TWO-DIMENSIONAL DIFFUSION ORTHOGONAL POLYNOMIALS ORDERED BY A WEIGHTED DEGREE

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Abstract. We study the following problem: describe the triplets (Ω, g, μ), $\mu = \rho\, dx$, where $g = (g^{ij}(x))$ is the (co)metric associated with the symmetric second order differential operator $\mathbf{L}(f) = \frac{1}{\rho}\sum_{ij}\partial_i(g^{ij}\rho\,\partial_j f)$ defined on a domain Ω of $\mathbb{R}^d$ and such that there exists an orthonormal basis of $\mathcal{L}^2(\mu)$ made of polynomials which are eigenvectors of $\mathbf{L}$, and the basis is compatible with the filtration of the space of polynomials with respect to some weighted degree.

In a joint paper with D. Bakry and M. Zani this problem was solved in dimension 2 for the usual degree. In the present paper we solve it still in dimension 2, but for a weighted degree with arbitrary positive weights.

1. Introduction

In [3] the following problem posed by Dominique Bakry is studied (see also [1], [2], [5], [6]): describe all triples $(\Omega, \mathbf{L}, \mu)$ where Ω is a domain in $\mathbb{R}^d$, $\mathbf{L}$ is an elliptic second order operator with real coefficients of the form

$$\mathbf{L}(f) = \sum_{i,j} g^{ij}(x)\partial_{ij} f + \sum_i b^i(x)\partial_i f \tag{1}$$

with g^{ij} and b^i continuous in Ω and μ a probability measure on Ω with $\mathcal{C}^1$-smooth density, such that there exists a polynomial orthogonal basis of $\mathcal{L}^2(\Omega,\mu)$ formed by eigenvectors of $\mathbf{L}$, which is also a basis (in the algebraic sense) of $\mathbb{R}[x]$, $x = (x_1, \dots, x_d)$. It is clear that in this case $\mathbf{L}$ is symmetric on the space of polynomials, i.e.,

$$\int_\Omega f_1 \mathbf{L} f_2\, d\mu = \int_\Omega f_2 \mathbf{L} f_1\, d\mu \tag{2}$$

for any two polynomial functions f_1 and f_2.

Suppose that $e_1, e_2, \dots$ is such a basis, and let V_n be the subspace spanned by $e_1, \dots, e_n$. Then we have an increasing sequence of $\mathbf{L}$-invariant subspaces $V_1 \subset V_2 \subset V_3 \subset \dots$ of $\mathbb{R}[x]$ whose union is $\mathbb{R}[x]$, and $\mathbb{R}[x]$ is dense in $\mathcal{L}^2(\Omega,\mu)$.

Vice versa, given an increasing sequence of finite-dimensional $\mathbf{L}$-invariant subspaces of $\mathbb{R}[x]$ whose union is $\mathbb{R}[x]$, one can always choose an orthogonal polynomial eigenbasis of $\mathcal{L}^2(\Omega,\mu)$ provided that $\mathbf{L}$ is symmetric on polynomials and the space of polynomials is dense in $\mathcal{L}^2(\Omega,\mu)$.

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It does not seem that this problem can be solved in such generality, without imposing a condition that the filtration $V_1 \subset V_2 \subset \dots$ is somewhat natural. So, in [3] the above problem was considered with an additional condition that for any n, the space of polynomials of degree $\le n$ is invariant under $\mathbf{L}$ and thus occurs among the V_i's. It was also assumed in [3] that, when Ω is not bounded, (2) holds for any pair of compactly supported functions (for bounded domains the latter condition easily follows from the symmetry of $\mathbf{L}$ on the polynomials combined with the density of $\mathbb{R}[x]$ in $\mathcal{L}(\Omega,\mu)$). Under these assumptions, a complete list of solutions of the above problem is given in [3] in dimensions 2. Up to affine transformation of $\mathbb{R}^2$, there is a one-parameter family of bounded domains and also 19 rigid domains (10 of them are bounded) for which there exists a solution. One of these 19 domains is omitted in [3] as well as some solutions for two other domains (see the coorections in §6.3 below).

Remark 1.1. In dimension 1, the only solutions under the aforementioned assumptions are the classical systems of orthogonal polynomials: Hermite, Laguerre, and Jacobi polynomials. They are obtained by Gram-Schmidt orthogonalization process for the measure densities, respectively, $Ce^{-x^2/2}$ on $\mathbb{R}$, $C_a x^{a-1}e^{-x}$ with $a > 0$ on $[0,\infty)$, and $C_{p,q}(1-x)^{p-1}(1+x)^{q-1}$ with $p, q > 0$ on $[-1,1]$ (here C, C_a, and $C_{p,q}$ are normalizing constants). The corresponding operators are $\partial^2 - x\partial$, $x\partial^2 + (a-x)\partial$, and $J_{p,q} = (1-x^2)\partial^2 - \big((p-q)+(p+q)x\big)\partial$.

It turns out that the filtration by the usual degree is too restrictive in dimension $d \ge 2$. Several natural systems of orthogonal polynomial are not covered. However, they can be obtained by this procedure if one considers a weighted degree instead (see [1]). As usually, the weighted degree of a polynomial $P = \sum_k a_k x^k$, $k = (k_1,\dots,k_d)$, with real positive weights $\mathbf{w} = (w_1,\dots,w_d)$ is defined as

$$\deg_{\mathbf{w}}(P) = \max_{a_k\ne 0}(w_1k_1 + \dots + w_dk_d).$$

In this paper (in §6), for any pair of positive weights $\mathbf{w} = (w_1, w_2)$, we give a complete list of two-dimensional solutions satisfying the condition that for any n, the set of polynomials P with $\deg_{\mathbf{w}}(P) \le n$ is invariant under $\mathbf{L}$.

Let us give precise definitions. We say that $\Omega \subset \mathbb{R}^d$ is a *natural domain* if it is a connected open set which coincides with the interior of its closure.

Definition 1.2. (cf. [1, 3]) Let $\Omega \subset \mathbb{R}^d$ be a natural domain, $\mathbf{L}$ be an elliptic second order differential operator of the form (1) with coefficients continuous in Ω, and $\mu(dx) = \rho(x)\,dx$ be a probability measure on Ω such that ρ is $\mathcal{C}^1$-smooth in Ω and the space of all polynomials is dense in $\mathcal{L}^2(\Omega,\mu)$. Let $\mathbf{w}$ be a d-tuple of positive real numbers. We say that the triple $(\Omega,\mathbf{L},\mu)$ is a solution of the *Diffusion Orthogonal Polynomial Problem with weights* $\mathbf{w}$ ($\mathbf{w}$-DOP problem for short) if, for any n, the space $\{P \in \mathbb{R}[x_1,\dots,x_d] \mid \deg_{\mathbf{w}}(P) \le n\}$ (considered as a subspace of $\mathcal{L}^2(\Omega,\mu)$) has an orthogonal basis formed by eigenvectors of $\mathbf{L}$. If in addition the equality (2) holds for any two smooth functions compactly supported in $\mathbb{R}^d$, then we say that $(\Omega,\mathbf{L},\mu)$ is a solution of the *Strong* $\mathbf{w}$-DOP problem ($\mathbf{w}$-SDOP problem).

It is shown in [3, Prop. 2.11] that if $(\Omega,\mathbf{L},\mu)$ is a solution of the $\mathbf{w}$-SDOP problem, then $\mathbf{L}$ is determined by the measure density ρ and by the cometric $g = (g^{ij})$ (the

g^{ij} are the coefficients in (1)), namely, it is the diffusion operator

$$\mathbf{L}(f) = \frac{1}{\rho} \sum_{i,j} \partial_i \Big(g^{ij} \rho \, \partial_j f \Big). \tag{3}$$

We shall then speak also of the triple (Ω, g, ρ) as a solution of the $\mathbf{w}$-SDOP problem. Notice that (3) is the Laplace-Beltrami operator for the riemannian metric $(g_{ij}) = g^{-1}$ in the case when $\rho = (\det g)^{-1/2}$.

As we mentioned above, any solution of the $\mathbf{w}$-DOP problem with a bounded domain Ω is a solution of the $\mathbf{w}$-SDOP problem (see [3, Prop. 2.12]). It is also shown in [3] that any solution of SDOP-problem is a solution of a certain algebraic problem formulated in terms of the metric g^{ij} only (AlgDOP-problem; see §2 below). This fact is proven in [3] for the usual degree but all the proofs extend without changes to any weighted degree as well. Then, to find all two-dimensional solutions of the weighted SDOP problem, we follow the same strategy as in [3]. Namely, we first find solutions of the algebraic problem over $\mathbb{C}$ using some basic properties of plane complex algebraic curves (see §§4–5), and then we look for Ω and ρ (see §6).

All the bounded domains Ω admitting a solution, appeared already in the literature except, maybe, one infinite family: the case (B4) with $m \neq n$ in Theorem 6.2.

2. Some general facts about solutions of weighted DOP/SDOP problem in an arbitrary dimension

2.1. The AlgDOP Problem.

Let $\mathbf{w} = (w_1, \dots, w_d)$ and let (Ω, g, ρ) be a solution of the $\mathbf{w}$-SDOP problem in $\mathbb{R}^d$. Let $\Delta = \det(g^{ij})$. Let $\mathcal{P}_{\mathbf{w}}(n; \mathbb{K})$ be the vector space of polynomials in $x_1, \dots, x_d$ with coefficients in a field $\mathbb{K}$ whose $\mathbf{w}$-weighted degree is at most n. When $\mathbb{K}$ is $\mathbb{R}$, we write just $\mathcal{P}_{\mathbf{w}}(n)$.

Let $I(\partial\Omega)$ be the ideal in $\mathbb{R}[x_1, \dots, x_d]$ of polynomials vanishing on $\partial\Omega$. The condition that Ω is a natural domain implies that $I(\partial\Omega)$ is a principal ideal (because $U \cap \partial\Omega$ cannot be of codimension ≥ 2 for any open U). Let Γ be a generator of $I(\partial\Omega)$, that is Γ is a minimal polynomial vanishing on the boundary of Ω. In particular, if Γ is not identically zero, then it is *square-free*, i.e., it does not have multiple factors. By convention we set $I(\varnothing) = \mathbb{R}[x]$, i.e., $\Gamma = 1$ in the case when Ω is the whole $\mathbb{R}^d$.

Proposition 2.1. (See [3, Thm. 2.21].)

(A1) $g^{ij} \in \mathcal{P}_{\mathbf{w}}(w_i + w_j)$ *for any* $i, j = 1, \dots, d$. *Hence* $\Delta \in \mathcal{P}_{\mathbf{w}}(2w_1 + \dots + 2w_d)$.
(A2) $\partial\Omega \subset \{\Delta = 0\}$, *hence* Γ *divides* Δ.
(A3) *For each* $i = 1, \dots, d$, *one has*

$$\sum_j g^{ij} \partial_j \Gamma = \Gamma S^i, \qquad S^i \in \mathcal{P}_{\mathbf{w}}(w_i). \tag{4}$$

Condition (A1) easily follows from the invariance of the weighted degree, (A3) is derived in [3] from the symmetry of $\mathbf{L}$, and (A2) follows from (A3).

This proposition leads us to the following definition (cf. [3, Definition 3.2]).

Definition 2.2. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $\mathbf{w} = (w_1, \dots, w_d)$ be a d-tuple of positive real numbers. A solution to the *Algebraic Counterpart of the* $\mathbf{w}$*-SDOP Problem* *over* $\mathbb{K}$ (**w**-AlgDOP Problem over $\mathbb{K}$ for short) is a pair (g,Γ) where $g=(g^{ij})$ is a symmetric $d\times d$ matrix with polynomial entries and Γ a polynomial such that

(A1) $g^{ij}\in\mathcal{P}_{\mathbf{w}}(w_i+w_j;\mathbb{K})$ for each $i,j=1,\dots,d$;
(A2) $\det g$ is not identically zero, and Γ is a square-free factor of $\det g$;
(A3) Γ divides $\sum_j g^{ij}\partial_j\Gamma$ for each $i=1,\dots,d$.

Thus Proposition 2.1 implies that if (Ω,g,ρ) is a solution of the **w**-SDOP Problem and Γ is a generator of $I(\partial\Omega)$, then (g,Γ) is a solution of the **w**-AlgDOP Problem over $\mathbb{R}$ and hence over $\mathbb{C}$. The following facts follow from the definition.

Proposition 2.3. *Let* $\mathbf{w}=(w_1,\dots,w_d)$ *and* $\mathbf{w}'=(w'_1,\dots,w'_d)$ *be two d-tuples of positive weights. If (g,Γ) is a solution of the* **w**-*AlgDOP Problem over* $\mathbb{K}$, *and* $g^{ij}\in\mathcal{P}_{\mathbf{w}'}(w'_i+w'_j;\mathbb{K})$ *for each* $i,j=1,\dots,d$, *then* (g,Γ) *is a solution of the* $\mathbf{w}'$-*AlgDOP Problem over* $\mathbb{K}$.

Proposition 2.4. *If* (g,Γ) *is a solution of the* **w**-*AlgDOP Problem over* $\mathbb{K}$ *and* Γ_1 *is a factor of* Γ, *then* (g,Γ_1) *is also a solution of the* **w**-*AlgDOP Problem over* $\mathbb{K}$.

If a square-free polynomial Γ is given, then it is easy to find all cometrics g such that (g,Γ) is a solution of the **w**-AlgDOP problem. Indeed, Condition (A3) of Definition 2.2 (in the form (4)) provides a system of linear equations for the coefficients of the polynomials g^{ij} and S^i. In §2.3 we show how to find all ρ for given Ω and g.

2.2. Admissible changes of variables.

A **w**-*admissible change of variables* is a bijective polynomial mapping $\Phi:\mathbb{R}^d\to\mathbb{R}^d$, $x\mapsto(y_1(x),\dots,y_d(x))$ with $\deg_{\mathbf{w}}(y_i)=w_i$, $i=1,\dots,d$. If $d=2$, then $(1,w)$-admissible automorphisms of $\mathbb{R}^2$ for $w=1$ are affine linear transformations, and for $w>1$, mappings of the form

$$(x,y)\mapsto(\alpha x+\beta,\ \gamma y+p(x)),\qquad \alpha\gamma\neq0,\ \deg(p)\le w. \tag{5}$$

The following proposition is similar to [3, Prop. 2.5] and we omit its proof.

Proposition 2.5. *(a). Let* $(\Omega,\mathbf{L},\mu)$ *be a solution of* **w**-*DOP (resp.* **w**-*SDOP) problem and* Φ *be a* **w**-*admissible change of variables. Let* $\Omega_1=\Phi(\Omega)$, $\mathbf{L}_1(f)=\mathbf{L}(f\circ\Phi)\circ\Phi^{-1}$, *and* $\mu_1(E)=\mu(\Phi^{-1}(E))$. *Then* $(\Omega_1,\mathbf{L}_1,\mu_1)$ *is also a solution of* **w**-*DOP (resp.* **w**-*SDOP) problem.*

(b). Let (g,Γ) *be a solution of the* **w**-*AlgDOP Problem over* $\mathbb{K}$ *and let* Φ *be a* **w**-*admissible change of variables. Then* $(\Phi_*(g),\Gamma\circ\Phi^{-1})$ *is also a solution of the* **w**-*AlgDOP Problem over* $\mathbb{K}$.

Example 2.6. Let $d=2$. For any $\mathbf{w}=(w_1,w_2)$, the mapping $\Phi:(x,y)\mapsto(x,-y)$ is **w**-admissible. Thus, if (g,Γ) with $g=\left(\begin{smallmatrix}a&b\\b&c\end{smallmatrix}\right)$ is a solution of the **w**-AlgDOP Problem over $\mathbb{K}$, then $(\Phi_*(g),\Gamma(x,-y))$ with

$$\Phi_*(g)=\begin{pmatrix}a(x,-y)&-b(x,-y)\\-b(x,-y)&c(x,-y)\end{pmatrix}$$

is also a solution of the **w**-AlgDOP problem over $\mathbb{K}$.

Example 2.7. More generally, let still $d = 2$. Any $(1, w)$-admissible change of variables for $w > 1$ is of the form (5) and it is a composition of the following variable changes $(x, y) \mapsto (X, Y)$:

$$T : (x, y) \mapsto (x + \beta, y), \quad H : (x, y) \mapsto (\alpha x, \gamma y), \quad S : (x, y) \mapsto (x, y + p(x)).$$

They transform $g = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ as follows (see [3, eq. (2.3)]): $T_*(g) = g(X - \beta, Y - \beta)$,

$$H_*(g) = \begin{pmatrix} a\alpha^2 & b\alpha\gamma \\ b\alpha\gamma & c\gamma^2 \end{pmatrix}_{\substack{x=X/\alpha \\ y=Y/\gamma}} \qquad S_*(g) = \begin{pmatrix} a & p'a + b \\ p'a + b & (p')^2 a + 2p'b + c \end{pmatrix}_{\substack{x=X \\ y=Y-p(X)}}$$

2.3. From AlgDOP to DOP/SDOP.

As we told in §2.1, any solution (Ω, g, μ) of the $\mathbf{w}$-SDOP problem gives a solution (Γ, g) of the $\mathbf{w}$-AlgDOP problem where Γ is a generator of the ideal $I(\partial\Omega)$. In this subsection we discuss how to find all possible (Ω, g, μ) from a given (Γ, g).

So, let (Γ, g) be a solution of the $\mathbf{w}$-AlgDOP problem over $\mathbb{R}$. First of all we find all connected components Ω of $\mathbb{R}^d \setminus \{\Gamma = 0\}$ such that g is positive definite on Ω. Then (see [3, Thm. 2.21]) it remains to find all measure densities ρ such that all polynomials are integrable and the operator $\mathbf{L}$ given by (3) is of the form (1) with $b^i \in \mathcal{P}_{\mathbf{w}}(w_i)$ for each $i = 1, \dots, d$. By comparing (1) with (3) we obtain

$$b^i = \sum_j \partial_j g^{ij} + \sum_j g^{ij} \partial_j h, \qquad h = \log \rho. \tag{6}$$

Hence

$$\partial_j h = \sum_i g_{ij} L^i, \qquad L^i = b^i - \sum_j \partial_j g^{ij}, \tag{7}$$

where $(g_{ij}) = g^{-1}$. We have $b^i \in \mathcal{P}_{\mathbf{w}}(w_i) \Leftrightarrow L^i \in \mathcal{P}_{\mathbf{w}}(w_i)$. The identities $\partial_k(\partial_j h) = \partial_j(\partial_k h)$ combined with (7) yield

$$\partial_k \Big(\sum_i g_{ij} L^i \Big) = \partial_j \Big(\sum_i g_{ik} L^i \Big) \tag{8}$$

which is a system of linear equations for the coefficients of all L^i. These observations lead to the following algorithm of finding all solutions for ρ. We start with polynomials L^i, $\deg_{\mathbf{w}} L^i = w_i$, whose coefficients we compute by solving the system of linear equations coming from (8). The solution may depend on several parameters. Then we find h by integrating the $\partial_j h$'s (expressed in (7) via the L^i's) and set $\rho = \exp(h)$. Finally we choose the values of the parameters such that $Q(x)\rho(x)$, is integrable over Ω for any polynomial Q (when Ω is bounded, it is enough to demand that $\int_\Omega \rho \, dx < \infty$).

These observations allow us to prove the following fact.

Proposition 2.8. *Let (Ω, g, ρ) be a solution of the $\mathbf{w}$-SDOP problem in $\mathbb{R}^d$ such that $g = \operatorname{diag}\big(g^{11}(x_1), \dots, g^{dd}(x_d)\big)$. Then there is a partition of the set of indices $\{1, \dots, d\} = J_1 \sqcup \dots \sqcup J_m$, $J_k = \{j_k(1), \dots, j_k(d_k)\}$, such that:*

- *$\rho = \rho_1(\mathbf{x}_1) \dots \rho_m(\mathbf{x}_m)$, where $\mathbf{x}_k = (x_{j_k(1)}, \dots, x_{j_k(d_k)})$, $k = 1, \dots, m$;*
- *if $d_k > 1$, then $w_i = w_j$ and g^{ii} is constant for any $i, j \in J_k$.*

Proof. We consider only the case $d = 2$. It is not difficult to derive from it the general case. Let $h = \log\rho$. It is enough to prove that either g is constant, or $\partial_{12}h = 0$ and then $h = h_1(x_1) + h_2(x_2)$ whence $\rho = \rho_1(x_1)\rho_2(x_2)$. Indeed, the inverse matrix of g is $\operatorname{diag}(g_{11}, g_{22})$ where $g_{ii} = 1/g^{ii}$. Then by (8) we have

$$\partial_{12}h = \frac{\partial_2 L^1(x_1,x_2)}{g^{11}(x_1)} = \frac{\partial_1 L^2(x_1,x_2)}{g^{22}(x_2)}, \qquad \deg_{\mathbf{w}} L^i \le w_i.$$

Therefore $\partial_{12}h$ is a polynomial, hence g^{ii} divides $\partial_j L^i$ for $i \neq j$. Thus, if $\partial_{12}h$ is nonzero, we have $0 \le \deg_{\mathbf{w}} g^{11} \le \deg_{\mathbf{w}} \partial_2 L^1 = \deg_{\mathbf{w}} L^1 - w_2 \le w_1 - w_2$ and, symmetrically, $0 \le \deg_{\mathbf{w}} g^{22} \le w_2 - w_1$, which implies $w_1 = w_2$ and $\deg_{\mathbf{w}} g = 0$, i.e., g is constant. $\square$

To conclude this subsection, we give some conditions (necessary or sufficient) on ρ to be compatible with given g and Ω. They are proven in [3] for the usual degree but the proofs can be easily adapted for any weighted degree.

Proposition 2.9. (See [3, Thm. 2.21].) *Let (g,Γ), $\Gamma = \Gamma_1 \dots \Gamma_s$, be a solution of* $\mathbf{w}$*-AlgDOP problem over $\mathbb{R}$ (recall that then Γ is squarefree). Let Ω be a bounded domain such that $\partial\Omega \subset \{\Gamma = 0\}$ and g is positive definite in Ω. Let $a_1, \dots, a_s$ be real numbers such that $\mu(\Omega) < \infty$ where μ is the measure with density $\rho = \prod_\nu \Gamma_\nu^{a_\nu}$ (for example, $a_\nu \ge 0$ for each ν). Then $(\Omega, \mathbf{L}, \mu)$ is a solution of* $\mathbf{w}$*-DOP problem, where $\mathbf{L}$ is given by (3).*

Definition 2.10. A solution (g,Γ) of the $\mathbf{w}$-AlgDOP problem is called *maximal* (and in this case Γ is called a *maximal boundary for* g) if Γ_1 divides Γ for any solution (g,Γ_1). By Proposition 2.4, a maximal solution is unique for any given g.

Proposition 2.11. (See [3, Props. 2.15, 2.17].) *Let (Ω, g, ρ) be a solution of* $\mathbf{w}$*-SDOP problem in $\mathbb{R}^d$ and $\Delta = \det(g)$. Let Γ be the maximal boundary for g and let $\Gamma_1, \dots, \Gamma_s$ be its irreducible (over $\mathbb{C}$) factors. Suppose that each factor Γ_k occurs in Δ with multiplicity 1, i.e., Γ_k^2 does not divide Δ. Then (see Remark 2.12)*

$$\rho = \Gamma_1^{p_1} \dots \Gamma_s^{p_s} \exp(Q) \tag{9}$$

for some $p_1, \dots, p_s \in \mathbb{C}$ and a polynomial Q such that, for each $j = 1, \dots, d$,

$$w_j \deg_{x_j}(Q\Delta) \le 2w_1 + \dots + 2w_d. \tag{10}$$

Remark 2.12. In Proposition 2.11, if $\lambda\Gamma_k$, $\lambda \in \mathbb{C}$, is real for some k, we always assume that Γ_k is real and positive on Ω; in this case p_k is real. Otherwise $\bar\Gamma_k$ is also a factor of Γ and it must occur in ρ with the power $\bar p_k$ because ρ is real. In this case $\Gamma_k^{p_k}\bar\Gamma_k^{\bar p_k}$ is understood as a single-valued branch of this function on Ω. Notice that choosing another single-valued branch we just change the constant term of Q.

Remark 2.13. Proposition 2.11 admits the following refinement. We still have (9) and (10) for a variable x_j even when there are multiple factors Γ_k of Δ but they are polynomials in variables $(x_i)_{i\in I}$ not including x_j, that is $j \notin I$. In this case Q is a polynomial in $(x_i)_{i\notin I}$ whose coefficients are rational functions in $(x_i)_{i\in I}$.

Corollary 2.14. *Under the hypothesis of Proposition 2.11, assume that* $mw_d = 2(w_1+\dots+w_d)$ *and* $w_d = w_i n_i$, $i = 1,\dots,d$, *for some* $m, n_1,\dots,n_d \in \mathbb{Z}$ (e.g. $d = 2$ and $(w_1, w_2) = (1, 2)$). *Suppose that* $\deg_{x_d} \Delta = m$. *Then* $Q = \mathrm{const}$ *in (9).*

Proof. By an admissible change of variables $y_d = x_d + \sum_{i=1}^{d-1} a_i x_i^{n_i}$ and $y_j = x_j$ for $j < d$, we may achieve that $w_j \deg_{y_j} \Delta = 2\sum w_i$ for each j. □

Recall that $(g_{ij}) = g^{-1}$, hence $g_{ij} = \hat g_{ij}/\det g$ with a polynomial $\hat g_{ij}$.

Proposition 2.15. (See [3, Cor. 2.19].) *Let* $U = \mathbb{R}\times\mathbb{B}^{d-1} = \{x_2^2+\dots+x_d^2 < 1\}$ *and* $U_+ = \mathbb{R}_+\times\mathbb{B}^{d-1} = U\cap\{x_1 > 0\}$. *Let* $M_1 = \max_j\big(\lfloor w_j/w_1\rfloor + \deg_{x_1}\hat g_{1j}\big)$. *Let* (Ω, g, ρ) *be a solution of the* $\mathbf{w}$*-SDOP problem,* $\Delta = \det g$. *Suppose that* $U\subset\Omega$ *(resp.* $U_+\subset\Omega$*). Then* $\deg_{x_1}\Delta < M_1$ *(resp.* $\deg_{x_1}\Delta < 1 + M_1$*).*

3. Weighted AlgDOP Problem in $\mathbb{C}^2$

3.1. Notation. In §§3–5 we study the $\mathbf{w}$-AlgDOP Problem over $\mathbb{C}$ in dimension 2 for any pair of weights $\mathbf{w}$. It is clear that a multiplication of $\mathbf{w}$ by a positive number does not change the problem. Therefore we assume throughout this section that (g,Γ), is a solution of the $\mathbf{w}$-AlgDOP Problem over $\mathbb{C}$ for $\mathbf{w} = (1, w)$ with a real $w > 1$ (the case $w = 1$ is already done in [3]).

We denote variables by (x, y) and we set $g = \left(\begin{smallmatrix} a & b\\ b & c\end{smallmatrix}\right)$. We denote the coefficient of y^m in $a(x, y)$ by $a_m(x)$ and the coefficient of $x^k y^m$ by a_{km} and we use similar notation for b and c. Sometimes we write $(a, b, c; \Gamma)$ instead of (g,Γ) when speaking of a solution of the $\mathbf{w}$-AlgDOP Problem.

As usual, given a polynomial $P(x, y) = \sum p_{km}x^k y^m$, we define its Newton polygon $\mathcal{N}(P)$ as the convex hull in $\mathbb{R}^2$ of the finite set $\{(k, m) \mid a_{km} \neq 0\}$. Condition (A1) of Definition 2.2 means that the Newton polygons of a, b, c, and Δ are contained in the polygons shown in Figures 1–2.

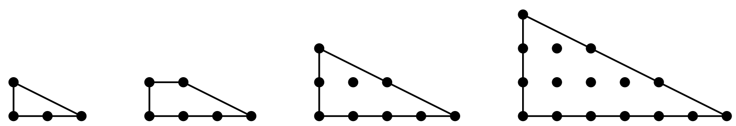

Figure 1. Polygons containing $\mathcal{N}(a), \mathcal{N}(b), \mathcal{N}(c), \mathcal{N}(\Delta)$ for $1 < w \le 2$.

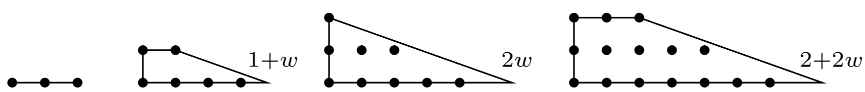

Figure 2. Polygons containing $\mathcal{N}(a), \mathcal{N}(b), \mathcal{N}(c), \mathcal{N}(\Delta)$ for $w > 2$.

3.2. Change of the weights and the $(1,\infty)$-AlgDOP Problem.
According to Proposition 2.3, any solution of $(1, w)$-AlgDOP Problem with $1 < w \le 2$ is a solution of $(1, 2)$-AlgDOP Problem (see Figure 1). Similarly (see Figure 2) any solution of $(1, w)$-AlgDOP Problem with $w > 2$ is a solution of $(1, w')$-AlgDOP problem for any $w' > w$.

So, we say that (g,Γ) is a solution of the $(1,\infty)$-AlgDOP Problem, if it is a solution of the $(1,w)$-AlgDOP Problem for some $w>2$, and a coordinate change (5) with an arbitrary polynomial $p(x)$ will be called $(1,\infty)$-admissible.

In the next two sections we find all solutions of the $(1,\infty)$- and $(1,2)$-AlgDOP Problems up to $(1,\infty)$- and $(1,2)$-admissible coordinate change.

3.3. Local branches of the curve $\Gamma=0$.

Let $P(x,y)$ be a squarefree polynomial. A *local branch* of P (or, equivalently, of the curve $P=0$) is a pair $\gamma=(\xi,\eta)$ of germs at 0 of meromorphic functions such that $P(\xi(t),\eta(t))$ is identically zero. Any meromorphic germ $t\mapsto\gamma(t)=(\xi(t),\eta(t))$ defines a *valuation* $v_\gamma:\mathbb{C}[x,y]\to\mathbb{Z}\cup\{\infty\}$, $v_\gamma(Q)=\operatorname{ord}_t Q(\xi(t),\eta(t))$ where $\operatorname{ord}_t(0)=\infty$ and $\operatorname{ord}_t f(t)=m$ if $f(t)=\sum_{k\ge m}p_kt^k$ and $p_m\ne0$.

Let $(a,b,c;\Gamma)$ be a solution of the $\mathbf{w}$-AlgDOP problem over $\mathbb{C}$ for some $\mathbf{w}=(1,w)$. Condition (A3) of Definition 2.2 reads

$$a\Gamma'_x+b\Gamma'_y=L_1\Gamma,\qquad b\Gamma'_x+c\Gamma'_y=L_2\Gamma. \tag{11}$$

It is easy to check that this condition is equivalent to

$$b(\xi,\eta)\dot\xi=a(\xi,\eta)\dot\eta,\qquad c(\xi,\eta)\dot\xi=b(\xi,\eta)\dot\eta \tag{12}$$

for any local branch of Γ. Condition (12) implies that

$$v_\gamma(a)-v_\gamma(b)=v_\gamma(b)-v_\gamma(c)=\operatorname{ord}_t(\dot\xi)-\operatorname{ord}_t(\dot\eta) \tag{13}$$

if both $\xi(t)$ and $\eta(t)$ are non-constant (see [3, Lemma 3.3]).

The following fact is well-known and immediately follows from the definitions.

Lemma 3.1. *Let F be a polynomial in (x,y) and let $(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}$. A local branch γ of F such that $\operatorname{ord}_t(\gamma)=(p,q)$ exists if and only if the vector (p,q) is orthogonal to some edge of $\mathcal{N}(F)$ and points from this edge inward $\mathcal{N}(\Gamma)$.* $\square$

Lemma 3.2. *Let $(a,b,c;\Gamma)$ is a solution of $(1,w)$-AlgDOP Problem with $w>1$. Suppose that Γ is divisible by x. Then a and b are divisible by x, hence $\deg_y a=0$ and $(a,b,c;\Gamma)$ is a solution of the $(1,\infty)$-AlgDOP Problem.*

Proof. We have $a(0,t)=b(0,t)=0$ by (12), hence x is a factor of a and b whence $\deg_y a=0$ (see Figure 1). $\square$

Lemma 3.3. *Let $w>1$. Suppose that Γ has a branch $\gamma=(\xi(t),\eta(t))$ such that $v_\gamma(x,y)=\operatorname{ord}_t\gamma=(p,q)$ with $q<0<p$. Then $b_1=b_{11}x$ and $a=a_{20}x^2$, hence $(g;\Gamma)$ is a solution of $(1,\infty)$-AlgDOP Problem.*

Proof. We have $\operatorname{ord}_t\dot\xi-\operatorname{ord}_t\dot\eta=p-q$, hence $v_\gamma(a)-v_\gamma(b)=v_\gamma(b)-v_\gamma(c)=p-q$ by (13). On the other hand we have $v_\gamma(c)\ge v_\gamma(y^2)=2q$, hence

$$v_\gamma(b)=v_\gamma(c)+p-q\ge p+q. \tag{14}$$

If b_{01} were non-zero, then we would have $v_\gamma(b)=v_\gamma(y)=q$ which contradicts (14) because $p>0$. Hence $b_1=b_{11}x$ (as required), and this fact implies that

$$v_\gamma(b)\ge\min(v_\gamma(1),v_\gamma(xy))=\min(0,p+q),$$

thus $v_\gamma(a)=v_\gamma(b)+p-q\ge\min(0,p+q)+p-q=p+\min(-q,p)>p$. Since the only monomials which may occur in a are $(y,1,x,x^2)$ and their γ-valuations are $(q,0,p,2p)$, we see that $a=a_{20}x^2$. $\square$

Lemma 3.4. *Let $1 < w \le 2$. Let $\gamma = (\xi(t), \eta(t))$ be a local branch of Γ, and $(p,q) = \operatorname{ord}_t \gamma = v_\gamma(x,y)$.*

(a). If $(p,q) = (2,1)$ then $a_{00} = a_{01} = b_{00} = 0$, and hence $\deg_y \Delta \le 2$ and $(a,b,c;\Gamma)$ is a solution of the $(1,\infty)$-AlgDOP problem.

(b). If $q > 0$, then $p \ne 3$.

(c). If $p = -3$ and $-4 \le q \le -1$, then $\deg b_0 \le 2$, $\deg c_0 \le 2$, and $\deg c_1 \le 1$, hence $(a,b,c;\Gamma)$ is a solution of the $(1,1)$-AlgDOP problem.

(d). If $q < 2p < 0$, then $a_{01} = 0$, hence $(a,b,c;\Gamma)$ is a solution of the $(1,\infty)$-AlgDOP problem.

Proof. (a). If $(p,q) = (2,1)$, then $\operatorname{ord}_t(\dot\xi, \dot\eta) = (1,0)$, hence by (13) we have $v_\gamma(b) = v_\gamma(c) + 1 \ge 1$ and $v_\gamma(a) = v_\gamma(b) + 1 = v_\gamma(c) + 2 \ge 2$, and the result follows from the fact that 1 (resp. y) is the only monomial of a and b of the γ-valuation equal to 0 (resp. to 1). The condition $a_{01} = 0$ implies that $(a,b,c;\Gamma)$ is a solution of the $(1,\infty)$-AlgDOP problem (see Figure 2).

(b). Suppose that $p = 3$ and $q > 0$. Then by the arguments similar to those in (a), we are going to show that $\deg_y a \le 0$ and hence $\deg_y \Delta \le 2$ which contradicts the condition $\operatorname{ord}_t \xi = 3$. Notice that $q \le \deg_x \Delta \le 6$.

If $q = 1$, the proof is the same as in (a).

If $q = 2$, then by (13) we have $v_\gamma(a) = v_\gamma(b) + 1$. Since $v_\gamma(a) \ne 1$ and $v_\gamma(b) \ne 1$, this is possible only when $v_\gamma(a) \ge 3$. By combining this fact with $v_\gamma(1, y, x, x^2) = (0, 2, 3, 6)$, we obtain $a_{00} = a_{01} = 0$ whence $\deg_y a \le 0$.

The case $q = 3$ is reduced to $q > 3$ by a change of variables $(x,y) \mapsto (x, y + \lambda x)$.

If $q = 4$, then $v_\gamma(a) \in \{0,3,4,6\}$, $v_\gamma(b) \in \{0,3,4,6,7,9\}$, $v_\gamma(c) \in \{0,3,4,6,\dots\}$. Then one can check that $v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = 1$ (this is (13)) is possible only when $v_\gamma(a) = 6$, hence $\deg_y a \le 0$.

If $q = 5$, then $v_\gamma(a) \in \{0,3,5,6\}$, $v_\gamma(b) \in \{0,3,5,6,8,9\}$, $v_\gamma(c) \in \{0,3,5,6,8,\dots\}$. Then $v_\gamma(c) - v_\gamma(b) = v_\gamma(b) - v_\gamma(a) = 2$ (this is (13)) is possible only when $v_\gamma(a) = 6$, hence $\deg_y a \le 0$.

The case $q = 6$ is reduced to $q > 6$ by a change of variables $(x,y) \mapsto (x, y + \lambda x^2)$.

(c). We need to show that x^3, x^4, and x^2y do not occur in b and in c.

If $p = -3$ and $-3 \le q \le -1$, then (13) implies $v_\gamma(c) \ge v_\gamma(b) \ge v_\gamma(a) \ge v_\gamma(x^2) = -6$. On the other hand, $v_\gamma(x^2y) = -6 + q \ge -7$, $v_\gamma(x^3) = -9$, $v_\gamma(x^4) = -12$, and these are the only monomials which might occur in b and in c with the respective γ-valuations.

If $(p,q) = (-3,-4)$, then (13) implies $v_\gamma(a) \ge v_\gamma(x^2) = -6$, $v_\gamma(b) = v_\gamma(a) - 1 \ge -7$, and $v_\gamma(c) = v_\gamma(b) - 1 \ge -8$. On the other hand, $v_\gamma(x^2y) = -10$, $v_\gamma(x^3) = -9$, $v_\gamma(x^4) = -12$, and these are the only monomials which might occur in b and in c with the respective γ-valuations.

(d). The condition $q < 2p < 0$ implies $v_\gamma(b) \ge v_\gamma(xy) = p + q$. It implies also $p - q > -p$. By (13) we have $v_\gamma(a) = v_\gamma(b) + p - q$. Hence

$$v_\gamma(a) = v_\gamma(b) + p - q > v_\gamma(b) - p \ge (p+q) - p = q = v_\gamma(y).$$

Therefore $a_{01} = 0$ because y is the only monomial which may occur in a with the γ-valuation equal to q. The condition $a_{01} = 0$ implies that $(a,b,c;\Gamma)$ is a solution of the $(1,\infty)$-AlgDOP problem (see Figure 2). $\square$

4. Solution of the $(1,w)$-AlgDOP Problem in $\mathbb{C}^2$ for $w>2$

In this section we give all solutions of the $(1,\infty)$-AlgDOP Problem over $\mathbb{C}$ up to $(1,\infty)$-admissible change of coordinates. As we explained in §3.2, this gives all solutions of the $(1,w)$-AlgDOP Problem for all $w>2$.

Lemma 4.1. *Let $(a,b,c;\Gamma)$ be a solution of the $(1,w)$-AlgDOP problem over $\mathbb{C}$ with $w>1$. Suppose that $\deg_y\Gamma=2$ and that Γ is not divisible by any non-constant polynomial in x. Then Γ is monic with respect to y, i.e., y^2 is the only monomial of Γ whose y-degree is 2.*

Proof. First, remark that $a\ne 0$. Indeed, otherwise $\Delta=b^2$, hence Γ would be a square-free factor of b which contradicts the condition $\deg_y\Gamma=2$. Suppose that the coefficient of y^2 in Γ is a non-constant polynomial in x. Without loss of generality we may assume that one of its roots is 0. Then Γ has a branch $\gamma=(\xi(t),\eta(t))$ such that $\operatorname{ord}_t\eta<0<\operatorname{ord}_t\xi$. Then, by Lemma 3.3, we have $a=a_{20}x^2$ and $b=b_{11}xy+b_0(x)$. Since $a\ne0$, we may assume that $a=x^2$.

If $b_{00}=0$, then x^2 is a factor of ac and of b^2, hence x^2 is a factor of Δ. Since Δ may not have have monomials y^2x^k with $k>2$, it follows that Γ is monic in y.

Now consider the case $b_{00}\ne0$. Then the constant term of Δ is $-b_{00}^2\ne0$. We have $a=x^2$ and $b_1=b_{11}x$, hence x^2y^2 is the only monomial of y-degree 2 which may occur in Δ. Therefore Δ cannot be divisible by any non-constant polynomial in x, i.e., $\Gamma=\Delta$.

The coefficients of y^2 in Δ and in $a\Delta'_x+b\Delta'_y$ are $d_{22}x^2$ and $2d_{22}(1+b_{11})x^3$ respectively where $d_{22}=c_{02}-b_{11}^2$. Then (11) implies $L_1=2(1+b_{11})x$. By plugging this back into (11), we obtain a contradiction. Indeed, we have:

$$\begin{aligned}
\Delta &= -b^2+O(x^2) = -b_{00}^2-2b_{00}(b_{10}+b_{11}y)x+O(x^2),\\
b\Delta'_y &= \big(b_{00}+O(x)\big)\big(-2b_{00}b_{11}x+O(x^2)\big) = -2b_{00}^2b_{11}x+O(x^2),\\
L_1\Delta &= -2b_{00}^2(1+b_{11})x+O(x^2).
\end{aligned}$$

Hence $a\Delta'_x+b\Delta'_y-L_1\Delta=2b_{00}^2x+O(x^2)\ne0$. $\square$

Lemma 4.2. *Let $\Gamma=y^2-p(x)$ and let $(a,b,c;\Gamma)$ be a solution of the $(1,w)$-AlgDOP problem with $w>2$. Then $b_0=c_1=0$, i.e., a and c are even with respect to y, and b is odd with respect to y (this means that the corresponding metric is invariant under the symmetry $y\mapsto -y$).*

Proof. Let us set $\hat a=a$, $\hat b(x,y)=-b(x,-y)$, $\hat c(x,y)=c(x,-y)$, and $\hat g=(\hat a,\hat b,\hat c)$. Since Γ is symmetric, Proposition 2.5 implies that $(\hat g;\Gamma)$ is also a solution to the $(1,w)$-AlgDOP problem (see Example 2.6). Hence both $(g;\Gamma)$ and $(\hat g;\Gamma)$ satisfy the equations (11) and, by linearity, $\big(\frac12(g-\hat g);\Gamma\big)$ satisfies it as well. Since $\frac12(g-\hat g)=(0,b_0,c_1y)$, this means $2yb_0=(y^2-p(x))L_1$ and $-b_0p'(x)+2y^2c_1=(y^2-p(x))L_2$. The first equation implies $b_0=0$. Plugging this into the second equation, we obtain $2y^2c_1=(y^2-p(x))L_2$. Note that $p\ne0$ because Γ cannot have multiple factors. Hence the last equations implies $c_1=0$, i.e., $g-\hat g=0$. $\square$

Proposition 4.3. *The following is a complete list of solutions $(g;\Gamma)$ of the $(1,\infty)$-AlgDOP problem over $\mathbb{C}$ up to $(1,\infty)$-admissible change of variables under condition* *that* $\deg_y \Gamma = 2$*:*

(i) $\Gamma = (1-x)^m(1+x)^n - y^2$, $m, n \ge 1$,

$$g = \begin{pmatrix} 1-x^2 & b_1 y \\ * & b_1^2(1-x)^{m-1}(1+x)^{n-1} - c_{02}\Gamma \end{pmatrix}, \quad b_1 = \frac{n-m}{2} - \frac{n+m}{2}x;$$

(ii) $\Gamma = x^n - y^2$, $n \ge 1$,

$$g = \begin{pmatrix} x & \frac{1}{2}ny \\ \frac{1}{2}ny & \frac{1}{4}n^2x^{n-1} - c_{02}\Gamma \end{pmatrix};$$

(iii) $\Gamma = \Gamma_2 x^k$ *where* $\Gamma_2 = (x_0 - x)^n - y^2$, $n \ge 0$; $k, x_0 \in \{0, 1\}$, $(n, c_{02}) \ne (0, 0)$,

$$g = \begin{pmatrix} x(x_0 - x) & -\frac{1}{2}nxy \\ -\frac{1}{2}nxy & \frac{1}{4}n^2x(x_0-x)^{n-1} - c_{02}\Gamma_2 \end{pmatrix};$$

(iv) $\Gamma = x^k(1-y^2)$, $g = \operatorname{diag}(x^k, -c_{02}(1-y^2))$, $k \in \{0, 1\}$, $c_{02} \ne 0$*;*

(v) $\Gamma = (1-x^2)(1-y^2)$, $g = \operatorname{diag}(1-x^2, -c_{02}(1-y^2))$, $c_{02} \ne 0$.

Proof. Let $\Gamma = \Gamma_0\Gamma_2$ where $\deg_y \Gamma_0 = 0$, $\deg_y \Gamma_2 = 2$, and Γ_2 is not divisible by any non-constant polynomial in x. Then Γ_2 is monic in y by Lemma 4.1 combined with Proposition 2.4, hence it can be reduced to the form $\Gamma_2 = y^2 + p(x)$ by a $(1, \infty)$-admissible change of coordinates. Since Γ cannot have any multiple factor, p is not identically zero. Moreover, by rescaling the y-coordinate, we may set the leading coefficient of p to any prescribed nonzero complex number. By Lemma 4.2, we have $a = a(x)$, $b = b_1(x)y$, and $c = c_2y^2 + c_0(x)$ (with $c_2 = c_{02} = \text{const}$). Also $a \ne 0$ (see the beginning of the proof of Lemma 4.1). By (11) we have

$$a(\Gamma_2)'_x + b(\Gamma_2)'_y = ap' + 2b_1y^2 = (y^2 + p)L_1.$$

Hence $L_1 = 2b_1$ and then

$$ap' = 2pb_1. \tag{15}$$

The second equation in (11) reads

$$b(\Gamma_2)'_x + c(\Gamma_2)'_y = b_1yp' + 2y(c_2y^2 + c_0) = (y^2 + p)L_2$$

whence $L_2 = 2c_2y$ and hence $c_0 = c_2p - \frac{1}{2}b_1p'$, i.e., $c = c_2\Gamma_2 - \frac{1}{2}b_1p'$. By combining this fact with (15), we see that

$$c = c_2\Gamma_2 - b_1^2p/a. \tag{16}$$

If $b = 0$, we have $c = c_{02}\Gamma_2$ by (16) and $p = \text{const}$ by (15). Then we may set $p = -1$ and we arrive to (iii)–(v) with $n = 0$ in (iii).

Now let $b \ne 0$. The equation (15) can be rewritten as $(\log p)' = 2b_1/a$. Since p, a, and b_1 are polynomials and $\deg(a) \le 2$, we conclude that, up to translation and rescaling the variable x, one of the following three cases occurs.

Case 1. $a = 1 - x^2$.

$$\frac{2b_1}{a} = \frac{p'}{p} = \frac{n}{1+x} - \frac{m}{1-x}, \qquad p = -(1-x)^m(1+x)^n, \qquad m, n > 0$$

(recall that the leading coefficient of p can be chosen arbitrarily). Therefore $b_1 = \frac{1}{2}(n-m) - \frac{1}{2}(n+m)x$. By combining this fact with (16) we obtain (i) as soon as $\Gamma = -\Gamma_2$. So, it remains to show that Γ coincides with Γ_2 up to a constant factor, that is $\Gamma_0 = \text{const}$. Indeed, if Γ_0 were non-constant, by Lemma 3.2 it would be a common factor of a and b, but this is impossible for our explicit form of a and b.

Case 2. $a = x$, $p'/p = 2b_1/a = n/x$, $p = x^n$, $n \geq 0$, hence $b_1 = n/2$. Since a and b are coprime, $\Gamma = \Gamma_2$ and we arrive to (ii) similarly to Case 1.

Case 3. $a = x(x_0 - x)$, $p'/p = 2b_1/a = -n/(x_0 - x)$, $p = -(x_0 - x)^n$, $n \geq 0$, hence $b_1 = -nx/2$. This yields (iii). □

Proposition 4.4. *The following is a complete list of solutions* $(a, b, c; \Gamma)$ *of the* $(1,\infty)$*-AlgDOP problem over* $\mathbb{C}$ *up to* $(1,\infty)$*-admissible change of variables under condition that* $\deg_y \Gamma = 1$*. Here* a_{10}*,* a_{20}*,* b_{11}*,* c_{02} *are constants and* a_0*,* b_1*,* c_0*,* c_1 *are polynomials in* x*;* $\deg a_0 \leq 2$*,* $\deg b_1 \leq 1$*.*

(i) $(x^2,\ -nxy,\ n^2y^2 - c_0\Gamma_1;\ x^k\Gamma_1)$, $\Gamma_1 = x^n y - 1$, $n \geq 1$, $k = 0,1$, $\Delta = -x^2 c_0 \Gamma_1$;
(ii) $(x^2,\ b_{11}\Gamma - 1,\ y^2 - c_0\Gamma;\ \Gamma)$, $\Gamma = xy - 1$, $\Delta = (xy + 1 - x^2c_0 - b_{11}^2\Gamma + 2b_{11})\Gamma$;
(iii) $(a_0,\ b_1 y,\ c_{02}y^2 + c_1 y;\ y)$;
(iv) $(a_{10}x + a_{20}x^2,\ b_{11}xy,\ c_{02}y^2 + c_1 y;\ xy)$;
(v) $(1 - x^2,\ 0,\ c_{02}y^2 + c_1 y;\ (1 - x^2)y)$;
(vi) $(0, 1 - xy, (1 - xy)c_0; 1 - xy)$.

We shall see in §6 that only (iii)–(v) provide a solution of the **w**-SDOP problem. Moreover, this appears only when (a, b, c) is $(a_{00}, 0, c_{01}y)$, $(a_{10}x, 0, c_{01}y)$, or $(1 - x^2, 0, c_{01}y)$ which corresponds to a product of one-dimensional solutions.

Proof. Let Γ_1 be the irreducible factor of Γ of y-degree 1, i.e., of the form $\Gamma_1 = \gamma_1 y - \gamma_0$ where γ_0 and γ_1 are polynomials in x and $\gamma_1 \neq 0$.

Case 1. $a \neq 0$ and γ_1 is not a constant. By Lemma 3.3, if x_0 is a root of γ_1, it is also a root of b_1 and a double root of a. Since a is a non-zero polynomial in x of degree at most 2, it has only one double root, hence γ also has only one root (maybe multiple) at x_0, hence we may assume that $x_0 = 0$ and $a = x^2$, $b_1 = b_{11}x$, and $\gamma_1 = x^n$, $n \geq 1$. Then γ_0/γ_1 is a Laurent polynomial, let us denote it by $p(x) = \sum_k p_k x^k$. By a change of variable $y = y_1 + p_0 + p_1 x + p_2 x^2 + \dots$ we may kill all the coefficients of non-negative powers of x. Thus we have $p = p_{-n}x^{-n} + \dots + p_{-1}x^{-1}$. The assumption that Γ_1 is irreducible implies $\gamma_0(0) \neq 0$, hence $p_{-n} \neq 0$, i.e., $\operatorname{ord}_x p = -n$. By rescaling the variable y we may assume $p_{-n} = 1$. The curve $\Gamma_1 = 0$ is parametrized by $x = x$, $y = p(x)$. Hence (12) yields

$$b(x,p) = x^2 p', \qquad c(x,p) = b(x,p)p', \qquad \text{and thus} \qquad c(x,p) = (xp')^2. \tag{17}$$

Since $b = b_{11}xy + b_0(x)$, the first equation in (17) reads

$$b_{11}\big(x^{-n+1} + \dots + p_{-2}x^{-1} + p_{-1}\big) + b_0 = -nx^{-n+1} - \dots - 2p_{-2}x^{-1} - p_{-1}. \tag{18}$$

Case 1.1. $n \geq 2$. Since b_0 is a polynomial, we derive from (18) that $b_{11} = -n$, $b_0 = -p_{-1} - b_{11}p_{-1} = (n-1)p_{-1}$ (a constant), and $p_{-n+1} = \dots = p_{-2} = 0$, i.e., $p = x^{-n} + p_{-1}x^{-1}$. Then the last equation in (17) reads

$$c_{02}\big(x^{-n} + p_{-1}x^{-1}\big)^2 + (x^{-n} + p_{-1})c_1 + c_0 = \big(nx^{-n} + p_{-1}x^{-1}\big)^2. \tag{19}$$

By comparing the coefficients of x^{-2n} and then those of x^{-n-1}, we obtain $c_{02} = n^2$ and then $p_{-1} = 0$. Hence $p = x^{-n}$ and $b_0 = 0$. By putting $p_{-1} = 0$ into (19), we get $n^2x^{-2n} + x^{-n}c_1 + c_0 = n^2x^{-2n}$, i.e., $c_1 = -x^n c_0$. Thus $(a, b, c) = (x^2,\ -nxy,\ n^2y^2 - x^n c_0 y + c_0)$ which corresponds to (i) with $k = 0$ when $\Gamma = \Gamma_1$. If Γ has another non-constant factor Γ_0 (a polynomial in x), the condition (11) implies that Γ_0 is a factor of a and b which corresponds to (i) with $k = 1$.

Case 1.2. $n = 1$ (thus $p = x^{-1}$). Then (18) reads $b_{11} + b_0 = -1$ and the last equation in (17) reads $c_{02}x^{-2} + x^{-1}c_1 + c_0 = x^{-2}$ whence $b_0 = -b_{11} - 1$, $c_{02} = 1$, and $c_1 = -xc_0$. Hence $b = b_{11}xy + b_0 = b_{11}xy - b_{11} - 1$ and $c = c_{02}y^2 + c_1y + c_0 = y^2 - xyc_0 + c_0$ which corresponds to (ii) when $\Gamma = \Gamma_1$. Once again, any additional non-constant factor Γ_0 of Γ should be a factor of a and b. Then $\Gamma_0 = x$ and it divides $b = b_{11}(xy - 1) - 1$. Therefore $b_{11} = -1$, thus $b = -xy$. This corresponds to (i) with $n = k = 1$.

Case 2. $\gamma_1 = \text{const}$. Then, up to an admissible change of variables, we may assume that $\Gamma_1 = y$. If $\Gamma = \Gamma_1$, this corresponds to (iii). Otherwise, as in the previous cases Γ/Γ_1 is a polynomial in x which divides a and b which corresponds to (iv) and (v).

Case 3. $a = 0$. Then Γ is a factor of $\Delta = -b^2$. By an admissible change of variables we can reduce b to y, xy, or $xy - 1$. This leads to (iii), (iv), or (vi). $\square$

5. Solution of the $(1, w)$-AlgDOP Problem in $\mathbb{C}^2$ for $1 < w \le 2$

In this section we find all solutions of the $(1, 2)$-AlgDOP Problem in $\mathbb{C}^2$. They include all solutions of the $(1, w)$-AlgDOP Problem for any w in the range $1 < w \le 2$ (see §3.2).

5.1. Compactification of $\mathbb{C}^2$.

Let $(a, b, c; \Gamma)$ be a solution of the $(1, 2)$-AlgDOP Problem in $\mathbb{C}^2$, and let $\Delta = ac - b^2$. The Newton polygon of Δ (and hence of Γ) is contained in the triangle $[(0, 0), (6, 0), (0, 3)]$ (see Figure 1). Therefore it is natural to consider $\mathbb{C}^2$ as the affine chart $Z \ne 0$ (with coordinates $x = X/Z$, $y = Y/Z^2$) of the *weighted projective plane* $\mathbb{P}^2_{1,2,1}$ which is the quotient of $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ by the equivalence relation $(X, Y, Z) \sim (\lambda X, \lambda^2 Y, \lambda Z)$, $\lambda \ne 0$ (we denote the class of (X, Y, Z) by $[X{:}Y{:}Z]$). This variety is smooth except at the point $[0{:}1{:}0]$.

A generic polynomial $P(x, y)$ with $\mathcal{N}(P) = [(0, 0), (6, 0), (0, 3)]$ defines an affine curve $\{P = 0\}$ in $\mathbb{C}^2$ whose closure in $\mathbb{P}^2_{1,2,1}$ is a smooth curve not passing through the singular point $[0{:}1{:}0]$, and the linear projection from this point

$$\mathbb{P}^2_{1,2,1} \setminus \{[0{:}1{:}0]\} \to \mathbb{P}^1, \quad [X{:}Y{:}Z] \mapsto (X{:}Z), \tag{20}$$

is a 3-fold branched covering of $\{P = 0\}$ onto $\mathbb{P}^1$. This is why $\mathbb{P}^2_{1:2:1}$ is an appropriate compactification of $\mathbb{C}^2$ in our setting. However the coefficients of Δ are not necessarily generic and the closure of $\{\Gamma = 0\}$ may have singularities and it may pass through $[0{:}1{:}0]$. To deal with such curves it is convenient to blow up the point $[0{:}1{:}0]$. This means that we consider $\mathbb{C}^2$ as an affine chart (x, y) of $\mathcal{F}_2$ – the *Hirzebruch surface of degree* 2 which is the smooth complex surface obtained by gluing together four copies of $\mathbb{C}^2$ with coordinates (x_k, y_k), $k = 0, \dots, 3$, (where $x_0 = x$, $y_0 = y$ are the coordinates on $\mathbb{C}^2$ we started with). The transition functions are:

$$\begin{aligned} &x_1 = 1/x, && x_2 = x, && x_3 = x_1 = 1/x, \\ &y_1 = y/x^2, && y_2 = 1/y, && y_3 = 1/y_1 = x^2/y. \end{aligned} \tag{21}$$

The set of real points of $\mathcal{F}_2$ is diffeomorphic to a torus. In Figure 3 we represent it as a rectangle with opposite edges identified. As an illustration of the chart gluing, we also show in Figure 3 how the closures of some two curves in $\mathcal{F}_2$ look like.

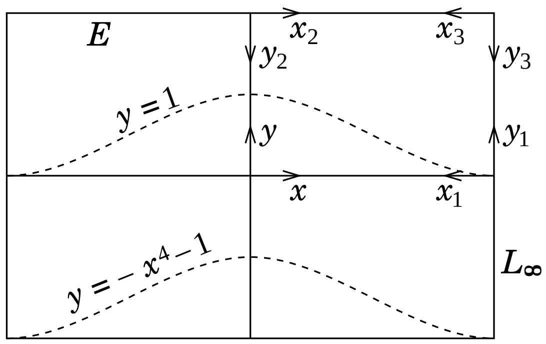

FIGURE 3. Coordinate axes for all the four charts on $\mathcal{F}_2$ and the curves $\{y=1\}=\{y_1=x_1^2\}$ and $\{y=-x^4-1\}=\{y_3=-x_3^2/(1+x_3^4)\}$.

The projection (20) extends to the projection $\pi:\mathcal{F}_2\to\mathbb{P}^1$ given in the affine charts by $(x_k,y_k)\mapsto(x_k:1)$ for $k=0,2$ and $(x_k,y_k)\mapsto(1:x_k)$ for $k=1,3$. It is a fibration with fiber $\mathbb{P}^1$. Its restriction to the closure of a curve $\{P(x,y)=0\}$ is a branched covering of degree $\deg_y P$. The strict transform of $[0:1:0]$ under the blowup (we denote it by E) is given by $y_2=0$ or $y_3=0$ in the respective charts. Its self-intersection is -2.

The set of $(1,2)$-admissible changes of variables coincides with the set of biregular automorphisms of $\mathcal{F}_2$ preserving $\mathbb{C}^2$.

5.2. The case when Δ is irreducible and $\deg_y\Delta=3$.

Let $(a,b,c;\Gamma)$ be a solution of the $(1,2)$-AlgDOP Problem such that $\Gamma=\Delta=ac-b^2$, Γ is irreducible, and $\deg_y\Gamma=3$. We assume that $(a,b,c;\Gamma)$ cannot be reduced to a solution of the $(1,1)$-AlgDOP Problem by a $(1,2)$-admissible change of coordinates. Let C be the closure of $\{\Gamma=0\}$ in $\mathcal{F}_2$. We identify $\mathbb{C}^2$ (where the affine curve $\Gamma=0$ sits) with the affine chart corresponding to the coordinate system (x,y). We shall call it *the main chart.* We denote the fiber $\{x_1=0\}$ by L_∞ (see Figure 3). The condition $\deg_y\Gamma=3$ implies that C is disjoint from E, and $\pi|_C$ is a 3-fold branched covering. Let $\nu:\tilde{C}\to C$ be the normalization (non-singular model) of C. This means $\tilde{C}$ is a smooth compact Riemann surface of genus $\mathbf{g}$ and ν a holomorphic mapping which is injective outside a finite number of points. There is a 1-to-1 correspondence between points of $\tilde{C}$ and local branches of C.

The genus formula for C reads

$$\mathbf{g}=1+\tfrac12 C(C+K_{\mathcal{F}_2})-\sum_{P\in C}\delta_P=4-\sum_{P\in C}\delta_P \qquad (22)$$

where $\delta_P=\delta_P(C)$ is the delta-invariant of (C,P), i.e. $2\delta_P=\sum m_i(m_i-1)$ where $m_1,m_2,\dots$ are the multiplicities of the infinitely near points of C at P (note that the "4" in (22) can be computed as the number of integral points in the interior of the triangle $[(0,0),(6,0),(0,3)]$). It is convenient to rewrite (22) in terms of local branches of C as it is done in [3, §3.2]. Namely, for a point $P\in C$ with local branches $\gamma_1,\dots,\gamma_r$ we set $n_P=\sum_{1\le i<j\le r}\gamma_1\cdot\gamma_j$. Then we have $\delta_P=n_P+\sum\delta(\gamma_i)$, hence the genus formula (22) takes the form

$$\mathbf{g}=4-n-\sum_{\gamma}\delta(\gamma),\qquad n=\sum_{P\in C}n_P \tag{23}$$

where the first sum is over all local branches of C.

For a local branch $t\mapsto\gamma(t)=(\xi(t),\eta(t))$ we denote the ramification index of $\pi\circ\gamma$ by $m_\pi(\gamma)$. The number $m_\pi(\gamma)$ can be also defined as the intersection number of γ with the fiber of π passing through the center of γ. If $\operatorname{ord}\xi\ge 0$ then $m_\pi(\gamma)=\operatorname{ord}_t(\xi(t)-\xi(0))$. If $\operatorname{ord}\xi<0$, then $m_\pi(\gamma)=-\operatorname{ord}_t\xi$. By Riemann-Hurwitz formula we have

$$2-2\mathbf{g}=6-\sum_{\gamma}(m_\pi(\gamma)-1). \tag{24}$$

Lemma 5.1.
(a). Let γ be a local branch of C at a point P. Then $m_\pi(\gamma)\le 3$ and we have:

- *if $m_\pi(\gamma)=3$, then $P\in L_\infty$ and γ is smooth*
  (we denote the number of such branches by β_3);
- *if $m_\pi(\gamma)=2$ and γ is smooth, then $P\in L_\infty$*
  (we denote the number of such branches by β_2; it is clear that $\beta_2+\beta_3\le 1$);
- *if γ is singular, then it is a singularity of type A_{2k} and $m_\pi(\gamma)=2$*
  (we denote the number of such branches by α_{2k});

(b). The curve C is rational (i.e., $\mathbf{g}=0$) and one of the following cases occurs (among the numbers n, α_k, β_k, we list only the non-zero ones):

(i) $\alpha_2=\alpha_4=\beta_3=n=1$;
(ii) $\alpha_2=4$;
(iii) $\alpha_2=3$ *and* $\beta_2=n=1$;
(iv) $\alpha_2=n=2$ *and* $\beta_3=1$.
(v) $\alpha_4=2$ *and* $\beta_3=1$;
(vi) $\alpha_2=\alpha_6=\beta_3=1$;
(vii) $\alpha_2=2$ *and* $\alpha_4=\beta_2=1$.

Proof. (a). Follows from Lemma 3.4. The only point which maybe needs some comments is the smoothness of γ in the case when $m_\pi(\gamma)=3$, and hence $P\in L_\infty$. Up to an admissible change of coordinates we may assume that P is at $(x_1,y_1)=(0,0)$ (see Figure 3). Then $v_\gamma(y_1)>0$ and the condition $m_\pi(\gamma)=3$ means that $v_\gamma(x_1)=3$, i.e., $v_\gamma(x)=-3$. Hence Lemma 3.4(c) implies that $v_\gamma(y)\notin[-4,-1]$. By (21) we have $v_\gamma(y_1)=v_\gamma(y)-2v_\gamma(x)=v_\gamma(y)+6$, thus $v_\gamma(y_1)\notin[2,5]$. It is easy to see that $v_\gamma(y_1)<6$. Hence $v_\gamma(y_1)=1$ whence the result.

(b). We have $\beta_2+\beta_3\le 1$ and the equations (23) and (24) imply

$$4=\mathbf{g}+n+\sum_{k\ge1}k\alpha_{2k},\qquad 2\mathbf{g}+4=\beta_2+2\beta_3+\sum_{k\ge1}\alpha_{2k}.$$

The only non-negative solutions are (i)–(vii). $\square$

Notice that if a curve in $\mathbb{P}^2$ is parametrized by $t\mapsto\big(\xi(t):\eta(t):\zeta(t)\big)$, then the projectively dual curve is parametrized by

$$t\mapsto(\dot\eta\zeta-\dot\zeta\eta:\dot\zeta\xi-\dot\xi\zeta:\dot\xi\eta-\dot\eta\xi). \tag{25}$$

Lemma 5.2. *The cases (v)–(vii) in Lemma 5.1 are unrealizable. In the other cases the curve C admits one of the following parametrizations $t \mapsto [X : Y : Z]$ in the weighted homogeneous coordinates introduced in §5.1:*

(i) $[\,32\,(t+1) : 256\,(5t+3)(t+3) : (t+3)^3\,]$, *thus*

$$\Gamma = y^3 - 20xy^2 + 16y^2 + 45x^3y - 40x^2y - 27x^5 + 25x^4; \tag{26}$$

(ii) $[\,t^2(t+1) : t^2(2t+1) : 3t+1+\alpha t^2(t+1)\,]$, *where* $\alpha^3 - 9\alpha^2 + 27\alpha \neq 0$;

(iii) $[\,(t-1)^2(t-\alpha) : (t-1)^3(2t^3+t^2-\alpha t^2+t-\alpha t-2\alpha) : (t+\alpha)^2(\alpha t+2t-2\alpha-1)\,]$, *where* $\alpha(\alpha^2-1)(\alpha^2+4\alpha+1) \neq 0$.

(iv) $[\,(t-2)^2(t+1) : (t-2)^3(3t^2+3\alpha t+2\alpha) : 1\,]$, *where* $\alpha \notin \{-3/2, 7/2\}$.

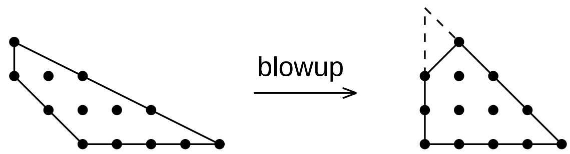

FIGURE 4

Proof. In each of the cases (i)–(vii) we may assume that C is singular at the origin of the main chart. Then we blow up this point and blow down the strict transform of the lines $x = 0$ and E. In coordinates this means that we consider the curve C_1 on $\mathbb{P}^2$ which is the projective closure of the affine curve $\Gamma(x, xy)/x^2 = 0$. We consider the homogeneous coordinates $(X_1 : Y_1 : Z_1)$ on $\mathbb{P}^2$ such that $x = X_1/Z_1$, $y = Y_1/Z_1$.

Then C_1 is a quartic curve tangent to the line $X_1 = 0$ at (0:1:0) (see Figure 4). If C has the A_{2k} singularity at the origin, then C_1 has the A_{2k-2} singularity somewhere on the line $X_1 = 0$ (when $k > 1$) or a simple tangency with this line (when $k = 1$). The line $Z_1 = 0$ is the strict transform of L_∞, thus C_1 has the tangency with it of the same nature as C has with L_∞. Up to a $(1, 2)$-admissible change of coordinates, the curve C is determined by $C_1 \cup \{X_1Z_1 = 0\}$. The weighted homogeneous coordinates (see §5.1) are expressed via $(X_1 : Y_1 : Z_1)$ as follows:

$$[X : Y : Z] = [X_1 : X_1Y_1 : Z_1]. \tag{27}$$

Now we separately consider the cases of Lemma 5.1.

Case (i). We assume that the node is at the origin. In this case the curve C_1 has singularities A_2 and A_4. An irreducible quartic curve with such singularities is unique up to an automorphism of $\mathbb{CP}^2$ and it is autodual (see e.g. [3, Cor. 3.10(iii)]). Such a curve has a single flex point P (because the dual has a single cusp) and the line $Z_1 = 0$ is the tangent to C_1 at P. Let Q be the other intersection point of C_1 with $\{Z_1 = 0\}$ (see Figure 5). Then $Q = $ (0:1:0) and the line $\{X_1 = 0\}$ is tangent to C_1 at Q. This means that $C_1 \cup \{X_1Z_1 = 0\}$ is uniquely determined up to automorphism of $\mathbb{P}^2$ whence the uniqueness of C. Thus it remains to check that the curve given in the statement of the lemma has the required properties. Indeed,

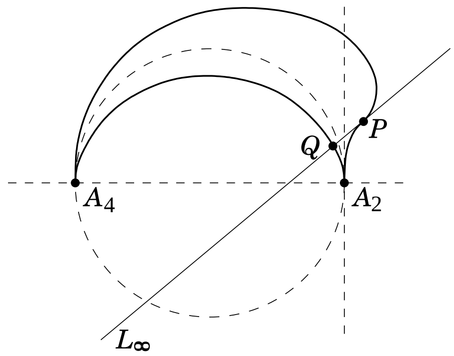

FIGURE 5. A quartic curve with A_2 and A_4.

it has a cusp at $[\frac{32}{27} : \frac{256}{81} : 1]$ ($t = 0$), an A_4 singulariry at [0:0:1] ($t = \infty$), a node at [1:1:1] ($t = -5 \pm 2\sqrt{5}$), and a flex point with tangent L_∞ at [1:0:0] ($t = -3$).

Case (ii). In this case the curve C_1 has three cusps A_2. The line $X_1 = 0$ is its bitangent. It is well known that such a curve (tricuspidal quartic) is unique up to automorphism of $\mathbb{CP}^2$, it has only one bitangent line, and the tangency points P and Q are interchangeable by an automorphism of $(\mathbb{P}^2, C_1)$. Thus C is determined by a choice of a line $Z_1 = 0$ passing through P. So, it depends on one parameter.

The curve C_1 is projectively dual to a nodal cubic, hence it has two real forms. We choose the one with the real bitangent (and hence with two complex conjugate cusps). In some homogeneous coordinates $(X_2 : Y_2 : Z_2)$, it has a parametrization

$$X_2 = t^2(t+1)^2, \qquad Y_2 = 2t + 1, \qquad Z_2 = (t+1)(3t+1).$$

Here the line $X_2 = 0$ is bitangent, the cusps correspond to $t = (-3 \pm i\sqrt{3})/6$ and $t = \infty$. According to the above discussion, we may choose $X_1 = X_2$, $Y_1 = Y_2$, and $Z_1 = Z_2 + \alpha X_2$. Plugging this into (27), we conclude. The roots of $\alpha^3 - 9\alpha^2 + 27\alpha$ are excluded because they correspond to the cases when the line L_∞ passes through a cusp which contradicts our assumptions (see Lemma 5.1(a)).

Case (iii). In this case C_1 has two cusps and one node. We assume that one of the cusps is at the origin. Then the line $X_1 = 0$ is bitangent. Let $C_2 = \check{C}_1$ be the curve projectively dual to C_1. Using Plücker's formulas (in the form given in [3, §3.2] or in [4]), one can check that the class of curves with two cusps, one node, and at least one bitangent is stable under the projective duality. Hence C_2 has two cusps and one node. Therefore, in some homogeneous coordinates $(X_3 : Y_3 : Z_3)$, it has a parametrization

$$X_3 = t^2, \qquad Y_3 = (t-1)(t-\alpha), \qquad Z_3 = t^2(t-1)(t-\alpha). \tag{28}$$

The cusps correspond to $t = 0$ and $t = \infty$. The node corresponds to $t = 1$ and $t = \alpha$. Being dual to C_2, the curve C_1 is parametrized in some coordinates by

$$t \mapsto \varphi(t) = (X_2 : Y_2 : Z_2) = \big(2(t-1)^2(t-\alpha)^2 : \alpha + 1 - 2t : (\alpha+1)t - 2\alpha\big)$$

(see (25)). The points where C_2 touches the bitangent line correspond to the local branches of C_3 at the node, which are at $\varphi(1)$ and $\varphi(\alpha)$, thus $X_1 = X_2$. In the coordinates $(X_1{:}Y_1{:}Z_1)$, the tangency points are at (0:0:1) and at (0:1:0). Up to rescaling the coordinate t, we may assume that $\varphi(1) = (0{:}1{:}0)$. Then the line $Z_1 = 0$ is uniquely determined by the condition that it passes through $(0{:}1{:}0)$ and is tangent to C_1 which gives

$$Z_1 = \tfrac{\alpha}{2}X_2 + (\alpha+1)Y_2 + \alpha^2(\alpha+1)Z_2.$$

The line $Y_1 = 0$ should pass through $(0{:}0{:}1)$. Then we may set $Y_1 = Y_2 + Z_2$ and using (27) we obtain the required parametrization of C.

The condition $\alpha \notin \{0, 1\}$ is clear from the construction. If $\alpha = -1$, then C_2 is a double conic because X_3, Y_3, and Z_3 are functions of t^2 (see (28)). If α is a root of $\alpha^2 + 4\alpha + 1$, then one of the cusps is on L_∞ which contradicts Lemma 5.1.

Case (iv). The condition $n = 2$ can be attained in three different ways.

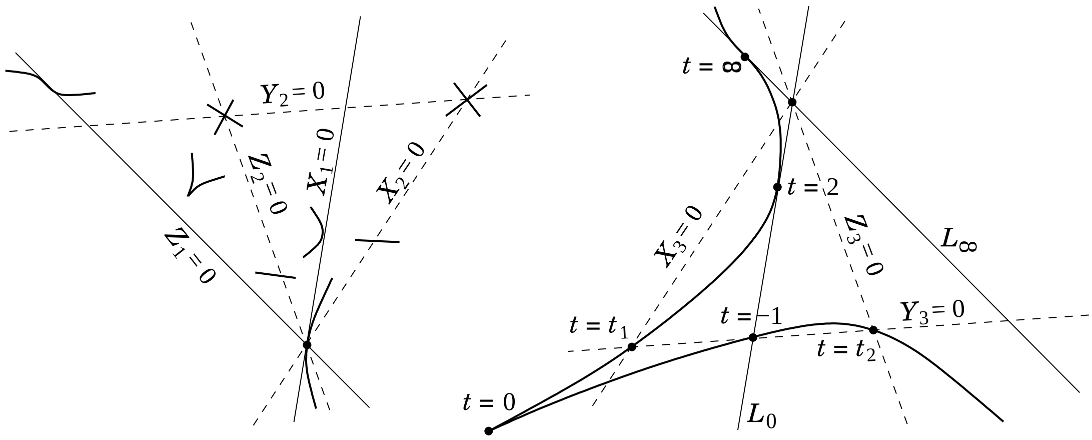

FIGURE 6. Case (iv$_1$) of Lemma 5.2.

Case (iv$_1$): $2A_2 + 2A_1$. The curve C has two cusps and two nodes. We assume that one of the cusps is at the origin. Then C_1 has one cusp and two nodes. The line $X_1 = 0$ is bitangent and the line $Z_1 = 0$ is tangent at a flex point. We choose the line $Y_1 = 0$ to pass through the both tangency points. Next we choose coordinates $(X_2 : Y_2 : Z_2)$ as shown on the left hand side of Figure 6 and perform the Cremona transformation $(X_2 : Y_2 : Z_2) \mapsto (X_3 : Y_3 : Z_3) = (Y_2Z_2 : Z_2X_2 : X_2Y_2)$. Then the lines $X_1 = 0$ and $Z_1 = 0$ are transformed into lines that we denote by L_0 and L_∞. The transform of C_1 is a cuspidal cubic C_3 shown on the right hand side of Figure 6 (two complex conjugate crossings with $Z_3 = 0$ are not shown). The lines $X_2 = 0$ and $Z_2 = 0$ cannot be tangent to the local branches of C_1 at the nodes because otherwise C_3 would have too many tangents in the pencil of lines through $(0 : 1 : 0)$ (see Figure 7).

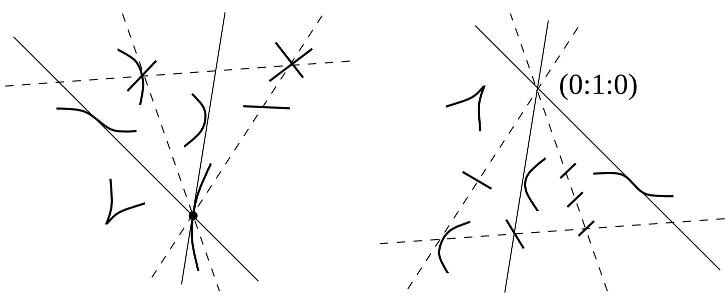

FIGURE 7. Unrealizable tangency at A_1 in Case (iv$_1$).

Let $(X_4 : Y_4 : Z_4)$ be coordinates such that C_3 is parametrized by $t \mapsto \varphi(t) = (t^2 : t^3 : 1)$. Up to rescaling we may assume that the point of tangency of C_3 with L_0 is $\varphi(2) = (4 : 8 : 1)$. Then $C_3 \cap L_0 = \varphi(-1) = (1 : -1 : 1)$ (see Figure 6) and the whole configuration is determined by a choice of the line $Y_3 = 0$ passing through $\varphi(-1)$, i.e., it is determined by a single parameter α such that $Y_3 = (Y_4 + Z_4) - \alpha(X_4 - Z_4)$. Let $\varphi(t_1)$ and $\varphi(t_2)$ be the other two points of $C_3 \cap \{X_3 = 0\}$. Then t_1 and t_2 are the roots of $(t^3 + 1) = \alpha(t^2 - 1)$ different from -1, i.e., the roots of $t^2 - (\alpha + 1)(t - 1) = 0$. Hence $t_2 = t_1/(t_1 - 1)$ and $\alpha = (t_1^3 + 1)/(t_1^2 - 1)$. The parametrization of C_3 is

$$(X_3{:}Y_3{:}Z_3) = \big(t^3 - t_1^3 - 3(t^2 - t_1^2) : (t+1)(t-t_1)(t-t_2) : t^3 - t_2^3 - 3(t^2 - t_2^2)\big)$$

and after routine computations we obtain the required parametrization of C. If $\alpha \in \{-3/2, 7/2\}$, this parametrization defines a curve covered by Case (i).

Case (iv$_2$): $A_2 + D_5$. The curve C has one cusp and a singular point of type D_5 (given by $u(u^2+v^3) = 0$ is some local curvilinear coordinates). We assume that the D_5 singularity is at the origin. We may also assume that the line $Y = 0$ is tangent to its cuspidal local branch and passes through the point of tangency of C with the line $Z = 0$. Then $\mathcal{N}(\Gamma) = [(4,0),(5,0),(0,3),(1,2)]$. Hence after blowing up the origin we obtain a curve in $\mathbb{P}^2$ with the Newton polygon $[(1,0),(2,0),(0,3),(0,2)]$ (cf. Figure 4). This is a cuspidal cubic which has a simple (quadratic) tangency with the line $X_1 = 0$ and a cubic tangency with the line $Z_1 = 0$. One easily checks that these conditions uniquely determine $C \cup \{X_1 Z_1 = 0\}$ up to automorphism of $\mathbb{P}^2$. Hence the curve C is unique up to an admissible change of coordinates. It remains to observe that the parametrization in the statement of the lemma gives the required curve when $\alpha = -1$. It has a cusp at $[0 : 0 : 1]$ and the D_5 singularity at $[4 : 16 : 1]$.

Case (iv$_3$): $2A_2 + A_3$. The curve C has two cusps A_2 and one tacnode A_3 (ordinary tangency of two smooth branches). We assume that one of the cusps is at the origin. Then the curve C_1 has one cusp A_2, one tacnode A_3 and at least one bitangent (the line $X_1 = 0$). Let us show that this combination is impossible for quartic curves in $\mathbb{P}^2$. It is convenient to apply Plücker's equations in the form given in [4, Thm. 1.3]. In the notation of [4] we have $d = 4$, $g = 0$, $n_v = 2$, $c_v = 1$, hence $\hat{d} = 5$ by [4, Eq. (1.6)]. Then the equations [4, (1.8)–(1.9)] read $\hat{n} + \hat{c} = 6$ and $2\hat{n} + 3\hat{c} = 16$ whence $\hat{n} = 2$.

It is easy to check that the dual branches of a tacnode form a tacnode of the dual curve and it contributes 2 to $\hat{n}$. Hence the dual curve does not have nodes, i.e., C_1 does not have bitangents. A contradiction.

Case (v). In this case C_1 has singularities A_2 (on the line $X_1 = 0$) and A_4. An irreducible quartic curve with such singularities is unique up to an automorphism of $\mathbb{CP}^2$ and it is autodual (cf. Case (i)). Hence it has a unique flex point because the dual curve has one cusp A_2. By the uniqueness, we may assume that C_1 is a small real perturbation of real double conic as explained in the remark in the proof of [3, Cor. 3.10] (see Figure 5). The line $Z_1 = 0$ is the tangent at the flex point P. The line $X_1 = 0$ is the tangent at the transverse crossing Q of $Z_1 = 0$ with C_1 and it must pass through the cusp A_2. We see in Figure 5 that this is impossible because the arc $A_2 Q A_4$ is convex.

Case (vi). We assume that the singularity of C at the origin is A_6. Then C_1 has A_2 and A_4 as singularities and the proof is the same as in Case (v) but with A_2 and A_4 exchanged.

Case (vii). We assume that the singularity of C at the origin is A_2. Then $X_1 = 0$ is a bitangent and C_1 has singular points A_2 and A_4. As we mentioned in Case (v), such a curve is autodual, hence it cannot have a bitangent because the dual curve does not have nodes. A contradiction. □

Proposition 5.3. *Let* $(g;\Gamma)$ *be a solution of the* $(1,2)$*-AlgDOP problem over* $\mathbb{R}$ *such that* Γ *is irreducible and* $\deg_y \Gamma = 3$*. Then, up to a* $(1,2)$*-admissible change of variables, either* $(g;\Gamma)$ *is a solution of the* $(1,1)$*-AlgDOP problem, or* Γ *is given by (26) and*

$$g = \begin{pmatrix} y + 8x - 9x^2 & 5(4y - 3xy - x^2) \\ 5(4y - 3xy - x^2) & -25(y^2 - 4xy + 3x^3) \end{pmatrix}. \tag{29}$$

Proof. Given a parametrization $(X(t), Y(t), Z(t))$ in the weighted homogeneous coordinates introduced in §5.1, the relation (12) applied to $\xi(t) = X(t)/Z(t)$, $\eta(t) = Y(t)/Z(t)^2$ yields a system of homogeneous linear equations for the coefficients a_{ij}, b_{ij}, c_{ij} of the entries of the cometric g in each of the cases (i)–(iv) of Lemma 5.2. Up to a constant factor, the only non-zero solution in Case (i) is (29).

In Cases (ii)–(iv) the system of equations polynomially depends on α. It has a non-zero solution if and only if the gcd of the determinants of the maximal minors vanishes. A straightforward computation shows that this gcd is a polynomial in α all whose roots are excluded in Lemma 5.2. For example, in Case (ii), the gcd is a power of α multiplied by a power of $\alpha^2 - 9\alpha + 27$. □

Remark 5.4. The solution of the $(1,2)$-DOP Problem given in [1, §8] transforms (up to a constant factor) into our solution given in Proposition 5.3 by the change of variables

$$\theta_1 = \frac{2+\sqrt{5}}{5}(x-1), \qquad \theta_2 = -\frac{\sqrt{5}}{125}(4y - 5x + 1).$$

5.3. The case when Γ has a factor of y-degree 2.

Proposition 5.5. *Let* $(g;\Gamma)$ *be a solution of the* $(1,2)$*-AlgDOP Problem such that* $\deg_y \Gamma = 2$*. Suppose that it is not a solution of the* $(1,\infty)$*-AlgDOP problem. Then, by a* $(1,2)$*-admissible change of variables,* $(g;\Gamma)$ *can be reduced either to a solution of the* $(1,1)$*-AlgDOP problem or to one of the following three cases:*

(i) $\Gamma = y^2 - x^3$ *and*

$$g = \begin{pmatrix} 4y & 6x^2 \\ 6x^2 & 9xy + \alpha\Gamma \end{pmatrix} + (\beta x + \mu) \begin{pmatrix} 4x & 6y \\ 6y & 9x^2 \end{pmatrix}. \tag{30}$$

Rescaling the coordinates, one can replace (α, β, μ) *by* $(\lambda\alpha, \lambda\beta, \lambda^{-1}\mu)$ *for any non-zero* λ*.*

(ii) $\Gamma = y(y - x^2)$, $g = g_{(\alpha,\beta,\mu)}$ *with* $(\alpha, \beta - \beta^2, \mu) \neq (0,0,0)$, $\mu \in \{0,1\}$*, where*

$$g_{(\alpha,\beta,\mu)} = (y - x^2)\begin{pmatrix} 1 & 0 \\ 0 & \alpha y \end{pmatrix} + (\beta x + \mu)\begin{pmatrix} x & 2y \\ 2y & 4xy \end{pmatrix}.$$

(iii) $\Gamma = y(y - x^2 + 1)$ *and* $g = g_{(\alpha,\beta)}$ *with* $(\alpha, \beta - \beta^2) \neq (0, 0)$*, where*

$$g_{(\alpha,\beta)} = (y - x^2 + 1)\begin{pmatrix} 1 & 0 \\ 0 & \alpha y \end{pmatrix} + \beta\begin{pmatrix} x^2 - 1 & 2xy \\ 2xy & 4x^2 y \end{pmatrix}. \tag{31}$$

Proof. Lemmas 3.2 and 3.3 imply that Γ does not have monomials of the form $x^k y^2$ with $k > 0$. Local branches of Γ correspond to edges of $\mathcal{N}(\Gamma)$ (see Lemma 3.1). Hence Lemma 3.4(d) implies that the slope of any upper edge is steeper than 1:2, hence $\mathcal{N}(\Gamma)$ is contained in the triangle $[(0,0),(4,0),(0,2)]$. This fact combined with Lemma 3.3 (which means that the affine curve $\Gamma = 0$ has no vertical tangent) leaves only five possibilities for Γ up to admissible change of coordinates: the three cases (i)–(iii) and also $y(y - x)$ and $y^2 - 1$.

In each case, the condition (12) yields a system homogeneous linear equations for the coefficients of a, b, and c (the entries of g). By solving these systems we obtain the result. In the last two cases we obtain $a_1 = 0$, thus these are solutions of the $(1,\infty)$-AlgDOP problem (see Figure 2). In the other cases we normalize the solutions so that $a_1 = 1$. The parameter μ in Case (ii) can be set to 0 or 1 by rescaling the coordinates (see Example 2.7). The conditions $(\alpha, \beta - \beta^2, \mu) \neq (0, 0, 0)$ (in Case (ii)) and $(\alpha, \beta - \beta^2) \neq (0, 0)$ (in Case (iii)) are equivalent to $\det g \neq 0$. □

Lemma 5.6. *Let* $(g; \Gamma)$ *be a solution of* $(1, 2)$*-AlgDOP problem which cannot be reduced to a solution of* $(1, 1)$*-AlgDOP problem. Let* $\Gamma = y(y - p_1(x))(y - p_2(x))$*. Then the polynomial* $p_1 p_2$ *has at most two roots (maybe multiple).*

Proof. Proposition 5.5 applied to $(g, y(y - p_k))$ implies that a_0 vanishes at the roots of p_k (recall that $g^{11} = a_{01}y + a_0(x)$). It remains to prove that a_0 cannot be identically zero. Indeed, if it is, then y would divide each entry of g, hence y^2 would divide $\det g$ which is impossible because $\det g = \Gamma$ in our case (see Figure 1) and Γ is squarefree.

Proposition 5.7. *Let* $(g; \Gamma)$ *be a solution of the* $(1, 2)$*-AlgDOP problem over* $\mathbb{C}$ *such that* Γ *is reducible and* $\deg_y \Gamma = 3$*. Then, up to a* $(1, 2)$*-admissible change of variables, either* $(g; \Gamma)$ *is a solution of the* $(1, 1)$*-AlgDOP problem, or* g *is given by (30) with* $(\alpha, \beta, \mu) = (-18, -3/2, 1/2)$ *and hence* $\Gamma = \Gamma_1\Gamma_2$ *where* $\Gamma_2 = y^2 - x^3$ *and* $\Gamma_1 = 8y - 3x^2 - 6x + 1$*.*

In the latter case the curve $\Gamma = 0$ *has singularities of the types* A_1*,* A_2*, and* A_5 *at* $(\frac{1}{9}, -\frac{1}{27})$*,* $(0, 0)$*, and* $(1, 1)$ *respectively.*

Proof. We have $\Gamma = \Gamma_1\Gamma_2$ with $\deg_y \Gamma_k = k$. By Proposition 2.4, (g, Γ_2) is also a solution of the $(1, 2)$-AlgDOP problem. Moreover, (g, Γ) reduces to a solution of a $(1, w)$-AlgDOP problem by a $(1, 2)$-admissible change if and only if the same is true for (g, Γ_2). Hence we may assume that (g, Γ_2), is as in Proposition 5.5.

Case 1. Γ_2 is irreducible. Then (g, Γ_2) is as in Proposition 5.5(i). A computation shows that $\det g = \Gamma_1\Gamma_2$ with $\Gamma_1 = \alpha y - f(x)$, hence $\alpha \neq 0$. Then, for the parametrization $\xi = t$, $\eta = f(t)/\alpha$ of $\{\Gamma_1 = 0\}$, the equations (12) take the form

$$a(\xi,\eta)\dot\eta - b(\xi,\eta)\dot\xi = 6FG/\alpha^2 = 0, \qquad b(\xi,\eta)\dot\eta - c(\xi,\eta)\dot\xi = FGH/\alpha^2 = 0,$$

where $F = At - 3B$, $A = (\alpha - 12\beta)(\alpha - 9\beta)$, $B = 18 + 5\alpha\mu - 36\beta\mu$, $G = (\beta t + \mu)^2 - t$, and $H = (9\beta - \alpha)t + 9\mu$. Since G cannot vanish identically, we have $A = B = 0$.

If $\alpha = 9\beta$, then $B = -27(2+\beta\mu)$, and we obtain a solution of the $(1,1)$-AlgDOP problem (the one discussed in [3, §4.10]).

If $\alpha = 12\beta$, then $B = -18(3+4\beta\mu)$, and we obtain the announced solution.

Case 2. Γ_2 is reducible. In this case $\{\Gamma = 0\} = L_1 \cup L_2 \cup L_3$ where $L_k = \{y = p_k(x)\}$, $k = 1,2,3$. Let $P_1 = L_1 \cap (L_2 \cup L_3)$, $P_2 = L_2 \cap (L_3 \cup L_1)$, and $P_3 = L_3 \cap (L_1 \cup L_2)$. Then Lemma 5.6 implies that each P_k has at most two points. By Proposition 5.5, if $k \neq m$, then L_k and L_m either are tangent, or cross at two points. Hence, up to a $(1,2)$-admissible change of coordinates, $\Gamma = y(y-p(x))(y-\lambda p(x))$ where $\lambda \notin \{0,1\}$ and $p(x)$ is x^2 or x^2-1. In this case g is as in Proposition 5.5 (ii) or (iii). Then one easily checks that (12) is not satisfied for the parametrization $\xi(t) = t$, $\eta(t) = \lambda p(t)$ of the curve $y = \lambda p(x)$. $\square$

6. Solution of the weighted DOP/SDOP problem in $\mathbb{R}^2$

6.1. Compact solutions.

In what follows we give the measure density ρ without the normalizing constant which is always assumed to be equal to $1/\int_\Omega \rho\, dx$.

Theorem 6.1. *Let (Ω, g, ρ) be a solution of the $(1,2)$-DOP problem in $\mathbb{R}^2$ which is not a solution of the $(1,\infty)$-DOP problem and such that Ω is bounded. Then, up to a $(1,2)$-admissible change of variables, either it is a solution of the $(1,1)$-DOP problem, or one of the following cases occurs (see Figure 8):*

(B1) *(dodecahedral quotient) g is given by (29), $\Gamma = -\frac{1}{25}\det g$ is given by (26), Ω is the bounded component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$, and $\rho = \Gamma^{p-1}$ with $p > \frac{3}{10}$;*

(B2) *(cuspidal cubic with cubically tangent parabola) g is as in Prop. 5.7, i.e.,*

$$g = g_{(-18,-\frac{3}{2},\frac{1}{2})} = \begin{pmatrix} 4(2y-3x^2+x) & 6(y-3xy+2x^2) \\ 6(y-3xy+2x^2) & 9(x^2+x^3+2xy-4y^2) \end{pmatrix},$$

$\frac{1}{36}\det g = \Gamma = \Gamma_1\Gamma_2$ with $\Gamma_1 = 8y-3x^2-6x+1$, $\Gamma_2 = x^3-y^2$, the domain Ω is the bounded component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$, and $\rho = \Gamma_1^{p-1}\Gamma_2^{q-1}$ with $p > 0$, $q > \frac{1}{6}$, $p+q > \frac{2}{3}$;

(B3) *(parabolic biangle) $g = g_{(\alpha,\beta)}$ is given by (31) with $\alpha < 0$ and $\beta \le 0$; $\Omega = \{x^2-1 < y < 0\}$, and $\rho = (-y)^{p-1}(y-x^2+1)^{q-1}$ with $p,q > 0$.*

The solution (B3) reduces to a solution of the $(1,1)$-problem by a $(1,2)$-admissible change if and only if either $\alpha = 4\beta$ (then it is already so), or $\alpha = 4\beta - 4$. In the latter case, the variable change is $(x,y) \mapsto (x, x^2-y-1)$ which transforms $g_{(4\beta-4,\beta)}$ into $-g_{(4-4\beta,1-\beta)}$. We have $g_{(4\beta,\beta)} = -\beta G'_{-1/\beta}$ in the notation of [3, §4.5].

Proof. According to Propositions 5.3, 5.5, and 5.7, these are the only solutions of the $(1,2)$-AlgDOP problem with bounded components of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$ which are not solutions of the $(1,\infty)$-AlgDOP problem. A direct computation (as explained in §2.3) shows that starting from any solutions of the $(1,2)$-AlgDOP being solution of the $(1,1)$-AlgDOP problem, we can obtain only those solutions of the $(1,2)$-DOP problem which are solutions of the $(1,1)$-DOP problem.

The imposed restrictions on α and β in Case (B3) are equivalent to the positive definiteness of g. Indeed, we have $g(0,y) = \operatorname{diag}\big(y-\beta+1, \alpha y(1+y)\big)$, thus the positivity of g on $\Omega \cap \{x = 0\}$ implies $\alpha < 0$ and $\beta \le 0$. Conversely, let $\alpha < 0$ and

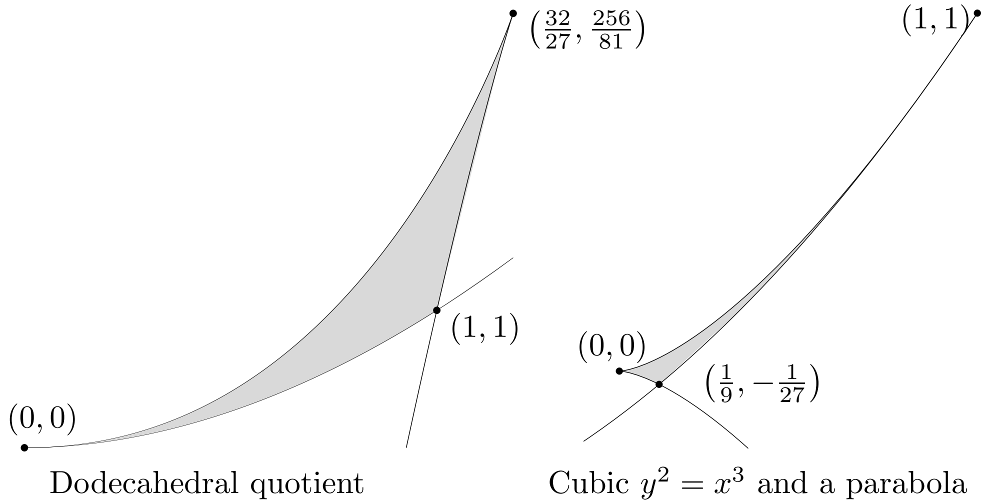

FIGURE 8. The first two domains in Theorem 6.1.

$\beta \le 0$. Then $g > 0$ at $(0, -\frac{1}{2})$, hence it is enough to show that Δ does not vanish in Ω. We have $\Gamma = y\Gamma_1$ and $\Delta := \det(g) = y\Gamma_0\Gamma_1$ where

$$\Gamma_0 = \alpha y + (4\beta - \alpha)(1-\beta)x^2 + \alpha(1-\beta), \qquad \Gamma_1 = y - x^2 + 1. \tag{32}$$

Thus it is enough to show that $\{\Gamma_0 = 0\} \cap \Omega = \varnothing$. If $\beta = 0$, then $\Gamma_0 = \alpha\Gamma_1$ and we are done. If $\beta > 0$, it is easy to check that the curve $\Gamma_0 = 0$ does not cross $\partial\Omega$.

The form of the measure density follows from Proposition 2.11 unless Δ has a multiple factor. This may happen only in Case (B3) and, as one can see from (32), only when $\beta = 0$ (then $\Delta = \alpha y \Gamma_1^2$) or $\beta = 1$ (then $\Delta = \alpha y^2 \Gamma_1$). In these two cases one can perform the computations described in the beginning of §2.3 (in fact, one case is reduced to the other one by the variable change indicated in the statement of this theorem). The inequalities for p and q are the integrability conditions (see [3, Remark 2.28]). $\square$

Theorem 6.2. *Let (Ω, g, ρ) be a solution of the $(1,\infty)$-DOP problem in $\mathbb{R}^2$ such that Ω is bounded. Then one of the following cases occurs up to a $(1,\infty)$-admissible change of variables.*

(B4) *(g,Γ) is as in Proposition 4.3(i) with $c_{02} < 0$, $\Omega = \{\Gamma > 0\} \cap \{x^2 < 1\}$ which is the only bounded component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$, and ρ is as follows:*

- *if one of m, n is odd, then $\rho = \Gamma^{p-1}$ with $p > \max\left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{m}\right)$;*
- *if m and n are both even, then*

$$\rho = \left((1-x)^{\frac{m}{2}}(1+x)^{\frac{n}{2}} + y\right)^{p-1}\left((1-x)^{\frac{m}{2}}(1+x)^{\frac{n}{2}} - y\right)^{q-1}$$

with positive p and q such that $p + q > \max\left(1 - \frac{2}{n}, 1 - \frac{2}{m}\right)$.

(B5) *(g,Γ) is as in Proposition 4.3(iii) with $k = x_0 = 1$, $n \ge 1$, and $c_{02} \le 0$, (i.e., $\Gamma = x\Gamma_2$, $\Gamma_2 = (1-x)^n - y^2$), Ω is $\{\Gamma > 0\} \cap \{0 < x < 1\}$ which is the only bounded component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$, and ρ is as follows:*

- *if n is odd, then $\rho = x^{r-1}\Gamma_2^{p-1}$ with $r > 0$ and $p > \max\left(0, \frac{1}{2} - \frac{1}{n}\right)$;*
- *if n even, then $\rho = x^{r-1}\big((1-x)^{n/2} + y\big)^{p-1}\big((1-x)^{n/2} - y\big)^{q-1}$ with positive p, q, and r such that $p + q > 1 - \frac{2}{n}$.*

Proof. According to Proposition 4.3, the only solutions of the $(1,\infty)$-AlgDOP problem with bounded components of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$ are the indicated ones.

Case (B4). We have

$$\Delta := \det(g) = \Gamma_0\Gamma, \qquad \Gamma_0 = \tfrac{1}{4}\Big((n-m) - (n+m)x\Big)^2 - c_{02}(1-x^2).$$

The condition $c_{02} < 0$ is equivalent to the fact that g is positive definite. Indeed, suppose that $c_{02} \geq 0$. Then

$$\Gamma_0(x_0) = -\frac{4c_{02}(c_{02} + mn)}{4c_{02} + (m+n)^2} \leq 0 \qquad \text{for} \quad x_0 = \frac{n^2 - m^2}{4c_{02} + (m+n)^2}.$$

Since $|x_0| < 1$, we have $(x_0, 0) \in \Omega$ and hence $\Gamma(x_0, 0) > 0$. Therefore $\Delta(x_0, 0) = \Gamma_0(x_0)\Gamma(x_0, 0) \leq 0$, thus g is not positive definite on Ω. Conversely, if $c_{02} < 0$, then $\Gamma_0(x) \geq -c_{02}(1 - x^2) > 0$ when $|x| < 1$, whence $\Delta|_\Omega > 0$ which implies $g|_\Omega > 0$ by Sylvester's criterion because $a|_\Omega > 0$.

The required form of ρ can be derived from Proposition 2.11. Indeed, Γ is maximal for g (by Proposition 4.3) and Δ is squarefree. Hence, by Proposition 2.11, ρ is as required but with an additional factor $\exp h$. We have $\deg_y h = 0$ by (10), i.e., $h'_y = 0$. Then (7) implies $\deg_{(1,w)} bh'_x \leq w$ whence $h'_x = 0$ because $\deg_{(1,w)} b > w$. Thus $h'_x = h'_y = 0$, i.e., h is constant, hence ρ is of the required form. The inequalities for p and q are the integrability conditions (see [3, Remark 2.28]).

Case (B5). The proof is almost the same as in Case (B4). We have $\Delta := \deg g = \Gamma_0\Gamma$ where $\Gamma_0 = \frac{1}{4}n^2x - c_{02}(1-x)$. The condition $c_{02} \leq 0$ is equivalent to $g > 0$. Indeed, if $c_{02} > 0$, then $\Gamma(x_0) = 0$ and $0 < x_0 < 1$ for $x_0 = c_{02}/(c_{02} + \frac{1}{4}n^2)$, which implies that $\Delta(x_0, 0) = 0$ whence $g|_\Omega$ is not positive definite. Conversely, if $c_{02} \leq 0$, then $\Gamma_0(x) > 0$ when $0 < x < 1$, hence $\Delta|_\Omega > 0$ whence $g|_\Omega > 0$ (because $a|_\Omega > 0$).

The form of ρ is established by the same arguments as in Case (B4). $\square$

6.2. Non-compact solutions.

Theorem 6.3. *Any solution (Ω, g, ρ) of the $(1,2)$-SDOP problem with an unbounded domain Ω can be reduced to a solution the $(1,1)$- or $(1,\infty)$-SDOP problem by a $(1,2)$-admissible change of variables.*

Proof. Let $\Delta = \deg g$ and Γ be the maximal boundary for g (see Definition 2.10). We assume that (g, Γ) does not reduce to a solution of the $(1,1)$-AlgDOP problem. Otherwise, using the classification [3] (see also §6.3), one can check (Ω, g, ρ) reduces to a solution of the $(1,1)$-SDOP problem. The integrability condition for the measure implies that the term $\exp(Q)$ in Proposition 2.11 is non-constant, hence $\deg_y \Gamma \leq \deg_y \Delta \leq 2$ by Corollary 2.14.

Case 1. $\deg_y \Gamma = 2$. Then g is as in Proposition 5.5. The condition $\deg_y \Delta < 3$ implies $\alpha = 0$ in all the three cases (i)–(iii). Then, using the algorithm of §2.3, we find that there is no exponential term in ρ.

Case 2. $\deg_y \Gamma = 1$. Then $\deg_y a = 1$ and $\Gamma = y$ up to a $(1,2)$-admissible coordinate change, because otherwise we have a solution of the $(1,\infty)$-AlgDOP problem (see Lemmas 3.2–3.3 and Figure 2) and one can check that it yields a solution of the $(1,\infty)$-SDOP problem. The fact that $\Gamma = y$ combined with (12) implies that y divides b and c. Since $\deg_y a = 1$ and $\deg_y b \le 1$, the condition $\deg_y(ac - b^2) < 3$ implies $\deg_y c = 1$, hence $(a,b,c) = (y + a_0, yb_1, yc_1)$ and $\Delta = (c_1 - b_1^2)y^2 + a_0c_1y$. If $\deg c_1 \le 1$, we obtain a solution of the $(1,1)$-SDOP problem. Let then $\deg c_1 = 2$.

Case 2.1. $a_0 \ne 0$. Then y^2 does not divide Δ, hence, by Proposition 2.11, $\rho = e^h y^\alpha$ for a polynomial h such that $\deg_y h \le 1$, thus $h = yh_1(x) + h_0(x)$. By (7) we have $L^k \in \mathcal{P}_{\mathbf{w}}(w_k)$, $\mathbf{w} = (1,2)$, where

$$L^1 = (y + a_0)(yh_1' + h_0') + yb_1h_1, \qquad L^2 = yb_1(yh_1' + h_0') + yc_1h_1.$$

The condition $L^1 \in \mathcal{P}_{\mathbf{w}}(w_1)$ means that $\deg_y L^1 = 0$, $\deg_x L^1 \le 1$. Hence $h_1' = 0$ (thus h_1 is a constant) and $h_0' = -h_1b_1$. The integrability condition implies that $h_1 < 0$ (we assume here that $\Omega = \{y > 0\}$). Up to rescaling y we may assume that $h_1 = -1$, hence $h_0' = b_1$. Then we have $L^1 = a_0b_1$ and $L^2 = (b_1^2 - c_1)y$. The condition $L^2 \in \mathcal{P}_{\mathbf{w}}(w_2)$ implies that $b_1^2 - c_1$ is a constant and hence $\deg b_1 = 1$ (recall that we have assumed that $\deg c_1 = 2$). Since $L^1 = a_0b_1$ and $\deg_x L^1 \le 1$, we conclude that a_0 is constant. Translating x, we may achieve $b_{01} = 0$ and we obtain $(a,b,c) = (y + \alpha, \beta xy, \beta^2x^2y + \gamma y)$ for some constants α, β, γ. The change $(x,y) \mapsto (x, y - \frac12\beta x^2)$ transforms it into $(y + \frac12\beta x^2 + \alpha, -\alpha\beta x, (\alpha\beta^2 + \frac12\beta\gamma)x^2 + \gamma y)$, which gives a solution of the $(1,1)$-SDOP problem (cf. [3, §6(2ii)]).

Case 2.2. $a_0 = 0$. Then $\Delta = y^2\Delta_0$, $\Delta_0 = c_1 - b_1^2$. We proceed as in the beginning of §2.3. Let

$$L^1 = p(x) = p_0 + p_1x, \qquad L^2 = q_1y + q(x) \tag{33}$$

(p_0, p_1, q_1 are constants). Then the equation (8) takes the form $e_2y^2 + e_1y + e_0 = 0$ where $e_2 = q_1\Delta_0'$. The equation $e_2 = 0$ yields $c_1 - b_1^2 = \text{const}$. Then we reduce to the $(1,1)$-problem by the same variable change as in Case 2.1 (cf. [3, §6(2iii)]).

Case 3. $\deg_y \Gamma = 0$. Then Ω contains a vertical strip Ω_0. We have $\deg_y a \le 1$. If $\deg_y a = 0$, then this gives a solution of the $(1,\infty)$-SDOP problem (see Figure 2). If $\deg_y a = 1$, then a vanishes at some point $P \in \Omega_0$, thus g cannot be positive definite in P. $\square$

Theorem 6.4. *Let (Ω, g, ρ) be a solution of the $(1,\infty)$-SDOP problem in $\mathbb{R}^2$ with an unbounded Ω. Then one of the following cases occurs up to a $(1,\infty)$-admissible change of variables.*

- (U1) *(g,Γ) is as in Proposition 4.3(ii) with $\Omega = \{x > 0\} \cap \{x^n > y^2\}$, $n \ge 1$, $c_{02} \le 0$, and ρ is as follows:*
  - *if n is odd, $\rho = (x^n - y^2)^{p-1}e^{-\lambda x}$, $\lambda > 0$, $p > \max(0, \frac12 - \frac1n)$;*
  - *if n is even, $\rho = (x^{\frac n2} + y)^{p-1}(x^{\frac n2} - y)^{q-1}e^{-\lambda x}$, $\lambda > 0$, $p + q > 1 - \frac2n$.*
- (U2) *(Ω, g, ρ) is a product of two one-dimensional solutions, i.e., $\Omega = \Omega_1 \times \Omega_2$, $g = \operatorname{diag}\big(\alpha g_1(x), \beta g_2(y)\big)$, $(\alpha, \beta) \ne (0,0)$, and $\rho = \rho_1(x)\rho_2(y)$ where each of (Ω_k, g_k, ρ_k) is one of the three one-dimensional solutions mentioned in Remark 1.1 (which correspond to Hermite, Laguerre, and Jacobi polynomials); see also §6.3.*

Proof. Let $w_{\min}$ be the minimal number w such that $w \geq 1$ and (Ω, g, ρ) is a solution of the $(1, w)$-AlgDOP problem. If $w_{\min} = 1$, the result can be derived from the classification in [3, §§5–6] (see also §6.3). So, we assume that $w_{\min} > 1$. Let $\Delta = \deg g$ and Γ be the maximal boundary for g (see Definition 2.10). Then $\partial\Omega \subset \{\Gamma = 0\}$. We have $\deg_y \Gamma \leq \deg_y \Delta \leq 2$ (see Figure 2).

Case 1. $\deg_y \Gamma = 2$. Then (Γ, g) realizes one of Cases (i)–(v) of Proposition 4.3. In Case (iii) with $n = 0$ and in Cases (iv)–(v) we obtain (U2) by Proposition 2.8. In Case (i), the same arguments as in the proof of Theorem 6.2 show the absence of the exponential factor in ρ, as well as in Case (iii) with $x_0 = 1$, $n > 0$.

In Case (iii) of Proposition 4.3 with $x_0 = 0$, $n > 0$, the above arguments (based on Proposition 2.11) do not apply literally, because Δ is no longer squarefree, but they give the result being combined with [3, Corollary 2.19].

The unrealizability of this case can be also proven as follows. We follow the algorithm from §2.3. Let $h = \log \rho$. Rewrite g replacing x by $-x$ and changing the sign:

$$g = \begin{pmatrix} x^2 & \frac{1}{2}nxy \\ \frac{1}{2}nxy & \frac{1}{4}n^2x^n - c_{02}\Gamma_2 \end{pmatrix}, \qquad \Gamma_2 = x^n - y^2, \qquad \Delta = \left(\tfrac{1}{4}n^2 - c_{02}\right)x^2\Gamma_2.$$

Let L^i be as in (33). Then (8) reads $np_0y - nxq(x) + 2x^2q'(x) = 0$. Hence $p_0 = 0$ and $q(x) = Cx^{n/2}$ ($C = 0$ when n is odd). Plugging the obtained solution into (7) and integrating h'_x and h'_y, we obtain $\rho = C_1x^\alpha(x^n - y^2)^\beta$ when n is odd, and $\rho = C_1x^\alpha(x^{n/2} + y)^{\beta_1}(x^{n/2} - y)^{\beta_2}$ when n is even, with some constants C_1, α, β, β_1, β_2. Thus ρ is not integrable on any component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$.

In Case (ii) of Proposition 4.3, we obtain (U1). Indeed, we have $(a, b, c) = (x, \frac{1}{2}ny, \frac{1}{4}n^2x^{n-1} - c_{02}\Gamma)$ and $\Delta = \Gamma_0\Gamma$ where $\Gamma_0 = \frac{1}{4}n^2 - c_{02}x$ and $\Gamma = x^n - y^2$. It is clear that $n > 0$ because if $n = 0$, then Δ has zeros in each component of $\mathbb{R}^2 \setminus \{\Gamma = 0\}$ which implies that g cannot be positive definite in Ω. The same argument leads to our choice of the real form of Γ and to the choice of Ω (up to the variable change $x \mapsto -x$). By Proposition 2.11, ρ is given by (9) and $\deg_y Q = 0$. Hence $\deg_x(aQ) \leq 2$ by (6) and then $\deg_x Q \leq 1$. The inequalities for p, q, and λ are the integrability conditions (see [3, Remark 2.28]), and $c_{02} \leq 0$ is equivalent to $\Gamma_0|_\Omega > 0$ and hence to $g|_\Omega > 0$. Thus we obtain (U1).

Case 2. $\deg_y \Gamma = 1$. Then (Γ, g) realizes one of Cases (i)–(vi) of Proposition 4.4.

Case 2(i). Then $k = 1$ because otherwise Γ is not g-maximal. We have $\Gamma = x\Gamma_1$, $\Gamma_1 = x^ny - 1$, $\Delta = -x^2c_0(x)\Gamma_1$, thus $c_0 \neq 0$. As above, we follow §2.3 to find ρ. Let L^i be as in (33). Then (8) takes the form

$$\Big((np + q_1x)xc'_0 + (p_0 + 2np + 2q_1x)nc_0\Big)y + nxqc_0 + x^2(qc'_0 - q'c_0) = 0,$$

Equating the coefficient of y^0 to zero, we obtain $q = C_1x^nc_0$ (here and below α, β, C_1, C_2, C_3 are some constants). Equating the coefficient of y^1 to zero, we obtain two solutions: the first one is $p_0 = 0$, $q_1 = -np_1$ (then c_0 is an arbitrary function); the second one is $c_0 = C_2(np + q_1x)/x^{2n+1}$. The first solution yields $\rho = C_3x^\alpha\Gamma_1^\beta$ which contradicts the integrability condition. The second solution is irrelevant because it cannot be a nonzero polynomial.

Case 2(ii). We have $\Gamma = xy - 1$ and $\Delta = (xy + 1 - x^2c_0 - b_{11}^2\Gamma + 2b_{11})\Gamma$. Then $c_0 \neq 0$ (otherwise $w_{\min} = 1$) and $b_{11} \neq -1$ (otherwise g is as in 2(i), hence Γ is not g-maximal). Hence Γ^2 does not divide Δ, thus we can apply Proposition 2.11 and write ρ in the form (9). We have $\deg_y Q = 0$ by (10), i.e., $Q'_y = 0$. Then (7) implies $\deg_{(1,w)} bh'_x \leq w$ whence $Q'_x = 0$ because $\deg_{(1,w)} b > w$. Thus Q is constant, hence ρ is not integrable.

Case 2(iii). We have $(a, b, c; \Gamma) = (a_0, b_1y, c_2y^2 + c_1y; y)$, $\Delta = (a_0c_2 - b_1^2)y^2 + a_0c_1y$. If $\deg c_1 < 2$, then $w_{\min} = 1$, thus $\deg c_1 \geq 2$. Suppose that $\deg a \geq 1$. Then $\deg_x ac \geq 3 > \deg_x b^2$ whence $\deg_x \Delta = \deg_x ac$ and we obtain a contradiction with Proposition 2.15 for $w = \deg c_1$. Thus $0 \neq a = \text{const}$ and we may set $a = 1$.

We have $ac_1 \neq 0$, hence y^2 does not divide Δ. Then Proposition 2.11 with $\mathbf{w} = (1, w)$, $w \gg 0$, implies that $\rho = y^pe^h$ with $\deg_y h \leq 2 - \deg_y \Delta$. If $\deg_y \Delta = 2$, this contradicts the integrability condition, hence $\deg_y \Delta = 1$. Then $c_2 = b_1^2$, in particular $b_1 = \text{const}$, thus $(a, b, c) = (1, \beta y, \beta^2y^2 + c_1y)$, $\beta \in \mathbb{R}$. Let $h = h_1(x)y + h_0(x)$. Then $\deg_y L^1 = 0$ for $L^1 = ah'_x + bh'_y = h'_1y + h'_0 + \beta yh_1$ (see (7)), hence $h'_1 = \beta h_1$. If $\beta \neq 0$, we obtain $h_1 = 0$ (since h_1 is a polynomial). If $\beta = 0$, the condition $\deg_{(1,w)} L^2 \leq w$ for $L^2 = bh'_x + ch'_y = yc_1h_1$ again implies $h_1 = 0$ because $\deg c_1 > 0$. Thus $\deg_y h = 0$ which contradicts the integrability condition.

Case 2(iv). Then $(a, b, c; \Gamma) = (x\tilde{a}_0,\, b_{11}xy,\, c_2y^2 + c_1y;\, xy)$, $\Delta = (\tilde{a}_0c_2 - b_{11}x)xy^2 + ac_1y$. As in Case 2(iii), we obtain $\deg c_1 \geq 2$, $\tilde{a}_0 = 1$, y does not divide Δ, hence (see Remark 2.13) $\deg_y \Delta = 1$, that is $c_2 = b_{11}^2x$. Thus we arrive to $(a, b, c) = (x, \beta xy, \beta y^2 + c_1y)$. Then $\rho = x^py^qe^h$ with $h = h_1y + h_0$ where h_k are rational functions of x. The rest of the proof is as in Case 2(iii).

Case 2(v). We obtain (U2) by Proposition 2.8.

Case 2(vi). $a = 0$ is impossible for a positive definite g.

Case 3. $\deg_y \Gamma = 0$, i.e., $\Gamma = \Gamma(x)$ is a polynomial in x only. Then $\Omega = I \times \mathbb{R}$ for a finite or infinite (from either side) interval I. Further, ρ is of the form (9) (see Remark 2.12) where $\deg_y Q \geq 2$ by the integrability condition, hence $\deg_y Q = 2$ and $\deg_y \Delta = 0$ by (10); we write $Q = h_2y^2 + h_1y + h_0$ where h_k are rational functions of x and $h_2 \neq 0$. Then (see (7))

$$\begin{array}{lll} \deg_{\mathbf{w}} L^1 = 1 & \text{for} & L^1 = a(h'_2y^2 + h'_1y + h'_0) + b(2h_2y + h_1), \\ \deg_{\mathbf{w}} L^2 = w & \text{for} & L^2 = b(h'_2y^2 + h'_1y + h'_0) + c(2h_2y + h_1). \end{array} \tag{34}$$

Since $\deg_y \Gamma = 0$, (12) implies that Γ divides a and b, thus $\deg_x \Gamma \leq 2$. Since $\Omega \subset \{\Gamma = 0\}$, Proposition 2.15 implies

$$\deg_x \Delta \leq \deg_x \Gamma - 1 + \max\big(\lfloor w_{\min} \rfloor + \deg_x b,\, 1 + \deg_x c\big). \tag{35}$$

Case 3.1. $\deg_x \Gamma = 2$. Then, up to rescaling, $a = \Gamma$ and $b = b_0 = \tilde{b}(x)\Gamma$ for some polynomial $\tilde{b}$. Then the variable change $y \mapsto y - p(x)$ with $p' = \tilde{b}$ (see Example 2.7) makes $b = 0$. Then $\deg_y \Delta = 0$ implies $c = c_0$. Then (34) yields $h'_2 = 0$ and $2h_2c_0 = \text{const}$. Since $h_2 \neq 0$, we conclude that $c_0 = \text{const}$, hence we obtain (U2).

Case 3.2. $\Gamma = x$. Then $a = x\tilde{a}$ and $b = x\tilde{b} = x(\beta y + \tilde{b}_0(x))$ ($\beta = b_{11}$) by (12), hence the coefficient of y^2 in Δ is $ac_2 - x^2\beta$. Since $\deg_y \Delta = 0$, $a \neq 0$, and $c_2 = \text{const}$, we have either $a = a_{20}x^2$ (then we may assume $a_{20} = 1$) or $\beta = 0$.

Case 3.2.1. $a = x^2$. We have $\Delta = x^2 d(x)$ for some polynomial $d = d(x)$, so we may write $c = \tilde{b}^2 + d$. If $\deg \tilde{b}_0 \le 1$, then either (35) fails or $w_{\min} = 1$. Thus we assume that $\deg \tilde{b}_0 \ge 2$. The change of variables $y \mapsto y + \lambda x^n$ transforms b into $b + \lambda(n-\beta)x^{n+1}$ (see Example 2.7). Hence we may kill all coefficients of b_0 unless $\beta = n \in \mathbb{N}$ in which case we kill all of them except $b_{n+1,0}$. Since $\deg \tilde{b}_0 \ge 2$, we then assume that $\tilde{b} = ny + \alpha x^n$, $n \ge 2$.

Then (see (34)) the coefficient of y^2 in L^1 is $x^2 h_2' + 2nxh_2$, hence $h_2' = -2nh_2/x$ whence $h_2 = Cx^{-2n}$ which contradict the facts that $h_2 \ne 0$ and h_2 is a rational function with denominator at most x (see [3, Prop. 2.15]).

Case 3.2.2. $\beta = 0$. Then the condition $\deg_y \Delta = 0$ implies $c_2 = c_1 = 0$, i.e., g does not depend on y. By the change of variables $y \mapsto y + p(x)$ we may achieve that $\deg_x \tilde{b} < \deg_x \tilde{a}$. If $\deg \tilde{a} = 0$, then $b = 0$ and the proof is the same as in Case 3.1. If $\deg \tilde{a} = 1$ and $b \ne 0$, we obtain a contradiction with (35).

Case 3.3. Γ is constant. Then $\Omega = \mathbb{R}^2$. Since $a|_\Omega > 0$, we have up to translation $a = x^2 + 1$ or $a = 1$.

Case 3.3.1. $a = x^2 + 1$. Recall that $\deg_y \Delta = 0$, hence $(x^2+1)c_2 = b_1^2$ and $ac_1 = 2b_1 b_0$ whence $b_1 = c_2 = c_1 = 0$. By a variable change $y \mapsto y - p(x)$ we may achieve that $\deg b \le 1$ (see Example 2.7) and we obtain a contradiction with (35).

Case 3.3.2. $a = 1$. If $b_1 = 0$, the proof is as in Case 3.1. If $b_1 \ne 0$, then (34) for L^1 implies $h_2' = -b_1 h_2$ which is impossible for nonzero polynomials. $\square$

6.3. Coorections to the paper [3].

(1). It is erroneously claimed in [3, §4.2] that each cometric solution on the square $[-1,1]^2$ is proportional to $\operatorname{diag}(1-x^2, 1-y^2)$. The correct answer is $g = \operatorname{diag}\big(\alpha(1-x^2), \beta(1-y^2)\big)$, $0 < \alpha \le \beta$. All the corresponding riemannian metrics $(g_{ij}) = g^{-1}$ are pairwise non-isometric.

(2). It is erroneously claimed in [3, §6(2iii)] that if $\partial\Omega = \{y = x^2\}$ and $\deg(\det g) = 2$, then the only cometric for which there exists a measure solution is $\begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}$. In fact, a solution exists for any g of the form [3, Prop. 3.21(3)], i.e.,

$$g = \begin{pmatrix} 1 & 2x \\ 2x & 4y + \gamma(y - x^2) \end{pmatrix}, \qquad \gamma \ne -4.$$

The admissible measure densities are $|y - x^2|^{p-1} \exp\big(\alpha x - \beta(4y + \gamma x^2)\big)$ with $p > 0$ and $\beta(4+\gamma) > 0$. By a linear change of variables preserving g, one can always reduce to $\alpha = 0$. If $\gamma + 4 > 0$ (resp. $\gamma + 4 < 0$), this is a solution of the $(1,1)$-SDOP problem on the convex domain $\Omega_+ = \{y > x^2\}$ (resp. on the non-convex domain $\Omega_- = \{y < x^2\}$). In particular, one should add Ω_- to the list of unbounded domains admitting a solution of the $(1,1)$-SDOP problem.

All these solutions can be transformed to a direct product of two one-dimensional solutions by the $(1,2)$-admissible change of variables $(x,y) \mapsto (x, y - x^2)$.

(3). The following solution for $\Omega = \mathbb{R}^2$ is lost in [3, Thm. 5.1]:[1]

$$g = \begin{pmatrix} 1 & -2\lambda x \\ -2\lambda x & 4\lambda^2 x^2 + 2 \end{pmatrix}, \qquad \rho = \frac{1}{\pi} \exp\Big(-x^2 - (y + \lambda x^2)^2 \Big). \tag{36}$$

[1]There is also a misprint in this theorem: X^2 and Y^2 should be exchanged in G.

This solution transforms to $g = \operatorname{diag}(1,2)$, $\pi\rho = \exp(-x^2 - y^2)$ by the $(1,2)$-admissible change $(x,y) \mapsto (x, y - \lambda x^2)$. The error in the proof of [3, Thm. 5.1] is in the assertion "this also requires that $(\partial_X + \partial_Y)P_3 = 0$" (p. 1060, l. 18).

Let us prove that affine linear transformations reduce any solution $(\mathbb{R}^2, g, \rho)$ of the $(1,1)$-SDOP problem to (36) or to the solution in [3, Thm. 5.1]. Let $\Delta = \det g$. Due to [3, §5.1], it is enough to prove that any solution with $\Delta = 1$ and non-constant g reduces to (36). By [3, §5.2, p. 1060] we may assume that $(g^{11}, g^{12}, g^{22}) = (\nu^2 l^2 + p_1, \nu l^2 + p_2, l^2 + p_3)$ where l is a linear form in x, y and $p_k = a_k l + b_k$ with $\nu, a_k, b_k \in \mathbb{R}$. By the change $(x,y) \mapsto (x - \nu y, y)$ we reduce to $\nu = 0$ (see Example 2.7) and eventually to $b_2 = 0$ by a translation. Then $\Delta = 1$ implies $a_1 = a_3 = 0$, $a_2^2 = b_1 = b_3^{-1}$. Rescaling x, l, and g we arrive to $g = \left(\begin{smallmatrix} 1 & l \\ l & l^2+1 \end{smallmatrix}\right)$.

We have $\rho = e^h$ where h is a polynomial, $\deg h \le 4$ [3, Prop. 2.15]. Write $h = h_0 + \dots + h_4$ where h_k is a form of degree k. Then (7) reads

$$\begin{pmatrix} h'_x \\ h'_y \end{pmatrix} = \begin{pmatrix} l^2+1 & -l \\ -l & 1 \end{pmatrix} \begin{pmatrix} l_1 + c_1 \\ l_2 + c_2 \end{pmatrix} \tag{37}$$

for some linear forms l_k and constants c_k. Hence $(h_4)'_y = 0$ and $(h_4)'_x = l_1 l^2$.

Suppose $l_1 = 0$. Then (37) gives $(h_4)'_y = (h_4)'_x = 0$ whence $h_4 = 0$. Then the integrability condition implies $h_3 = 0$ and hence (37) implies $l_2 = c_1 l$ whence $h'_y = c_2$, i.e., $h = c_2 y + f(x)$ which contradicts the integrability. Thus $l_1 \neq 0$. Then $(h_4)'_y = 0$ and $(h_4)'_x = l_1 l^2$ imply $l = \alpha x$, $\alpha \neq 0$. Solving the equations (8) and rescaling the coordinates, we obtain (36).

6.4. All solutions up to $(1,w)$-admissible changes for any fixed w.

For $w = 1$, all solutions up to affine linear changes are given in [3] and Remark 6.5. For $w > 1$, a complete list up to $(1,w)$-admissible change is the following (here p and q are positive numbers, λ is any number):

- all direct products of one-dimensional solutions;
- the images of $\big(\mathbb{R}^2, \operatorname{diag}(1,n), e^{-x^2-y^2}\big)$ and $\big(\mathbb{R}\times\mathbb{R}_+, \operatorname{diag}(1,ny), y^{q-1}e^{x^2-2y}\big)$ through $(x,y) \mapsto (x, y + \lambda x^n)$, $n = \lfloor w \rfloor + 1$, and, if $w \ge n - \frac{1}{2}$, also the images of $\big(\mathbb{R}_+ \times \mathbb{R}, \operatorname{diag}(x,n), x^{p-1}e^{-2x-y^2}\big)$, $\big(\mathbb{R}_+^2, \operatorname{diag}(x,ny), x^{p-1}y^{q-1}e^{-2x-2y}\big)$;
- if $w \le 2$, the solutions in [3, §4.7, §4.10–11, §6(2ii)], (B3) (with $\alpha = 4\beta$ when $w < 2$), and the image of $\big(\mathbb{R} \times \mathbb{R}_+, \operatorname{diag}(1,y), y^{p-1}e^{-x^2-2y}\big)$ through $(x,y) \mapsto (x + \lambda y, y)$;
- if $3/2 \le w \le 2$, (B1) and (B2);
- (B4) with $m + n \le 2w$ (cf. [3, §4.3]) and (B5), (U1) with $n \le 2w$;
- (B4) with $m + n \le 2w + 1$ and $4c_{02} = -(m+n)^2$ (cf. [3, §4.8]);
- the image of (B4) through $(x,y) \mapsto (x, y + \lambda x^k)$, $k \in \mathbb{Z}$, with $c_{02} = -k^2$ and either $w < k = m = n \le w + 1$ (cf. [3, §4.5]) or $2w < 2k = m + n \le 2w + 1$;
- the image of (B4) through $(x,y) \mapsto \big(x, \frac{1}{2}(y + (1 - x^2)^k)\big)$, $k \in \mathbb{Z}$, with $m = n = 2k \le w + 2$ and $c_{02} = -n^2$; in this case we have

$$g = \begin{pmatrix} 1 - x^2 & -nxy \\ -nxy & n^2 y\big((1-x^2)^{k-1} - y\big) \end{pmatrix}, \qquad \det g = n^2 y\big((1-x^2)^k - y\big);$$

- (B5) (resp. (U1)) with $n \le 2w + 1$ and $4c_{02} = -n^2$ (resp. $c_{02} = 0$);
- the image of (B5) (resp. of (U1)) through $(x,y) \mapsto (x, y + \lambda x^k)$, $k \in \mathbb{Z}$, with $2w < n = 2k \le 2w + 1$ and $c_{02} = -k^2$ (resp. $c_{02} = 0$);

- the image of (B5) (resp. (U1)) through $(x,y)\mapsto\bigl(x,\frac12(y+(1-x)^k)\bigr)$ (resp. $(x,y)\mapsto(x,\frac12(y+x^k))$), $k\in\mathbb{Z}$, with $n=2k\le 2w+2$ and $c_{02}=-k^2$ (resp. $c_{02}=0$); in these cases (cf. (Ω_6,Γ_6) in [1]) we have
$$g=\begin{pmatrix} x(1-x) & -kxy\\ -kxy & k^2y\bigl((1-x)^{k-1}-y\bigr)\end{pmatrix}\quad\text{and}\quad g=\begin{pmatrix} x & ky\\ ky & k^2x^{k-1}y\end{pmatrix}$$
respectively, and $\det g=k^2xy((1-x)^k-y)$ (resp. $\det g=k^2y(x^k-y)$).

7. Realization of solutions as images of the Laplace operator

Let M_1 and M_2 be smooth manifolds and $\Phi: M_1\to M_2$ be a smooth mapping which is a submersion at a generic point of M_1. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be differential operators on M_1 and on $\Omega=\Phi(M_1)$ respectively. We say that $\mathbf{L}_2$ is *the image of* $\mathbf{L}_1$ *through* Φ if $\mathbf{L}_2(f)=\mathbf{L}_1(f\circ\Phi)$. Notice that the image of $\mathbf{L}_1$ through Φ may or may not exist, moreover, it does not exist for generic $\mathbf{L}_1$ and Φ unless Φ is injective. However it does exist when $\mathbf{L}_1$ and Φ are invariant under an action of a group G on M_1, and then Φ identifies Ω with the orbit space M_1/G.

For example, for half-integer p and q, the Jacobi operator $J_{p,q}$ on the interval $(-1,1)$ (see Remark 1.1) is the image of the Laplace operator $\mathbf{\Delta}_{\mathbb{S}^n}$ on the sphere $\mathbb{S}^n$, $n=2p+2q-1$, through the mapping
$$(x_1,\dots,x_{2p},y_1,\dots,y_{2q})\mapsto 2(x_1^2+\dots+x_{2p}^2)-1$$
(see [3, §2.1], the end of [3, §4.1], and references therein).

In [3, §4] similar interpretations are given for many values of parameters of each compact solution of (1,1)-DOP problem. They are interpreted as images of the Laplace (Casimir) operators on $\mathbb{S}^n$, $\mathbb{R}^n$, $SO(n)$, or $SU(n)$.

Each quotient of $\mathbb{S}^2$ or $\mathbb{R}^2$ by a reflection group can be identified with a compact domain Ω, $\partial\Omega\subset\{\Gamma(x,y)=0\}\subset\mathbb{R}^2$, so that $(\Omega,\mathbf{L},\Gamma^{-1/2}dx)$ is a solution of $(1,w)$-DOP problem where $\mathbf{L}$ is the image of $\mathbf{\Delta}_{\mathbb{S}^2}$ or $\mathbf{\Delta}_{\mathbb{R}^2}$ through the quotient map. Under this identification, the vertices of the fundamental polygon of the group action are in bijection with singular points of $\partial\Omega$ so that the angle π/n corresponds to the singularity A_{n-1} (given by $u^2=v^n$ in some local coordinates). Explicit formulas for the mappings $\mathbb{S}^2\to\mathbb{R}^2$ and $\mathbb{R}^2\to\mathbb{R}^2$ which realize the operators $\mathbf{L}$ as images of the Laplace operator are given in [3] and [1]; see detailed references in Table 1.

If $\mathbf{L}$ lifts to $\mathbf{\Delta}_{\mathbb{S}^2}$ or $\mathbf{\Delta}_{\mathbb{R}^2}$, then the corresponding metric is of constant non-negative curvature. Moreover, the classification obtained in [3] implies that the curvature is constant and non-negative for all solutions with $\det(g)$ of maximal degree. For maximal degree solutions, Lev Soukhanov [5], [6] proved the constancy of curvature (but not its non-negativity) not using the classification. He also proved this fact for any weighted degree (which well agrees with the classification in the present paper) as well as some its generalizations to an arbitrary dimension.

The curvature in solution (B4) in Theorem 6.2 is constant if and only if $m=n$ and $c_{02}=-n^2$. In (B5) it is constant if and only if $c_{02}=-\frac14n^2$. If $m\ne n$ in (B4), the curvature cannot be constant because there are no biangles of constant curvature with different angles. For $m=n$ in (B4) and for (B5), the curvature is
$$\frac{-\lambda n^2\bigl(2(n^2+\alpha)x^k+\alpha\bigr)}{\bigl((n^2+\alpha)x^k-\alpha\bigr)^2},\qquad (k,\lambda,\alpha)=\begin{cases}(2,2,c_{02}), & \text{(B4)},\ m=n,\\ (1,\frac12,4c_{02}), & \text{(B5)},\end{cases}$$

	Angles	Boundary of Ω	w	Reference
$\mathbb{R}^2$	2,2,2,2	Rectangle (see also Remark 6.5)	1	[3,§4.2]
	2,4,4	Parabola with two tangents	1	[3,§4.7]
	3,3,3	Deltoid: $x = 2\cos\theta + \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$	1	[3,§4.12]
	2,3,6	Cubic $y^2 = x^3$ with a cubically tangent parabola	2	[3, end of §4.12]
$\mathbb{S}^2$	—	Circle	1	[3,§4.3]
	2,2	Coaxial parabolas: $y^2 = (1-x^2)^2$	1	[3,§4.5]
	n, n	$y^2 = (1-x^2)^n$, $n \geq 2$	n	[1,§6]: $\Omega_1^{(n)}$
	2,2,2	Triangle	1	[3,§4.4]
	2,2,3	$(y^2 - x^3)(x-1) = 0$	1	[3,§4.9]
	2,2,4	$y(y-x^2)(x-1) = 0$	1	[3,§4.6]
	2,2,n	$(y^2 - x^n)(x-1) = 0$, $n \geq 2$	n	[1,§6]: $\Omega_3^{(n)}, \Omega_6^{(\frac{n}{2})}$
	2,3,3	Swallow tail: $\mathrm{discrim}_t(t^4 - t^2 + xt + y) = 0$	1	[3,§4.11]
	2,3,4	Cubic $y^2 = x^3$ with a tangent line	1	[3,§4.10]
	2,3,5	Dodecahedral quotient, see Theorem 6.1(B1)	2	[1,§8]: Ω_{21}

TABLE 1. Quotients of $\mathbb{R}^2$ and $\mathbb{S}^2$ by reflection groups; $a, b, \dots$ in the "Angles" cell means: the angles of the fundamental domain are $\frac{\pi}{a}, \frac{\pi}{b}, \dots$

which is evidently non-constant when $\alpha \neq -n^2$ (cf. [3, §4.5]). The similarity between the formulas for the curvature of (B4) and (B5) is not occasional: up to adjusting the constants, (B5) is the image of (B4) through $(x, y) \mapsto (x^2, y)$.

According to our classification, the only bounded domains which admit solutions but are not covered by Table 1 are those in Theorem 6.2(B4) for $m \neq n$. The simplest one is the nodal cubic $y^2 = x^2 - x^3$ which corresponds to $(m, n) = (1, 2)$. This solution is realized in [3, §4.8] as the image of $\mathbf{\Delta}_{\mathbb{S}^{3c}}$, $c = 1, 2, 4, 8$. The realization for $c = 1$ immediately extends to any (m, n) as follows (it could be interesting to do the same for $c = 2, 4, 8$). We consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ with coordinates (z_1, z_2). Then the image of $\frac{1}{4}\mathbf{\Delta}_{\mathbb{S}^3}$ through

$$(z_1, z_2) \mapsto (X, Y) = \big(|z_1|^2, \, \mathrm{Re}(z_1^n \bar{z}_2^m)\big)$$

is an operator on the domain bounded by the curve $(1-X)^m X^n - Y^2 = 0$. Its image through the affine change $(X, Y) \mapsto (x, y) = (\,2X - 1, \, 2^{(m+n)/2} Y\,)$ is the operator corresponding to (B4) with $p = q = \frac{1}{2}$ and $c_{02} = -\frac{1}{4}(m+n)^2$.

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