive.

Let us denote the number of primitive lattice triangulations of a polygon P by $f^*(P)$. When P sits in the strip $\{0 \leq x \leq m\}$, we also define $f(P)$ as the number of primitive lattice triangulations of P which do not have any interior edge whose projection onto the x -axis is the whole segment $[0, m]$ (we choose a simpler notation for a more complicated notion because the numbers $f(P)$ will be used more often than $f^*(P)$).

The following lemma is the inclusion-exclusion formula in our setting. The proof is the same as for [4, Lemma 2.2].

Lemma 2.2. *For any φ_0 -admissible shape S , we have*

$$f^*(S) = \sum_{S'} (-1)^{\#(S', S)-1} f^*(S'), \quad \text{and} \quad f(S) = \sum_{S'} (-1)^{\#(S', S)-1} f(S'),$$

where the left sum is taken over all proper φ_0 -admissible subshapes of S , and the right sum is taken over those proper φ_0 -admissible subshapes of S whose upper part of the boundary contains a point from $\mathbb{Z}^2 \cap \{0 < x < m\}$.

Example 2.3. Let $m = 2$ and $\varphi_0 = 0$. For non-negative integers a, b, c , let $S_{a,b,c}$ be the φ_0 -admissible shape bounded from above by the segment $[(0, a), (1, b)]$ and $[(1, b), (2, c)]$. Let $f_{a,b,c} = f(S_{a,b,c})$. We set also $f_{a,b,c} = 0$ when $\min(a, b, c) < 0$. Then (see Figure 2) the recurrent formula of Lemma 2.2 reads

$$f_{a,b,c} = \begin{cases} f_{a-1,b,c} + f_{a,b-1,c} + f_{a,b,c-1} - f_{a-1,b,c-1} & \text{if } (a, b, c) \neq (0, 0, 0), \\ 1 & \text{if } (a, b, c) = (0, 0, 0). \end{cases}$$

Let $F(x, y, z) = \sum_{a,b,c} f_{a,b,c} x^a y^b z^c$ be the generating function. Then, by summing the recurrent relation over all triples $(a, b, c) \neq (0, 0, 0)$, we obtain

$$\begin{aligned} F(x, y, z) - 1 &= \sum f_{a-1,b,c} x^a y^b z^c + \sum f_{a,b-1,c} x^a y^b z^c + \dots \\ &= \sum f_{a,b,c} x^{a+1} y^b z^c + \sum f_{a,b,c} x^a y^{b+1} z^c + \dots \\ &= F(x, y, z)(x + y + z - xz) \end{aligned}$$

whence $F(x, y, z) = 1/(1 - x - y - z + xz)$.

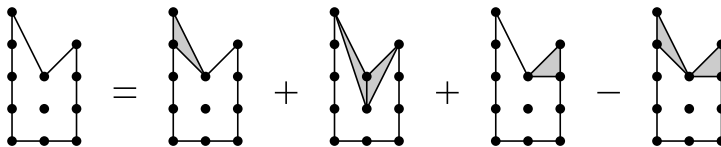


FIGURE 2

Example 2.4. Let φ_0 and $S_{a,b,c}$ be as in Example 2.3. For non-negative a, c such that $a \equiv c + 1 \pmod{2}$, we define $S'_{a,c}$ as the φ_0 -admissible shape bounded from above by the segment $[(0, a), (2, c)]$. Let $f_{a,b,c}^* = f^*(S_{a,b,c})$ and $g_{a,c}^* = f^*(S'_{a,c})$. We

set also $f_{a,b,c}^* = 0$ when $\min(a, b, c) < 0$ and $g_{a,c}^* = 0$ when $\min(a, c) < 0$ or $a \equiv c \pmod 2$. Then, for $(a, b, c) \neq (0, 0, 0)$, the recurrent formula of Lemma 2.2 applied to $S_{a,b,c}$ reads

$$f_{a,b,c}^* = f_{a-1,b,c}^* + f_{a,b-1,c}^* + f_{a,b,c-1}^* - f_{a-1,b,c-1}^* + \chi_{a,b,c} g_{a,c}^*$$

where $\chi_{a,b,c} = 1$ if $2b = a + c + 1$, and $\chi_{a,b,c} = 0$ otherwise. Let $F^*(x, y, z)$ and $G^*(x, z)$ be the respective generating functions. Then (cf. Example 2.3) we have

$$F^*(x, y, z) - 1 = F^*(x, y, z)(x + y + z - xz) + \sum \chi_{a,b,c} g_{a,c}^* x^a y^b z^c$$

and the last sum is equal to

$$\sum_{a,c} g_{a,c}^* x^a y^{(a+c+1)/2} z^c = y^{1/2} \sum_{a,c} g_{a,c}^* (xy^{1/2})^a (y^{1/2}z)^c = y^{1/2} G^*(xy^{1/2}, y^{1/2}z)$$

which gives us the relation

$$F^*(x, y, z)(1 - x - y - z + xz) = 1 + y^{1/2} G^*(xy^{1/2}, y^{1/2}z).$$

Now let us apply the recurrent relation to $S'_{a,c}$. The only admissible subshape of $S'_{a,c}$ is $S_{a,(a+c-1)/2,c}$, hence the relation for $S'_{a,c}$ reads $g_{a,c}^* = f_{a,(a+c-1)/2,c}^*$. In terms of the generating functions this means that

$$\begin{aligned} G^*(x, z) &= \sum_{a,c} f_{a,(a+c-1)/2,c}^* x^a z^c = \text{coef}_{u^0} \left[\sum_{a,b,c} f_{a,b,c}^* x^a u^{2b-a-c+1} z^c \right] \\ &= \text{coef}_{u^0} \left[u \sum_{a,b,c} f_{a,b,c}^* (x/u)^a (u^2)^b (z/u)^c \right] = \text{coef}_{u^0} (uF^*(x/u, u^2, z/u)). \end{aligned}$$

2.2. Some exact values of $f(m, n)$.

The recurrent relations in Lemma 2.2 provide an algorithm of computation of exact values of $f(m, n)$ for small m and n . The algorithm is similar to the one described in [4, §2.2]. We performed computations using this algorithm and one can see in Table 1 that we advanced much further with respect to the computations in [4]. There are three reasons for this which have more or less equal impact.

Table 1

Capacities computed in [4]		Capacities computed in this paper		
$c_1 = 2.0000$	$c_{4,32} = 2.0557$	$c_1 = 2.0000$	$c_{4,200} = 2.0946$	$c_{7,20} = 2.0813$
$c_{2,375} = 2.0441$	$c_{5,12} = 2.0175$	$c_2 = 2.0526$	$c_{5,115} = 2.1045$	$c_{8,13} = 2.0669$
$c_{3,60} = 2.0275$	$c_{6,7} = 1.9841$	$c_3 = 2.0838$	$c_{6,50} = 2.1024$	$c_{9,9} = 2.0490$

The first reason (an evident one) is that the computers became more powerful. The second reason is that we used another definition of admissible shapes which allowed us to divide the amount of used memory by 3^{m-1} which is rather important when $m = 9$ (as it is pointed out in [4], for this kind of algorithms, “*the bottleneck in the computations is always memory*”). The third reason is that instead of long

arithmetics, we used computations mod different primes and then recovered the results with the Chinese Remainder Theorem. This trick allowed us to “convert” memory to time whose lack was not so crucial.

We have computed $f(3, n)$ till $n = 600$ and $f(4, n)$ till $n = 200$. The exact value of $f(3, 600)$ has 1127 digits and it yields $c_{3,600} = 2.07966\dots$. Comparing this with the limit value $c_3 = 2.08385$ we see that the convergence is very slow. For $m = 4$, the last computed exact value is

$$\begin{aligned}
 f(4, 200) = & 262199334303965073140522141167072596609151907003573304927487 \\
 & 419128543906730659218480439253346584137204205604500628092962 \\
 & 697997426095545403404830271634194339979807927812812142668569 \\
 & 097560203843935394728621308903256950859658838687531965864231 \\
 & 570521446370439565640979852878302993978768696718322811686043 \\
 & 307749541067654061321020767838164602474781629699981105797912 \\
 & 385346265396601164596410043968216134349971638142523003353406 \\
 & 530183843913302635663917084864069175263416748948835535483336 \\
 & 4717309018125451550646500; \quad c(4, 200) = 2.09455\dots
 \end{aligned}$$

In Tables 2–6 we present some other results of computations in the same format as in [4]. All the computed exact values are available on the webpage <https://www.math.univ-toulouse.fr/~orevkov/tr.html>

Table 3

n	# primitive triangulations of rectangle $6 \times n$	$c(6, n)$
1	924	1.6419
2	2801708	1.7848
3	12244184472	1.8617
4	61756221742966	1.9088
5	341816489625522032	1.9415
6	1999206934751133055518	1.9655
7	12169409954141988707186052	1.9840
8	76083336332947513655554918994	1.9987
9	484772512167266688498399632918196	2.0107
10	3131521959869770128138491287826065904	2.0206
11	20443767611927599823217291769468449488548	2.0289
12	134558550368400096364589064704536849131736024	2.0360
13	891513898740246853038326950483812868791208442016	2.0421
14	5938780824869668513059568892370775952933721743377354	2.0474
15	39738456660509411434285642370153959115525603844258515860	2.0521
\vdots	\vdots	\vdots
50	733088849377871573475229677373109896289395791929 288892292779893207423013116473882328714681504398 803902969400882970235141773360945092837017232937 1864995986534063127990363531908201551410584718	2.1023

Table 2

n	# primitive triangulations of rectangle $5 \times n$	$c(5, n)$
1	252	1.5954
2	182132	1.7474
3	182881520	1.8297
4	208902766788	1.8802
5	260420548144996	1.9155
6	341816489625522032	1.9415
7	464476385680935656240	1.9615
8	645855159466371391947660	1.9773
9	913036902513499041820702784	1.9902
10	1306520849733616781789190513820	2.0008
11	1887591165891651253904039432371172	2.0098
12	2747848427721241461905176361078147168	2.0174
13	4024758386310801427793602374466243714608	2.0240
14	5924744736041718687622958191829471010847132	2.0298
15	8757956199571261116690226598764501142088496860	2.0348
16	12991215957916577635251095613859465176216530106080	2.0394
17	19327902156972014645215931908930612218954616366464668	2.0434
18	28828843648796117963238681180919362090157971920576213992	2.0470
\vdots	\vdots	\vdots
115	18700706608364882730712710491937598381242505216572196 74626658766824095096227084981348969054292582022965697 97536209347455134357618461876316197344892595460029612 59669310339853198410108464789290118181041289819323068 31435995596306245022821112218622320544399050742600358 31426475886050757674088153732325783413307209633451618 73035677107305109076541667755690839416820326596	2.1044

2.3. Convexity conjecture for the numbers $f(m, n)$. The following conjecture is confirmed by all the computed exact values of the numbers $f(m, n)$ (we set by convention $f(m, 0) = 1$).

Conjecture 2.5. One has $f(m, n - 1)f(m, n + 1) \geq f(m, n)^2$ for any $m, n \geq 1$.

Proposition 2.6. *If Conjecture 2.5 holds true, then $c_m \geq (n + 1)c(m, n + 1) - nc(m, n)$ for any $m, n \geq 1$. In particular, Conjecture 2.5 would imply that $c \geq c_{115} \geq 5c(115, 5) - 4c(115, 4) = 2.1684837\dots$*

Proof. Let us set $d(m, n) = \log_2 f(m, n + 1) - \log_2 f(m, n)$. Then Conjecture 2.5 implies $d(m, n) \leq d(m, n + 1) \leq d(m, n + 2) \leq \dots$ whence $\log_2 f(m, n + k) - \log_2 f(m, n) \geq kd(m, n)$. Dividing by km and passing to the limit when $k \rightarrow \infty$, we obtain $c_m \geq d(m, n)/m = (n + 1)c(m, n + 1) - nc(m, n)$. \square

3. THE EXACT VALUE OF c_2 (PROOF OF THEOREM 1)

For $a, c \geq 0$, $a \equiv c \pmod 2$, let $g_{a,c}^*$ be the number of primitive lattice triangulations of the trapezoid $T(a, c)$ spanned by $(0, 0)$, $(a, 0)$, $(1, 2)$, $(1 + c, 2)$ (if $a = 0$ or $c = 0$, then $T(a, c)$ degenerates to a triangle). When $a \not\equiv c \pmod 2$, we set $g_{a,c}^* = 0$.

Table 4

n	# primitive triangulations of rectangle $7 \times n$	$c(7, n)$
1	3432	1.6778
2	43936824	1.8134
3	839660660268	1.8862
4	18792896208387012	1.9307
5	464476385680935656240	1.9615
6	12169409954141988707186052	1.9840
7	332633840844113103751597995920	2.0014
8	9369363517501208819530429967280708	2.0152
9	269621109753732518252493257828413137272	2.0264
10	7880009979020501614060394747170100093057300	2.0357
11	233031642883906149386619647304562977586311372556	2.0435
12	6953609830304518024125545674642770582274167760568260	2.0501
13	208980994833103266855771653608680330159883854051275967612	2.0559
\vdots	\vdots	\vdots
20	52066212145180734892042606757684021681422119	
	85233630730198914071476153736678384063983252	2.0813

Table 5

n	# primitive triangulations of rectangle $8 \times n$	$c(8, b)$
1	12870	1.7064
2	698607816	1.8362
3	58591381296256	1.9056
4	5831528022482629710	1.9480
5	645855159466371391947660	1.9773
6	76083336332947513655554918994	1.9987
7	9369363517501208819530429967280708	2.0152
8	1191064812882685539785713745400934044308	2.0282
9	155023302820254133629368881178138076738462112	2.0388
10	20527337238769032315796332007167102984745417344046	2.0476
11	2753810232976351788081274786378733309236298426977203848	2.0550
12	373119178357778061717948099980013460229206030805799398500854	2.0613
13	509513267535377736964009580351904	
	45392087069512323700346738258636	2.0668

We also set $g_{0,0}^* = 1$. Let $G^*(x, z)$ be the generating function for $g_{a,c}^*$:

$$\begin{aligned}
G^*(x, z) &= \sum_{a,c \geq 0} g_{a,c}^* x^a z^c \\
&= 1 + (x^2 + xz + z^2) + (6x^4 + 10x^3z + 12x^2z^2 + 10xz^3 + 6z^4) + \dots
\end{aligned}$$

(here the definition of $g_{a,c}^*$ and G^* is not the same as in Example 2.4). Let g_n^* be the coefficient of x^{2n} in the series $G^*(x, x) = \sum_{n \geq 0} g_n^* x^{2n}$, i.e.

$$g_n^* = g_{0,2n}^* + g_{1,2n-1}^* + g_{2,2n-2}^* + \dots + g_{2n,0}^*.$$

Then Theorem 1 follows immediately from Lemmas 3.1 and 3.2 below.

Table 6

n	# primitive triangulations of rectangle $9 \times n$	$c(9, n)$
1	48620	1.7299
2	11224598424	1.8547
3	4140106747178292	1.9214
4	1835933384812941453312	1.9621
5	913036902513499041820702784	1.9902
6	484772512167266688498399632918196	2.0107
7	269621109753732518252493257828413137272	2.0264
8	155023302820254133629368881178138076738462112	2.0388
9	91376512409462235694151119897052344522006298310908	2.0489

Lemma 3.1. $\lim_{n \rightarrow \infty} f(2, n)^{1/n} = \lim_{n \rightarrow \infty} (g_n^*)^{1/n}$.

Proof. The rectangle $2 \times (n - 1)$ can be placed into $T(n, n)$, hence $f(2, n - 1) < g_n^*$. On the other hand, the union of $T(a, c)$ with its image under the central symmetry with center $(\frac{1}{2}(a + c + 1), 1)$ is $T(a + c, a + c)$, and it can be placed into the rectangle $2 \times (a + c + 1)$, hence $(g_{a,c}^*)^2 < f(2, a + c + 1)$. Therefore

$$\frac{g_n^*}{2n} = \sum_{a+c=2n} \frac{g_{a,c}^*}{2n} \leq \max_{a+c=2n} g_{a,c}^* \leq f(2, 2n + 1)^{1/2} \leq (g_{2n+2}^*)^{1/2}$$

whence $\frac{1}{n}(\log g_n^* - \log(2n)) \leq \frac{1}{2n} \log f(2, 2n + 1) \leq \frac{1}{2n} \log g_{2n+2}^*$ and the result follows because $\frac{1}{n} \log(2n) \rightarrow 0$. \square

Lemma 3.2. $\lim_{n \rightarrow \infty} (g_n^*)^{1/n} = \alpha$ where α is as in Theorem 1.

Proof. For $a, c \geq 0$, $a \equiv c \pmod{2}$, let $g_{a,c}$ be the number of those primitive lattice triangulations of the trapezoid $T(a, c)$ which do not contain interior edges of the form $[(k, 0), (l, 2)]$, in other words, primitive lattice triangulations which agree with the subdivision of $T(a, c)$ into two triangles and two trapezoids depicted in Figure 3(left). If $a + c$ is odd, we set $g_{a,c} = 0$. By convention, we set $g_{0,0} = 0$. Let $G(x, z) = \sum_{a,c \geq 0} g_{a,c} x^a z^c$ be the generating function.

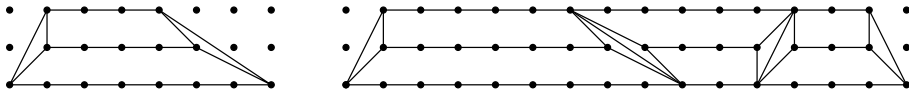


FIGURE 3

The edges of the form $[(k, 0), (l, 2)]$ of any primitive lattice triangulation cut $T(a, c)$ into smaller trapezoids. They can be transformed into $T(a_i, c_i)$'s with $\sum a_i = a$ and $\sum c_i = c$ by uniquely determined lattice automorphisms of the form $(x, y) \mapsto (x + p_i y + q_i, y)$ with $p_i, q_i \in \mathbb{Z}$ (see Figure 3). Hence

$$g_{a,c}^* = \sum_{\substack{a_1 + \dots + a_k = a \\ c_1 + \dots + c_k = c}} \prod_{j=1}^k g_{a_j, c_j}, \quad \text{thus} \quad G^*(x, z) = \frac{1}{1 - G(x, z)}. \quad (2)$$

Easy to see (cf. (1)) that the number of primitive lattice triangulations of the narrow (i.e. of width 1) trapezoids in Figure 3 are binomial coefficients, hence $G(x, z) = (x^2 + xz + z^2) + (5x^4 + 8x^3z + 9x^2z^2 + 8xz^3 + 5z^4) + \dots$

One can also check that $g_{a,c} = f_{a,(a+c)/2-1,c}$ where $f_{a,b,c} = f(S_{a,b,c})$ are the numbers discussed in Example 2.3. Hence (cf. Example 2.4)

$$\begin{aligned} G(x, z) &= \sum_{a,c} f_{a,(a+c)/2-1,c} x^a z^c = \text{coef}_{u^0} \left[\sum_{a,b,c} f_{a,b,c} x^a u^{2b-a-c+2} z^c \right] \\ &= \text{coef}_{u^0} \left[u^2 \sum_{a,b,c} f_{a,b,c} (x/u)^a u^{2b} (z/u)^c \right] = \text{coef}_{u^{-1}} \left[uF(x/u, u^2, z/u) \right]. \end{aligned}$$

Since the function $1/(1-x-y-z+xz) = 1/((1-x)(1-z)-y)$ is analytic in the domain $\max(|x|, |2y|, |z|) < 1/2$, its power series $\sum f_{a,b,c} x^a y^b z^c$ (see Example 2.3) converges to it in this domain. Therefore, for $0 < \varepsilon \ll r < 1/2$, the Laurent series of $F(x/u, u^2, z/u)$ converges in the domain $\max(|x|, |z|) < \varepsilon$, $r - \varepsilon < |u| < r + \varepsilon$. Hence, for x small enough, we have

$$G(x, x) = \text{coef}_{u^{-1}} [F(x/u, u^2, x/u)] = \frac{1}{2\pi i} \oint_{|u|=r} \frac{u \, du}{(1-x/u)^2 - u^2}$$

and

$$\begin{aligned} \frac{u}{(1-x/u)^2 - u^2} &= -\frac{u}{2(u^2 + u - x)} - \frac{u}{2(u^2 - u + x)} \\ &= \sum_{j=1}^2 \frac{1}{2(u_j^+ - u_j^-)} \left(\frac{u_j^+}{u - u_j^+} + \frac{u_j^-}{u - u_j^-} \right), \end{aligned}$$

where, for $|x|$ small enough,

$$u_1^\pm = -\frac{1}{2}(1 \pm \sqrt{1+4x}), \quad u_2^\pm = \frac{1}{2}(1 \pm \sqrt{1-4x}); \quad |u_j^+| > r, \quad |u_j^-| < r.$$

Thus

$$G(x, x) = \sum_{j=1}^2 \text{Res}_{u=u_j^-} (\dots) = \sum_{j=1}^2 \frac{u_j^-}{2(u_j^+ - u_j^-)} = \frac{1}{4\sqrt{1-4x}} + \frac{1}{4\sqrt{1+4x}} - \frac{1}{2}.$$

The graph of the function $y = G(x, x)$ sits in the algebraic curve

$$(2y+1)^2(16x^2-1)(4x^2+(y^2+y)(16x^2-1))+x^2=0.$$

By (2), the poles of $G^*(x, x)$ are the x -coordinates of the intersections of this curve with the line $y = 1$, i.e., the roots of $5184x^4 - 611x^2 + 18$ (the smallest ones being $\pm\sqrt{1/\alpha}$), and the branching points are $\pm 1/4$. Hence the radius of convergence of the series $G^*(x, x) = \sum g_n^* x^{2n}$ is $\sqrt{1/\alpha}$ whence $\lim_{n \rightarrow \infty} (g_n^*)^{1/n} = \alpha$. \square

4. COMPUTATION OF c_3 (PROOF OF THEOREM 2)

4.1. Preparation. For $a, d \geq 0$ such that $a \not\equiv d + 1 \pmod 3$, let $h_{a,d}^*$ be the number of primitive lattice triangulations of the trapezoid $T_3(a, d)$ spanned by $(0, 0), (1, 3), (1 + d, 3), (a, 0)$. We set $h_{0,0}^* = 1$ and $h_{a,d}^* = 0$ when $a \equiv d + 1 \pmod 3$, and we consider the generating function

$$H^*(x) = \sum_n h_n^* x^n = \sum_{a,d \geq 0} h_{a,d}^* x^{a+d} = 1 + x + 3x^2 + 19x^3 + 125x^4 + \dots$$

Similarly to the beginning of the proof of Lemma 3.2, we define $h_{a,d}$ as the number of the triangulations of $T_3(a, d)$ which do not have edges of the form $[(k, 0), (l, 3)]$ and we consider the generating function

$$H(x) = \sum_n h_n x^n = \sum_{a,d \geq 0} h_{a,d} x^{a+d} = x + 2x^2 + 14x^3 + 86x^4 + 712x^5 + \dots$$

These functions satisfy the relation similar to (2) specialized for $x = z$:

$$H^*(x) = 1/(1 - H(x))$$

Indeed, the edges of the form $[(k, 0), (l, 3)]$ cut $T_3(a, d)$ into smaller trapezoids. Each of them can be mapped to a standard one by a unique lattice automorphism of the form $(x, y) \mapsto (x + py + q, y \text{ or } 3 - y)$ with $p, q \in \mathbb{Z}$ (in contrary to §2, here the upper and lower horizontal sides of the trapezoids are mixed, so we do not have (2) for two-variable generating functions). In Figure 4 we illustrate the relation

$$h_3^* = h_{03}^* + h_{12}^* + h_{30}^* = h_{01}^3 + 2h_{01}(h_{11} + h_{20}) + (h_{03} + h_{12} + h_{30}) = h_1^3 + 2h_1 h_2 + h_3.$$

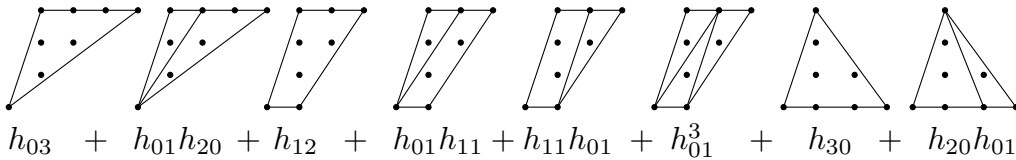


FIGURE 4

Similarly to Lemma 3.1, we have $\lim_n f(3, n)^{1/n} = \lim_n (h_{2n}^*)^{1/n} = 1/\beta^2$ where β is the real positive root of the equation $H(x) = 1$, hence $c_3 = -\frac{2}{3} \log_2 \beta$.

4.2. Recurrent relations. Using the notation introduced in §2, let us set $m = 3$, $\varphi_0(x) = \frac{1}{3}x - 1$, and

$$F(x, y, z, w) = \sum_{a,b,c,d} f_{a,b,c,d} x^a y^b z^c w^d,$$

$$G_1(x, z, w) = \sum_{a,c,d} g_{a,c,d}^{(1)} x^a z^c w^d, \quad G_2(x, y, w) = \sum_{a,b,d} g_{a,b,d}^{(2)} x^a y^b w^d$$

$$H_k(x, w) = \sum_{a,d} h_{a,d}^{(k)} x^a w^d, \quad (k = 1, 2)$$

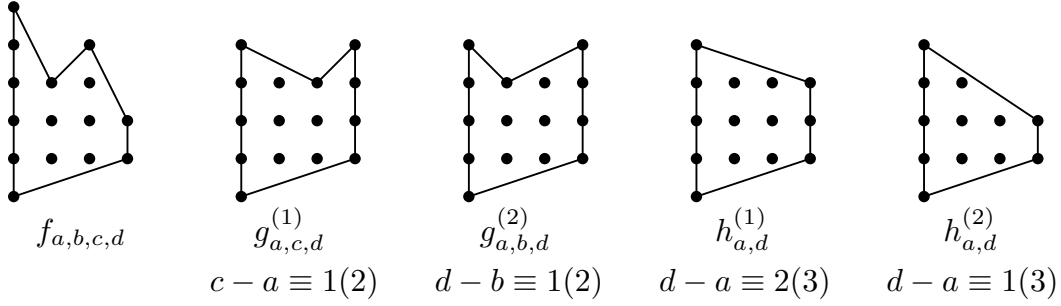


FIGURE 5

where all the coefficients are of the form $f(S)$ (see §2) for the φ_0 -admissible shapes in Figure 5 where $(0, a)$, $(1, b)$, $(2, c)$, and $(3, d)$ (if present) are the coordinates of integral points on the upper part of the boundary of S . The lower corners of S are at the points $(0, -1)$ and $(3, 0)$. If the congruences given in Figure 5 are not satisfied, then the corresponding numbers are zero. If $\min(a + 1, b, c, d) < 0$, they are also zero (this case does not correspond to any φ_0 -admissible shape). By convention, we also set $h_{-1,0}^{(2)} = 0$ (the case when S degenerates to a segment).

In terms of the generating functions, the recurrent relations in Lemma 2.2 read (cf. Examples 2.3, 2.4):

$$\begin{aligned} F(x, y, z, w)(1 - x - y - z - w + xz + yw + xw) \\ = y^{1/2}(1 - w)G_1(xy^{1/2}, y^{1/2}z, w) + z^{1/2}(1 - x)G_2(x, yz^{1/2}, z^{1/2}w), \end{aligned}$$

$$\begin{aligned} G_1(x, z, w)(1 - w) &= \text{coef}_{u^{-1}} [F(x/u, u^2, z/u, w)(1 - w)] + x^{-1}, \\ G_2(x, y, w)(1 - x) &= \text{coef}_{u^{-1}} [F(x, y/u, u^2, w/u)(1 - x)] \end{aligned}$$

(the asymmetry between G_1 and G_2 is caused by the asymmetry of φ_0),

$$\begin{aligned} H_1(x, w) &= \text{coef}_{u^{-1}} [G_1(x/u, u^3, w/u^2)], \\ H_2(x, w) &= \text{coef}_{u^{-1}} [G_2(x/u^2, u^3, w/u)]. \end{aligned}$$

Notice that in this subsection, by generating functions we mean formal series. Let us consider the symmetrized generating functions

$$\begin{aligned} \tilde{F}(x, y, z, w) &= F(x, y, z, w) + F(w, z, y, x), \\ \tilde{G}(x, z, w) &= G_1(x, z, w) + G_2(w, z, x), \\ \tilde{H}(x, w) &= H_1(x, w) + H_2(w, x). \end{aligned}$$

The above relations for F, G_1, G_2, H_1, H_2 imply immediately:

$$\begin{aligned} \tilde{F}(x, y, z, w)(1 - x - y - z - w + xz + yw + xw) \\ = y^{1/2}(1 - w)\tilde{G}(xy^{1/2}, y^{1/2}z, w) + z^{1/2}(1 - x)\tilde{G}(x, yz^{1/2}, z^{1/2}w), \end{aligned} \quad (3)$$

$$\tilde{G}(x, z, w)(1 - w) = \text{coef}_{u^{-1}} [\tilde{F}(x/u, u^2, z/u, w)(1 - w)] + x^{-1} \quad (4)$$

$$\tilde{H}(x, w) = \text{coef}_{u^{-1}} [\tilde{G}(x/u, u^3, w/u^2)]. \quad (5)$$

4.3. The equation. We are going to obtain an equation for $\tilde{G}(xt^{-1/2}, t^{3/2}, t^{-1}x)$ by expressing \tilde{F} via \tilde{G} from (3) and plugging it to (4). To this end we need to divide power series by polynomials. However, when some variables appear with powers varying from $-\infty$ to $+\infty$, the meaning of such division should be precised. To illustrate a possible ambiguity, let us consider the expression $\text{coef}_{u^{-1}}[1/(x-uy)]$. It can be understood either as

$$\text{coef}_{u^{-1}}\left[\frac{x^{-1}}{1-uyx^{-1}}\right] = \frac{1}{x}\text{coef}_{u^{-1}}\left[1 + \frac{uy}{x} + \frac{u^2y^2}{x^2} + \dots\right] = 0$$

or as

$$\text{coef}_{u^{-1}}\left[-\frac{(uy)^{-1}}{1-x(uy)^{-1}}\right] = -\text{coef}_{u^{-1}}\left[\frac{1}{uy}\left(1 + \frac{x}{uy} + \frac{x^2}{u^2y^2} + \dots\right)\right] = -\frac{1}{y}.$$

To avoid this kind of ambiguity, we introduce a new formal variable q and consider the formal series

$$\begin{aligned} F_q(x, y, z, w) &= F(xq, yq^2, zq^2, wq), \\ G_{1,q}(x, z, w) &= G_1(xq^2, zq^3, wq), \\ G_{2,q}(x, y, w) &= G_2(xq, yq^3, wq^2), \\ H_{k,q}(x, w) &= H_k(xq^3, wq^3), \quad k = 1, 2, \end{aligned}$$

and all the generating functions will be treated as elements of the ring

$$\mathbb{Z}[x^{\pm 1}, y^{\pm 1/2}, z^{\pm 1/2}, w^{\pm 1}, u^{\pm 1/2}, t^{\pm 1/2}]((q))$$

of formal power series in q (starting, maybe, with a negative power) whose coefficients are Laurent polynomials in $x, y^{1/2}, \dots$

The geometric meaning of an exponent of q is twice the doubled signed area of the φ_0 -admissible shape corresponding a monomial, i.e., $2 \int_0^3 \varphi(x) dx$ where the graph of φ is the upper boundary of the shape. One can easily check by hand that

$$\begin{aligned} F_q &= (xq)^{-1} + (1 + x^{-1}w) + (x + w + x^{-1}y + x^{-1}z + x^{-1}w^2)q + \dots \\ G_{1,q} &= x^{-1}q^{-2} + w(xq)^{-1} + w^2x^{-1} + w^3x^{-1}q + (x + w^4x^{-1})q^2 + \dots \\ G_{2,q} &= x^{-1}wq + (w + yx^{-1})q^2 + (wx + 2y)q^3 + (wx^2 + 4xy)q^4 + \dots \\ H_{1,q} &= wx^{-1} + xq^3 + 4w^2q^6 + (30wx^2 + 24w^4x^{-1})q^9 + \dots \\ H_{2,q} &= wq^3 + 5(x^2 + w^3x^{-1})q^6 + 32w^2xq^9 + \dots \end{aligned}$$

Further, we define $\tilde{F}_q, \tilde{G}_q, \tilde{H}_q$ by the same formulas as in §4.2 but with the subscript q everywhere. For example,

$$\tilde{G}_q(x, z, w) = \frac{1}{xq^2} + \frac{w}{xq} + \frac{w^2}{x} + \left(\frac{w^3}{x} + \frac{x}{w}\right)q + \left(2x + \frac{w^4}{x} + \frac{z}{w}\right)q^2 + \dots$$

Then the relations (3)–(5) take the form

$$\tilde{F}_q = \frac{qy^{1/2}(1-wq)\tilde{G}_q(xy^{1/2}, y^{1/2}z, w) + qz^{1/2}(1-xq)\tilde{G}_q(wz^{1/2}, z^{1/2}y, x)}{1-xq-yq^2-zq^2-wq+xzq^3+ywq^3+xwq^2}, \quad (6)$$

$$\tilde{G}_q(x, z, w) = \text{coef}_{u^{-1}} [q\tilde{F}_q(x/u, u^2, z/u, w)] + \frac{1}{x(1-wq)q^2}, \quad (7)$$

$$\tilde{H}_q(x, w) = \text{coef}_{u^{-1}} [q\tilde{G}_q(x/u, u^3, w/u^2)]. \quad (8)$$

Let us set

$$\begin{aligned} g_q(x, t) &= t^{1/2}x^2q^2\tilde{G}_q(x^2t^{-1/2}, x^3t^{3/2}, xt^{-1}) \\ &= t + xq + t^{-1}x^2q^2 + (t^{-2} + t)x^3q^3 + (t^{-3} + 2 + t^3)x^4q^4 + \dots \end{aligned}$$

The parity condition on the indices of nonzero coefficients of G_1 and G_2 (see Figure 5) ensures that the series $g_q(x, t)$ does not have fractional powers. Moreover, x and q appear in each monomial of g_q with the same power, thus we have $g_q(x, t) = g(xq, t)$ with $g(x, t) \in \mathbb{Z}[t^{\pm 1}]((x))$.

By plugging (3) into (4), denoting the denominator in (6) by $Q_q(x, y, z, w)$, and observing that

$$\text{coef}_{u^{-1}} [\mathcal{F}(x, t, u)] = \text{coef}_{u^{-1}} [xt^{-1/2}\mathcal{F}(x, t, uxt^{-1/2})] \quad (9)$$

for any formal Laurent series in u , we obtain

$$\begin{aligned} g_q(x, t) &\stackrel{(7)}{=} \text{coef}_{u^{-1}} \left[t^{1/2}x^2q^3\tilde{F}_q\left(\frac{x^2}{ut^{1/2}}, u^2, \frac{x^3t^{3/2}}{u}, \frac{x}{t}\right) \right] + \frac{t^2}{t-xq} \\ &\stackrel{(9)}{=} \text{coef}_{u^{-1}} \left[x^3q^3\tilde{F}_q\left(\frac{x}{u}, \frac{x^2u^2}{t}, \frac{x^2t^2}{u}, \frac{x}{t}\right) \right] + \frac{t^2}{t-xq} \\ &\stackrel{(6)}{=} x^2q^2\text{coef}_{u^{-1}} \left[\frac{\frac{u}{t}(1-\frac{xq}{t})g_q(x, t) + \frac{t}{u}(1-\frac{xq}{u})g_q(x, u)}{Q_q(x/u, x^2u^2/t, x^2t^2/u, x/t)} \right] + \frac{t^2}{t-xq} \\ &= x^2q^2\text{coef}_{u^{-1}} \left[\frac{u^3(t-xq)g_q(x, t) + t^3(u-xq)g_q(x, u)}{P(xq, t, u)} \right] + \frac{t^2}{t-xq} \end{aligned}$$

where

$$P(x, t, u) = u^2t^2 - (u+t)utx + (1-t^3-u^3)utx^2 + (t^4+u^4)x^3. \quad (10)$$

We see that the variables x and q are ‘‘synchronized’’ in the right hand side of the obtained equation: they occur with the same power in each monomial of each power series in this expression. Hence we obtain the following identity in the ring $\mathbb{Z}[t^{\pm 1}, u^{\pm 1}]((x))$:

$$g(x, t)\Psi(x, t) = \frac{t^2}{t-x} + \text{coef}_{u^{-1}} \left[\frac{t^3x^2(u-x)g(x, u)}{P(x, t, u)} \right] \quad (11)$$

where

$$\Psi(x, t) = 1 - x^2(t-x)\Phi(x, t), \quad \Phi(x, t) = \text{coef}_{u^{-1}} [u^3/P(x, t, u)].$$

Here are several initial terms of these series:¹

$$\begin{aligned} \Phi(x, t) &= t^{-2}x^2 + (t^{-3} + 1)x^3 + (t^{-4} + 2t^{-1} + t^2)x^4 + (t^{-5} + 3t^{-2} + 3t)x^6 + \dots \\ \Psi(x, t) &= 1 - t^{-1}x^4 - tx^5 - (1+t^3)x^6 - (t^{-1} + 2t^2)x^7 - (6t^{-2} + 3t)x^8 - \dots \end{aligned} \quad (12)$$

Having found g from (11), we can compute $\tilde{H}(x, x)$. Indeed, by (5) we have $x\tilde{H}_q(x^3, x^3) = \text{coef}_{t^0} [tx\tilde{G}_q(x^3/t, t^3, x^3/t^2)] = \text{coef}_{t^0} [t^{1/2}x\tilde{G}_q(x^3/t^{1/2}, t^{3/2}, x^3/t)]$.

Replacing t by x^2t (cf. (9)) and setting $q = 1$, we obtain

$$x\tilde{H}(x^3, x^3) = \text{coef}_{t^0} [g(x, t)]. \quad (13)$$

¹All coefficients of Φ and $1 - \Psi$ that I have computed are positive. If they are really all positive, it would be interesting to find their combinatorial meaning.

4.4. Computation. In this subsection we study the analytic functions defined by the series discussed in the previous subsection.

By §4.1, we need to find the smallest positive pole of $H^*(x)$, that is the smallest positive zero β of $1 - H(x)$. One can check that

$$H(x) = x\tilde{H}(x, x). \quad (14)$$

Being the sum of a power series with positive coefficients, the function $x\tilde{H}(x, x)$ is increasing when $x > 0$, thus it is enough know how to compute with any given precision the value of $\tilde{H}(x, x)$ for any fixed x in an interval containing β . By (13), this can be done by numerical integration of the function $g(x^{1/3}, t)$ along a suitable contour Γ_x (cf. the proof of Lemma 3.2). Thus we need to be able to compute $g(x, t)$ for any $x \in [0, x_0^+]$ and $t \in \Gamma_x$ for some $x_0^+ > x_0 = \beta^{1/3}$. This can be done because for a fixed x , after replacing $\text{coef}_{u^{-1}}[\dots]$ by $\frac{1}{2\pi i} \int_{\Gamma_x} (\dots) du$, the equation (11) becomes a Fredholm equation for the function g restricted to Γ_x . Now we pass to more detailed explanations.

Let

$$\begin{aligned} \Gamma &= \{(x, t, u) \in \mathbb{R} \times \mathbb{C}^2 \mid 0 < x < 1/2, |t| = |u| = 1\}, \\ \Gamma' &= \{(x, t) \in \mathbb{R} \times \mathbb{C} \mid 0 < x < 1/2, |t| = 1\}. \end{aligned}$$

Lemma 4.1. *The polynomial $P(x, t, u)$ defined in (10) does not vanish on Γ . For any fixed $(x, t) \in \Gamma'$, the polynomial $P(x, t, u)$ has two simple roots $u_k(x, t)$, $k = 1, 2$, in the unit disk $|u| < 1$ and two simple roots outside it.*

Proof. The first statements can be checked using any software for symbolic computations. This can be done, for example, as follows. Let S^1 be the unit circle in \mathbb{C} . Then $\Gamma = (0, 1/2) \times S^1 \times S^1$. We can identify S^1 with \mathbb{RP}^1 by some rational parametrization. Then $\text{Re } P$ and $\text{Im } P$ become real rational functions on the variety Γ and, by computing resultants, discriminant, etc., one can check that the real algebraic curve given by the equations $\text{Re } P = \text{Im } P = 0$ does not enter in the layer $0 < r < 1/2$. More precisely, let $p(x, T, U)$ and $q(x, T, U)$ be real polynomials such that

$$P(x, \zeta(T), \zeta(U)) = \frac{p(x, T, U) + iq(x, T, U)}{(i+T)^4(i+U)^4}, \quad \zeta(X) = \frac{i-X}{i+X}.$$

Note that $\zeta(\mathbb{R}) = S^1 \setminus \{-1\}$, hence (x, T, U) are coordinates on the affine chart $\Gamma \setminus \{(t+1)(u+1) = 0\}$ of Γ . The projection of the real algebraic curve $\Gamma \cap \{P = 0\}$ onto the plane (x, T) is given by the equation $R(x, T) = 0$ where $R(x, T)$ is the resultant of p and q with respect to U . To prove that the curve $R(x, T) = 0$ does not have real points with $0 < x < 1/2$, we compute the real roots of $D(x) = 0$ on this interval where $D(x)$ is the discriminant of R with respect to T , and we check that the equations $R(x_k, T) = 0$ for each $k = 1, \dots, 2n+1$ do not have real roots where $0 < x_1 < \dots < x_{2n+1} < 1/2$ and x_k with even k are all the real roots of $D(x)$ on the interval $0 < x < 1/2$. This computation shows that $P(x, u, t) \neq 0$ when $(x, u, t) \in \Gamma$ and $(t+1)(u+1) \neq 0$. Then we check that $P(x, \zeta(T), -1) \neq 0$, $P(x, -1, \zeta(U)) \neq 0$, and $P(x, -1, -1) \neq 0$ for $0 < x < 1/2$, $T \in \mathbb{R}$.

Similarly one can check that for any fixed $(x, t) \in \Gamma'$, the discriminant of P with respect to the variable u does not vanish, hence for any fixed $(x, t) \in \Gamma'$, all the four roots of P (viewed as a polynomial in u) are pairwise distinct.

Therefore, the number of roots of P in the unit disk $|u| < 1$ is constant. Thus, to prove the second statement, it is enough to check it for some value of x and t , for example, for $t = 1$ and a very small x . \square

Lemma 4.2. (a). *The formal power series $1/P(x, t, u) \in \mathbb{Z}[t^{\pm 1}, u^{\pm 1}]((x))$ converges to the function $1/P(x, t, u)$ in a neighborhood of $\Gamma \cap \{x < \frac{1}{4}\}$.*

(b). *The formal power series $\Phi(x, t) \in \mathbb{Z}[t^{\pm 1}]((x))$ converges to an analytic function (which we also denote by $\Phi(x, t)$) in a neighborhood of $\Gamma' \cap \{x < \frac{1}{4}\}$. The function $\Phi(x, t)$ admits an analytic continuation to a neighborhood of Γ' defined by the Cauchy integral*

$$\Phi(x, t) = \frac{1}{2\pi i} \oint_{|u|=1} \frac{u^3 du}{P(x, t, u)} = \sum_{k=1}^2 \frac{u_k(x, t)^3}{P'_u(x, t, u_k(x, t))} \quad (15)$$

where $u_1(x, t)$ and $u_2(x, t)$ are the roots of P in the unit disk $|u| < 1$; see Lemma 4.1.

Proof. The power series $1/P(x, t, u)$ involved in the definition $\Phi(x, t)$ is a power series expansion with respect to x , hence $1/P = a_0^{-1}(1 + X + X^2 + \dots)$ where $X = (a_1 + a_2 + a_3)/a_0$ and $a_k = x^k \text{coef}_{x^k}[P]$. If $(x, t, u) \in \Gamma$, then $|a_0| = 1$, $|a_1| \leq 2x$, $|a_2| \leq 3x^2$, $|a_3| \leq 2x^3$, and thus $|X| \leq 2x + 3x^2 + 2x^3$. Therefore $|X| < 1$ for $x < 1/4$, whence the convergence of $1/P$ in the required domain. This fact combined with Lemma 4.1 implies all the other assertions of the lemma. \square

Mathematica function `Psi` in Figure 7 computes $\Psi(x, t)$ for $(x, t) \in \Gamma'$ with any given precision.

Notice that one of the functions $u_1(x, t)$ or $u_2(x, t)$ has a ramification point at $(x, t) = (1/2, 1)$, and hence the functions Φ and Ψ are ramified in this point as well. The Laurent-Puiseux expansion of $\Psi(x, 1)$ in powers of $s = \sqrt{1/2 - x}$ is

$$\Psi(x, 1) = -\frac{1}{4\sqrt{6}}s^{-1} + \frac{12-\sqrt{2}}{8} - \frac{3}{8\sqrt{6}}s - \frac{3}{8\sqrt{2}}s^2 + \frac{103}{96\sqrt{6}}s^3 - \frac{87}{32\sqrt{2}}s^4 + \frac{2635}{192\sqrt{6}}s^5 + \dots$$

Let $x_0^- = \frac{16}{33}$ and $x_0^+ = \frac{17}{35}$. We shall see later that $x_0 \in [x_0^-, x_0^+]$; in fact, x_0^\pm are given by initial segments of the continued fraction of x_0 .

Using the expansion of Ψ at $(\frac{1}{2}, 1)$ and computing the values of $\Psi(x, t)$ (with the program in Figure 7) on a sufficiently dense grid on Γ' , one can check that Ψ does not vanish on $\Gamma' \cap \{x < x_0^+\}$ and

$$\min_{0 \leq x \leq x_0^+, |t|=1} |\Psi(x, t)| = \min_{0 \leq x \leq x_0^+, |t|=1} \text{Re } \Psi(x, t) = \Psi(x_0^+, 1) = 0.44768\dots \quad (16)$$

See the level lines of $\text{Re } \Psi$ in Figure 6; we omit the details of the error estimate.

Using Lemma 5.2 applied to the function $|P(x/4, e^{i\tau}, e^{i\theta})|^2$ with an appropriately chosen h , we find

$$\min_{x < x_0^+, |t|=|u|=1} |P| = P(x_0^+, 1, 1) = 0.02183\dots \quad (17)$$

(here we rescaled x to equilibrate the partial derivatives). The computation can be fastened by choosing different grid in different zones of Γ . In our computation, the grid step varied from $h = 1/300$ near the point of minimum to $h = 1/20$ far from

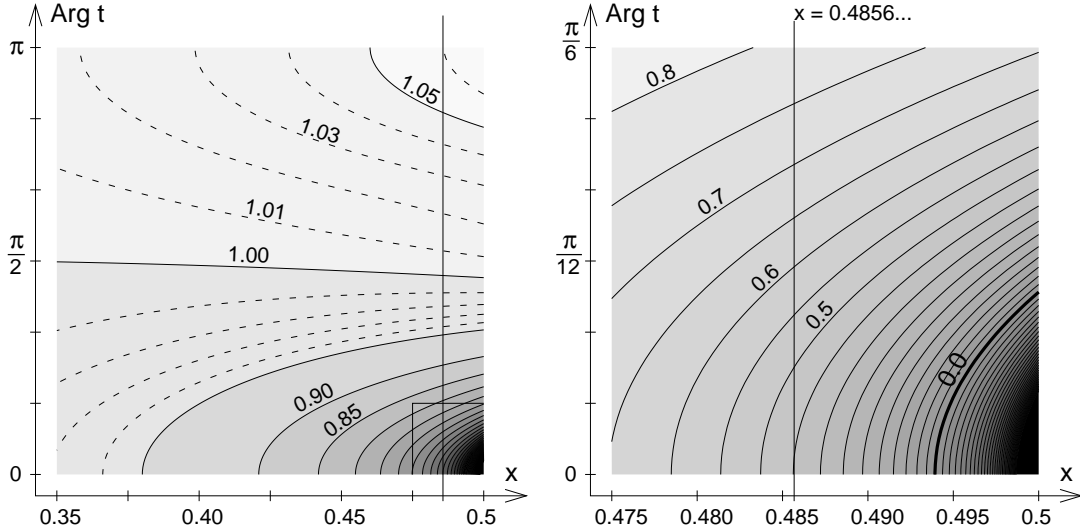


FIGURE 6. Level lines of $\operatorname{Re} \Psi(x, t)$ for $|t| = 1$. The shown vertical line is $x = x_0$ or $x = x_0^+$ (no difference with this resolution).

it. To estimate the error, we used evident coarse bounds for the fourth derivatives and, using them, computed finer upper bounds for the second derivatives in each zone again using Lemma 5.2.

Lemma 4.3. *The formal series $g(x, t)$ (introduced in §4.2) converges in some neighborhood of $\Gamma' \cap \{|x| < 2^{-3/2}\}$.*

Proof. By Anclin's theorem [1], the number of primitive lattice triangulations of a lattice polygon Π is bounded above by 2^N for $N = \#(\Pi \cap (\mathbb{Z}^2 \setminus \frac{1}{2}\mathbb{Z}^2))$, and it is easy to derive from Pick's formula that $N < 3\operatorname{Area}(\Pi) - 3/2$. The area of the shape corresponding to $g_{a,c,d}^{(k)}$ is $(2a + 3c + d + 3)/2$. Hence $\tilde{g}_{a,c,d} < c_0 2^{3(2a+3c+d)/2}$ for some constant c_0 and, for $|t| = 1$, we obtain

$$\begin{aligned} |g(x, t)| &\leq x^2 \sum_{a,c,d} |\tilde{g}_{a,c,d} x^{2a} x^{3c} x^d| \\ &\leq c_0 x^2 \sum_{a,c,d} |2^{3(2a+3c+d)/2} x^{2a+3c+d}| = c_0 x^2 \sum_n 2^{3/2n} A_n x^n, \end{aligned}$$

where $A_n = \#\{(a, c, d) \in \mathbb{Z}_+^3 \mid 2a + 3c + d = n\}$. Since A_n is bounded by a polynomial function of n , the series converges for $x < 2^{-3/2}$. \square

Lemmas 4.2 and 4.3 combined with (11) and (16) imply that the function $g(x, t)$ is analytic in a neighborhood of $\Gamma' \cap \{x < 2^{-3/2}\}$, and it satisfies the condition

$$g(x, t) = \frac{t^2}{(t-x)\Psi(x, t)} + \frac{1}{2\pi i} \oint_{|u|=1} \frac{x^2 t^3 (u-x) g(x, u) du}{P(x, t, u) \Psi(x, t)}. \quad (18)$$

For any fixed x , this is a Fredholm equation of the second kind for $g(x, t)$ considered as a function of t .

Lemma 4.4. *The function $g(x, t)$ analytically extends to a neighborhood of $\Gamma' \cap \{x < x_0^+\}$ and it satisfies the equation (18) in this domain.*

Proof. Let us rewrite (18) in a more conventional form

$$\varphi_g(x, \tau) = f(x, \tau) + \int_0^1 K(x, \tau, \theta) \varphi_g(x, \theta) d\theta \quad (19)$$

where we set $t = e^{2\pi i\tau}$, $u = e^{2\pi i\theta}$, and

$$\varphi_g(x, \tau) = g(x, t), \quad f(x, \tau) = \frac{t^2}{(t-x)\Psi(x, t)}, \quad K(x, \tau, \theta) = \frac{x^2 t^3 u(u-x)}{P(x, t, u)\Psi(x, t)}.$$

As we already pointed out, g satisfies (18) and thus φ_g satisfies (19) for small x . Thus, thanks to the Identity Theorem for analytic functions, it is enough to show that for any $x \in [0, x_0^+]$ there exists a unique solution of (19) and that it is analytic with respect to (x, τ) . Hence, by Lemma 5.6, it suffices to show that 1 is not an eigenvalue of \mathcal{K}_x for any $x \in [0, x_0^+]$ where $\mathcal{K}_x : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$ is the the Fredholm integral operator which takes $\varphi(\tau)$ to $\psi(\tau) = \int_0^1 K(x, \tau, \theta) \varphi(\theta) d\theta$. The latter fact, in its turn, follows from the bound

$$\max_{0 \leq x \leq x_0^+} \mathcal{N}_2(x) = \mathcal{N}_2(x_0^+) = 0.88525$$

where $\mathcal{N}_2(x) = \int_{[0,1]^2} |K(x, \tau, \theta)|^2 d\tau d\theta$. This bound is computed by numerical integration. To estimate the approximation error, one needs upper bounds of partial derivatives of K . They can be easily obtained using the lower bounds (16) and (17) of $|\Psi|$ and $|P|$, and upper bounds of the derivatives of Ψ obtained from its integral form in (15). For upper bounds of the derivatives of polynomials involved in the definition of K one can use just the sums of upper bounds of monomials. \square

Replacing the integrals by integral sums, equation (18) can be solved with any given precision. Then, due to (13) and (14) we can numerically compute $H(x)$ using the Cauchy integral

$$H(x^3) = \frac{x^2}{2\pi i} \oint_{|t|=1} \frac{g(x, t) dt}{t} = x^2 \int_0^1 \varphi_g(x, \tau) d\tau \quad (20)$$

(recall that $\varphi_g(x, \tau) := g(x, e^{2\pi i\tau})$; see (19)). We can summarize the content of this section as follows (recall that $f(m, n)$ is the number of primitive lattice triangulations of the rectangle $m \times n$).

Proposition 4.5. $\lim_{n \rightarrow \infty} f(3, n)^{1/n} = 1/x_0^2$ where:

- x_0 is a unique solution of the equation $H(x^3) = 1$ on the interval $[0, x_0^+]$ with $x_0^+ = \frac{17}{35}$;
- $H(x)$ is defined via $g(x, t)$ by (20) and it is monotone on $[0, x_0^+]$;
- $g(x, t)$ is the solution of the Fredholm equation (18) whose ingredients P and Ψ are defined by (10) and by $\Psi(x, t) = 1 - x^2(t-x)\Phi(x, t)$ with Φ defined by (15); for any $x \in [0, x_0^+]$ the equation (18) has a unique solution.

In Figure 7 we present a Mathematica function \mathbf{H} which computes $H(x)$ with any given precision. The approximating error can be estimated using Lemma 5.4. One can check that the functions $P(x, t, u)$ and $\Psi(x, t)$ do not vanish when $x < x_0^+$,

```

P = u^2*t^2-(u+t)u*t*x+(1-t^3-u^3)u*t*x^2+(t^4+u^4)x^3;
Psi = Function[{x0,t0,prec},Module[{P0,u0,i},
  P0=P/.{x->x0,t->t0}; u0 = NRroots[0==P0,u,prec];
  u0=Sort[Table[{Abs[u0[[i,2]]],u0[[i,2]]},{i,4}]];
  1-x0^2(t0-x0)Sum[(u^3/D[P0,u])/u->u0[[i,2]],{i,2}]];
H = Function[{x3,n,prec},
  Module[{x0,z,P0,Id,K,F,G,j,k,Tj,Uk,PsiTj,Pjk},
    x0=N[x3^(1/3),prec]; z=N[Exp[2Pi*I/n],prec];
    K=Id=IdentityMatrix[n]; F=K[[1]]; P0=P/.x->x0;
    Do[ Tj = z^j; PsiTj=Psi[x0,Tj,prec];
      F[[j]] = Tj^2/(Tj-x0)/PsiTj;
      Do[ Uk = z^k; Pjk=P0/.{t->Tj,u->Uk};
        K[[j,k]] = x0^2*Tj^3(Uk-x0)Uk/Pjk/PsiTj/n,
        {k,n}],
      {j,n}];
    G = Inverse[Id-K].F; x0^2*(Plus@@G)/n]];

```

FIGURE 7. Mathematica code for computation of $H(x)$

$|u| = 1$, and $\frac{10}{13} < |t| < \frac{13}{10}$. In Figures 8 and 9 we show the image of the annulus $\frac{10}{13} < |t| < \frac{13}{10}$ under the mapping $t \mapsto \Psi(x_0, t)$. Thus we can apply the error estimate (27) with $r = 10/13$ and hence $a = -\frac{\log r}{2\pi} = 0.04176$. When estimating the error of $H(x)$ with $x \approx x_0$, we can set in (27)

$$C \leq 1; \quad \frac{1}{n}\|B\|_1 \leq 3.05; \quad M \leq 3910; \quad M' \leq 94.6; \quad M_f \leq 258.$$

Then we obtain the error estimate presented in the last column of Table 7. We see that it is reasonably close to the actual error which is given in the 4th column.

Table 7

n	prec.	time (sec.)	n -th approx. of $H(x_0) - 1$	error estimate
100	24	0.299391	1.44×10^{-10}	6.95×10^{-4}
200	36	6.759046	5.01×10^{-22}	5.60×10^{-15}
300	48	21.77949	1.73×10^{-33}	3.39×10^{-26}
400	60	51.22560	6.02×10^{-45}	1.82×10^{-37}
500	72	115.5499	2.09×10^{-56}	9.19×10^{-49}
600	84	231.5893	7.26×10^{-68}	4.45×10^{-60}
700	96	380.6020	2.52×10^{-79}	2.09×10^{-71}
800	108	608.9937	8.78×10^{-91}	9.65×10^{-83}
900	120	869.7188	3.06×10^{-102}	4.38×10^{-94}
1000	132	1072.923	1.06×10^{-113}	1.96×10^{-105}
1100	144	1456.021	3.72×10^{-125}	8.70×10^{-117}
1200	156	1852.763	1.29×10^{-136}	3.83×10^{-128}

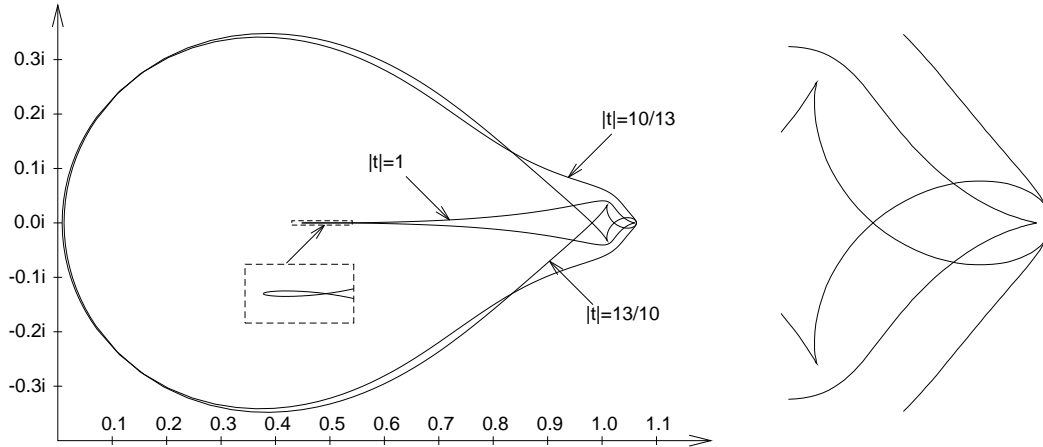


FIGURE 8. A realistic drawing of the image of the circles $|t| = \frac{10}{13}$, $|t| = 1$, and $|t| = \frac{13}{10}$ by the mapping $t \mapsto \Psi(x_0, t)$. The left zoom is stretched in the vertical direction.

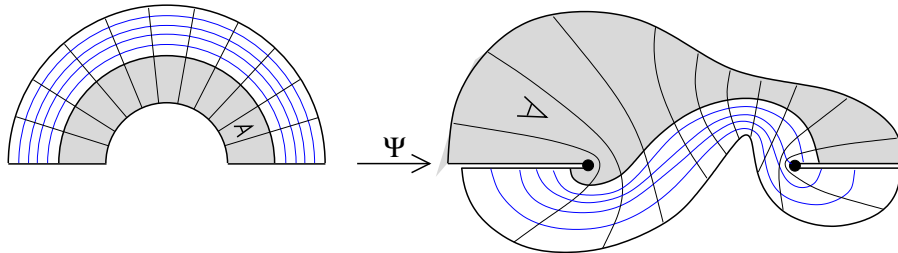


FIGURE 9. A schematic drawing of the image of the upper half-annulus $\{\frac{10}{13} \leq |t| \leq \frac{13}{10}, \text{Im } t \geq 0\}$ by the mapping $t \mapsto \Psi(x_0, t)$. The lower half-annulus is mapped symmetrically. The gray part is $|t| < 1$.

5. APPROXIMATE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS WITH ANALYTIC KERNELS

5.1. Error estimates. Generalities. The notation in this subsection is independent of the notation in the rest of the paper.

Lemma 5.1. *Let f be a holomorphic function in a neighborhood of the annulus $R_1 < |z| < R_2$, and let $f(z) = \sum_{n \in \mathbb{Z}} c_n z^n$ be its Laurent series. Then, for $R_1 < r < R_2$ and for any $n > 0$,*

$$\left| c_0 - \frac{1}{n} \sum_{k=1}^n f(r\omega^k) \right| = \left| \int_0^1 f(re^{2\pi it}) dt - \frac{1}{n} \sum_{k=1}^n f(r\omega^k) \right| \leq \frac{M_1 q_1^n}{1 - q_1^n} + \frac{M_2 q_2^n}{1 - q_2^n} \quad (21)$$

where $\omega = e^{2\pi i/n}$, $q_1 = R_1/r$, $q_2 = r/R_2$, and $M_j = \max_{|z|=R_j} |f(z)|$ for $j = 1, 2$.

Proof. We have

$$\sum_{k=1}^n f(r\omega^k) = \sum_{k=1}^n \sum_{m \in \mathbb{Z}} c_m (r\omega^k)^m \quad \text{and} \quad \sum_{k=1}^n \omega^{km} = \begin{cases} n, & n \text{ divides } m, \\ 0, & \text{otherwise} \end{cases}$$

hence the left hand side of (21) is equal to $|\sum_{p \in \mathbb{Z} \setminus \{0\}} c_{pn} r^{pn}|$ and the coefficients can be estimated using the Cauchy integrals. \square

Lemma 5.2. *Let $h > 0$ and $D \subset \mathbb{R}^d$ be a product of segments $[0, n_1 h] \times \cdots \times [0, n_d h]$ with positive integers n_1, \dots, n_d . Let $f : D \rightarrow \mathbb{R}$ be a function of class C^2 and $M = \max_{i,j} \max_D |\partial_i \partial_j f|$. Then*

$$\min_D f \geq \min_{D \cap h\mathbb{Z}^d} f - \frac{1}{8} M d^2 h^2$$

where $h\mathbb{Z}^d = \{h\vec{n} \mid \vec{n} \in \mathbb{Z}^d\}$. A similar estimate holds for $\max_D f$.

Proof. Induction on d . Let the minimum be attained at $x_0 \in D$. If x_0 is in the interior of D , then we estimate $|f(x) - f(x_0)|$ for the nearest to x_0 grid point x using the Taylor–Lagrange formula for the second order expansion of $f(x_0 + t(x - x_0))$ at $t = 0$. If x_0 is on the boundary of D , then we apply the induction hypothesis to the restriction of f to the facet of D containing x_0 . \square

5.2. Error estimates for approximate solutions of Fredholm equations.

Let $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ be a continuous solution of the Fredholm integral equation

$$\varphi(x) = \int_0^1 K(x, y) \varphi(y) dy + f(x) \quad (22)$$

with analytic complex-valued functions K and f which are (bi)-periodic with period 1, i.e., $K(x, y) = K(x + 1, y) = K(x, y + 1)$ and $f(x) = f(x + 1)$. Assume that K and f extend to complex analytic functions in a neighborhood of $(D \times \mathbb{R}) \cup (\mathbb{R} \times D_1)$ in \mathbb{C}^2 and in a neighborhood of D in \mathbb{C} respectively where

$$D = \{z \in \mathbb{C} \mid -a \leq \operatorname{Im} z \leq a\}, \quad D_1 = \{z \in \mathbb{C} \mid -a_1 \leq \operatorname{Im} z \leq a_1\}, \quad 0 < a_1 < a.$$

Let us set

$$C = \int_0^1 |\varphi(x)| dx, \quad M = \max_{D \times \mathbb{R}} |K|, \quad M'_1 = \max_{\mathbb{R} \times D_1} |K|, \quad M_f = \max_D |f|.$$

Lemma 5.3. *The function φ analytically extends to a neighborhood of D and*

$$M_\varphi := \max_{D_1} |\varphi| \leq \frac{a(CM + M_f)}{a - a_1}. \quad (23)$$

Proof. For any $(x_0, y_0) \in \mathbb{R}^2$ and any n , we have

$$\left| \partial_x^n K(x_0, y_0) \right| \leq \left| \frac{n!}{2\pi i} \int_{|z|=a} \frac{K(x_0 + z, y_0) dz}{z^{n+1}} \right| \leq \frac{Mn!}{a^n}$$

and similarly $|f^{(n)}(x_0)| \leq M_f n! / a^n$. Then, differentiating (22) n times with respect to x , we obtain

$$\left| \varphi^{(n)}(x_0) \right| = \left| \int_0^1 \partial_x^n K(x_0, y) \varphi(y) dy + f^{(n)}(x_0) \right| \leq \frac{(CM + M_f)n!}{a^n}. \quad (24)$$

Hence the Taylor series of φ at x_0 converges in the disk $|z - x_0| < a$ and, for $|z - x_0| \leq a_1$, we have

$$|\varphi(z)| = \left| \sum_{n \geq 0} \frac{\varphi^{(n)}(x_0)}{n!} (z - x_0)^n \right| \leq \sum_{n \geq 0} \frac{(CM + M_f)a_1^n}{a^n} = \frac{a(CM + M_f)}{a - a_1}$$

whence the required bound for M_φ . \square

For a positive integer n , let us see what happens if we replace the integral in (22) by the n -th integral sum. Namely, consider the vectors $\varphi^{[n]} = (\varphi_1^{[n]}, \dots, \varphi_n^{[n]})$, $f^{[n]} = (f_1^{[n]}, \dots, f_n^{[n]})$, and the $n \times n$ matrix $K^{[n]} = (K_{jk}^{[n]})_{jk}$ defined by

$$\varphi_j^{[n]} = \varphi(j/n), \quad f_j^{[n]} = f(j/n), \quad K_{jk}^{[n]} = \frac{1}{n}K(j/n, k/n).$$

Let $\hat{\varphi}^{[n]} = (\hat{\varphi}_1^{[n]}, \dots, \hat{\varphi}_n^{[n]})$ be a solution of the equation

$$\hat{\varphi}^{[n]} = K^{[n]}\hat{\varphi}^{[n]} + f^{[n]}. \quad (25)$$

This equation is a discretization of (22) and it is natural to expect that $\hat{\varphi}^{[n]}$ well approximates φ . Now, following the approach from [5], we estimate the rate of the convergence. Our final purpose is to find a good upper bound for the approximating error

$$E_n := \left| \int_0^1 \varphi(x) dx - \frac{1}{n} \sum_{j=1}^n \hat{\varphi}_j^{[n]} \right|.$$

We define the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, on \mathbb{C}^n in the usual way. For a square matrix $A = (a_{jk})_{jk}$ with complex entries we set

$$\|A\|_1 = \sum_{j,k} |a_{jk}|, \quad \|A\|_2 = \left(\sum_{j,k} |a_{jk}|^2 \right)^{1/2}.$$

Lemma 5.4. (a). *Suppose that the matrix $A^{[n]} = I - K^{[n]}$ is invertible and denote its inverse by $B^{[n]}$. Then*

$$E_n \leq \frac{2a(CM + M_f)r_1^n}{(a - a_1)(1 - r_1^n)} \left(1 + \frac{1}{n} \|B^{[n]}\|_1 M'_1 \right), \quad r_1 = e^{-2\pi a_1}. \quad (26)$$

If K analytically extends to a neighborhood of $\mathbb{R} \times D$ and $M' = \max_{\mathbb{R} \times D} K$,

$$E_n \leq 4\pi e(CM + M_f) \left(1 + \frac{1}{n} \|B^{[n]}\|_1 M' \right) \frac{nar^n}{1 - er^n}, \quad r = e^{-2\pi a}. \quad (27)$$

For $n > \alpha_n$, we have

$$C \leq \frac{\|\hat{\varphi}^{[n]}\|_1 + \alpha_n M_f}{n - \alpha_n} \quad \text{where} \quad \alpha_n = \frac{2M'_1 a \|B^{[n]}\|_1 r_1^n}{(a - a_1)(1 - r_1^n)} + \frac{1}{4a}. \quad (28)$$

(b). *Suppose that $\|K^{[n]}\|_2 = M_2 < 1$. Then $A^{[n]}$ is invertible and $\frac{1}{n} \|B^{[n]}\|_1 \leq 1/(1 - M_2)$ which implies in particular that $\alpha_n < \alpha_0$ for some constant $\alpha_0 = \alpha_0(a, a_1, M, M'_1, M_2, M_f)$ and hence C can be estimated using (28) for $n > \alpha_0$.*

Proof. (a). Let $J = \int_0^1 \varphi(x) dx$, $S = \frac{1}{n} \sum_j \varphi(j/n)$, $\hat{S} = \frac{1}{n} \sum_j \hat{\varphi}(j/n)$, $\rho = \varphi^{[n]} - \hat{\varphi}^{[n]}$, and $\sigma = A^{[n]}\rho$. In this notation, $E_n = |J - \hat{S}|$. We have

$$\|\sigma\|_\infty = \|A^{[n]}\varphi^{[n]} - A^{[n]}\hat{\varphi}^{[n]}\|_\infty \stackrel{(25)}{=} \|A^{[n]}\varphi^{[n]} - f^{[n]}\|_\infty = \|K^{[n]}\varphi^{[n]} - (\varphi^{[n]} - f^{[n]})\|_\infty$$

By (22), we have $\varphi_j^{[n]} - f_j^{[n]} = \int_0^1 K(j/n, y)\varphi(y) dy$, and the j -th component of the vector $K^{[n]}\varphi^{[n]}$ is the n -th integral sum for this integral. Hence, applying Lemma 5.1 to the functions $K(j/n, z(\zeta))\varphi(z(\zeta))$ after the change of variable $\zeta = e^{2\pi iz}$, we obtain $\|\sigma\|_\infty \leq M'_1 C_1$ with $C_1 = 2M_\varphi r_1^n / (1 - r_1^n)$ and then

$$\|\rho\|_1 = \|B^{[n]}\sigma\|_1 \leq \|B^{[n]}\|_1 \times \|\sigma\|_\infty \leq M'_1 C_1 \|B^{[n]}\|_1. \quad (30)$$

Lemma 5.1 applied to $\varphi(z(\zeta))$ yields $|J - S| \leq C_1$. We also have $|S - \hat{S}| \leq \frac{1}{n} \|\rho\|_1$, hence

$$E_n = |J - \hat{S}| \leq |J - S| + |S - \hat{S}| \leq C_1 + \frac{1}{n} \|\rho\|_1 \leq C_1 + \frac{1}{n} M'_1 C_1 \|B^{[n]}\|_1$$

which yields (26) after applying (23). Setting $a_1 = a - \frac{1}{2\pi n}$ (hence $r_1 = e^{1/n} r$) and $M'_1 < M'$ in (26), we obtain (27).

Let us prove (28). It is easy to check that

$$nC \leq \|\varphi^{[n]}\|_1 + \frac{1}{4} \max_{\mathbb{R}} |\varphi'| \leq \|\hat{\varphi}^{[n]}\|_1 + \|\rho\|_1 + \frac{1}{4} \max_{\mathbb{R}} |\varphi'|.$$

Using the estimates (30) and (24) for $\|\rho\|_1$ and $|\varphi'|$ respectively, we obtain

$$nC \leq \|\hat{\varphi}^{[n]}\|_1 + \frac{2M_1 M_\varphi \|B^{[n]}\|_1 r_1^n}{1 - r_1^n} + \frac{CM + M_f}{4a} \stackrel{(23)}{\leq} \|\hat{\varphi}^{[n]}\|_1 + (CM + M_f)\alpha_n.$$

(b). Suppose now that $\|K^{[n]}\|_2 = M_2 < 1$. Then $\|B^{[n]}\|_2 = \|(I - K^{[n]})^{-1}\|_2 = \|I + K^{[n]} + (K^{[n]})^2 + \dots\|_2 \leq 1/(1 - M_2)$. By Cauchy Inequality we also have $\|B^{[n]}\|_1 \leq n\|B^{[n]}\|_2$ \square

5.3. A numerical criterion of existence and uniqueness of solutions. Here we keep the above assumptions about $K(x, y)$ and $f(x)$ except that we no longer assume a priori that equation (22) has a continuous solution φ . Let $\mathcal{K} : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$ be the Fredholm integral operator with kernel $K(x, y)$, i.e., the operator $\varphi \mapsto \psi$ where $\psi(x) = \int_0^1 K(x, y)\varphi(y) dy$.

Lemma 5.5. (cf. [5, Ch.II, §1, Eq. (26)]). *Suppose that there exists n such that the matrix $I - K^{[n]}$ is invertible and $\alpha_n < n$ where α_n is defined in (28) (note that neither f nor φ is used in the definition of α_n). Then 1 is not an eigenvalue of \mathcal{K} and hence, for any given continuous function f , equation (22) has a unique continuous solution φ .*

Proof. Let n be such that $\alpha_n < n$. Let us apply Lemma 5.4(a) when $f = 0$ and hence $\hat{\varphi}^{[n]} = 0$. Then (28) reads $C \leq 0$ which means that there are no non-zero solutions of the equation $\mathcal{K}\varphi = \varphi$, i.e. 1 is not an eigenvalue of \mathcal{K} . By Fredholm Theorem [2], in this case (22) has a unique continuous solution for any f . \square

5.4. Analyticity of solutions with respect to a parameter. Let Λ be a domain in \mathbb{C} and $U = \{z \in \mathbb{C} \mid -a \leq \operatorname{Im} z \leq a\}$, $a > 0$. Let $K(\lambda, x, y)$ be an analytic function in a neighborhood of $\Lambda \times U^2$ in \mathbb{C}^3 and $f(\lambda, y)$ be an analytic function in a neighborhood of $\Lambda \times U$ in \mathbb{C}^2 . We assume that $K(\lambda_0, x, y)$ is $(1, 1)$ -biperiodic and $f(\lambda_0, x)$ is 1-periodic for any fixed $\lambda_0 \in \Lambda$.

For $\lambda \in \Lambda$, let $\mathcal{K}_\lambda : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$ be the Fredholm integral operator $\varphi \mapsto \psi$, $\psi(x) = \int_0^1 K(\lambda, x, y)\varphi(y)dy$. The next lemma immediately follows from Fredholm's results in his seminal paper [2] (a more general fact is proven in [11]).

Lemma 5.6. *Suppose that 1 is not an eigenvalue of \mathcal{K}_λ for any $\lambda \in \Lambda$. Then, for any $\lambda \in \Lambda$, there exists a unique solution $\varphi(\lambda, x)$ of the equation*

$$\varphi(\lambda, x) = \int_0^1 K(\lambda, x, y)\varphi(\lambda, y) dy + f(\lambda, x) \quad (31)$$

and the function $\varphi(\lambda, x)$ is analytic in a neighborhood of $\Lambda \times U$.

Proof. By Fredholm's results [2] (see also [6]), for any $\lambda \in \Lambda$, the solution $\varphi(\lambda, x)$ is unique under our assumptions and it can be written as

$$\varphi(\lambda, x) = f(\lambda, x) + \int_0^1 \frac{D(\lambda, x, y)}{D(\lambda)} f(\lambda, y) dy$$

where

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n A_n(\lambda)}{n!}, \quad D(\lambda, x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n(\lambda, x, y)}{n!}, \quad (32)$$

$$A_n(\lambda) = \int_{[0,1]^n} K(\lambda, \mathbf{x}, \mathbf{x}) d\mathbf{x}, \quad B_n(\lambda, x, y) = \int_{[0,1]^n} K(\lambda, x, \mathbf{x}, y, \mathbf{x}) d\mathbf{x},$$

$$K(\lambda, x_1, \dots, x_n, y_1, \dots, y_n) = \det (K(\lambda, x_i, y_j))_{i,j=1}^n.$$

It is shown in [2] that $D(\lambda)$ does not vanish on U (because 1 is not an eigenvalue of \mathcal{K}_λ for any $\lambda \in \Lambda$). It is clear that the functions A_n and B_n are analytic in Λ and in $\Lambda \times U^2$ respectively and the Hadamard Inequality $|\det N| \leq n^{n/2} \max_{i,j} |N_{ij}|$ implies the upper bounds (cf. [2, p. 368, line 4]):

$$|A_n(\lambda)| \leq n^{n/2} M(\lambda)^n, \quad |B_{n-1}(\lambda, x, y)| \leq n^{n/2} M(\lambda)^n$$

where $M(\lambda) = \sup_{(x,y) \in U^2} |K(\lambda, x, y)|$. Hence the series (32) converge to analytic functions whence the result. \square

6. NON-PRIMITIVE LATTICE TRIANGULATIONS

Denote the number of all (not necessarily primitive) lattice triangulations of the $m \times n$ rectangle by $f^{\text{np}}(m, n)$, and set

$$c^{\text{np}} = \lim_{n \rightarrow \infty} \frac{\log_2 f^{\text{np}}(n, n)}{n^2}.$$

Proposition 6.1. $c^{\text{np}} \leq 4.735820221\dots$

Proof. Let $N = n^2$. Any lattice triangulation can be subdivided up to a primitive lattice triangulation. Hence a lattice triangulation is completely determined by a choice of a primitive lattice triangulation and a set of its edges to be removed. Let $f_k^{\text{np}}(n, n)$ be the number of lattice triangulations of the $n \times n$ square with k interior vertices and hence with $\approx 3k$ edges. Then

$$f_k^{\text{np}}(n, n) \leq \binom{3N}{3k} 2^{cN} \quad (33)$$

(recall that 2^{cN} is a bound for the number of primitive lattice triangulations). On the other hand the number of triangulations with vertices in an arbitrary fixed set of k points on a plane is bounded above by 30^k (see [10]), hence

$$f_k^{\text{np}}(n, n) \leq \binom{N}{k} 30^k. \quad (34)$$

Combining (33) and (34) with Stirling formula, we obtain

$$c^{\text{np}} \leq \max_{0 \leq x \leq 1} \min(3h(x) + c, h(x) + x \log_2 30), \quad (35)$$

$$h(x) = -x \log_2 x - (1 - x) \log_2(1 - x).$$

Using the bound $c \leq 4 \log_2 \frac{1+\sqrt{5}}{2}$ (see [7], [12], [13]), we obtain the result (the maximum in (35) is attained at $x = 0.83206855$). \square

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STEKLOV MATHEMATICAL INSTITUTE, GUBKINA 8, MOSCOW, RUSSIA

IMT, L'UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, TOULOUSE, FRANCE

AGHA LABORATORY, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, RUSSIA
E-mail address: `orevkov@math.ups-tlse.fr`