SEPARATING SEMIGROUP OF HYPERELLIPTIC CURVES AND OF GENUS 3 CURVES

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Abstract. A rational function on a real algebraic curve C is called separating if it takes real values only at real points. Such a function defines a covering $\mathbb{R}C \to \mathbb{RP}^1$. Let $A_1, \dots, A_n$ be connected components of $\mathbb{R}C$. In a recent paper, M. Kummer and K. Shaw defined the separating semigroup of C as the set of all sequences $(d_1(f), \dots, d_n(f))$ where f is a separating function and d_i is the degree of the restriction of f to A_i.

We describe the separating semigroup for hyperelliptic curves and for genus 3 curves.

1. Introduction

By a *real algebraic curve* we mean a complex algebraic curve C endowed with an antiholomorphic involution $\mathrm{conj} : C \to C$ (the complex conjugation involution). In this case we denote the *real locus* $\{p \in C \mid \mathrm{conj}(p) = p\}$ by $\mathbb{R}C$. A real curve is of *dividing type* (or of *type I*) if $\mathbb{R}C$ divides C into two halves exchanged by the complex conjugation. All curves considered here are smooth and irreducible.

A sufficient condition for C to be of dividing type is the existence of a *separating morphism* $f : C \to \mathbb{P}^1$, that is a morphism such that $f^{-1}(\mathbb{RP}^1) = \mathbb{R}C$. It follows from Ahlfors' results [1] that this condition is also necessary: any real curve of dividing type admits a separating morphism. The restriction of a separating morphism to $\mathbb{R}C$ is a covering over $\mathbb{RP}^1$. If we fix the numbering of connected components $A_1, \dots, A_n$ of $\mathbb{R}C$, we may consider the sequence $d(f) = (d_1, \dots, d_n)$ where d_i is the covering degree of f restricted to A_i.

Kummer and Shaw studied in [2] the following problem. Given a curve C of dividing type, which sequences are realizable as $d(f)$ for separating morphisms $f : C \to \mathbb{P}^1$? It is easy to see that the set of all realizable sequences is an additive semigroup (see [2, Proposition 2.1]). Following [2], we call it the *separating semigroup* of C and denote by $\mathrm{Sep}(C)$.

Several interesting properties of $\mathrm{Sep}(C)$ are established in [2]. In particular, it is shown that $\mathrm{Sep}(C) = \mathbb{N}^{g+1}$ for an M-curve C (a curve C of genus g is called an M-curve if $\mathbb{R}C$ has $g + 1$ connected components which is the maximal possible number for genus g curves). Also it is shown in [2] that sometimes the separating semigroup does depend on the numbering of the components. The simplest example is a hyperbolic quartic curve in $\mathbb{RP}^2$ (a plane curve is called *hyperbolic* if the linear projection from some point is a separating morphism). Let C be such a curve. Then $\mathbb{R}C$ consists of two ovals one inside another. If we number them so that the inner

Partially supported by RFBR grant no. 17-01-00592-a

oval is first, then we have $(1,2) \in \mathrm{Sep}(C)$ but $(2,1) \notin \mathrm{Sep}(C)$; see [2, Example 3.7]. Moreover, $\mathrm{Sep}(C)$ is almost computed in [2]: it is shown that $\mathbb{N} \times \mathbb{N}_{\geq 2} \subset \mathrm{Sep}(C)$. We complete this computation:

Theorem 1. *Let C be a nonsingular real hyperbolic quartic curve in $\mathbb{RP}^2$ whose ovals are numbered so that the inner one is first. Then* $\mathrm{Sep}(C) = \mathbb{N} \times \mathbb{N}_{\geq 2}$.

The proof relies on two facts: Theorem 2 below and Natanzon's theorem [3, Theorem 2.3] which states that two branched coverings over a disk are (left-right) topologically equivalent if and only if they are equivalent over the boundary circle.

Theorem 2. *Let C be a real hyperelliptic curve of genus $g \geq 2$ of dividing type but not an M-curve. Then*

$$\mathrm{Sep}(C) = \begin{cases} (1,1)\mathbb{N} \cup (\mathbb{N}_{\geq (g+1)/2})^2 & \text{if } g \text{ is odd,} \\ 2\mathbb{N} \cup \mathbb{N}_{\geq g} & \text{if } g \text{ is even.} \end{cases}$$

Note that any real genus three curve of dividing type is either an M-curve, or hyperelliptic, or a plane hyperbolic quartic. Thus the results of [2] completed by our Theorems 1 and 2 provide separating semigroups of all real curves of dividing type up to genus 3.

2. Dual Vandermonde system of equations

Let $x_1, x_2, \dots, x_n$ be real numbers. We consider the homogeneous system of linear equations with indeterminates $h_1, \dots, h_n$ (the dual Vandermonde system):

$$\sum_{i=1}^{n} x_i^k h_i = 0, \qquad k = 0, \dots, g-1. \tag{1}$$

This condition on $(h_1, \dots, h_n)$ can be equivalently rewritten as follows

$$\sum_{i=1}^{n} h_i F(x_i) = 0 \qquad \text{for any } F \in \mathbb{R}[x] \text{ with } \deg F < g.$$

Given a sequence of real numbers $h = (h_1, \dots, h_n)$, we define $\mathrm{ch}(h)$ as *the number of changes of sign* of h, i. e., the number of pairs (i,j) such that $1 \leq i < j \leq n$, $h_i h_j < 0$, and $h_k = 0$ if $i < k < j$.

Proposition 2.1. *Let $x_1 < \dots < x_n$, $n > 0$. A sequence $s = (s_1, \dots, s_n)$ with $s_i \in \{-1, 0, 1\}$ is the sequence of signs of a non-zero solution to the system (1) if and only if* $\mathrm{ch}(s) \geq g$.

Proof. ($\Rightarrow$). Suppose that $h = (h_1, \dots, h_n)$ is a solution to (1), and $\mathrm{ch}(h) < g$. Then we can choose a polynomial F of degree less than g such that $F(x_i) \neq 0$ and $h_i F(x_i) \geq 0$ for any $i = 1, \dots, n$. Then $h_1 F(x_1) + \dots + h_n F(x_n) = 0$ and each term in this sum is non-negative. Hence $h = (0, \dots, 0)$.

($\Leftarrow$). Let $\mathrm{ch}(s) \geq g$. Let $I = \{i_0, \dots, i_g\} \subset \{1, \dots, n\}$ be such that $\mathrm{ch}(s_{i_0}, \dots, s_{i_g}) = g$. Let $h'_I = (h'_{i_0}, \dots, h'_{i_g})$ be a non-zero solution to the system (1) with $\sum_{1 \leq i \leq n}$ replaced by $\sum_{i \in I}$. By "($\Rightarrow$)" part, we have $\mathrm{ch}(h'_I) = g$. Thus, changing the sign of h'_I if necessary, we have $\mathrm{sign}\, h'_{i_j} = s_{i_j} \neq 0$ for all $j = 0, \dots, g$.

Let us choose $(h_i)_{i\notin I}$ such that $\operatorname{sign} h_i = s_i$ and $|h_i| < \varepsilon \ll 1$. Set $h_{i_0} = h'_{i_0}$. Since the Vandermonde $(g \times g)$-determinant (corresponding to the columns numbered by $I \setminus \{i_0\}$) is non-zero, the remaining numbers $h_{i_1}, \dots, h_{i_g}$ are uniquely determined by (1). Moreover, if ε is small enough, then $h_I = (h_i)_{i\in I}$ is close to h'_I, thus $\operatorname{sign} h_i = \operatorname{sign} h'_i = s_i$ for all $i \in I$. □

Corollary 2.2. *Let $x_1, \dots, x_n$ be real numbers, not necessarily distinct. For $x \in \mathbb{R}$ we set $I(x) = \{i \mid x_i = x\}$. Let $(h_1, \dots, h_n)$ be a real solution to the system (1) such that $h_i \neq 0$ for all $i = 1, \dots, n$. Then at least one of the following two possibilities takes place:*

(i) $\sum_{i\in I(x)} h_i = 0$ *for any $x \in \mathbb{R}$, in particular, each x_i occurs at least twice in the sequence $(x_1, \dots, x_n)$;*

(ii) *the sequence $(h_1, \dots, h_n)$ contains at least $[(g+1)/2]$ positive and at least $[(g+1)/2]$ negative members.*

3. Separating semigroup of hyperelliptic curves

In this section we prove Theorem 2.

Lemma 3.1. *Let C be a (complex) hyperelliptic curve of genus g and f a meromorphic function on C such that the zero divisor $(f)_0$ is special (this is so, for example, when $\deg f < g$). Then $f = f_1 \circ \pi$ where $\pi : C \to \mathbb{P}^1$ is the hyperelliptic projection and f_1 a meromorphic function on $\mathbb{P}^1$.*

Proof. If D and D' are two effective divisors on a curve, then the embedding ϕ_D defined by the complete linear system $|D|$ is a composition of $\phi_{D+D'}$ with a linear projection. Let $D = (f)_0$ and let D' be an effective divisor such that $D + D' \sim K_C$ (such D' exists since D is special). Thus ϕ_D is a projection of the canonical embedding which is known to factor through the hyperelliptic projection. □

Lemma 3.2. *Let C be a real algebraic curve of genus $g > 0$ of dividing type. Let $\omega_1, \dots, \omega_g$ be a base of holomorphic 1-forms on C.*

(a). Let $f : C \to \mathbb{P}^1$ be a separating morphism and $\{p_1, \dots, p_n\} = f^{-1}(p)$ for some point $p \in \mathbb{RP}^1$. Then there exist real positive (with respect to a fixed complex orientation) tangent vectors $v_1, \dots, v_n$ (v_i tangent at p_i) such that

$$\sum_{i=1}^{n} \omega_k(v_i) = 0 \qquad \text{for each } k = 1, \dots, g. \tag{2}$$

(b). Conversely, let $p_1, \dots, p_n$ be distinct points on $\mathbb{R}C$ and $v_1, \dots, v_n$ be positive real tangent vectors (v_i tangent at p_i) such that (2) holds. Suppose in addition that the divisor $D = p_1 + \dots + p_n$ is non-special i. e., $h^0(K_C - D) = 0$. Then there exists a separating morphism with fiber D.

Proof. (a). Follows from Abel-Jacobi Theorem.

(b). Follows from Abel-Jacobi Theorem combined with [2, Lemma 2.10]. Indeed, consider the Abel-Jacobi mapping $\varphi : \mathrm{Sym}^n(C) \to \mathcal{J}(C)$. The condition (2) means that $v = (v_1, \dots, v_n)$ considered as a tangent vector to $\mathrm{Sym}^n(C)$ at D is in the kernel of the differential of φ at D. The non-specialness of D means that φ is a submersion near D, hence v is tangent to $\varphi^{-1}(\varphi(D)) = |D|$ at D. Hence there exists a path $[0,t_0]\to|D|$, $t\mapsto D_t$, such that $D_0=D$ and $\left(\frac{d}{dt}D_t\right)_{t=0}=v$. Then, for any t, $0<t\le t_0$, there exists a meromorphic function $f_t:C\to\mathbb{P}^1$ such that $D=(f_t)_0$ and $D_t=(f_t)_\infty$. If t is small enough, then the condition of positivity of the v_i's implies that the zeros and poles of f_t interlace along $\mathbb{R}C$, thus f_t is a separating morphism by [2, Lemma 2.10]. □

Proof of Theorem 2. Let C be a real hyperelliptic curve of genus $g\ge 2$ of dividing type, which is not an M-curve. Then it is given by an equation $y^2=G(x)$ where $G(x)$ is a real polynomial of degree $2g+2$, without multiple roots and positive everywhere on $\mathbb{R}$. We consider the standard base of holomorphic 1-forms $\omega_1,\dots,\omega_g$ where $\omega_k=x^{k-1}dx/y$. The hyperelliptic projection is given by $(x,y)\mapsto x$. Its restriction to $\mathbb{R}C$ is an unramified two-fold covering over $\mathbb{RP}^1$ which is trivial for even g and non-trivial for odd g. We choose the complex orientation on $\mathbb{R}C$ such that $dx>0$ on positive tangent vectors.

Let $f:C\to\mathbb{P}^1$ be a separating morphism and let $\{p_1,\dots,p_n\}=f^{-1}(p)$ for a generic $p\in\mathbb{RP}^1$. We set $p_i=(x_i,y_i)$, $i=1,\dots,n$. By Lemma 3.2(a) there exist positive tangent vectors $v_1,\dots,v_n$ such that (2) holds. Let $a_i=dx(v_i)$. The positivity of v_i means $a_i>0$. Then (2) takes the form (1) for $h_i=a_i/y_i$, and Theorem 2 follows from Corollary 2.2 and Lemma 3.1.

4. Separating semigroup of genus three curves

In this section we prove Theorem 1. Let C be a plane hyperbolic quartic curve. We have $\mathbb{N}\times\mathbb{N}_{\ge 2}\subset\operatorname{Sep}(C)$, see [2, Example 3.7]. Let us prove the inverse inclusion.

It is shown in [2, Example 2.8] that $(1,1)\notin\operatorname{Sep}(C)$. Suppose there exists a separating morphism $f_0:C\to\mathbb{P}^1$ with $d(f_0)=(n,1)$, $n\ge 2$. Let C^+ be one of the two halves into which $\mathbb{R}C$ divides C. Then the restriction of f_0 to C^+ is a branched covering over a disk Δ which is one of the halves of $\mathbb{CP}^1\setminus\mathbb{RP}^1$. By perturbing f_0 (together with C) we may assume that all critical values are simple, i. e., $f^{-1}(p)$ has at least n points for any $p\in\Delta$.

Let $f_1:C\to\mathbb{P}^1$ be a separating morphism with $d(f_1)=(1,n)$ which exists by [2, Example 3.7]. It can be chosen so that all its critical values are simple. Then, by Natanzon's result [3, Theorem 2.3], there exists a continuous family of branched coverings $f_t:C^+\to\Delta$, $0\le t\le 1$, which connects f_0 with f_1. Let C_t^+ be C^+ endowed with the complex structure lifted from Δ by f_t, and let C_t be C_t^+ glued along the boundary with its complex conjugate copy. Then f_t extends to a separating morphism $C_t\to\mathbb{P}^1$ which we also denote by f_t. So, we obtain a continuous family of separating morphisms f_t of genus three curves C_t.

By continuity, we have $d(f_t)=(1,n)$ for a suitable numbering of the components of $\mathbb{R}C_t$. Hence, by Theorem 2, the curve C_t cannot be hyperelliptic for any t. It is well-known that any non-hyperelliptic genus three curve is isomorphic to a smooth quartic curve in $\mathbb{P}^2$. Thus there exists a continuous family of embeddings $\iota_t:C_t\to\mathbb{P}^2$ such that $\iota_t(C_t)$ is a smooth real quartic curve, and we have a continuous family of separating morphisms of them onto $\mathbb{P}^1$. The interior and exterior ovals cannot interchange in this family which contradicts the fact that $d(f_0)\ne d(f_1)$ and the embedding to $\mathbb{P}^2$ is unique up to projective equivalence.

References

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