QUASIPOSITIVE LINKS AND CONNECTED SUMS

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Abstract. We prove that the connected sum of two links is quasipositive if and only if each summand is quasipositive. The proof is based on the filling disk technique.

1. Introduction

An n-braid is called *quasipositive* if it is a product of conjugates of the standard (Artin's) generators $\sigma_1, \dots, \sigma_{n-1}$ of the braid group B_n. A braid is called *strongly quasipositive* if it is a product of braids of $\sigma_{j,k+1} = \tau_{k,j}\sigma_j\tau_{k,j}^{-1}$ for $j \le k$ where $\tau_{k,j} = \sigma_k\sigma_{k-1}\dots\sigma_j$. Such braids are called *band generators* (they are also known as the generators in the Birman-Ko-Lee presentation of B_n, see [3]).

All links in this paper are assumed to be oriented links in the 3-sphere S^3. A link is called *(strongly) quasipositive* if it is the braid closure of a (strongly) quasipositive braid. This terminology was introduced by Lee Rudolph (see [13, 14]) and now it has become standard in the knot theory.

The main result of this note is the (a) statement of the following theorem.

Theorem 1. *Let $L = L_1 \# L_2$ be the connected sum of two links in S^3. Then:*

(a). L is quasipositive if and only if L_1 and L_2 are.

(b). L is strongly quasipositive if and only if L_1 and L_2 are.

Note that the (b) statement of this theorem is an almost immediate corollary of the main result of [14] (see §3). I added it just for the sake of completeness as well as the following two theorems.

Theorem 2. *Let $L = L_1 \sqcup L_2$ be the split sum of two links in S^3. Then:*

(a). L is quasipositive if and only if L_1 and L_2 are.

(b). L is strongly quasipositive if and only if L_1 and L_2 are.

Let $\operatorname{sh}_m : B_n \to B_{m+n}$ be the homomorphism defined on the generators by $\sigma_k \mapsto \sigma_{m+k}$ (the m-shift). If links L_1 and L_2 are represented by braids $X_1 \in B_m$ and $X_2 \in B_n$, then $L_1 \sqcup L_2$ and $L_1 \# L_2$ can be represented by the braids

$$X_1 \operatorname{sh}_m(X_2) \in B_{m+n} \qquad \text{and} \qquad X_1 \operatorname{sh}_{m-1}(X_2) \in B_{m+n-1} \tag{1}$$

respectively. So, the next result is a braid-theoretic counterpart of Theorem 2.

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Theorem 3. *Let $X_1 \in B_m$ and $X_2 \in B_n$. Let $X = X_1 \operatorname{sh}_m(X_2) \in B_{m+n}$. Then:*

(a). ([12, Thm. 3.2]) *X is quasipositive if and only if X_1 and X_2 are.*

(b). X is strongly quasipositive if and only if X_1 and X_2 are.

Conjecture 4. *Let $X_1 \in B_m$ and $X_2 \in B_n$. Let $X = X_1 \operatorname{sh}_{m-1}(X_2) \in B_{m+n-1}$. Then X is quasipositive if and only if X_1 and X_2 are.*

Remark 5. A particular case of Conjecture 4 is the main result of [11] which states that an n-braid X is quasipositive if and only if the $(n+1)$-braid $X\sigma_n$ is.

Remark 6. In [11, Question 1] I asked if the minimal braid index representative of a quasipositive link is necessarily a quasipositive braid. In virtue of Theorem 1(a), an affirmative answer to this question combined with arguments from [9] would imply Conjecture 4. Indeed, the self-linking number (the algebraic length minus the number of strings) of a quasipositive braid is maximal over all braids representing the same link type, see e.g. [15]. It follows that, in the setting of Conjecture 4, it is maximal for each X_j, $j = 1, 2$. Then, as shown in the proof of [9, Thm. 1.2], X_j can be transformed into a braid X'_j with the minimal number of strings by conjugations and positive (de)stabilizations only. Hence Theorem 1(a) combined with an affirmative answer to [11, Question 1] would imply the quasipositivity of X'_j which is equivalent to that of X_j by [11, Thm. 1]. See also [8, 9] for some interesting results related to [11, Question 1].

Remark 7. In particular, Theorem 3(a) implies that *an n-braid is quasipositive, if so is the m-braid given by the same braid word for some $m \geq n$* (see [12, Thm. 3.1]). This fact is also an immediate formal consequence from the invariance of quasipositivity under destabilizations (see [11] and Remark 5). Indeed, if $X \in B_n$ and the $(n+1)$-braid given by the same braid word is quasipositive, then evidently so is $X\sigma_n \in B_{n+1}$ whence, by [11], X as well.

Remark 8. All the discussed statements concerning (not strongly) quasipositive braids are purely combinatorial whereas their proofs are based on the (almost) complex analysis, PDE, etc. The only particular case where I know a combinatorial proof is the statement in Remark 7; see [12, §3.3].

Remark 9. Conjecture 4 is a braid-theoretical counterpart of Theorem 1(a). Using the Birman-Ko-Lee version of Garside's theory [3], one can easily prove several analogs of Theorem 1(b) for braids. However, they do not seem to be of any interest because the strong quasipositivity is not invariant under conjugations.

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2. Quasipositive links: proof of the (a) cases of the theorems

The proofs of the (a) cases of the theorems are inspired by Eroshkin's paper [7] which is based on the filling disk technique [1, 6, 10]. In fact, Theorems 2(a) and 3(a) are almost immediate consequences from [7] (see [12, Thm. 3.2] for more details). Our proof of Theorem 1(a) is a combination of a more precise version of Bedford-Klingenberg-Kruzhilin's Theorem (with the smoothness of the Levi-flat hypersurface), the smoothing of pseudoconvex domains, and the result from [4] which states that the boundary link of an algebraic curve in a pseudoconvex ball is a quasipositive link.

Lemma 10. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary. Let $B \subset \Omega$ be a smooth Levi-flat hypersurface transverse to $\partial\Omega$, and A be a smooth two-dimensional surface transverse to both B and $\partial\Omega$. Suppose that $\Omega \setminus B$ has two connected components Ω_1 and Ω_2. Then for each $j = 1, 2$, there exists a strictly pseudoconvex domain $\Omega_j^- \subset \Omega_j$ with smooth boundary such that the pairs $(\Omega_j, A \cap \Omega_j)$ and $(\Omega_j^-, A \cap \Omega_j^-)$ are homeomorphic.*

Proof. We identify $\mathbb{C}^2$ with an affine chart of $\mathbb{CP}^2$. Let $j = 1$ or 2. For $z \in \Omega_j$, we define $d(z)$ as the distance from p to $\partial\Omega_j$ with respect to the Fubini-Study metric on $\mathbb{CP}^2$. By [16, Thm. 1], $h := -\log d$ is a plurisubharmonic function on Ω_j. Let φ be a smooth function in $\mathbb{C}^2$ supported on the unit ball and positive on it, and let $\varphi_\varepsilon(z) = \varphi(z/\varepsilon)/\varepsilon^4$ (thus φ_ε tends to a delta-function as $\varepsilon \to 0$). Let $U_\varepsilon = \{p \in \Omega_j \mid d(p) > \varepsilon\}$. Let h_ε be the convolution $h * \varphi_\varepsilon$, it is smooth and plurisubharmonic on U_ε (see e.g. [5, Thm. 5.5]). Then, for $a \gg 1$ and $0 < \varepsilon \ll \exp(-a)$, the domain $\Omega_j^- = \{z \in \mathbb{C}^2 \mid h_\varepsilon(z) + \varepsilon\|z\|^2 < a\}$ has the required properties. $\square$

Proof of Theorem 1(a). ($\Leftarrow$) Follows from (1).

($\Rightarrow$) Suppose that L is a quasipositive link in S^3 which we identify with the unit sphere in $\mathbb{C}^2$. By [13] we may assume that $L = S^3 \cap A$ where A is an algebraic curve transverse to S^3. Let $\Gamma \subset S^3$ be a smooth embedded 2-sphere which defines the decomposition of L into the connected sum $L_1 \# L_2$. This means that Γ divides S^3 into two 3-balls B_1 and B_2, and there is an embedded segment $I \subset \Gamma$ such that $(L \cap B_j) \cup I = L_j$, $j = 1, 2$. It follows from [1, Thm. 1] that, after perturbing Γ if necessary, one can find a smoothly embedded Levi-flat 3-ball B transverse[1] to S^3 and such that $\partial B = \Gamma$. Moreover (see also [6]), there exists a smooth function $F : B \to \mathbb{R}$ with non-vanishing gradient, whose restriction to Γ (we denote it by f) is a Morse function, and all whose level surfaces are unions of holomorphic disks and (for some critical levels) isolated points.

Let $\{p_1, p_2\} = \partial I = \Gamma \cap L$, we number the points p_1 and p_2 so that $f(p_1) \le f(p_2)$. Then the linking numbers of L with the level lines of f are:

$$\operatorname{lk}\big(L, f^{-1}(c)\big) = \begin{cases} 1 & \text{if } f(p_1) \le c \le f(p_2), \\ 0 & \text{otherwise.} \end{cases}$$

Hence $F^{-1}(c)$ meets A transversally at a single point when $f(p_1) \le c \le f(p_2)$, and $F^{-1}(c) \cap A = \varnothing$ otherwise. Therefore $B \cap A$ is an unknotted arc in B with the endpoints $p_1, p_2 \in \Gamma = \partial B$ (it is crucial here that the fibers $F^{-1}(c)$ are unions of disks because otherwise an arc cutting each fiber at most once might be knotted).

Let Ω_1 and Ω_2 be the two domains into which B divides the unit ball in $\mathbb{C}^2$. Let $j = 1$ or 2. We have $\partial\Omega_j = B \cup B_j$, and the arc I is isotopic to $B \cap A$ in B relative to the boundary whence the homeomorphisms (see Figure 1):

$$(S^3, L_j) = (S^3, (B_j \cap A) \cup I) \cong (\partial\Omega_j, (B_j \cap A) \cup I) \cong (\partial\Omega_j, \partial(\Omega_j \cap A)).$$

By Lemma 10, we may approximate Ω_j by a strictly pseudoconvex domain Ω_j^- with smooth boundary diffeomorphic to the 3-sphere. Then, by Eliashberg's result

[1] The transversality is not stated in [1, Thm. 1], however, the proof of the smoothness of B is nothing else than a proof of its transversality to S^3.

[6, Thm. 5.1], Ω_j^- is diffeomorphic to the 4-ball, and then the quasipositivity of $(\partial\Omega_j, \partial(\Omega_j \cap A))$ follows from [4, Thm. 2]. □

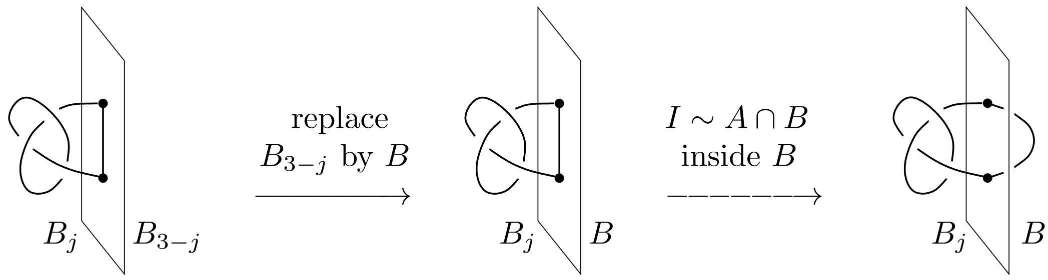

Figure 1.

3. Strongly quasipositive links: proof of the (b) cases of the theorems

Lemma 11. *Let L be $L_1 \# L_2$ or $L_1 \sqcup L_2$. Let S be a Seifert surface of L of maximal Euler characteristic. Then the sphere S^3 can be presented as a union of embedded 3-balls $S^3 = B_1 \cup B_2$ such that:*

(i) $B_1 \cap B_2 = \partial B_1 = \partial B_2$;
(ii) $\partial(S \cap B_j)$ *is* L_j *for* $j = 1, 2$;
(iii) $S \cap \partial B_1$ *is an embedded segment if* $L = L_1 \# L_2$ *and empty if* $L = L_1 \sqcup L_2$.

Proof. The lemma follows from the standard arguments used in the proof of the additivity of the knot genus. Namely, let $S^3 = B_1 \cup B_2$ be a splitting of S^3 involved in the definition of the split or connected sum. We assume that ∂B_1 is transverse to S. If (B_1, B_2) is not as required, we choose a closed curve C in $S \cap \partial B_1$ which bounds a disk D in $\partial B_1 \setminus S$. If C bounds a disk D' in S, then $D \cup D'$ bounds a 3-ball $B \subset B_j$, $j = 1$ or 2, and we may replace B_{3-j} with a thickening of $B_{3-j} \cup B$ which reduces the number of components of $S \cap \partial B_1$. Otherwise we attach a 2-handle to S along D and remove a closed component if it appears. This operation increases $\chi(S)$ which contradicts its maximality. □

If an n-braid X is a product of c band generators, then its braid closure admits a Seifert surface of a special form which has Euler characteristic $n - c$. It is obtained by attaching c positively half-twisted bands to n parallel disks so that a band corresponding to $\sigma_{i,j}$ connects the i-th disk with the j-th disk; see details in [14]. Such a surface is called a *quasipositive Seifert surface.*

Proof of Theorems 1(b) and 2(b). (⇐) Follows from (1).

(⇒) Suppose that L is strongly quasipositive. Let S be a quasipositive Seifert surface for L. By Bennequin's inequality [2, Ch. II, Thm. 3] the Euler characteristic of S is maximal among all Seifert surfaces of L. Hence the sphere S^3 can be cut into two 3-balls $S^3 = B_1 \cup B_2$ as in Lemma 11. Then $B_j \cap S$ is a Seifert surface of L_j which is a full subsurface of S (see the definition in [14, p. 231]), thus $B_j \cap S$ is isotopic to a quasipositive surface by the main result of [14], whence the strong quasipositivity of L_j. □

Proof of Theorem 3(b). (⇐) Evident.

(⇒) Let S be the quasipositive Seifert surface. By the same reasons as in the proof of Theorems 1(b) and 2(b), S is a disjoint union $S_1 \cup S_2$ with $\partial S_j = L_j$. Then each S_j is a quasipositive Seifert surface by construction. □

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