SOLUTION OF THE WORD PROBLEM IN THE SINGULAR BRAID GROUP

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Abstract. Singular braids are isotopy classes of smooth strings which are allowed to cross each other pairwise with distinct tangents. Under the usual multiplication of braids, they form a monoid. The *singular braid group* was introduced by Fenn-Keyman-Rourke as the quotient group of the singular braid monoid. We give a solution of the word problem for this group. It is obtained as a combination of the results by Fenn-Keyman-Rourke and some simple geometric considerations based on the mapping class interpretation of braids. Combined with Corran's normal form for the singular braid monoid, our algorithm provides a computable normal form for the singular braid group.

1. Introduction. Let X be any set. Let us denote $X\times\{1,\dots,n-1\}$ by X_n. We shall denote an element (x,i) of X_n by x_i. Let $\Sigma_n=\{\sigma_1,\dots,\sigma_{n-1}\}$. The *singular braid group* $B_n(X)_G$ is the group generated by $X_n\cup\Sigma_n$ and subject to the relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i,\quad \sigma_i x_j=x_j\sigma_i,\quad x_iy_j=y_jx_i,\qquad |i-j|\ge 2,\quad x,y\in X; \tag{1}$$

$$\sigma_i\sigma_j\sigma_i=\sigma_j\sigma_i\sigma_j,\quad x_i\sigma_j\sigma_i=\sigma_j\sigma_i x_j,\qquad |i-j|=1,\quad x\in X; \tag{2}$$

$$\sigma_i x_i = x_i\sigma_i \qquad x\in X. \tag{3}$$

In this paper we give a solution of the word problem for $B_n(X)_G$.

Let $\Sigma_n^{-1}=\{\sigma_1^{-1},\dots,\sigma_{n-1}^{-1}\}$. The *singular braid monoid* $B_n(X)_M$ is the monoid generated by $X_n\cup\Sigma_n\cup\Sigma_n^{-1}$ and subject to the relations (1) – (3) and

$$\sigma_i\sigma_i^{-1}=\sigma_i^{-1}\sigma_i=1. \tag{4}$$

When $X=\varnothing$, both $B_n(X)_G$ and $B_n(X)_M$ coincide with the usual braid group.

In the case when X is a one-element set $\{\tau\}$, the singular braid monoid was introduces by Baez [1] and Birman [2]. Corran solved the word [3] and conjugacy [4] problems for this monoid (and for its natural generalization for any Artin group). She did it when $X=\{\tau\}$, however the same proofs work for any X.

The singular braid group was introduced by Fenn, Keyman, and Rourke [7] (for $X=\{\tau\}$ but their arguments are valid for any X). They proved that $B_n(X)_M$ embeds into $B_n(X)_G$. This result is derived in [7] from Theorem 1 formulated below.

Let $X^{-1}=\{x^{-1}\,|\,x\in X\}$. There is a natural homomorphism of monoids

$$\iota: B_n(X\cup X^{-1})_M\to B_n(X)_G.$$

2000 *Mathematics Subject Classification* Primary 20F36; Secondary 20F10, 57M25.
Key words and phrases. Singular braid group, word problem.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 1. An element $\alpha \in B_n(X \cup X^{-1})_M$ is called *irreducible* if it cannot be written as $\alpha = \beta x_i x_i^{-1} \gamma$ or $\alpha = \beta x_i^{-1} x_i \gamma$.

Theorem 1. ([7; Corollary 3.3]). *If α and β are irreducible elements of $B_n(X \cup X^{-1})_M$ and $\iota(\alpha) = \iota(\beta)$ then $\alpha = \beta$.*

Thus, due to Theorem 1, we can solve the word problem in $B_n(X)_G$ as soon as we know a *reduction algorithm*, i.e. an algorithm which computes an irreducible word representing a given element of $B_n(X)_G$. Indeed, to compare two elements α and β of $B_n(X)_G$, we compute a reduced word γ, representing $\alpha\beta^{-1}$. If γ contains letters from $X_n \cup X_n^{-1}$ then $\gamma \neq 1$, hence $\alpha \neq \beta$. Otherwise, we apply any of numerous known algorithms to decide if γ is trivial in the usual braid group. Moreover, combined with Corran's normal form [3] for elements of $B_n(X \cup X^{-1})_M$, a reduction algorithm provides a computable normal form for elements of $B_n(X)_G$.

We shall give a reduction algorithm in Section 4. It is based on the life discs introduced in [7] and the ideas from [6]. Modified in Section 6 according to Dynnikov (see [5; Ch.III, 4.19–4.23]), this algorithm turns out to be of biquadratic time (quadratic time if one considers additions and comparings of integers as elementary operations).

2. Geometric singular braids. Let $\mathbb{D}$ be the closed unit disc in $\mathbb{C}$ and $I = [-1, 1]$. Let $P_n = \{p_1, \dots, p_n\} \subset I$, where $p_0 = -1 < p_1 < \dots < p_n < p_{n+1} = 1$.

A *geometric X-braid* (or *geometric singular braid* if X is not precised) α is a union of smooth closed curves (called *strings*) in the cylindre $\mathbb{D} \times [0, 1]$, such that:

- (*i*) The projection of any string onto $[0, 1]$ is a diffeomorphism.
- (*ii*) $\alpha \cap (\mathbb{D} \times \{0\}) = P_n \times \{0\}$ and $\alpha \cap (\mathbb{D} \times \{1\}) = P_n \times \{1\}$.
- (*iii*) Strings meet each other only pairwise and with distinct tangents at each crossing.

In fact, the condition (*iii*) implies that the number of crossings is finite. To each crossing is associated an element of X (its *color*). A disc $\mathbb{D} \times \{t\}$, $t \in [0, 1]$ will be called *level*. A union of levels $\mathbb{D} \times \{t\}$ for $t \in [a, b] \subset [0, 1]$ will be called *layer*.

Two geometric X-braids α_0 and α_1 are called *isotopic* if there exists a smooth family $\{\alpha_t\}_{t \in [0,1]}$ of geometric X-braids relating α_0 to α_1. The elements of the monoid $B_n(X)_M$ are in one-to-one correspondence with the isotopy classes of geometric X-braids (see [1, 2, 7] for details). Under this correspondence, the generators σ_i, σ_i^{-1} and x_i, $x \in X$, correspond to geometric braids whose projections onto $I \times [0, 1]$, $(z, t) \mapsto (\operatorname{Re} z, t)$, are as in Figure 1.

σ_i $\qquad$ σ_i^{-1} $\qquad$ x_i

FIGURE 1

3. Life discs and youth discs. The following definition is taken from [7]. Let α be a geometric $(X \cup X^{-1})$-braid. A *life disc* for α is a disc D embedded in $\mathbb{D} \times [0,1]$ such that:

(i) The interior of D is disjoint from α.

(ii) The boundary of D lies on β and contains precisely two crossings q and q' (called *birth* and *death*) colored by x and x^{-1} for some $x \in X$.

(iii) D is contained in the layer between the levels of q and q' (inclusive). It meets the levels of q and q' precisely in q and q' respectively and it meets each level strictly between q and q' transversally in an arc.

Lemma 1. [7]. *A geometric $(X \cup X^{-1})$-braid is reduced if and only if there is no life disc for it.* $\square$

Let us define a *youth disc* (we are trying to keep the terminology style proposed by Fenn, Keyman, and Rourke) for a singular geometric braid α as a disc embedded in $\mathbb{D} \times [0,1]$ which can be completed to a life disc of $\alpha\beta$ for some singular geometric braid β. If D is a youth disc then the curve $\Gamma = \mathrm{pr}_1(D \cap (\mathbb{D} \times \{1\}))$ is called the *final curve* of D where $\mathrm{pr}_1 : \mathbb{D} \times [0,1] \to \mathbb{D}$ is $(z,t) \mapsto z$.

The following definitions are inspired by [6]. Let P_n and I be as in Section 2. Let Γ be an embedded curve in $\mathbb{D}$ whose endpoints belong to P_n and no interior point belong to P_n. We shall say that Γ is *transversal* to I if it is either really transversal or it coincides with one of the segments $[p_i, p_{i+1}]$. Let curves Γ and Γ' be transversal to I. They are called I-equivalent if they are isotopic via an ambient isotopy which is fixed on P_n and which preserves I.

Suppose that Γ is transversal to I. A component Δ of $\mathbb{D} \setminus (\Gamma \cup I)$ is calls a *digon between* Γ and I if Δ is homeomorphic to an open disc and is bounded by an open segment of Γ, an open segment of $I \setminus P_n$, and two points (any of which may, or may not, belong to P_n). We say that Γ is *reduced* if it is transversal to I and there is no digon between Γ and I. Let us say that a youth disc is *reduced* if its final curve is reduced.

It it easy to see that any curve Γ can be reduced by an isotopy which is the identity on P_n (see, e.g. [6]). When $\Gamma = D \cap (\mathbb{D} \times \{1\})$ for a youth disc D, such an isotopy can be extended to a neighbourhood of $\mathbb{D} \times \{1\}$ up to an isotopy of D. Thus, we have

Lemma 2. *If there exists a youth disc for a singular braid α, born at some crossing q, then there exist a reduced youth disc for α born at q.* $\square$

Lemma 3. *Let D_1 and D_2 be two youth discs for the same geometric singular braid α born at the same crossing. Then their final curves are isotopic by an ambient isotopy which is fixed on P_n.*

Proof. This can be proved by a kind of standard argument like "Let us choose the maximal t such that $D_1 \cap (\mathbb{D} \times \{t\})$ and $D_2 \cap (\mathbb{D} \times \{t\})$ are isotopic and extend the isotopy a little bit further...". One can also proceed as in [7; Sect.4, Case (3)]. $\square$

Combining Lemmas 2 and 3, we get

Lemma 4. *If there exists a youth disc for a singular braid α, born at some crossing q, then there exist a reduced youth disc for α born at q and its final curve is uniquely determined by α and q up to I-equivalency.* $\square$

4. Prolongation of youth discs. Let α be a geometric $(X\cup X^{-1})$-braid and D a youth disc for it born at a crossing q of a color x^ε, $\varepsilon=\pm1$. Suppose that D is reduced and let Γ be its final curve. Let β be a geometric braid representing a standard generator, i.e. $\beta\in\Sigma\cup\Sigma^{-1}\cup X\cup X^{-1}$. The following lemma is simple and we omit its proof.

Lemma 5. (a). *Let $\beta=\sigma_i^{\pm1}$. Then there exists a reduced youth disc for $\alpha\beta$ born at q. Its final curve is obtained from Γ by the standard action of β by a diffeomorphism (see, e.g.,* [6]*).*

(b). *Suppose that $\beta=y_i^\delta$, $y\in X$, $\delta=\pm1$. Then:*

- (b1). *A youth disc D' for $\alpha\beta$ born at q exists if and only if $\Gamma\cap[p_i,p_{i+1}]=\varnothing$. In this case the final curve of D' is I-equivalent to Γ*
- (b2). *A life disc for $\alpha\beta$ born at q exists if and only if $\Gamma=[p_i,p_{i+1}]$, $y=x$ and $\delta=-\varepsilon$.*

Now we are ready to formulate the reduction algorithm. We consider one by one all the crossings and check for each of them if there exists a life disc born at it transforming the final curve of the youth disc according to Lemma 5. To make this algorithm to be very fast, in the next section we apply the idea due to Dynnikov.

5. Lamination coordinates of final curves. Let us denote by I^* the union of I with all the segments $[p_i,\pm\sqrt{-1}\,]$, $i=1,\dots,n$. We define that Γ is transverse to I^* and reduced with respect to I^* in the same way as in Section 3 (no digons between Γ and I^*). So, let Γ be reduced with respect to I^*. We define the *lamination coordinates of* Γ as the sequence $(c_0,a_1,b_1,c_1,a_2,b_2,c_2,\dots,a_n,b_n,c_n)$,

$$a_i=\#(\Gamma\cap]p_i,\sqrt{-1}[\,),\quad b_i=\#(\Gamma\cap]p_i,-\sqrt{-1}[\,),\quad c_i=\#(\Gamma\cap]p_i,p_{i+1}[\,).$$

If $\Gamma=[p_i,p_{i+1}]$, we set $c_i=-1$. Let us set also $a_0=b_0=a_{n+1}=b_{n+1}=0$.

Lemma 6. (Compare with Dynnikov's formulas [5; III.4.20]). *Let α be a singular braid and let $(c_0,a_1,b_1,c_1,\dots,a_n,b_n,c_n)$ be the lamination coordinates of the final curve of some youth disc for α born at q. Then the lamination coordinates of the final curve of a youth disc for $\alpha\sigma_i^\varepsilon$, $\varepsilon=\pm1$, born at q, are $(c_0',a_1',b_1',c_1',\dots,a_n',b_n',c_n')$, where $a_k'=a_k$, $b_k'=b_k$ for $k\neq i,i+1$, $c_k'=c_k$ for $k\neq i-1,i+1$, and the numbers $c_{i-1}',a_i',b_i',a_{i+1}',b_{i+1}',c_{i+1}'$ are defined as follows. If $\varepsilon=1$ then*

$$c_{i-1}'=\max(c_{i-1}+b_{i+1},c_i+b_{i-1})-b_i,\quad c_{i+1}'=\max(c_i+a_{i+2},c_{i+1}+a_i)-a_{i+1};$$

$$\begin{cases} a_i'=\max(c_{i-1}'+a_i,c_i+a_{i-1})-c_{i-1}\\ b_{i+1}'=\max(c_{i+1}'+b_{i+1},c_i+b_{i+2})-c_{i+1}\end{cases}\qquad \begin{cases} a_{i+1}'=a_i\\ b_i'=b_{i+1}\end{cases}$$

If $\varepsilon=-1$ then (we swap a and b)

$$c_{i-1}'=\max(c_{i-1}+a_{i+1},c_i+a_{i-1})-a_i,\quad c_{i+1}'=\max(c_i+b_{i+2},c_{i+1}+b_i)-b_{i+1};$$

$$\begin{cases} b_i'=\max(c_{i-1}'+b_i,c_i+b_{i-1})-c_{i-1}\\ a_{i+1}'=\max(c_{i+1}'+a_{i+1},c_i+a_{i+2})-c_{i+1}\end{cases}\qquad \begin{cases} b_{i+1}'=b_i\\ a_i'=a_{i+1}\end{cases}$$

6. The reduction algorithm. Let $\alpha = \alpha_0 u_0 u_1 \cdots \in B_n(X \cup X^{-1})$ with $u_k \in \Sigma \cup \Sigma^{-1} \cup X \cup X^{-1}$ and $u_0 = x_i^\varepsilon$ for some $x \in X$, $\varepsilon = \pm 1$. We must check if there exists a life disc for α born at u_0. We shall compute the lamination coordinates and the endpoints p and q of the final curves of youth discs born at u_0 (if they exist) successively for $\alpha_0 u_0$, $\alpha_0 u_0 u_1$, etc., using Lemmas 5 and 6. For $\alpha_0 u_0$, we have $p = p_i$, $q = p_{i+1}$, $c_i = -1$ and all the other coordinates are zero. When we add u_k to our word, we do the following:

If $u_k = \sigma_j^{\pm 1}$, we compute new coordinates (by Lemma 6) and new endpoints p, q (transposing p_j and p_{j+1}), and we pass to u_{k+1}.

If $u_k = y_j^\delta$ and $c_j = -1$, then we finish the computation and conclude that the life disc exists (if $y = x$ and $\delta = -\varepsilon$, in this case u_k is the death point) or does not exist (otherwise).

If $u_k = y_j^\delta$, $c_j = 0$, and $\{p, q\} \cap \{p_j, p_{j+1}\} = \varnothing$, we do nothing and pass to u_{k+1}.

If $u_k = y_j^\delta$, $c_j \geq 0$, and either $c_j \geq 1$ or $\{p, q\} \cap \{p_j, p_{j+1}\} \neq \varnothing$, we finish the computation and conclude that the life disc does not exist.

Remark. In fact, the enpoints p, q are determined by the lamination coordinates. To find them, it suffices to find the two triples among (a_i, a_{i+1}, c_i) and (b_i, b_{i+1}, c_i), $i = 0, \ldots, n$, for which the triangle inequality fails.

If we have treated all the u_k's and the youth disc has survived, we conclude that the life disc does not exist.

If we found a life disc born at u_0 and died at u_k, we just remove u_0 and u_k from our word and continue the reduction. If there is no life disc, then the singular braid is reduced by Lemma 1.

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