DIFFUSION ORTHOGONAL POLYNOMIALS IN 3-DIMENSIONAL DOMAINS BOUNDED BY DEVELOPABLE SURFACES

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Abstract. The following problem is studied: describe the triplets (Ω, g, μ), $\mu = \rho\, dx$, where $g = (g^{ij}(x))$ is the (co)metric associated with the symmetric second order differential operator $\mathbf{L}(f) = \frac{1}{\rho}\sum_{ij}\partial_i(g^{ij}\rho\,\partial_j f)$ defined on a domain Ω of $\mathbb{R}^n$ and such that there exists an orthonormal basis of $\mathcal{L}^2(\mu)$ made of polynomials which are eigenvectors of $\mathbf{L}$, and the basis is compatible with the filtration of the space of polynomials with respect to some weighted degree.

In a joint paper with D. Bakry and M. Zani this problem was solved in dimension 2 for the usual degree. In the author's subsequent paper this problem was solved in dimension 2 for any weighted degree. In the present paper this problem is solved in dimension 3 for the usual degree under the condition that $\partial\Omega$ contains a piece of a tangent developable surface. The proof is based on Plücker-like formulas in the form given by Ragni Piene. All the found solutions are generalized for any dimension.

1. Introduction

This paper continues the study of the diffusion orthogonal polynomials started in [3] (see also [1], [7], [11], [12]). It is devoted to the following problem posed by Dominique Bakry (we refer to [1] and to the introduction of [3] for the motivation): describe all triples $(\Omega, \mathbf{L}, \mu)$ where Ω is a domain in $\mathbb{R}^n$ such that $\Omega = \operatorname{Int}\overline{\Omega}$, $\mathbf{L}$ is a *diffusion operator*, that is an elliptic second order operator of the form

$$\mathbf{L}(f) = \sum_{i,j} g^{ij}(x)\partial_{ij} f + \sum_i b^i(x)\partial_i f \tag{1}$$

with g^{ij} and b^i continuous in Ω, and $\mu = \rho\, dx$ a probability measure on Ω with $\mathcal{C}^1$-smooth density ρ, and such that there exists a polynomial orthogonal basis of $\mathcal{L}^2(\Omega, \mu)$ formed by eigenvectors of $\mathbf{L}$, which is also a basis (in the algebraic sense) of $\mathbb{R}[x]$, $x = (x_1, \dots, x_n)$, and which is compatible with the filtration of $\mathbb{R}[x]$ by the degree (a variant: by a weighted degree, see [1], [7]). The latter condition means that the space $\mathcal{P}_m$ of polynomials of degree $\le m$ is $\mathbf{L}$-invariant for any m. We say that such a triple $(\Omega, \mathbf{L}, \mu)$ is a solution of the *Diffusion Orthogonal Polynomial problem* (DOP problem for short). If in addition $\int_\Omega f_1 \mathbf{L} f_2\, d\mu = \int_\Omega f_2 \mathbf{L} f_1\, d\mu$ for any pair of compactly supported functions (for bounded domains this condition follows from the other ones), we say that $(\Omega, \mathbf{L}, \mu)$ is a solution of the *strong DOP problem* (SDOP problem for short). In this case

$$\mathbf{L}(f) = \frac{1}{\rho}\sum_{i,j}\partial_i\Big(g^{ij}\rho\,\partial_j f\Big), \tag{2}$$

thus $\mathbf{L}$ is determined by $g = (g^{ij})$ and ρ, and we therefore talk about (Ω, g, ρ) as a solution of the SDOP problem. If $\rho = (\det g)^{-1/2}$, then $\mathbf{L}$ given by (2) is the Laplace-Beltrami operator for the metric $(g_{ij}) = g^{-1}$.

As shown in [3, Thm. 2.21], (Ω, g, ρ) is a solution of the SDOP problem (and hence of the DOP problem when Ω is bounded) if and only if there exists a squarefree polynomial Γ such that:

(A1) $g^{ij} \in \mathcal{P}_2$ for each $i, j = 1, \dots, n$;
(A2) Γ divides $\det g$;
(A3) Γ divides $\sum_j g^{ij}\partial_i\Gamma$ for each $i = 1, \dots, n$;
(A4) $\partial\Omega \subset \{\Gamma = 0\}$ and $g|_\Omega$ is positive definite;
(A5) $\sum_j g^{ij}\partial_i \log\rho \in \mathcal{P}_1$ for each $i = 1, \dots, n$;
(A6) polynomials are dense in $\mathcal{L}^2(\Omega, \rho\, dx)$.

Condition (A3) is equivalent to the fact that for any germ $\xi : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^n, x)$ of rank $n-1$ such that $\Gamma \circ \xi = 0$, one has

$$\xi^*(\omega_i) = 0, \quad \omega_i = \sum_j (-1)^j g^{ij}\, dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n, \quad i = 1, \dots, n. \tag{3}$$

Note that Conditions (A1)–(A3) are purely algebraic and they make sense for polynomials with coefficient in any field $\mathbb{K}$. If they are satisfied for a field $\mathbb{K}$, we say that (g, Γ) is a solution of the *algebraic counterpart of the DOP problem over* $\mathbb{K}$ (AlgDOP/$\mathbb{K}$ problem for short).

In dimension 2, all solutions of the DOP problem are found in [3] for the usual degree and in [7] for any weighted degree. In the present paper we attack the classification of the solutions in dimension 3 for the usual degree. By (A3), $\partial\Omega$ sits on an algebraic hypersurface of degree at most $2n$, thus on a quartic curve when $n = 2$. The arguments in [3] essentially rely on the Plücker-like formulas relating the singularities of this curve and those of its projectively dual curve. It seems that this approach can be also applied at least in dimension 3. Here we take the first step in this direction. Namely, we describe all irreducible surfaces Σ in $\mathbb{R}^3$ whose projective dual has dimension 1 (i.e., is a curve, which we denote by $\check{C}$) and such that a relatively open piece of Σ appears in $\partial\Omega$ for some solution (Ω, g, ρ) of the SDOP problem. Moreover, in the case when $\check{C}$ is not contained in any plane (in this case Σ is the tangent developable of another curve C called the dual curve of $\check{C}$), we describe all such solutions (Ω, g, ρ) (Theorems 5.1 and 5.2). If $\check{C}$ is contained in some plane, then Σ is a cylinder or a cone over some planar curve A. If Σ is a cylinder, then it is easy to show that a piece of A occurs in the boundary of a two-dimensional solution. If Σ is a cone, we prove in Theorem 7.1 that $\deg A = 2$, thus Σ is a standard quadratic cone. In the conical case there indeed exist some solutions (see Remark 7.2) but we do not know if this list is exhaustive.

To prove Theorems 5.1 and 5.2, we follow the strategy similar to that in [3, §3]. Condition (3) yields equations for the coefficients of the polynomials g^{ij} and those of local parametrizations of the curve C. By solving them we obtain in §3 rather strong restrictions on a priori possible types of local branches (real and complex) of C. Then in §4, using Plücker-like formulas due to Ragni Piene [8] (introduced in §2), we find all solutions of the AlgDOP problem over $\mathbb{C}$, and then (in §5) we find Ω, ρ, and the real form of g satisfying the remaining conditions (A4)–(A6).

In §6, for each bounded domain in Theorem 5.1, we show that the Laplace-Beltrami solution is the image of Euclidean or spherical Laplace operator through on an appropriate realization of quotient of $\mathbb{R}^3$ or $\mathbb{S}^3$ by a Coxeter group, and we generalize this construction to any dimension. In §7 we prove the aforementioned result about conical surfaces.

2. Tangent developables.

Let $C = \nu(\widetilde{C})$ be an irreducible algebraic curve in $\mathbb{P}^3$ of genus g which is not contained in a plane (here $\widetilde{C}$ is a smooth compact Riemann surface and $\nu : \widetilde{C} \to \mathbb{P}^3$ an analytic mapping). Let Σ be the *tangent developable surface of* C, i.e., Σ is the union of all lines tangent to C.

Following [8], we introduce the following notation. For any point $p \in \widetilde{C}$ there exists a local affine chart of $\mathbb{P}^3$ centered at $\nu(p)$ such that the corresponding local branch of C is parametrized by $t \mapsto (t^{m_0}, t^{m_1}, t^{m_2})$ with

$$(m_0, m_1, m_2) = (1 + l_0,\ 2 + l_0 + l_1,\ 3 + l_0 + l_1 + l_2), \qquad l_j = l_j(p) \geq 0.$$

Then we say that p is a point of type (m_0, m_1, m_2) and we set $k_j = k_j(C) = \sum_{p\in\widetilde{C}} l_j(p)$, $j = 0, 1, 2$. We also denote the *osculating* plane at p by O_p. In the above coordinates, this is the plane spanned by $(1, 0, 0)$ and $(0, 1, 0)$.

The curve $\check{C}$ in the dual projective space $\check{\mathbb{P}}^3$ parametrized by $\check{\nu} : \widetilde{C} \to \check{\mathbb{P}}^3$, $p \mapsto O_p$ is called the *dual curve* of C. Let r_0, r_1, r_2 be the degrees of C, Σ, $\check{C}$ respectively (see [8] for a more uniform definition). It is immediate to check that the dual of $\check{C}$ is C and $k_j(\check{C}) = k_{2-j}(C)$, $r_j(\check{C}) = r_{2-j}(C)$, $j = 0, 1, 2$. The classical Plücker–Cayley equations in the form given by Ragni Piene in [8], [9, Eq. (1)] read as follows:

$$\begin{aligned} r_1 &= 2r_0 + 2g - 2 - k_0, & r_1 &= 2r_2 + 2g - 2 - k_2,\\ r_2 &= 3(r_0 + 2g - 2) - 2k_0 - k_1, & r_0 &= 3(r_2 + 2g - 2) - 2k_2 - k_1,\\ k_2 &= 4(r_0 + 3g - 3) - 3k_0 - 2k_1, & k_0 &= 4(r_2 + 3g - 3) - 3k_2 - 2k_1. \end{aligned} \tag{4}$$

Any three of these equations imply the others.

Proposition 2.1. *If $r_1 \leq 6$, then one of the cases listed in Table 1 takes place.*

Proof. If $r_0 \leq 3$ (recall that $r_0 = \deg C$), then the only non-planar curve (up to automorphism of $\mathbb{P}^3$) is the rational cubic parametrized by $t \mapsto (1 : t : t^2 : t^3)$. It corresponds to Case 1°. Then assume that $r_0 \geq 4$ and (by the duality) $r_2 \geq 4$.

We assume for simplicity that $l_0(p) + l_1(p) + l_2(p) \leq 1$ for each point p of C, i. e., each point of C contributes at most 1 to $k_0 + k_1 + k_2$. It is not difficult to adapt the proof for the general case.

The degree of the cuspidal edge of Σ is $r_0 + k_1$ (see [9, p. 112, l. 15 ff]), hence the genus formula for a generic plane section of Σ yields

$$r_0 + k_1 \leq (r_1 - 1)(r_1 - 2)/2. \tag{5}$$

If $k_0 > 0$, then we consider the plane projection of C from one of its cusps. Its degree is $r_0 - 2$ and it has $k_0 - 1$ cusps. Hence, by the genus formula,

$$k_0 > 0 \quad \Rightarrow \quad g + k_0 - 1 \leq (r_0 - 3)(r_0 - 4)/2. \tag{6}$$

One can easily check that in Table 1 there listed all non-negative integer solutions $(g, k_0, k_1, k_1, r_0, r_1, r_2)$ of the equations (4) combined with the inequalities (5), (6), $r_0 \geq 4$, $r_2 \geq 4$, and $3 \leq r_1 \leq 6$. □

no.		g	k_0	k_1	k_2	r_0	r_1	r_2
1°	Twisted cubic	0	0	0	0	3	4	3
2°	Cuspidal quartic	0	1	0	1	4	5	4
3°	Once inflected quartic	0	0	1	2	4	6	5
4°	Twice inflected quartic	0	0	2	0	4	6	4
5°	Inflected bicuspidal quintic	0	2	1	0	5	6	4
6°	Generic quartic	0	0	0	4	4	6	6
7°	Non-inflected bicuspidal quintic	0	2	0	2	5	6	5
8°	Four-cuspidal sextic	0	4	0	0	6	6	4

TABLE 1. Curves whose tangent developables have degree at most 6

Lemma 2.2. *Suppose that C is rational. Let $p,q\in\widetilde{C}$, $p\neq q$, be points of the types (m_0,m_1,m_2) and (m_0',m_1',m_2'). Recall that $r_0=\deg C$.*

(a). If $m_2'\le m_1'+m_0=m_0'+m_1=m_2=r_0$, then C has parametrization $t\mapsto\big(\sum_{j=0}^{r_0-m_2'}a_jt^j : t^{m_0} : t^{m_1} : t^{m_2}\big)$, $a_0a_{r_0-m_2'}\neq 0$, in some homogeneous coordinates.

(b). If $r_0=4$ and $\nu(p)=\nu(q)$, then C has parametrization $t\mapsto\big(1+t^4 : t : t^2 : t^3\big)$ in some homogeneous coordinates.

Proof. We have $\widetilde{C}=\mathbb{P}^1$ and we may assume that $p=(0:1)$, $q=(1:0)$, and the mapping ν is given by $(t:s)\mapsto(f_0(t,s):\dots:f_3(t,s))$, where f_j are homogeneous polynomials of degree r_0 and $\operatorname{ord}_t(f_0,\dots,f_3)=(0,m_0,m_1,m_2)$.

(a). The condition $m_2=r_0$ implies that $f_3=t^{m_2}$ up to rescaling. By a coordinate change $f_j\to f_j-c_jf_3$, $j=0,1,2$, we may attain $\operatorname{ord}_s f_j\ge m_0'$ for $j\ge 2$. Then the condition $m_0'+m_1=r_0$ implies that $f_2=t^{m_1}s^{m_0'}$ up to rescaling. Proceeding in this way we arrive to the required parametrization.

(b). The condition $\nu(p)=\nu(q)$ implies $f_1(q)=f_2(q)=f_3(q)=0$, i.e., $\deg_t f_j\le 3$ for $j=1,2,3$. Then by coordinate changes $f_i\to f_i-c_{ij}f_j$, $i<j$, we arrive to the required parametrization. □

3. Restrictions on local branches

Let the notation be as in §2 but we fix an affine chart in $\mathbb{P}^3$ with coordinates (x,y,z). We denote the plane at infinity by P_∞. Let $\Gamma(x,y,z)=0$ be the equation of Σ. Suppose that there exists a cometric $g=(g^{ij})$ such that (g,Γ) is a solution of the SDOP problem. We denote the coefficient of $x^ky^lz^m$ in g^{ij} by g^{ij}_{klm}. Let

$$t\mapsto\gamma(t)=(\xi_1(t),\xi_2(t),\xi_3(t))$$

be a local meromorphic branch of C at a finite or infinite point. Then Σ admits parametrization

$$(t,u)\mapsto\big(\hat\xi_1(t,u),\hat\xi_2(t,u),\hat\xi_3(t,u)\big),\qquad \hat\xi_j=\xi_j+u\dot\xi_j,$$

at a neighbourhood of the line tangent to C at $\gamma(0)$. Then the equations (3) take the form $E_1=E_2=E_3=0$ where

$$E_i=\sum_{j=1}^{3}\frac{\partial(\hat\xi_{j+1},\hat\xi_{j-1})}{\partial(t,u)}g^{ij}(\hat\xi_1,\hat\xi_2,\hat\xi_3)=u\sum_{j=1}^{3}\big(\ddot\xi_{j+1}\dot\xi_{j-1}-\ddot\xi_{j-1}\dot\xi_{j+1}\big)g^{ij}(\hat\xi_1,\hat\xi_2,\hat\xi_3)$$

(here the indices are considered mod 3). We have $E_i = \sum_{\alpha=\alpha_0}^{\infty} t^\alpha \sum_{\beta=1}^{3} E_{\alpha,\beta,i} u^\beta$ where the $E_{\alpha,\beta,i}$ are linear forms in g^{ij}_{klm} whose coefficients are polynomial functions of the coefficients of the ξ_i's.

In the following Lemmas 3.1–3.12, for several a priori possible values of $\mathrm{ord}_t(\gamma)$, we either exclude them or (in Lemma 3.3) show that the given value implies a certain explicit form of C. In all the proofs (except those of Lemmas 3.1–3.2) we assume that γ is parametrized by $t \mapsto (x, y, z)$,

$$x = t^{j_1}, \qquad y = t^{j_2} + \sum_{j>j_2} b_j t^j, \qquad z = t^{j_3} + \sum_{j>j_3} c_j t^j, \qquad j_1 > j_2 > j_3,$$

and, moreover, $b_0 = b_{j_1} = c_0 = c_{j_1} = c_{j_2} = 0$. The latter condition can be easily achieved by the change of variables $(y, z) \to (y_1, z_1)$, $y_1 = y - b_0 - b_{j_1} x$, $z_1 = z - c_0 - c_{j_2} y_1 - c_{j_1} x$. Then we solve a system of some n linear equations $E_{\alpha,\beta,i} = 0$ for some n unknowns g^{ij}_{klm} whose determinant is a nonzero constant. The number n and the choice of the equations and unknowns is indicated in each proof. In most cases the solution plugged into g implies that x^2 divides $\det g$ which contradicts the condition $\deg \Gamma \geq 5$. In other cases we then solve some few additional equations.

Lemma 3.1. *If* $\deg \Gamma \geq 5$*, then* $\mathrm{ord}_t(\gamma) \neq (1, 3, 4)$*.*

Proof. We choose a parametrization of the form $x = t$, $y = t^3 + \sum_{\nu \geq 4} b_\nu t^\nu$, $z = t^4 + \sum_{\nu \geq 5} c_\nu t^\nu$. By the change of variables $y \to y - b_4 z$ we make $b_4 = 0$.

All variables g^{ij}_{klm} except g^{11}_{klm} with $k+l+m = 2$, g^{12}_{0lm} with $l+m = 2$, g^{13}_{002}, and g^{13}_{101} (thus 49 variables) can be found by solving the following system of 49 equations: $E_{1,\beta,i}$ (all β, i); $E_{3,2,i}$, $E_{3,3,i}$, $E_{4,2,i}$, $E_{4,3,i}$, $E_{5,3,i}$ ($i = 1, 2, 3$); $E_{2,1,i}$, $E_{2,2,i}$, $E_{2,3,i}$, $E_{5,1,i}$, $E_{5,2,i}$, $E_{6,1,i}$, $E_{6,2,i}$, $E_{7,1,i}$ ($i = 1, 2$); $E_{6,3,i}$, $E_{7,3,i}$ ($i = 2, 3$); $E_{4,1,1}$, $E_{7,2,2}$, $E_{8,2,2}$, $E_{8,3,2}$, $E_{9,3,2}$. The determinant of this system is a non-zero constant. By plugging the solution to $E_{8,3,3}$ we obtain the equation $36 c_5 g^{13}_{101} = 0$. This equation implies that z^2 divides $\det g$ which contradicts the condition $\deg \Gamma \geq 5$. □

Lemma 3.2. *If* $\deg \Gamma \geq 5$*, then* $\mathrm{ord}_t(\gamma) \neq (1, 2, 4)$*.*

Proof. We choose a parametrization of the form $x = t$, $y = t^2 + \sum_{\nu \geq 3} b_\nu t^\nu$, $z = t^4 + \sum_{\nu \geq 5} c_\nu t^\nu$. By the change of variables $y \to y - b_4 z$ we make $b_4 = 0$. We solve 40 equations for 40 unknowns. The equations: $E_{0,\beta,i}$ (all β, i); $E_{1,2,i}$, $E_{1,3,i}$, $E_{2,3,i}$, $E_{3,2,i}$, $E_{3,3,i}$, $E_{4,3,i}$ ($i = 1, 2, 3$); $E_{2,1,i}$, $E_{2,2,i}$, $E_{4,1,i}$, $E_{4,2,i}$, $E_{5,1,i}$ ($i = 1, 2, 3$); $E_{3,1,1}$, $E_{5,2,3}$, $E_{5,3,3}$. The unknowns: g^{ij}_{kl0} ($1 \leq i \leq j \leq 2$, $k, l \neq 2$), g^{12}_{200}, g^{22}_{200}, and all the g^{i3}_{klm} except g^{13}_{011}, g^{13}_{002}, g^{23}_{002}, g^{33}_{002}. The determinant is a nonzero constant. Plugging the solution to g, we obtain that z^2 divides $\det g$ which contradicts $\deg \Gamma \geq 5$. □

Lemma 3.3. *If* $\deg \Gamma \geq 5$ *and* $\mathrm{ord}_t(\gamma) = (-1, 1, 2)$ *(i.e., γ is a generic branch transverse to P_∞), then C is parametrized by*

$$t \mapsto (t^{-1} + t,\ 3t - t^3,\ 2t^2 - t^4) \tag{7}$$

in some affine coordinates.

Proof. $n = 58$. The equations: $E_{-3,\beta,i}$, $E_{-1,\beta,i}$, $E_{0,\beta,i}$ (all β, i); $E_{-7,3,i}$, $E_{-5,2,i}$, $E_{-5,3,i}$, $E_{-4,3,i}$, $E_{-2,3,i}$, $E_{1,2,i}$ ($i = 1, 2, 3$); $E_{-6,3,i}$, $E_{-4,2,i}$, $E_{-2,1,i}$, $E_{1,1,i}$, $E_{1,3,i}$,

$E_{2,1,i}$ $(i=2,3)$; $E_{2,3,3}$. The unknowns are all the g^{ij}_{klm} except g^{33}_{002} and g^{22}_{011} which we denote by h and h_1 respectively. Plugging the solution into g, we see that x^2 divides $\det g$ when $h_1=0$. This contradicts $\deg\Gamma\ge 5$, hence we may set $h_1=1$. Then $E_{-2,2,3}$ yields $b_4=0$. Putting this into $E_{1,3,1}$, $E_{2,2,1}$, $E_{2,2,2}$, $E_{2,3,1}$ we obtain a linear system with constant coefficients for c_3, c_4, b_5, b_6 which yields

$$c_3=16b_3h-\tfrac{3}{2}b_3^2,\qquad b_5=-\tfrac{8}{3}b_3h-\tfrac{1}{2}b_3^2,\qquad c_4=b_6=0.$$

Plugging this solution into $E_{2,3,2}$ and $E_{3,2,2}$, we obtain a linear system with constant coefficients for the unknowns c_5 and b_7 which yields

$$c_5=b_3^3-\tfrac{728}{5}b_3^2h+\tfrac{1648}{45}b_3h^2,\qquad b_7=\tfrac{1}{2}b_3^3+40b_3^2h-\tfrac{80}{9}b_3h^2.$$

Putting this into $E_{3,2,3}$, we obtain the equation $b_3h^2(3b_3-2h)=0$. If $h=0$, then $\det g=0$. If $b_3=0$, then x^2 divides $\det g$. Hence $b_3\ne 0$ and we may set $b_3=2$ by rescaling the parameter t. Then $h=3$ and this gives us all coefficients of g.

Thus the curve C is uniquely determined up to an affine linear change of variables. It remains to observe that (7) has the required branch at $t=0$, and to check that (7) gives a solution of the AlgDOP problem. $\square$

Lemma 3.4. *If* $\deg\Gamma\ge 5$, *then* $\operatorname{ord}_t(\gamma)\ne(-1,2,3)$, *i.e.,* γ *cannot be of type* $(1,3,4)$ *(flex) with* $\gamma\cdot P_\infty=1$.

Proof. $n=45$. The equations: $E_{-6,3,i}$, $E_{-4,2,i}$, $E_{-3,3,i}$, $E_{-2,1,i}$, $E_{-2,3,i}$, $E_{-1,2,i}$, $E_{-1,3,i}$, $E_{0,2,i}$, $E_{0,3,i}$ $(i=1,2,3)$; $E_{-5,3,i}$, $E_{-3,2,i}$, $E_{-1,1,i}$, $E_{1,3,i}$, $E_{2,3,i}$, $E_{3,3,i}$ $(i=2,3)$; $E_{0,1,3}$, $E_{1,1,2}$, $E_{2,2,2}$, $E_{3,2,2}$. The unknowns: g^{ij}_{klm} $(1\le i\le j\le 2)$, g^{13}_{klm} with $(k,l,m)\notin\{200,110\}$, and g^{23}_{klm} with $(k,l,m)\notin\{020,110,200\}$. Plugging the solution into $E_{4,2,2}$ and $E_{4,3,1}$, we obtain the equations $(58b_4+23c_1)g^{33}_{002}=0$ and $(8b_4+c_1)g^{33}_{002}=0$. If $g^{33}_{002}=0$ or $b_4=c_1=0$, then x^2 divides $\det g$. $\square$

Lemma 3.5. *If* $\deg\Gamma\ge 5$, *then* $\operatorname{ord}_t(\gamma)\ne(-1,1,3)$, *i.e.,* γ *cannot be of type* $(1,2,4)$ *(flat branch) with* $\gamma\cdot P_\infty=1$.

Proof. $n=42$. The equations: $E_{-3,\beta,i}$ (all β, i); $E_{-7,3,i}$, $E_{-5,2,i}$, $E_{-5,3,i}$, $E_{-2,2,i}$, $E_{-1,2,i}$ $(i=1,2,3)$; $E_{-4,2,i}$, $E_{-2,1,i}$, $E_{-1,1,i}$, $E_{0,1,i}$ $(i=2,3)$; $E_{-1,3,i}$, $E_{0,2,i}$, $E_{1,3,i}$ $(i=1,2)$; $E_{1,1,2}$, $E_{1,2,2}$, $E_{2,1,2}$, $E_{2,2,1}$. The unknowns: g^{33}_{002}, g^{33}_{001}, and g^{ij}_{klm} with $(i,j)\ne(3,3)$ except g^{12}_{200}, g^{13}_{200}, g^{13}_{110}, g^{22}_{200}, g^{22}_{110}, g^{23}_{200}, g^{23}_{110}, g^{23}_{011}, g^{23}_{101}, g^{23}_{100}. The solution implies that x^2 divides $\det g$. $\square$

Lemma 3.6. *If* $\deg\Gamma\ge 5$, *then* $\operatorname{ord}_t(\gamma)\ne(-1,1,4)$, *i.e.,* γ *cannot be of type* $(1,2,5)$ *(doubly flat branch) with* $\gamma\cdot P_\infty=1$.

Proof. $n=31$. The equations: $E_{-3,\beta,i}$ (all β, i); $E_{-7,3,i}$, $E_{-5,2,i}$, $E_{-5,3,i}$, $E_{-2,3,i}$ $(i=1,2,3)$; $E_{-4,3,i}$, $E_{-2,2,i}$, $E_{-1,2,i}$, $E_{0,1,i}$ $(i=2,3)$; $E_{0,2,1}$, $E_{0,3,1}$. The unknowns: g^{11}_{100}, g^{11}_{110}, g^{1j}_{101} $(j=1,2,3)$, g^{2j}_{000}, g^{2j}_{001}, g^{2j}_{011}, g^{2j}_{002} $(j=2,3)$, and all the g^{1j}_{0lm}. The solution implies that x^2 divides $\det g$. $\square$

Lemma 3.7. *If* $\deg\Gamma\ge 5$, *then* $\operatorname{ord}_t(\gamma)\ne(-2,-1,1)$, *i.e.,* γ *cannot be of type* $(1,2,3)$ *(generic branch) with* $\gamma\cdot P_\infty=2$.

Proof. $n=43$. The equations: $E_{-12,3,i}$, $E_{-11,3,i}$, $E_{-10,3,i}$, $E_{-9,2,i}$, $E_{-9,3,i}$, $E_{-8,2,i}$, $E_{-8,3,i}$, $E_{-6,1,i}$, $E_{-6,2,i}$, $E_{-6,3,i}$ $(i=1,2,3)$, $E_{-8,1,i}$, $E_{-7,1,i}$, $E_{-7,2,i}$, $E_{-7,3,i}$, $E_{-5,1,i}$ $(i=2,3)$, $E_{-5,2,1}$, $E_{-5,3,1}$, $E_{-4,2,1}$. The unknowns: g^{1j}_{klm} $(j=1,2,3)$, g^{j3}_{001}, g^{j3}_{011}, g^{j3}_{002}, g^{j3}_{020} $(j=2,3)$, g^{22}_{001}, g^{22}_{010}, g^{22}_{101}, g^{22}_{011}, g^{22}_{002}. We obtain that x^2 divides $\det g$. $\square$

Lemma 3.8. *If* $\deg\Gamma \ge 5$, *then* $\mathrm{ord}_t(\gamma) \ne (-2,-1,2)$, *i.e.,* γ *cannot be of type* $(1,2,4)$ *(flat branch) with* $\gamma \cdot P_\infty = 2$.

Proof. $n = 29$. The equations: $E_{-12,3,i}$, $E_{-11,3,i}$, $E_{-10,3,i}$, $E_{-9,2,i}$, $E_{-8,3,i}$, $E_{-8,2,i}$, $E_{-7,3,i}$, $E_{-6,1,i}$ $(i = 1,2,3)$; $E_{-9,3,i}$, $E_{-7,2,i}$ $(i = 2,3)$; $E_{-5,2,1}$. The unknowns: g^{1j}_{101}, g^{1j}_{110} $(j = 1,2,3)$, g^{2j}_{002}, g^{2j}_{011} $(j = 2,3)$, g^{11}_{100}, and all the g^{1j}_{0lm}. The solution implies that x^2 divides $\det g$. □

Lemma 3.9. *If* $\deg\Gamma \ge 5$, *then* $\mathrm{ord}_t(\gamma) \ne (-2,-1,3)$, *i.e.,* γ *cannot be of type* $(1,2,5)$ *(doubly flat branch) with* $\gamma \cdot P_\infty = 2$.

Proof. $n = 43$. The equations: $E_{-12,3,i}$, $E_{-11,3,i}$, $E_{-10,3,i}$, $E_{-9,2,i}$, $E_{-8,2,i}$, $E_{-7,3,i}$, $E_{-6,1,i}$, $E_{-6,3,i}$, $E_{-4,2,i}$ $(i = 1,2,3)$; $E_{-8,3,i}$, $E_{-6,2,i}$, $E_{-5,2,i}$, $E_{-5,3,i}$, $E_{-4,1,i}$, $E_{-3,\beta,i}$ $(i = 2,3)$. The unknowns: g^{ij}_{001}, g^{ij}_{010}, g^{ij}_{011}, g^{ij}_{002} $(2 \le i \le j \le 3)$; g^{i3}_{020}, g^{2i}_{101} $(i = 2,3)$; and all the g^{1j}_{klm} with $k \ne 2$. We obtain that x^2 divides $\det g$. □

Lemma 3.10. *If* $\deg\Gamma \ge 5$, *then* $\mathrm{ord}_t(\gamma) \ne (-2,1,2)$, *i.e.,* γ *cannot be of type* $(2,3,4)$ *(cusp) with* $\gamma \cdot P_\infty = 2$.

Proof. $n = 43$. The equations: $E_{-4,\beta,i}$ (all β, i), $E_{-10,3,i}$, $E_{-7,2,i}$, $E_{-7,3,i}$, $E_{-6,3,i}$ $(i = 1,2,3)$, $E_{-9,3,i}$, $E_{-6,2,i}$, $E_{-5,2,i}$, $E_{-5,3,i}$, $E_{-3,1,i}$ $(i = 2,3)$, $E_{-3,2,i}$, $E_{-3,3,i}$ $(i = 1,2)$, $E_{-2,\beta,2}$, $E_{-1,\beta,2}$ $(\beta = 1,2,3)$, $E_{-2,3,1}$, $E_{0,2,2}$. The unknowns: g^{ij}_{klm} $(1 \le i \le j \le 2)$, g^{13}_{klm} with $(k,l,m) \notin \{001, 011, 002\}$, and g^{23}_{klm} with $(k,l,m) \notin \{100, 110, 020, 200\}$. Plugging the solution into $E_{-1,3,1}$, we obtain the equation $c_{-1} g^{33}_{002} = 0$. If $g^{33}_{002} = 0$, then $\det g = 0$. Hence $c_{-1} = 0$ and we may set $g^{33}_{002} = 1$. Putting this into $E_{0,3,1}$, we obtain $b_3 = 0$. Then x^2 divides $\det g$. □

Lemma 3.11. *If* $\deg\Gamma \ge 5$, *then* $\mathrm{ord}_t(\gamma) \ne (-2,1,3)$, *i.e.,* γ *cannot be of type* $(2,3,5)$ *(flat cusp) with* $\gamma \cdot P_\infty = 2$.

Proof. $n = 36$. The equations: $E_{-4,\beta,i}$ (all β, i), $E_{-10,3,i}$, $E_{-7,2,i}$, $E_{-7,3,i}$, $E_{-5,3,i}$ $(i = 1,2,3)$; $E_{-8,3,i}$, $E_{-5,2,i}$, $E_{-3,1,i}$, $E_{-2,1,i}$ $(i = 2,3)$; $E_{-2,2,i}$, $E_{-2,3,i}$ $(i = 1,2)$; $E_{-3,3,2}$, $E_{-1,1,2}$, $E_{-1,2,2}$. The unknowns: g^{1j}_{101} $(j = 1,2,3)$, g^{1j}_{100}, g^{1j}_{110} $(j = 1,2)$, g^{2j}_{000}, g^{2j}_{001}, g^{2j}_{011}, g^{2j}_{002} $(j = 2,3)$, g^{22}_{010}, g^{22}_{020}, g^{22}_{101}, and all the g^{1j}_{0lm}. Plugging the solution into $E_{-3,2,3}$ and $E_{-1,3,2}$, we obtain the equations $(40b_2 - 7c_{-1})g^{33}_{002} = 0$ and $(10b_2 - c_{-1})g^{33}_{002} = 0$. If $g^{33}_{002} = 0$ or $b_4 = c_{-1} = 0$, then x^2 divides $\det g$. □

Lemma 3.12. *If* $\deg\Gamma \ge 5$, *then* $\mathrm{ord}_t(\gamma) \ne (-3,-1,1)$, *i.e.,* γ *cannot be of type* $(2,3,4)$ *(cusp) with* $\gamma \cdot P_\infty = 3$.

Proof. $n = 42$. The equations: $E_{-15,3,i}$, $E_{-13,3,i}$, $E_{-11,2,i}$, $E_{-11,3,i}$, $E_{-10,2,i}$, $E_{-9,2,i}$, $E_{-9,3,i}$, $E_{-7,1,i}$ $(i = 1,2,3)$; $E_{-12,2,i}$, $E_{-9,1,i}$, $E_{-8,1,i}$, $E_{-8,2,i}$ $(i = 2,3)$; $E_{-7,2,i}$, $E_{-7,3,i}$, $E_{-6,2,i}$ $(i = 1,2)$; $E_{-6,1,2}$, $E_{-5,1,2}$, $E_{-6,3,1}$, $E_{-4,2,1}$. The unknowns: g^{ij}_{klm} with $1 \le i \le j \le 2$ (except g^{22}_{100} and g^{22}_{200}); g^{13}_{klm} (except g^{13}_{100} and g^{13}_{200}); g^{23}_{001}, g^{23}_{002}, g^{23}_{020}, g^{23}_{011}, g^{23}_{101}, g^{33}_{011}. Plugging the solution into $E_{-6,3,1}$, we obtain $c_{-2} g^{33}_{002} = 0$. Then x^2 divides $\det g$. □

4. Tangent developables which admit solutions of the AlgDOP problem

Let the notation be as in §3. Thus C is an irreducible curve in $\mathbb{C}^3$ not lying in any plane, and $\Gamma(x,y,z) = 0$ is the equation of its tangent developable.

Proposition 4.1. *Suppose that there exists $g = (g^{ij})$ such that (g, Γ) is a solution of the AlgDOP problem over $\mathbb{C}$. Then C admits one of the following parametrizations in some affine coordinates in $\mathbb{C}^3$:*

(i) $t \mapsto (t, t^2, t^3)$;
(ii) $t \mapsto (t^{-1}, t, t^2)$;
(iii) $t \mapsto (t^2,\, 2t^3,\, 3t^4)$; *cusp at* $t = 0$;
(iv) $t \mapsto (t^{-1} + t,\, 3t - t^3,\, 2t^2 - t^4)$ *(cf. Lemma 3.3)*; *cusps at* $t = \pm 1$;
(v) $t \mapsto (3t - t^3,\, 4t^2 - 2t^4,\, 5t^3 - 3t^5)$; *cusps at* $t = \pm 1$;
(vi) $\theta \mapsto (3\cos\theta + \cos 3\theta,\, 3\sin\theta - \sin 3\theta,\, 6\cos 2\theta)$; *cusps at* $\theta = 0, \pi, \pm\pi/2$.

They correspond respectively to Cases $1°$, $1°$, $2°$, $5°$, $7°$, $8°$ *of Table 1.*

Proof. By Proposition 2.1 one of the cases in Table 1 takes place.

If $\deg C = 3$, then C is the rational normal curve, i.e., it admits a parametrization $t \mapsto (1 : t : t^2 : t^3)$ in some projective coordinates. In these coordinates, H_∞ is uniquely determined by the divisor D which it cuts on C. Thus there are only three possibilities: $D = 3p_1$ (then Case (i) occurs); $D = p_1 + 2p_2$ (then Case (ii) occurs); $D = p_1 + p_2 + p_3$ and then there is no solution (we compute g by solving a linear system (3), and see that $\det g = 0$).

Let $\deg C \geq 4$ and then $\deg \Gamma \geq 5$ (see Table 1). We assume also that Case (iv) does not occur. We see in Table 1 that either C or $\check{C}$ has degree at most 5. Hence each branch of C has type (m_1, m_2, m_3) with $m_3 \leq 5$ (i.e. contributes at most 2 into $k_0 + k_1 + k_2$) and $m_3 = 5$ (i.e. the contribution 2) is possible in Case $7°$ only. Then Lemmas 3.3–3.11 imply that C does not have any branch γ such that $\gamma \cdot P_\infty = 1$ or 2. Thus one of the following three cases occurs.

Case 1. $\deg C = 4$ *and* C *has a branch* γ *such that* $\gamma \cdot P_\infty = 4$. Cases $6°$, $3°$, and $4°$ of Table 1 are impossible because C cannot have branches of type $(1, 3, 4)$ or $(1, 2, 4)$ (flex or flat branch) in $\mathbb{C}^3$ by Lemmas 3.1 and 3.2. In Case $2°$ we obtain (iii) by Lemma 2.2(a).

Case 2. $\deg C = 5$ *and* C *has a branch* γ *such that* $\gamma \cdot P_\infty = 5$. As above, Case $5°$ of Table 1 is impossible by Lemmas 3.1 and 3.2, thus Case $7°$ takes place. Then γ is of the type $(3, 4, 5)$, $(2, 3, 5)$, or $(1, 2, 5)$ which corresponds to $(l_0, l_2) = (2, 0)$, $(1, 1)$, or $(0, 2)$ respectively. If there is an affine branch with $l_0 + l_2 = 2$, i.e., of type $(m_1, m_2, m_3) = (1, 2, 5)$, $(2, 3, 5)$, or $(3, 4, 5)$, then Lemma 2.2(a) implies that C is parametrized by $t \mapsto (t^{m_1}, t^{m_2}, t^{m_3})$. Solving the corresponding linear systems (3), we obtain that $\det g = 0$ in all the three cases.

Thus C has two affine branches γ_1 and γ_2, each contributing 1 to $k_0 + k_2$. By Lemma 3.2, C does not have any ordinary flat branch in $\mathbb{C}^3$, hence γ is of type $(1, 2, 5)$, and γ_1, γ_2 are of type $(2, 3, 4)$. Then the dual branches $\check{\gamma}$, $\check{\gamma}_1$, $\check{\gamma}_2$ of $\check{C}$ are of type $(3, 4, 5)$, $(1, 2, 4)$, $(1, 2, 4)$. By Lemma 2.2(a) applied to $\check{\gamma}$ and $\check{\gamma}_1$, the dual curve $\check{C}$ is parametrized by $t \mapsto (1+t : t^3 : t^4 : t^5)$ in some homogeneous coordinates. Thus $\check{C}$ (and hence C as well) is uniquely determined up to automorphism of $\mathbb{P}^3$. The choice of P_∞ is also unique because it is the osculating plane at γ. It remains to check that the curve in (v) gives a solution of the AlgDOP problem and its non-generic branches are of the required types.

Case 3. $\deg C = 6$ *and* C *has branches* γ_1, γ_2 *such that* $\gamma_1 \cdot P_\infty = \gamma_2 \cdot P_\infty = 3$. Then C is a 4-cuspidal sextic (see. Table 1). Since $\deg \check{C} = 4$, all cusps are ordinary. Then Lemma 3.12 implies that γ_1 and γ_2 are of type $(1, 2, 3)$ and P_∞ is the the osculating plane for each of them. Hence $\check{C}$ has a point with two local branches.

Then Lemma 2.2(b) implies that $\check{C}$ is uniquely determined, and we obtain (vi) by the same argument as in Case 2. $\square$

Proposition 4.2. *Suppose that C is real and there exists $g = (g^{ij})$ such that (g, Γ) is a solution of the AlgDOP problem over $\mathbb{R}$. Then C admits one of the following parametrizations in some affine coordinates in $\mathbb{R}^3$:* (i)–(vi) of Proposition 4.1 or

- (iv′) $t \mapsto (t^{-1} - t,\ 3t + t^3,\ 2t^2 + t^4)$; cusps at $t = \pm i$;
- (v′) $t \mapsto (3t + t^3,\ 4t^2 + 2t^4,\ 5t^3 + 3t^5)$; cusps at $t = \pm i$;
- (vi′) $t \mapsto (3t^{-1} + t^3,\ 3t^{-2} + 3t^2,\ t^{-3} + 3t)$; cusps at $t = \pm 1, \pm i$;
- (vi″) $t \mapsto (3t^{-1} - t^3,\ 3t^{-2} - 3t^2,\ t^{-3} - 3t)$; cusps at the roots of $t^4 + 1$.

Proof. It is easy to see that the real form of C is determined by the involution of complex conjugation on $\widetilde{C}$. The latter must preserve the set of points of each type and the divisor $\nu^*(P_\infty)$ on $\widetilde{C}$. This implies that the list is exhaustive. A computation shows that all the cases are realizable. $\square$

The tangent developable of the twisted cubic $t \mapsto (t, t^2, t^3)$ is given by the equation $\Gamma_4 = 0$ where Γ_4 is the discriminant of $P(u) = u^3 + 3xu^2 + 3yu + z$, i.e.,

$$\Gamma_4 = 3x^2y^2 - 4y^3 - 4x^3z + 6xyz - z^2. \tag{8}$$

Lemma 4.3. (cf. Prop. 4.1(i)) *Let (g, Γ) be a solution of the AlgDOP problem over $\mathbb{R}$ such that the surface $\Gamma = 0$ contains the tangent developable of the curve C parametrized by $t \mapsto (t, t^2, t^3)$. Then*

$$g = a\begin{pmatrix} 0 & 0 & 0 \\ * & 2(x^2-y) & 3(xy-z) \\ * & * & 18(y^2-xz) \end{pmatrix} + b\begin{pmatrix} 1 & 2x & 3y \\ * & 4x^2 & 6xy \\ * & * & 9y^2 \end{pmatrix} + c\begin{pmatrix} x & 2x^2 & 3xy \\ * & 5xy-z & 6y^2 \\ * & * & 9yz \end{pmatrix}$$
$$+ d\begin{pmatrix} x & 2y & 3z \\ * & 3xy+z & 6xz \\ * & * & 9yz \end{pmatrix} + e\begin{pmatrix} x^2 & 2xy & 3xz \\ * & 4y^2 & 6yz \\ * & * & 9z^2 \end{pmatrix} + f\begin{pmatrix} 2(x^2-y) & xy-z & 0 \\ * & 2(y^2-xz) & 0 \\ * & * & 0 \end{pmatrix},$$

$\det g = \Gamma_4\Gamma_2$ *where Γ_4 is as in (8) and*

$$\begin{gathered}\Gamma_2 = a^2b + a^2cx - 2abcx + a^2dx + 2abdx - 2ac^2x^2 + a^2ex^2 + 2abex^2 + 2a^2fx^2 + 4abfx^2 + bc^2y - 2bcdy + 2ad^2y \\ + bd^2y - 2abey - 2a^2fy - 2abfy + c^3xy - c^2dxy - 2acexy - bcexy + 2adexy + bdexy + 2acfxy - 2bcfxy \\ + 4adfxy + 2bdfxy - 2c^2fy^2 + 4aefy^2 + 2befy^2 + 2af^2y^2 + bf^2y^2 - cd^2z + d^3z + bcez - bdez - 2adfz + c^2exz \\ - 2cdexz + d^2exz + 2d^2fxz - 2aefxz - 2befxz - 2af^2xz - 2cefyz + 2defyz + cf^2yz + df^2yz + ef^2z^2\end{gathered}$$

and one of the following cases occurs up to affine linear change of coordinates:

- (i_1) $\Gamma = \Gamma_4$ *and* $(a, c - d, f) \neq (0, 0, 0)$, $(b, c + d, e, af - d^2) \neq (0, 0, 0, 0)$*; in this case Γ_2 is a non-zero constant if and only if $c = d = e = f = 0$ and $ab \neq 0$;*
- (i_2) $b = d = f = 0$, $(c, ae) \neq (0, 0)$, *and $\Gamma = x\Gamma_4$, then we have $\Gamma_2 = x\Gamma_1$ where $\Gamma_1 = a^2c + a(ae - 2c^2)x + c(c^2 - 2ae)y + c^2ez$; in this case Γ_1 cannot be a nonzero constant, and we have $\Gamma_1 = x$ if and only if $c = 0$ and $ae \neq 0$;*
- (i_3) $a = b = c = 0$, $(d, ef) \neq (0, 0)$, *and $\Gamma = z\Gamma_4$, then we have $\Gamma_2 = z\Gamma_1$ where $\Gamma_1 = d^3 + d^2(e + 2f)x + df(2e + f)y + ef^2z$; in this case Γ_1 is a nonzero constant if and only if $e = f = 0$ and $d \neq 0$; we have $\Gamma_1 = z$ if and only if $d = 0$ and $ef \neq 0$;*
- (i_4) $(a, \dots, f) = (0,\ 0,\ 0,\ 1,\ -1,\ 0)$, $\Gamma = (x - 1)z\Gamma_4$*;*
- (i_5) $(a, \dots, f) = (1, 1, 0, 0, -1, -1)$, $\Gamma = P(1)P(-1)\Gamma_4$, *therefore $\{\Gamma_2 = 0\}$ is the union of two osculating planes of C; recall that $\Gamma_4 = \mathrm{discr}_u P(u)$;*
- (i_6) $(a, \dots, f) = (2\alpha, 1, 0, 0, \pm 1, 0)$, $\alpha \neq 0$, $\Gamma = (\alpha + 1)x^2 - y \pm \alpha$.
- (i_7) $(a, \dots, f) = (1,\ 0,\ 1,\ 1,\ 0,\ 0)$, $\Gamma = (x - x^2 + y)\Gamma_4$*;*

Proof. Step 1. We find g by solving the system of linear equations (3). If $\Gamma = \Gamma_4$, we arrive to (i_1) where the indicated condition on $(a, \dots, f)$ is equivalent to $\det g \neq 0$. We have

$$\Gamma_2(t, t^2, t^3) = (b + ct + dt + et^2)(a - ct + dt + ft^2)^2.$$

Hence $\{\Gamma_2 = 0\}$ is disjoint from the curve C (in $\mathbb{C}^3$) if and only if $c = d = e = f = 0$, i.e., if and only if Γ_2 is a non-zero constant.

Step 2. The variable changes $\varphi_\mu : (x, y, z) \mapsto (x,\, y + 2\mu x,\, z + 3\mu y + 3\mu^2 x)$ and $\psi_\lambda : (x, y, z) \mapsto (\lambda x, \lambda^2 y, \lambda^3 z)$ preserve $\{\Gamma_4 = 0\}$ and replace $(a, \dots, f)$ with

$$\big(\, a + \mu(c - d) + \mu^2 f,\; b - \mu(c + d) + \mu^2 e,\; c + \mu(f - e),\; d - \mu(f + e),\; e,\; f \,\big) \tag{9}$$

and $(\lambda^2 a, \lambda^2 b, \lambda c, \lambda d, e, f)$ respectively. Thus, if $f \neq 0$ or $c - d \neq 0$, we may assume that $a = 0$; if $e \neq 0$ or $c + d \neq 0$, we may assume that $b = 0$.

Step 3. Here we suppose that $\Gamma = \Gamma_4\Gamma_1$ with $\deg \Gamma_1 = 1$. Any affine plane cuts the curve C. Hence, up to affine change of coordinates, we may assume that the plane $P = \{\Gamma_1 = 0\}$ passes through the origin.

Case 3.1. P is transverse to C at the origin, i.e., P is parametrized by $(t, u) \mapsto (x, y, z) = (At + Bu, t, u)$. Then (3) has three solutions:

- $A = B = b = d = f = 0$ (this is (i_2));
- $b = e = 0$, $c = -d = aA$, $f = aA^2$, $B = -\frac{1}{3}A^2$ (then $\det g = 0$);
- $b = f = B = 0$, $c = d = -aA$, $e = 2aA^2$.

In the latter case we have $\Gamma_2 = 2a^3 A(x - Ay)$, thus $A \neq 0$, and the variable change φ_μ followed by ψ_λ, $\lambda = \mu = -1/(2A)$ (see Step 2) gives (i_6) with $\alpha = -1$.

Case 3.2. P has an ordinary tangency with C at the origin, i.e., up to rescaling of the coordinates, $\Gamma_1 = z - y$. Then (3) does not have any non-zero solution.

Case 3.3. P is the osculating plane of C at the origin, i.e., $\Gamma_1 = z$. Then the only non-zero solution of (3) is (i_3).

Step 4. Suppose that $\deg \Gamma = 6$ and $\Gamma_2 = \Gamma_1\tilde\Gamma_1$ with $\deg \Gamma_1 = \deg \tilde\Gamma_1 = 1$. According to the result of Step 3, each of the planes $\{\Gamma_1 = 0\}$, $\{\tilde\Gamma_1 = 0\}$ is either an osculating plane for C (Case (i_3)) or a plane of the form $\{x = x_0\}$ (Case (i_2)). Any two distinct points on C can be mapped to any fixed positions by an affine linear automorphism which preserves C (see Step 2). Thus Γ_2 is as in (i_4) or (i_5) unless $\Gamma_2 = x^2 - 1$ or $\Gamma_2 = xz$. In the latter two cases the system (3) does not have any nonzero solution. Notice that this fact can be checked without any computations. Indeed, $\Gamma_2 = x^2 - 1$ would imply that the g^{i1} are divisible by $x^2 - 1$ and $\Gamma_2 = xz$ would imply that the g^{i1} (resp. g^{i3}) are divisible by x (resp. by z). It is immediately seen from the form of g that this is impossible.

Step 5. Suppose that $\deg \Gamma_2 = 2$ and Γ_2 is irreducible.

Case 5.1. $ef \neq 0$. Then $\deg_z \Gamma_2 = 2$ and its coefficient of z^2 is ef^2 (a nonzero constant). By the result of Step 2 we may assume that $a = 0$. Then we compute the remainders of the division of $g^{11}\partial_x\Gamma_1 + g^{12}\partial_y\Gamma_1 + g^{13}\partial_z\Gamma_1$ (viewed as a polynomial in z) by Γ_2 and equate its coefficients to zero (see (A3) in §1). The obtained system of equations has only two solutions with $ef \neq 0$. These are: (S1) $b = c = 0$, $e = f$ and (S2) $b = c = d = 0$. In both cases Γ_2 reducible.

Case 5.2. $ef = a = 0$. Then $\Gamma_2 = p_0(y, z) + p_1(y, z)x$.

Case 5.2.1. $p_1(y,z)=0$. If $f=0$, then $p_1=((d-c)((c^2-be)y+(c-d)ez))$. If it is zero, then $d=c$ (then $\Gamma_2=0$) or $c=e=0$ (then $\deg\Gamma_2<2$). If $e=0$ and $f\neq 0$, then $p_1=(d-c)(c^2-2bf)y-2d^2fz$. If it is zero, then $cd=0$ which implies that $\Gamma_2=\Gamma_1^{(1)}\Gamma_1^{(2)}$, $\deg\Gamma_1^{(k)}\leq 1$.

Case 5.2.2. $p_1(y,z)\neq 0$. Then we solve the system (3) for the parametrization $(t,u)\mapsto\big(p_0(t,u)/p_1(t,u),\,t,\,u\big)$ of $\{\Gamma_2=0\}$. If $f=0$, the solutions are: (S1) $d=c$; (S2) $b\neq 0$, $e=cd/b$; (S3) $b=c=0$; (S4) $b=d=0$. If $e=0$, the solutions are: (S5) $c=d=0$; (S6) $c=f=0$; (S7) $d=0$, $f\neq 0$, $b=c^2/(2f)$; (S8) $f=d-c=0$; (S9) $d=f=0$. In all these cases we have $\Gamma_2=\Gamma_1^{(1)}\Gamma_1^{(2)}$, $\deg\Gamma_1^{(k)}\leq 1$.

Case 5.3. $ef=0$ and $a\neq 0$. By the result of Step 1 we know that C and $\{\Gamma_2=0\}$ have a common point in $\mathbb{C}^3$. Suppose first that there is a real common point. By an affine linear change of coordinates in $\mathbb{R}^3$ we can achieve that this is the origin. Since $\Gamma_2(0,0,0)=a^2b$, we then have $b=0$. By the result of Step 2 we may assume that $f=0$ and $d=c$ (otherwise we reduce to Case 5.2). Then $c\neq 0$ because otherwise $\Gamma_2=x^2$. Thus we have $b=f=0$ and $d=c\neq 0$. One can check that this is a solution of AlgDOP problem. If $e=0$ we obtain (i$_7$) by the coordinate change ψ_λ (see Step 2) with $\lambda=c/a$. If $e\neq 0$, the coordinate change φ_μ with $\mu=c/e$ followed by ψ_λ with $\lambda=e/c$, we obtain (i$_6$) with $(a,\dots,f)=(-ae/c^2,1,0,0,-1,0)$. In the case when C and $\{\Gamma_2=0\}$ do not have common real points, one can show that $(a,\dots,f)=(2\alpha,1,0,0,1,0)$, $\alpha\in\mathbb{R}$ (we omit the details). In both cases we have $\alpha\neq 0$ (otherwise $\Gamma_2=0$). $\square$

The following lemma is a direct computation.

Lemma 4.4. (cf. Prop. 4.1(ii)) *Let (g,Γ) be a solution of the AlgDOP problem over $\mathbb{R}$ such that the surface $\Gamma=0$ contains the tangent developable of the curve $t\mapsto(t^{-1},t,t^2)$. Then*

$$g=a\begin{pmatrix}0&0&0\\ *&2(1-xy)&3(y-xz)\\ *&*&18(y^2-z)\end{pmatrix}+b\begin{pmatrix}x^2&-xy&-2xz\\ *&y^2&2yz\\ *&*&4z^2\end{pmatrix},\qquad ab\neq 0,$$

$\det g=9a^2b\,x^2(3y^2-4xy^3-4z+6xyz-x^2z^2)$. *The coordinate change $(x,y,z)\mapsto(\lambda^{-1}x,\lambda y,\lambda^2 z)$ preserves* $\det g$ *and transforms (a,b) into $(a,\lambda^2 b)$. Thus we can reduce to $(a,b)=(1,\pm1)$.* $\square$

The tangent developable of the cuspidal quartic curve $t\mapsto(t^2,2t^3,3t^4)$ is given by the equation $\Gamma_5=0$ where Γ_5 is the discriminant of $P(u)=u^4-6xu^2-4yu-z$, i.e.,

$$\Gamma_5=-54x^3y^2+27y^4+81x^4z-54xy^2z+18x^2z^2+z^3. \tag{10}$$

Lemma 4.5. (cf. Prop. 4.1(iii)) *Let (g,Γ) be a solution of the AlgDOP problem over $\mathbb{R}$ such that the surface $\Gamma=0$ contains the tangent developable of the curve $t\mapsto(t^2,2t^3,3t^4)$, i.e., Γ_5 a factor of Γ. Then*

$$g=a\begin{pmatrix}4x&6y&8z\\ *&3(9x^2+z)&36xy\\ *&*&144(y^2-xz)\end{pmatrix}+b\begin{pmatrix}4y&2(9x^2+z)&24xy\\ *&36xy&36y^2\\ *&*&48yz\end{pmatrix}+c\begin{pmatrix}4x^2&6xy&8xz\\ *&9y^2&12yz\\ *&*&16z^2\end{pmatrix},$$

$\det g=\Gamma_5\Gamma_1$ *where $\Gamma_1=3a^3+3(a^2c-3ab^2)x+3(b^3-abc)y+b^2cz$, and one of the following cases occurs up to rescaling of the coordinates:*

(iii$_1$) $(a,b)\neq(0,0)$, $\Gamma=\Gamma_5$, *in this case Γ_1 is a nonzero constant if and only if $a\neq 0$ and $b=c=0$;*

(iii$_2$) $(a,b,c)=(3,1,-1)$, $\Gamma=\Gamma_1\Gamma_5$, *in this case* $\{\Gamma_1=0\}$ *is the osculating plane at* $t=3$, *and we have* $\Gamma_1=P(3)$ *(recall that* $P(u)=u^4-6xu^2-4yu-z$*);*

(iii$_3$) $(a,b,c)=(1,0,\pm1)$, $\Gamma=(x\pm1)\Gamma_5$.

Proof. We find g by solving the linear system of equations (3). Then $\det g$ is as stated. It vanishes identically if and only if $a=b=0$. Since Γ_5 divides Γ and Γ divides $\det g$, we have either $\Gamma=\Gamma_5$ or $\Gamma=\det g=\Gamma_5\Gamma_1$. In the former case everything is done. So, we suppose that $\Gamma=\Gamma_5\Gamma_1$.

The change $(x,y,z)\mapsto(\lambda^2x,\lambda^3y,\lambda^4z)$ transforms (a,b,c) to $(a,\lambda b,\lambda^2c)$. Thus, if $abc\neq0$, we may assume that $(a,b)=(3,1)$. Then the remainder of the division of $g^{11}\partial_x\Gamma_1+g^{12}\partial_y\Gamma_1+g^{13}\partial_z\Gamma_1$ (viewed as a polynomial in z) by Γ_1 is equal to $(c+1)q(x,y)$ where $q(x,y)$ is a polynomial in x,y such that $q(0,0)\neq0$, and we arrive to solution (iii$_2$).

If $abc=0$, we may rescale the coordinates so that each of a,b,c is 0 or ±1. In each case we check if (3) is satisfied. $\square$

The following lemma is a direct computation based on Proposition 4.2. In Cases (v), (v$'$) instead of C we consider its image under $(x,y,z)\mapsto(x,\frac{3}{2}y,2z)$.

Lemma 4.6. *Let* (g,Γ) *be a solution of the AlgDOP problem over* $\mathbb{R}$ *such that the surface* $\Gamma=0$ *contains the tangent developable of the curves in Props. 4.1(iv)–(vi) and 4.2(iv$'$)–(vi$''$). Then* $\Gamma=\det g$ *and one of the following cases takes place:*

(iv) *g is given by (11) with* $\varepsilon=1$*:*

$$\begin{pmatrix} x^2 & 3xy-12 & 4xz-4y \\ * & 9y^2-12z & 12yz \\ * & * & 16z^2 \end{pmatrix}+\varepsilon\begin{pmatrix} -4 & 0 & 0 \\ * & 72-24xy & 24y-36xz \\ * & * & 32y^2-144z \end{pmatrix}; \tag{11}$$

(iv$'$) *g is given by (11) with* $\varepsilon=-1$*;*

(v) *g is given by (12) with* $\varepsilon=1$*:*

$$\begin{pmatrix} 4y-9x^2 & 2z-12xy & -15xz \\ * & -16y^2 & -20yz \\ * & * & -25z^2 \end{pmatrix}+\varepsilon\begin{pmatrix} 24 & 32x & 40y \\ * & 16(6x^2-5y) & 120(xy-z) \\ * & * & 400(y^2-xz) \end{pmatrix}, \tag{12}$$

$\frac14\det(5g)$ *is the discriminant of* $u^5-10u^3-10xu^2-5yu-z$*;*

(v$'$) *g is given by (12) with* $\varepsilon=-1$*;*

(vi)

$$g=\begin{pmatrix} 8+y^2+4z-2x^2 & -3xy & 12x-2xz \\ * & 8-4z+x^2-2y^2 & -12y-2yz \\ * & * & 16+8x^2+8y^2-4z^2 \end{pmatrix},$$

$\frac14\det g$ *is the discriminant of* $u^4-su^3+zu^2-\bar{s}u+1$, $s=x+iy$*;*

(vi$'$) *g is given by (13) with* $\varepsilon=-1$*:*

$$\begin{pmatrix} 3x^2-8y & 2xy-12z & xz \\ * & 4y^2-8xz & 2yz \\ * & * & 3z^2 \end{pmatrix}+\varepsilon\begin{pmatrix} 0 & 0 & 16 \\ 0 & 16 & 12x \\ 16 & 12x & 8y \end{pmatrix}; \tag{13}$$

(vi$''$) *g is given by (13) with* $\varepsilon=1$.

5. Solutions of SDOP problem bounded by tangent developables

5.1. Bounded solutions.

Theorem 5.1. *Up to affine linear change of coordinates, the following is a complete list of solutions (Ω, g, ρ) of DOP problem in $\mathbb{R}^3$ such that Ω is a bounded domain whose boundary $\partial\Omega$ contains a piece of a tangent developable surface. In each case g is as in the corresponding case of Lemmas 4.3, 4.5, 4.6 (sometimes with additional restrictions on the parameters) and Ω is the only bounded component of the complement of $\{\det g = 0\}$; see Figures 1–5 and comments on them in §5.3.*

(i_4) $\rho = \Gamma_4^{p-1} z^{q-1}(1-x)^{r-1}$, $6p > 1$, $q > 0$, $r > 0$, $2p + q > 1$;
(i_5) $\rho = \Gamma_4^{p-1} P(1)^{q-1} P(-1)^{r-1}$, $6p > 1$, $q > 0$, $r > 0$, $2p + q > 1$, $2p + r > 1$;
(i_6) $\alpha > 0$ *and* $e = -1$, $\rho = \Gamma_4^{p-1}\Gamma_2^{q-1}$, $6p > 1$, $q > 0$ *(see Remark 5.3)*;
(iii_2) $\rho = \Gamma_5^{p-1}\Gamma_1^{q-1}$, $4p > 1$, $q > 0$, $2p + q > 1$;
(iii_3) $c = -1$, $\rho = \Gamma_5^{p-1}(1-x)^{q-1}$, $4p > 1$, $q > 0$;
(v,vi) $\rho = (\det g)^{p-1}$, $4p > 1$;

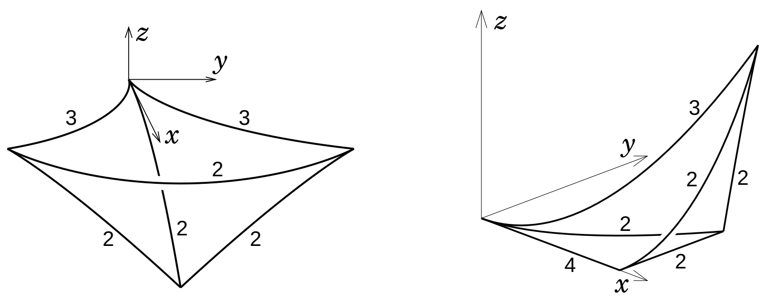

FIGURE 1. (iii_3) and (i_4): the quotients of $\mathbb{S}^3$ by the reflection groups $A_1 + A_3$ (the truncated swallow tail) and $A_1 + B_3$.

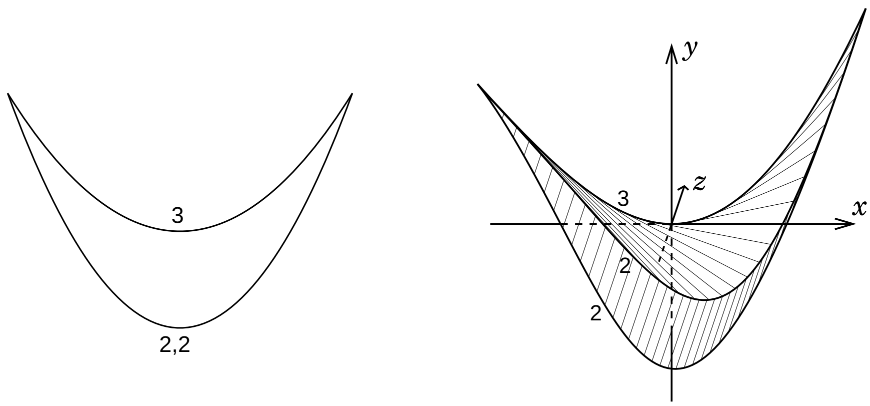

FIGURE 2. (i_6): the quotient of $\mathbb{S}^3$ by the reflection group $A_1 + A_2$ (the projection on the xy-plane is on the left hand side).

Proof. Boundedness of Ω. Let us show that $\mathbb{R}^3 \setminus \Sigma$ where $\Sigma = \{\Gamma = 0\}$ does not have any bounded component in all other cases of Lemmas 4.3–4.6. For (i_1), (i_2), (i_3), (iii_1) this fact is evident because Γ is quasihomogeneous. In other cases we consider the projection $\pi : (x, y, z) \mapsto (x, y)$ and find the regions on the xy-plane

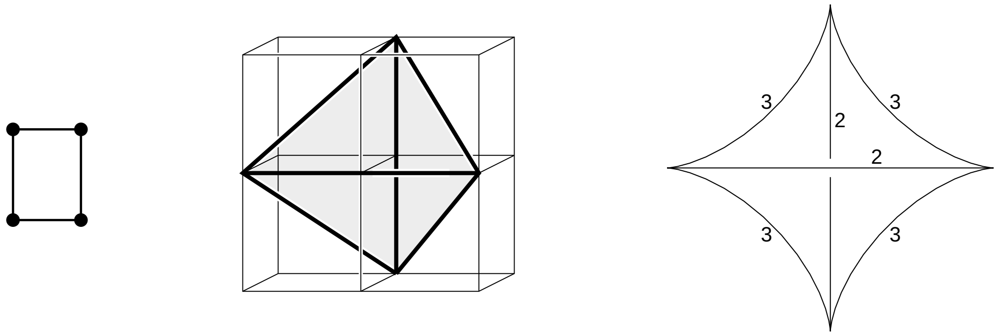

FIGURE 3. (vi): the quotient of $\mathbb{R}^3$ by the affine reflection group $\widetilde{A}_3$.

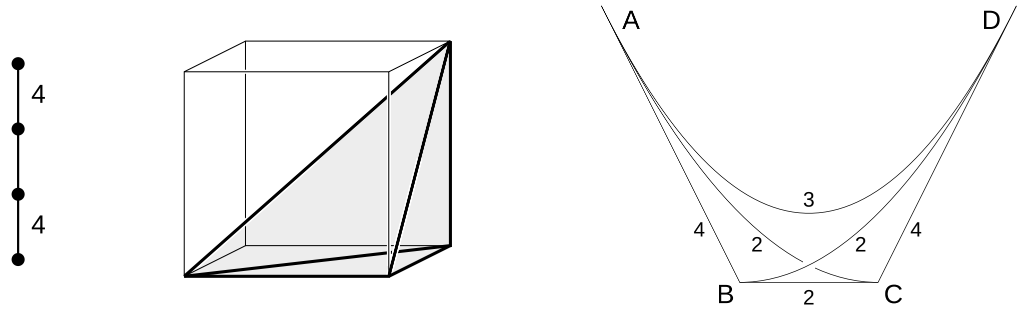

FIGURE 4. (i_5): the quotient of $\mathbb{R}^3$ by the affine reflection group $\widetilde{C}_3$. The faces ABC and BCD are on the osculating planes at A and D resp.

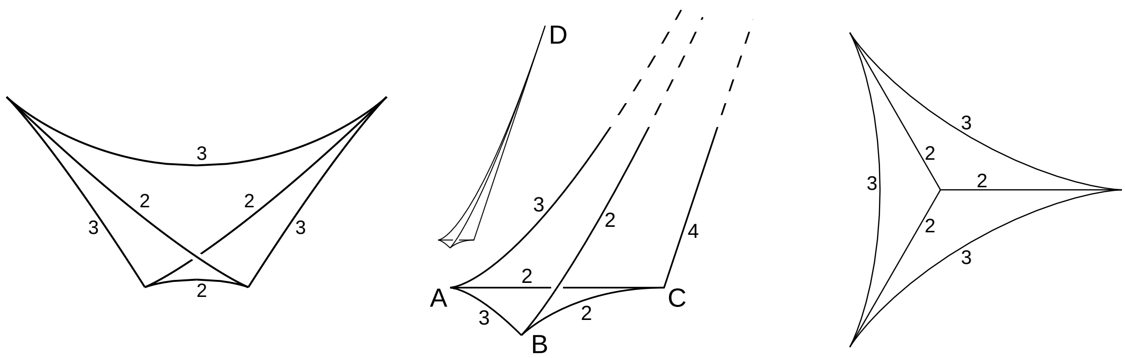

FIGURE 5. (v), (iii_2), §6.5: the quotients of $\mathbb{S}^3$ by the reflection groups A_4, B_4, D_4. The face BCD belongs to the osculating plane at D.

over which Σ is a disjoint union of graphs of smooth functions (i.e. over which $\pi|_\Sigma$ is a covering). This is the complement of the real curve $R = \{D_z(x,y)C_z(x,y) = 0\}$ where D_z is the discriminant of Γ with respect to z, and C_z is the coefficient of the highest power of z in Γ. This curve is depicted in Figure 6 in the respective cases. The dashed line represents $\pi(B_-)$ where B_- is the part of the curve of self-intersection of (the complexification of) Σ such that two non-real local branches of Σ cross at points of B_- (that is Σ has the equation $u^2 + v^2 = 0$ in some local real analytic coordinates (u, v, w) near each point of B_-). We see in Figure 6 that all components of $\mathbb{R}^2 \setminus (R \setminus \pi(B_-))$ are unbounded. Hence so are all components of $\mathbb{R}^3 \setminus \Sigma$. It remains to exclude Case (i_6) when $\alpha < 0$ or $e = 1$. In this case we have $D_z = y - x^2$ and $C_z = (\alpha+1)x^2 - y + e\alpha$, $e = \pm 1$. If $e = 1$, then all components of

$\mathbb{R}^2 \setminus R$ are unbounded. If $e = -1$ and $\alpha < 0$, then there is a bounded component Ω, but $\pi^{-1}(\Omega) \cap \Sigma$ is empty.

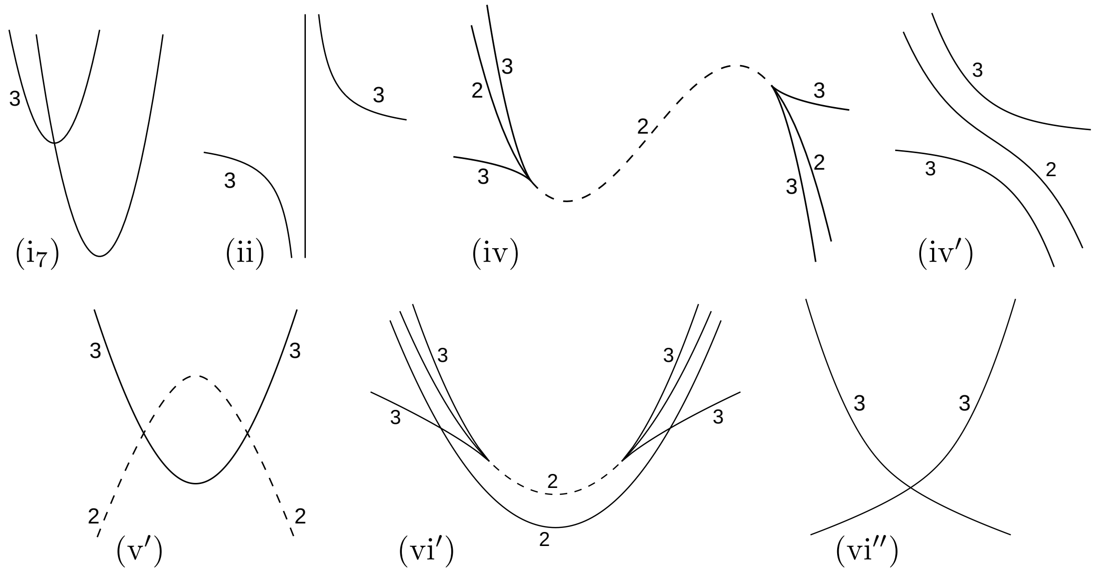

FIGURE 6. Irrelevant solutions of the AlgDOP problem.

Integrability of ρ. In Case (i_4), the integrability conditions at the origin, at cuspidal edge, and at the x-axes (the line of tangency) are, respectively, $2p+q > 1$, $p > 1/6$, and $p + q > 1/2$ but the last condition follows from the first one because $q > 0$. Let us prove the first condition. Let $\Omega_1 = \{(y, z) \mid (1, y, z) \in \Omega\}$. We have $\Gamma_4(x, y, z) = x^6\Gamma_4(1, y/x^2, z/x^3)$, hence, using the variable change $y = x^2\eta$, $z = x^3\zeta$, we obtain

$$\int_{\Omega\cap\{x<\varepsilon\}} z^{q-1}\Gamma_4^{p-1}\,dx\,dy\,dz = \int_0^{\varepsilon} dx \int_{\Omega_1} (x^3\zeta)^{q-1}\big(x^6\Gamma_4(1, \eta, \zeta)\big)^{p-1} x^5 d\eta\, d\zeta,$$

which is finite if and only if $6(p-1) + 3(q-1) + 5 > -1$ (i.e., $2p + q > 1$) and $\int_{\Omega_1} \zeta^{q-1}\Gamma_4(1, \eta, \zeta)^{p-1} d\eta\, d\zeta$ is finite. The integrability conditions in dimension 2 are obtained in the same way (in [3, Remark 2.28] they are stated as an evident fact). In our case they are $6p > 1$ and $2p + 2q > 1$. The same or similar arguments work in Cases (i_5), (i_6), (iii_3) as well. In the remaining three cases the surface is not quasihomogeneous, however, one can show that the restrictions are the same as in the quasihomogeneous case (we omit the proof). □

5.2. Unbounded solutions.

Theorem 5.2. *Up to affine linear change of coordinates, the following is a complete list of solutions (Ω, g, ρ) of SDOP problem in $\mathbb{R}^3$ such that Ω is an unbounded domain whose boundary $\partial\Omega$ contains a piece of a tangent developable surface. In each case g is as in the corresponding case of Lemmas 4.3, 4.5 with additional restrictions on the parameters.*

- (i_1) $(a, \dots, f) = (2\alpha, 1, 0, 0, 0, 0)$, $\alpha > 0$; *Ω is the component of* $\mathbb{R}^3 \setminus \{\det g = 0\}$ *containing* $(0, -1, 0)$ *(i.e., the domain in Figure 2 is* $\Omega \cap \{y \geq (\alpha+1)x^2 - \alpha\}$*),* $\rho = \Gamma_4^{p-1} \exp\big(\lambda y - \lambda(1 + \alpha)x^2\big)$, $p > 1/6$, $\lambda > 0$;

(i$_3$) $(a,\dots,f)=(0,0,0,1,0,0)$; Ω *is the only component of* $\mathbb{R}^3\setminus\{\det g=0\}$ *such that* $\Omega\cap\{x=1\}$ *is bounded (i.e.,* $\Omega\cap\{x\le 1\}$ *is the right domain in Figure 1),* $\rho=\Gamma_4^{p-1}z^{q-1}\exp(-\lambda x)$, $p>1/6$, $q>0$, $2p+q>1$, $\lambda>0$;

(iii$_1$) $(a,b,c)=(1,0,0)$; Ω *is the only component of* $\mathbb{R}^3\setminus\{\det g=0\}$ *such that* $\Omega\cap\{x=1\}$ *is bounded (i.e.,* $\Omega\cap\{x\le 1\}$ *is the left domain in Figure 1),* $\rho=\Gamma_5^{p-1}\exp(-\lambda x)$, $p>1/4$, $\lambda>0$.

Proof. Let Γ be the minimal polynomial vanishing on $\partial\Omega$. Then (g,Γ) is a solution of the AlgDOP problem. Set $\Delta=\det g$ and $\Sigma=\{\Gamma=0\}$. If Ω is unbounded and Δ does not have multiple components, then $\deg\Delta<6$. This fact excludes all the cases of Lemmas 4.3–4.6 for (g,Γ) except those considered below.

(i$_1$). Then $\partial\Omega\subset\Sigma_4=\{\Gamma_4=0\}$ (recall that Γ_4 is given in (8)). Let π be the projection $\mathbb{R}^3\to\mathbb{R}^2$, $(x,y,z)\mapsto(x,y)$. Then Σ_4 cuts $\mathbb{R}^3$ into two unbounded components Ω_+, Ω_-. One of them (let it be Ω_+) is projected by π onto the non-convex set $\{y<x^2\}$, and $\pi^{-1}(\pi(p))$ is a finite interval for any $p\in\Omega_+$ (see Figure 2). Since Ω is one of Ω_+, Ω_-, we have $\partial\Omega=\Sigma_4$.

Let us show that any affine plane P intersects each of Ω_+, Ω_-. The curve C (whose tangent developable is Σ_4) has only one point at infinity: the infinite point of the z-axis. If the projective closure of P does not pass through this point, then P cuts C in some finite real point because the degree of C is odd. Otherwise $P=\pi^{-1}(L)$ where L is a line in $\mathbb{R}^2$, hence P cuts Σ because $\pi(\Sigma)=\{y<x^2\}$. In both cases P cuts each of $\Omega_\pm$. This fact implies that $\Delta=\Gamma_4$ (up to a scalar factor). Indeed, recall that either Δ has a multiple component, or $\deg\Delta<6$. Thus, if $\deg\Delta>4$, then in both cases Δ would vanishes on some plane P. This is impossible because $\Delta|_\Omega\ne 0$ and $P\cap\Omega\ne\varnothing$.

By solving the system of linear equations (3), we obtain the required form of ρ, maybe, multiplied by $e^{\lambda_1 x}$, however, this factor can be killed by the transformation φ_μ with a suitable μ; see (9). We have $\Omega=\Omega_+$ because Ω_- contains cylinders parallel to the z-axis, which contradicts the integrability condition for the measure of this form. The positive definiteness of g in Ω implies that $a>0$ and $b>0$. Then we may set $b=1$, $a=2\alpha$, $\alpha>0$. The integrability conditions near C and at the infinity are, respectively, $p>1/6$ (see [3, Remark 2.28]) and $\lambda>0$.

(i$_2$). Then $\deg\Delta=6$, and Δ has multiple factors of Δ if and only if and $b=c=d=f=0$, $ae\ne 0$. In this case $\Delta=x^2\Gamma_4$. No exponential factor of ρ.

(i$_3$) with $a=b=c=e=f=0$, $d\ne 0$. The solution is antisymmetric under the rotation $(x,y,z)\mapsto(-x,y,-z)$, hence we may set $d=1$. Then g is positive definite only in the indicated domain. Solving the linear equations, we obtain that the measure is of the required form. The integrability condition at the infinity is $\lambda>0$. The others are the same as in Theorem 5.1(i$_4$).

(i$_3$) with $a=b=c=d=0$, $ef\ne 0$. Solving the linear equations, we obtain that ρ has an exponential factor only when $e=f$, and it is $\exp(\lambda y/z)$. Since $\Gamma_4(\lambda x,\lambda^2 y,\lambda^3 z)=\lambda^6\Gamma_4(x,y,z)$, using the variable change $y_1=y/x^2$, $z_1=z/x^3$, one can easily show that the integrability condition fails for any choice of Ω.

(i$_6$) with $\alpha=-1$. No exponential factor of ρ.

(iii$_1$) with $b=c=0$. Straightforward; see the bound for p in Theorem 5.1(iii$_3$). $\square$

Remark 5.3. The solutions (i$_1$) and (i$_6$) with different values of the parameter α (and in the latter case even the underlying domains) cannot be transformed to each other by any affine linear transformation. However, these solutions are also solutions of the weighted DOP problem (see [1], [7]) with weights $(1, 2, 3)$ and the $(1, 2, 3)$-admissible change of variables (see the definition in [7, §2.2])

$$(x, y, z) \mapsto \big(x,\, (x^2 - y)/\alpha,\, (2x^3 - 3xy + z)/(2\alpha^{3/2})\big)$$

transforms (i_1) and (i_6) into, respectively,

(i_1^*) $\Omega = \{y^3 > z^2\}$, $g = g_0$ (see below), and $\rho = (y^3 - z^2)^{p-1} e^{-\lambda\alpha(x^2+y)}$;

(i_6^*) Ω is the only bounded component of $\mathbb{R}^3 \setminus \{(y^3 - z^2)(1 - x^2 - y) = 0\}$, $g = g_0 - g_1$, $\rho = (y^3 - z^2)^{p-1}(1 - x^2 - y)^{q-1}$, where

$$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4y & 6z \\ 0 & 6z & 9y^2 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} x^2 & 2xy & 3xz \\ 2xy & 4y^2 & 6yz \\ 3xz & 6yz & 9z^2 \end{pmatrix}$$

(g_1 is the coefficient of e in the matrix in Lemma 4.3). Thus we have one-parameter families of pairwise non-equivalent solutions of the DOP problem such that the members of each family become equivalent to each other when they are considered as solutions of the weighted DOP problem with suitable weights. The same phenomenon was observed in dimension 2 in [3, §4.5], [7, Remark 6.5].

5.3. Comments on the figures. In Figures 1–5 we show the bounded domains appearing in Theorem 5.1 and the domain discussed in §6.5. All the domains are curvilinear polyhedra (all but one being tetrahedra), so we present them by the planar projections of their edges. When the axes are not shown, the projection is assumed to be $(x, y, z) \mapsto (x, y)$ (or $(x, y, z) \mapsto (x, z)$ in Figure 5 on the right). The number n near an edge means that the surface is given by the equation $v^2 = u^n$ in some local curvilinear coordinates (u, v, w) in a neighbourhood of this edge. In all the cases the metric $(g_{ij}) = g^{-1}$ is of constant non-negative curvature and the boundary of Ω is totally geodesic (which well agrees with Soukhanov's results [11], [12]). Thus Ω can be identified with the quotient of $\mathbb{R}^3$ or $\mathbb{S}^3$ by a group generated by reflections (a Coxeter group) and $\mathbf{L}$ is the image of the Laplace operator. The types of the groups are indicated in the figure captions. The explicit formulas for these identifications are given in §6. Note that if an edge of Ω is marked by a number n, then the angle at the corresponding edge of the fundamental polyhedron in $\mathbb{R}^3$ or $\mathbb{S}^3$ is π/n. For affine Coxeter groups (Figures 3 and 4) we also present the fundamental tetrahedra and the corresponding Coxeter graphs. Notice also that the curves on the right hand sides of Figures 3 and 5 are $(4, 1)$- and $(3, 1)$-hypocycloids.

6. Higher dimensional solutions of the DOP problem on the quotients of $\mathbb{S}^n$ or $\mathbb{R}^n$ by Coxeter groups

Using the approach from [3, §4], in this section we realize each solution from Theorem 5.1 as an image of the Laplace operator on $\mathbb{S}^3$ or $\mathbb{R}^3$ through the quotient by a discrete group generated by reflections (a Coxeter group). Moreover, we include each of these solutions into an infinite series of solutions in all dimensions.

6.1. Generalities. With a second order differential operator $\mathbf{L}$ with no 0-order term on a manifold M, is associated the operator "carré du champ"

$$\mathbf{\Gamma_L}(f_1, f_2) = \frac{1}{2}\Big(\mathbf{L}(f_1 f_2) - f_1\mathbf{L}(f_2) - f_2\mathbf{L}(f_1)\Big).$$

(see [2]). Notice that the operator $\mathbf{\Gamma_\Delta}$ (for the Laplace operator on $\mathbb{R}^n$) plays a key rôle in [10] where it is denoted by $\langle df_1, df_2\rangle$. If $\mathbf{L}$ is given by (1) in some coordinates $(x_1,\dots,x_n)$, then $g^{ij} = \Gamma_{\mathbf{L}}(x_i,x_j)$ and $b^i = \mathbf{L}(x_i)$. Let $\mathbf{f} : M \to \mathbb{R}^n$, $p \mapsto (f_1(p),\dots,f_n(p))$ be a mapping such that $\Gamma_{\mathbf{L}}(f_i,f_j) = G^{ij}\circ\mathbf{f}$ and $\mathbf{L}(f_i) = B^i\circ\mathbf{f}$ for some functions G^{ij} and B^i defined on $\mathbf{f}(M)$. Then a direct computation shows that the operator

$$\mathbf{f}_*(\mathbf{L}) = \sum_{i,j} G^{ij}\partial_{ij} + \sum_i B^i\partial_i$$

is such that $\mathbf{f}_*(\mathbf{L})(\varphi) = \mathbf{L}(\varphi\circ\mathbf{f})$ for any smooth $\varphi : \mathbf{f}(M) \to \mathbb{R}$. We say that $\mathbf{L}_*(\mathbf{f})$ is the *image of* $\mathbf{L}$ *through* $\mathbf{f}$.

Let G be a discrete group generated by orthogonal reflections acting on $\mathbb{R}^n$ (see [4] for a general introduction to the subject). We discuss here only bounded solutions of the DOP problem. Therefore, when G is finite (a *spherical Coxeter group* or just *Coxeter group*), we assume that the origin is a fixed point and we restrict the action form $\mathbb{R}^n$ to the unit sphere $\mathbb{S}^{n-1}$. If G is infinite (an *affine Coxeter group*), we assume that it contains a full rank subgroup of translations. So, in both cases the orbit space M/G is compact (M is $\mathbb{R}^n$ or $\mathbb{S}^{n-1}$).

If G is finite, it is known (see [5], [4, Ch. V, §§5–6]) that the ring of invariant polynomials is freely generated by some invariant homogeneous forms $I_1,\dots,I_n$. The choice of the invariants I_j's is not unique (see, e.g., [6], [10] for different concrete choices) but their degrees $d_1,\dots,d_n$ are uniquely determined. These numbers (called *exponents* in [4]) for each Coxeter group can be found in Tables (Planches) I–X in [4]. One of the basic invariants is (if the action is irreducible) or can be chosen to be $x_1^2+\dots+x_n^2$. Let it be I_1. Then [3, Eq. (4.5)] implies that the image of the Laplace operator $\mathbf{\Delta}_{\mathbb{S}^{n-1}}$ for $\mathbf{f} : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$, $p \mapsto (I_2(p),\dots,I_n(p))$, is a solution of the weighted DOP problem (see [1], [7] for the definition) with weights $(d_2,\dots,d_n)$ on $\mathbf{f}(\mathbb{S}^{n-1})$. However, in §§6.2–6.8 we show that for the Coxeter groups of types A_n, B_n, and their direct products, as well as for D_4, one can choose the basic invariants so that the image of the Laplace operator is a solution of the DOP problem (with weights $(1,\dots,1)$).

Consider now the case when G is an affine Coxeter group acting on $E = \mathbb{R}^n$. It is shown in [4, Ch. VI, §3.4] that the ring of invariant Fourier polynomials is freely generated by certain elements $f_1,\dots,f_n$ which are explicitly described via the fundamental weights $\omega_1,\dots,\omega_n$ corresponding to some Weyl chamber C. One can check that the image of $\mathbf{\Delta}_E$ through $p \mapsto (f_1(p),\dots,f_n(p))$ is a solution of the weighted DOP problem with the weights $\alpha(\omega_1),\dots,\alpha(\omega_n)$ where α is any linear function positive on C. In §§6.9–6.11 we show that these are also solutions of the DOP problem (with weights $(1,\dots,1)$) for the affine Coxeter groups of types $\tilde{A}_n$ and $\tilde{C}_n$. It seems plausible that the quotients by other spherical or affine Coxeter groups never give a solution of the DOP problem. In dimension 2 this fact follows from the classification in [3].

For each solution (Ω, g, ρ) obtained as the image of a Laplace operator through the quotient by a Coxeter group, $\mathbf{L}$ is the Laplace-Beltrami operator for the metric g^{-1}, hence $\rho = (\det g)^{-1/2}$.

6.2. Quotient of $\mathbb{S}^{n-2}$ by the Coxeter group A_{n-1}.

Let $E = \mathbb{R}^n$ with coordinates $x_1,\dots,x_n$, let $H \subset E$ be the hyperplane $x_1+\dots+x_n = 0$, and $\mathbb{S}^{n-2}$ be the unit sphere in H. The Coxeter group A_{n-1} acting on $\mathbb{S}^{n-2}$ is generated by the orthogonal reflections in the hyperplanes $x_i = x_j$. The ring of invariants is freely generated by the elementary symmetric polynomials $s_2, \dots, s_n$. So, we consider the mapping $\Phi : \mathbb{S}^{n-2} \to \mathbb{R}^{n-2}$, $(x_1, \dots, x_n) \mapsto (s_3, \dots, s_n)$ where $P(u) = (u+x_1)\dots(u+x_n) = \sum_{k=0}^n s_k u^{n-k}$. Note that $(s_0, s_1, s_2)|_{\mathbb{S}^{n-2}} = (1, 0, -\frac{1}{2})$ and we set $s_k = 0$ for $k \notin [0, n]$. Then $\Omega = \Phi(\mathbb{S}^{n-2})$ is bounded by the hypersurface

$$\operatorname{discr}_u(u^n - \tfrac{1}{2}u^{n-2} + X_3 u^{n-3} + \dots + X_{n-1}u + X_n) = 0.$$

(cf. Thm. 5.1(v)). Here $(X_3, \dots, X_n)$ are the coordinates in the target space $\mathbb{R}^{n-2}$. Let $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbb{S}^{n-2}}$ and let $\mathbf{\Gamma}$ be the corresponding carré du champ. We are going to check that $\Phi_*(\mathbf{\Delta})$ is a Laplace-Beltrami solution of the DOP problem on Ω. We have $\mathbf{\Gamma}(s_k, s_m) = \mathbf{\Gamma}_H(s_k, s_m) - km s_k s_m$ (see [3, Eq. (4.5)]) and $\mathbf{\Gamma}_H(s_k, s_m)$ is the coefficient of $u^{n-k}v^{n-m}$ in $\mathbf{\Gamma}_H(P(u), P(v))$. We have

$$\mathbf{\Delta}_E = \mathbf{\Delta}_H + \tfrac{1}{n}\partial_0^2 \quad \text{where} \quad \partial_0 = \sum \partial_i. \tag{14}$$

Hence $\mathbf{\Gamma}_H(f_1, f_2) = \mathbf{\Gamma}_E(f_1, f_2) - \frac{1}{n}(\partial_0 f_1)(\partial_0 f_2)$. It is clear that

$$\partial_0 P(u) = \sum_i \frac{P(u)}{u + x_i} = P'(u), \tag{15}$$

thus $\mathbf{\Gamma}_H(P(u), P(v)) = \mathbf{\Gamma}_E(P(u), P(v)) - \frac{1}{n}P'(u)P'(v)$. Finally, by [3, p. 1033],

$$\begin{aligned}
\mathbf{\Gamma}_E(P(u), P(v)) &= \sum_{i,j} \big(\partial_i P(u)\big)\big(\partial_j P(v)\big)\mathbf{\Gamma}_E(x_i, x_j) = \sum_i \big(\partial_i P(u)\big)\big(\partial_i P(v)\big) \\
&= \sum_i \frac{P(u)P(v)}{(u+x_i)(v+x_i)} = \frac{P(u)P(v)}{v-u}\sum_i \Big(\frac{1}{u+x_i} - \frac{1}{v+x_i}\Big) \\
&\overset{(15)}{=} \frac{P'(u)P(v) - P'(v)P(u)}{v-u} = \sum_{k,m}(n-k)s_k s_m \frac{u^{n-k-1}v^{n-m} - v^{n-k-1}u^{n-m}}{v-u} \\
&= \sum_{k,m}(n-k)s_k s_m \left(\sum_{l=0}^{k-m} u^{n-k+l-1}v^{n-m-l-1} - \sum_{l=1}^{m-k-1} u^{n-k-l-1}v^{n-m+l-1}\right),
\end{aligned}$$

hence for $a \le b$ we have

$$\mathbf{\Gamma}(s_a, s_b) = (a-1)(1 - \tfrac{b-1}{n})s_{a-1}s_{b-1} - ab s_a s_b + \sum_{l \ge 1}(a - b - 2l)s_{a-l-1}s_{b+l-1}.$$

Thus the coefficients g^{ab}, $a \le b$, of $\Phi_*(\mathbf{\Delta})$ are given by the same expression where $s_0, s_1, \dots, s_n$ are replaced by $1, 0, -\frac{1}{2}, X_3, \dots, X_n$ and s_k is set to zero when $k \notin [0, n]$. Here $X_3, \dots, X_n$ are coordinates in the target space $\mathbb{R}^{n-2}$.

By [3, Eq. (4.5)] we have $\mathbf{\Delta}(s_a) = \mathbf{\Delta}_H(s_a) - a(n+a-3)s_a$. Counting the number of monomials, one obtains

$$\partial_0(s_a) = (n - a + 1)s_{a-1}. \tag{16}$$

These formulae combined with (14) and with $\mathbf{\Delta}_E(s_a) = 0$ yield

$$\mathbf{\Delta}(s_a) = -\tfrac{1}{n}(n - a + 1)(n - a + 2)s_{a-2} - a(n + a - 3)s_a. \tag{17}$$

6.3. Quotient of $\mathbb{S}^{n-1}$ by the Coxeter group B_n.

Let E be $\mathbb{R}^n$ with coordinates $x_1, \dots, x_n$ and $\mathbb{S}^{n-1}$ be the unit sphere in E. The Coxeter group B_n acting on E is generated by the reflections in the hyperplanes $x_i = x_j$ and $x_i = 0$. The ring of polynomial invariants is generated by the elementary symmetric polynomials in x_i^2. We consider the mapping $\Phi : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$, $(x_1, \dots, x_n) \mapsto (s_2, \dots, s_n)$ where $P(u) = (u + t_1) \dots (u + t_n) = \sum_{k=0}^{n} s_k u^{n-k}$, $t_i = x_i^2$. We have $(s_0, s_1)|_{\mathbb{S}^{n-1}} = (1, 1)$ and we set $s_k = 0$ for $k \notin [0, n]$. Then $\Phi(\mathbb{S}^{n-1})$ is bounded by the hypersurface

$$X_n \operatorname{discr}_u(u^n + u^{n-1} + X_2 u^{n-2} + \dots + X_{n-1} u + X_n) = 0$$

(cf. Thm. 5.1(iii$_2$) and Figure 5). Its component $X_n = 0$ is the image of the hyperplanes $x_i = 0$ and the other component is the image of the planes $x_i = x_j$.

Let $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbb{S}^{n-1}}$ and let $\mathbf{\Gamma}$ be the corresponding carré du champ. $\mathbf{\Gamma}(s_k, s_m)$ is the coefficient of $u^{n-k} v^{n-m}$ in $\mathbf{\Gamma}(P(u), P(v))$. The s_k are homogeneous of degree $2k$, hence (see [3, Eq. (4.5)]) $\mathbf{\Gamma}(s_k, s_m) = \mathbf{\Gamma}_E(s_k, s_m) - 4km s_k s_m$ and

$$\mathbf{\Gamma}_E(t_i, t_j) = \mathbf{\Gamma}_E(x_i^2, x_j^2) = 4x_i x_j \mathbf{\Gamma}_E(x_i, x_j) = 4t_i \delta_{ij}.$$

Then (cf. [10, Prop. 2.2.2])

$$\frac{1}{4}\mathbf{\Gamma}_E\big(P(u), P(v)\big) = \frac{1}{4}\sum_{i,j} \big(\partial_{t_i} P(u)\big)\big(\partial_{t_j} P(v)\big)\mathbf{\Gamma}_E(t_i, t_j) = \sum_i \frac{t_i P(u) P(v)}{(u + t_i)(v + t_i)}$$

$$= \frac{P(u)P(v)}{u - v} \sum_i \Big(\frac{u}{u + t_i} - \frac{v}{v + t_i}\Big) = \frac{uP'(u)P(v) - vP'(v)P(u)}{u - v}$$

$$= \sum_{k,m} (n - k) s_k s_m \frac{u^{n-k} v^{n-m} - v^{n-k} u^{n-m}}{u - v}$$

$$= \sum_{k,m} (n - k) s_k s_m \left(\sum_{l=1}^{m-k} u^{n-k-l} v^{n-m+l-1} - \sum_{l=1}^{k-m} u^{n-k+l-1} v^{n-m-l} \right)$$

Hence for $a \le b$ we have

$$\mathbf{\Gamma}(s_a, s_b) = -4ab\, s_a s_b + \sum_{l \ge 1} 4(b - a + 2l - 1) s_{a-l} s_{b+l-1}. \tag{18}$$

The coefficients g^{ab}, $a \le b$, of $\Phi_*(\mathbf{\Delta})$ are given by the same expression where $s_0, s_1, \dots, s_n$ are replaced by $1, 1, X_2, \dots, X_n$ and s_k is set to zero when $k \notin [0, n]$.

We have $\mathbf{\Delta}(s_a) = \mathbf{\Delta}_E(s_a) - 2a(n + 2a - 2)s_a$ (see [3, Eq. (4.5)]). Similarly to (16) one obtains $\mathbf{\Delta}_E(s_a) = 2(n - a + 1)s_{a-1}$, hence

$$\mathbf{\Delta}(s_a) = 2(n - a + 1)s_{a-1} - 2a(n + 2a - 2)s_a. \tag{19}$$

6.4. Quotient of $\mathbb{S}^{n-1}$ by the Coxeter group B_n (another mapping).
In this subsection we compute $\Phi_*(\mathbf{\Delta}_{\mathbb{S}^{n-1}})$ for another polynomial mapping Φ invariant under the action of B_n. This time $\Omega = \Phi(\mathbb{S}^{n-1})$ is bounded by

$$\{X = (X_2,\dots,X_n) \mid P_X(1)\,\mathrm{discr}_u P_X(u) = 0\}, \quad P_X(u) = u^n + \sum_{k=2}^n X_k u^{n-k}.$$

Up to rescaling of the coordinates, we obtain the solution in Thm. 5.1(iii$_2$) (see Figure 5) when $n = 4$, and the solution in [3, §4.10] when $n = 3$.

The mapping $\Phi : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ is given by $(x_1,\dots,x_n) \mapsto (s_2,\dots,s_n)$ where $P(u) = (u+t_1)\dots(u+t_n) = \sum_{k=0}^n s_k u^{n-k}$, $t_i = nx_i^2 - 1$. It factors through $\Phi_1 : (x_2,\dots,x_n) \mapsto (t_2,\dots,t_n)$ and $\Phi_1(\mathbb{S}^{n-1})$ is the $(n-1)$-simplex σ given by $\sum t_i = 0$, $t_i \ge -1$. The image of $\partial\sigma$ is the hyperplane $P_X(1) = 0$, and the image of the $(n-2)$-planes $\sigma \cap \{t_i = t_j\}$ is the discriminantal hypersurface. This solution is obtained from the one in §6.3 by the change of variable $u \mapsto u - \frac{1}{n}$ which corresponds to an evident affine linear transformation in the coefficient space. It seems, however, that it is easier to recompute $\Phi_*(\mathbf{\Delta})$ rather than to perform this change of variables. Let us do it. By linearity, $\mathbf{\Gamma}(s_k,s_m)$ is the coefficient of $u^{n-k}v^{n-m}$ in $\mathbf{\Gamma}(P(u),P(v))$. We have (see [3, Eq. (4.2)])

$$\mathbf{\Gamma}(t_i,t_j) = \mathbf{\Gamma}(nx_i^2, nx_j^2) = 4n^2 x_i x_j(\delta_{ij} - x_i x_j) = 4n^2\delta_{ij}x_i^2 - 4n^2 x_i^2 x_j^2.$$

Hence

$$\begin{aligned}
\frac{\mathbf{\Gamma}(P(u),P(v))}{4n^2} &= \sum_{i,j} \big(\partial_{t_i}P(u)\big)\big(\partial_{t_j}P(v)\big)\frac{\mathbf{\Gamma}(t_i,t_j)}{4n^2} \\
= \sum_{i,j} \frac{P(u)P(v)\mathbf{\Gamma}(t_i,t_j)}{4n^2(u+t_i)(v+t_j)} &= \sum_i \frac{P(u)P(v)x_i^2}{(u+t_i)(v+t_i)} - \sum_{i,j}\frac{P(u)P(v)x_i^2x_j^2}{(u+t_i)(v+t_j)} \\
&= \frac{P(u)P(v)x_i^2}{v-u}\sum_i\Big(\frac{1}{u+t_i} - \frac{1}{v+t_i}\Big) - \sum_{i,j}\frac{P(u)x_i^2P(v)x_j^2}{(u+t_i)(v+t_j)} \\
&= \frac{Q(u)P(v) - Q(v)P(u)}{v-u} - Q(u)Q(v)
\end{aligned}$$

where

$$\begin{aligned}
Q(u) &= \sum_i \frac{P(u)x_i^2}{u+t_i} = \frac{1}{n}\sum_i \frac{P(u)(t_i+1)}{u+t_i} = \frac{1}{n}\sum_i\Big(P(u) - \frac{(u-1)P(u)}{u+t_i}\Big) \\
&= P(u) - \frac{1}{n}(u-1)P'(u) = \frac{1}{n}\sum_k s_k\Big(ku^{n-k} + (n-k)u^{n-k-1}\Big)
\end{aligned}$$

Thus $\big(Q(u)P(v) - Q(v)P(u)\big)/(v-u)$ is equal to

$$\begin{aligned}
&\sum_{k,m}\frac{s_ks_m}{n}\bigg(k\,\frac{u^{n-k}v^{n-m} - v^{n-k}u^{n-m}}{v-u} + (n-k)\frac{u^{n-k-1}v^{n-m} - v^{n-k-1}u^{n-m}}{v-u}\bigg) \\
&= \sum_{k,m}\frac{s_ks_m}{n}\Bigg\{k\left(\sum_{l=1}^{k-m}u^{n-k+l-1}v^{n-m-l} - \sum_{l=1}^{m-k}u^{n-k-l}v^{n-m+l-1}\right) \\
&\qquad + (n-k)\left(\sum_{l=0}^{k-m}u^{n-k+l-1}v^{n-m-l-1} - \sum_{l=1}^{m-k-1}u^{n-k-l-1}v^{n-m+l-1}\right)\Bigg\}
\end{aligned}$$

If $a \le b$, then

$$\tfrac14\Gamma(s_a,s_b)=\sum_{l\ge1} n(a-b-2l)s_{a-l-1}s_{b+l-1}+\sum_{l\ge1} n(b-a+2l-1)s_{a-l}s_{b+l-1}$$
$$+\,n(n-b+1)s_{a-1}s_{b-1}-\Big(as_a+(n-a+1)s_{a-1}\Big)\Big(bs_b+(n-b+1)s_{b-1}\Big).$$

The coefficients g^{ab}, $a \le b$, of $\Phi_*(\mathbf{\Delta})$ are given by the same expression where $s_0, s_1, \dots, s_n$ are replaced by $1, 0, X_2, \dots, X_n$ and s_k is set to zero when $k \notin [0, n]$.

We have $\mathbf{\Delta} = \mathbf{\Delta}_E - (r\partial_r)^2 - (n-2)r\partial_r$ where $r\partial_r = \sum x_i\partial_{x_i}$ (see [3, Eq. (4.4)]) and $\mathbf{\Delta}_E P = 2nP'$, $r\partial_r P = 2nP + 2(1-u)P'$, hence

$$(r\partial_r)^2P = 4n^2P + 4(2n-1)(1-u)P' + 4(1-u)^2P'',$$

$$\mathbf{\Delta}P = 2nP' - 2(3n^2-n)P - 2(5n-3)(1-u)P' - 4(1-u)^2P'',$$

and we obtain

$$\mathbf{\Delta}s_a = 2a(2-2a-n)s_a + 8(n-a+1)(1-a)s_{a-1} - 4(n-a+1)(n-a+2)s_{a-2}.$$

6.5. Quotient of $\mathbb{S}^{n-1}$ by the Coxeter group D_n. Let the notation be as in §6.3. The Coxeter group D_n acting on E is generated by the reflections in the hyperplanes $x_i \pm x_j = 0$. The ring of polynomial invariants is generated by $s_1, \dots, s_{n-1}$ and $\hat s_n = \sqrt{s_n} = x_1 \dots x_n$. The values of $\mathbf{\Gamma}(s_a, s_b)$ and $\mathbf{\Delta}(s_a)$ are already computed in §6.3, and we have (recall that $(s_0, s_1)|_{\mathbb{S}^{n-1}} = (1,1)$)

$$\begin{aligned}
\mathbf{\Gamma}(s_a,\hat s_n) &= \mathbf{\Gamma}(s_a, s_n^{1/2}) = \tfrac12 s_n^{-1/2}\mathbf{\Gamma}(s_a,s_n) \overset{(18)}{=} -2ans_a\hat s_n + 2(n-a+1)s_{a-1}\hat s_n,\\
\mathbf{\Gamma}(\hat s_n,\hat s_n) &= \mathbf{\Gamma}(s_n^{1/2}, s_n^{1/2}) = \tfrac14 s_n^{-1}\mathbf{\Gamma}(s_n,s_n) \overset{(18)}{=} -n^2\hat s_n^2 + s_{n-1},\\
\mathbf{\Delta}(\hat s_n) &= \mathbf{\Delta}_E(\hat s_n) - 2n(n-1)\hat s_n = -2n(n-1)\hat s_n \qquad \text{(by [3, Eq. (4.5)])}.
\end{aligned}$$

Thus, for a given n, the image of $\mathbf{\Delta}$ is a solution of the DOP problem if and only if, for any $a, b < n$, $\mathbf{\Gamma}(s_a, s_b)$ does not contain any monomial of the form $s_k s_n$ with $2 \le k \le n$. This is the case for $n = 4$. The corresponding matrix g is

$$\begin{pmatrix}
-16x^2+4x+12y & -24xy+8y+16z^2 & -16xz+6z\\
-24xy+8y+16z^2 & -36y^2+4xy+12z^2 & -24yz+4xz\\
-16xz+6z & -24yz+4xz & -16z^2+y
\end{pmatrix}$$

The projection of the cuspidal edge of the surface $\deg g = 0$ onto the xz-plane is the deltoid (see Figure 5 and [3, §4.12]) up to affine transformation of $\mathbb{R}^2$.

If $n \ge 5$, then $\Phi_*(\mathbf{\Delta})$ is not a solution of the DOP problem because, for example,

$$\mathbf{\Gamma}(s_4, s_{n-1}) = -16(n-1)s_4s_{n-1} + 4(n-4)s_3s_{n-1} + 4(n-2)s_2\hat s_n^2$$

has a monomial of degree 3. However, for any n, it is, evidently, a solution of the weighted DOP problem with weights $(1, \dots, 1, \frac12)$.

6.6. Direct products of Coxeter groups.

Let finite Coxeter groups G_α, $\alpha = 1,\dots,m$, act on vector spaces E_α, $\dim E_\alpha = n_\alpha$. We assume that these representations are irreducible or trivial. Consider the diagonal action of $G = G_1 \times \cdots \times G_m$ on $E = \bigoplus_\alpha E_\alpha$. Let $n = \sum_\alpha n_\alpha$. We denote the Laplace operator and the corresponding "carré du champ" on the unit sphere in E (resp. in E_α) by $\mathbf{\Delta}$ and $\mathbf{\Gamma}$ (resp. by $\mathbf{\Delta}_\alpha$ and $\mathbf{\Gamma}_\alpha$). Let $I_{\alpha,k}$, $k = 1,\dots,n_\alpha$, be sets of basic invariant homogeneous polynomials for the respective group actions, $d_{\alpha,k} = \deg I_{\alpha,k}$. We assume that $d_{\alpha,1}$ is minimal among the $d_{\alpha,k}$'s. Then $d_{\alpha,1} = 2$ unless G_α is trivial.

Let $g_\alpha^{ij}\big(x_{\alpha,1},\dots,x_{\alpha,n_\alpha}\big)$ and $b_\alpha^i\big(x_{\alpha,1},\dots,x_{\alpha,n_\alpha}\big)$ be the polynomials such that

$$\mathbf{\Gamma}_\alpha(I_{\alpha,i}, I_{\alpha,j}) = g_\alpha^{ij}\big(I_{\alpha,1},\dots,I_{\alpha,n_\alpha}\big), \qquad \mathbf{\Delta}_\alpha(I_{\alpha,i}) = b_\alpha^i\big(I_{\alpha,1},\dots,I_{\alpha,n_\alpha}\big).$$

We assume that $\deg g_\alpha^{ij} \le 2$ and $b_\alpha^i \le 1$ for any $i, j, \alpha \ge 1$ (notice that this condition is fulfilled for A_n and B_n, but not for D_4; see §§6.2–6.5).

First construction. Suppose that $d_{1,1} = \cdots = d_{m,1} = 2$, i.e. all the $I_{\alpha,1}$ are positive definite quadratic forms. Let $\mathbb{S}^{n-1}$ be the sphere in E given by the equation $\sum_\alpha I_{\alpha,1} = 0$ and let $\Phi : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ be the mapping defined by $p \mapsto \big(\tilde I_1(p), I_2(p),\dots,I_m(p)\big)$, where $I_\alpha = (I_{\alpha,1},\dots,I_{\alpha,n_\alpha})$ and $\tilde I_1 = (I_{1,2},\dots,I_{1,n_1})$. It is easy to see that the image of $\mathbf{\Delta}$ through Φ is a solution of the DOP problem. Denote the coordinates in the target space $\mathbb{R}^{n-1}$ by $(\tilde{\mathbf{x}}_1, \mathbf{x}_2,\dots,\mathbf{x}_m)$ where $\mathbf{x}_\alpha = (x_{\alpha,1},\dots,x_{\alpha,n_\alpha})$ and $\tilde{\mathbf{x}}_\alpha = (x_{\alpha,2},\dots,x_{\alpha,n_\alpha})$. Then the corresponding matrix g is the block matrix $(g_{\alpha\beta})_{\alpha,\beta=1}^m$ with the block dimensions $(n_1 - 1, n_2,\dots,n_m)$ and the blocks $g_{\alpha\beta} = \big(g_{\alpha\beta}^{ij}(\tilde{\mathbf{x}}_1,\mathbf{x}_2,\dots,\mathbf{x}_m)\big)_{i,j}$ defined by

$$g_{\alpha\beta}^{ij} = \begin{cases} g_1^{ij}(1 - x_{2,1} - \cdots - x_{m,1}, \tilde{\mathbf{x}}_1), & \alpha = \beta = 1,\\ g_\alpha^{ij}(\mathbf{x}_\alpha), & \alpha = \beta \ge 2,\\ -d_{\alpha,i}\, d_{\beta,j}\, x_{\alpha,i}\, x_{\alpha,j}, & \alpha \ne \beta. \end{cases}$$

Up to affine linear change of coordinates, $\Phi_*(\mathbf{\Delta})$ does not depend on the order of the summands. For example, if we exchange E_1 and E_2, then the resulting solution is obtained from the initial one by the affine linear change of coordinates

$$(\tilde{\mathbf{x}}_1, \mathbf{x}_2,\dots,\mathbf{x}_m) \mapsto (\tilde{\mathbf{x}}_2,\ 1 - x_{2,1} - \cdots - x_{m,1}, \tilde{\mathbf{x}}_1,\ \mathbf{x}_3,\dots,\mathbf{x}_m).$$

Second construction. Suppose now that G_1 is trivial, $d_{2,1} = \cdots = d_{m,1} = 2$, and $\dim E_m = 1$. Then $I_{1,1},\dots,I_{1,n_1}$ are just linear coordinates on E_1 and $d_{1,1} = \cdots = d_{1,n_1} = 1$. Let $\mathbb{S}^{n-1}$ be the sphere in E given by the equation $\sum_i I_{1,i}^2 + \sum_{\alpha\ge 2} I_{\alpha,1} = 0$ and let $\Phi : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ be the mapping defined by $p \mapsto \big(I_1(p),\dots,I_{m-1}(p)\big)$, where $I_\alpha = (I_{\alpha,1},\dots,I_{\alpha,n_\alpha})$. Then $\Phi_*(\mathbf{\Delta})$ is a solution of the DOP problem. Denote the coordinates in the target space $\mathbb{R}^{n-1}$ by $(\mathbf{x}_1,\dots,\mathbf{x}_{m-1})$ where $\mathbf{x}_\alpha = (x_{\alpha,1},\dots,x_{\alpha,n_\alpha})$. Then the corresponding matrix g is the block matrix $\big(g(\alpha,\beta)\big)_{\alpha,\beta=1}^m$ with the block dimensions $(n_1, n_2,\dots,n_{m-1})$ and with the blocks $g(\alpha,\beta) = \big(g_{\alpha\beta}^{ij}\big)_{i,j}$ defined by

$$g_{\alpha\beta}^{ij} = \begin{cases} \delta^{ij} - x_{1,i}\, x_{1,j}, & \alpha = \beta = 1,\\ g_\alpha^{ij}(\mathbf{x}_\alpha), & 2 \le \alpha = \beta \le m-1,\\ -d_{\alpha,i}\, d_{\beta,j}\, x_{\alpha,i}\, x_{\alpha,j}, & \alpha \ne \beta. \end{cases}$$

6.7. Quotient of $\mathbb{S}^{n-1}$ and $\mathbb{S}^n$ by the Coxeter group $A_1 + A_{n-1}$.

Let the notation be as in §6.2. Let $\mathbb{S}^{n-1}$ be the unit sphere in $H_+ = \mathbb{R} \oplus H \subset \mathbb{R} \oplus E$ (we denote the coordinate on $\mathbb{R}$ by x_0). Let $\mathbf{\Delta}_+$ be the Laplace operator on $\mathbb{S}^{n-1}$. Consider the product G of the Coxeter groups A_1 and A_{n-1} diagonally acting on H_+. According to §6.6 (first construction), the image of $\mathbf{\Delta}_+$ through the mapping $\Phi_+ : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$, $(x_0, x_1, \dots, x_n) \mapsto (s_2, \dots, s_n)$, provides a solution of the DOP problem on the domain $\Phi_+(\mathbb{S}^{n-1})$, which is bounded by the hypersurface

$$(1 + 2X_2)F = 0, \qquad F = \operatorname{discr}_u\big(u^n + X_2 u^{n-2} + \dots + X_{n-1}u + X_n\big) = 0. \quad (20)$$

Its component $1 + 2X_2 = 0$ is the image of $H \cap \mathbb{S}^{n-1}$. For $n = 4$, this is the solution in Thm. 5.1(iii$_3$) (see Figure 1) up to rescaling of the coordinates. The entries of the matrix g are given by the formulas in §6.3 with $s_0, s_1, \dots, s_n$ replaced by $1, 0, X_2, \dots, X_n$. We have $\mathbf{\Delta}_+(s_a) = \mathbf{\Delta}(s_a) - a s_a$ with $\mathbf{\Delta}(s_a)$ as in (17).

Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R} \oplus H_+$. We denote the newly added coordinate by $\hat{x}_0$. Extend the above action of G to $\mathbb{R} \oplus H_+$ assuming that it acts trivially on the first component. Consider the image of $\mathbf{\Delta}_{\mathbb{S}^n}$ through $(\hat{x}_0, x_0, x_1, \dots, x_n) \mapsto (\hat{x}_0, s_2, \dots, s_n)$. According to §6.6 (second construction), it gives a solution of the DOP problem in the domain in $\mathbb{R}^n$ with coordinates $(X_1, \dots, X_n)$ bounded by the hypersurface $(1 + 2X_2 - X_1^2)F = 0$; see (20). We have $g^{1b} = \delta^{1b} - bX_1X_b$, g^{ab} for $2 \le a \le b$ are as above, $\mathbf{\Delta}_{\mathbb{S}^n}(s_a) = \mathbf{\Delta}(s_a) - 2a s_a$ ($\mathbf{\Delta}(s_a)$ is as in (17)), and $\mathbf{\Delta}_{\mathbb{S}^n}(\hat{x}_0) = -n\hat{x}_0$ For $n = 3$ we obtain (i$_6^*$) in Remark 5.3 up to rescaling.

The solution (i$_6$) in Theorem 5.1 and its generalization for higher dimensions can be obtained as the image of $\mathbf{\Delta}_{\mathbb{S}^n}$ through a quotient by $A_1 + A_{n-1}$ using a more direct (and somewhat more natural) construction as follows. Let the notation still be as in §6.2. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R} \oplus E$. Consider the mapping $\mathbb{S}^n \to \mathbb{R}^n$, $(x_0, x_1, \dots, x_n) \mapsto (s_1, \dots, s_n)$. Its image is bounded by the hypersurface

$$(1 + 2X_2 - X_1^2) \operatorname{discr}_u\big(u^n + X_1 u^{n-1} + \dots + X_{n-1}u + X_n\big) = 0.$$

Using the computations in §6.2, for $1 \le a \le b \le n$, we obtain

$$g^{ab} = (n - b + 1)s_{a-1}s_{b-1} - ab s_a s_b + \sum_{l \ge 1}(a - b - 2l)s_{a-l-1}s_{b+l-1}$$

with $s_0, \dots, s_n$ replaced by $1, X_1, \dots, X_n$ and $s_k = 0$ for $k \notin [0, n]$. We have $\Delta_{\mathbb{S}^n}(s_a) = -a(n+a-1)s_a$. When $n = 3$, we obtain the solution in Theorem 5.1(i$_6$) with $\alpha = 1/2$ (see Figure 2) after rescaling $(x, y, z) = (3^{-1/2}X_1, X_2, 3^{3/2}X_3)$.

6.8. Quotient of $\mathbb{S}^n$ by the Coxeter group $A_1 + B_n$.

Let the notation be as in §6.3. Let $\mathbb{S}^n$ be the unit sphere in $E_+ = \mathbb{R} \oplus E$ and let $\mathbf{\Delta}_+$ be the Laplace operator on $\mathbb{S}^n$. We denote the coordinate on $\mathbb{R}$ by x_0 (recall that the coordinates on E are $x_1, \dots, x_n$). Consider the product of the Coxeter groups A_1 and B_n diagonally acting on E_+. It is generated by the reflections in the hyperplanes $x_i = 0$ $(0 \le i \le n)$ and $x_i = x_j$ $(1 \le i < j \le n)$.

According to §6.6 (first construction), the image of $\mathbf{\Delta}_+$ through the mapping $\Phi_+ : \mathbb{S}^n \to \mathbb{R}^n$, $(x_0, x_1, \dots, x_n) \mapsto (s_1, \dots, s_n)$, provides a solution of the DOP problem on the domain $\Phi_+(\mathbb{S}^n)$, which is bounded by the hypersurface

$$X_n(1 - X_1) \operatorname{discr}_u\big(u^n + X_1 u^{n-1} + \dots + X_{n-1}u + X_n\big) = 0.$$

Its component $X_1 = 1$ is the image of $E \cap \mathbb{S}^n$. The other components are as in §6.3. In the case $n = 3$, this is the solution in Thm. 5.1(i$_4$) (see Figure 1) up to rescaling of the coordinates. The entries of g are as in §6.3, but with $s_0, s_1, \dots, s_n$ replaced by $1, X_1, \dots, X_n$; $\mathbf{\Delta}_+(s_a) = \mathbf{\Delta}(s_a) - 2a s_a$ (with $\mathbf{\Delta}(s_a)$ as in (19)).

6.9. Quotient of $\mathbb{R}^n$ by the affine Coxeter group $\widetilde{C}_n$.

Let $E = \mathbb{R}^n$ with coordinates $\theta_1, \dots, \theta_n$. The ring of invariant Fourier polynomials for the affine Coxeter group $\widetilde{B}_{n-1}$ is freely generated by $s_1, \dots, s_n$ where $P(u) = (u+t_1)\dots(u+t_n) = \sum_{k=0}^n s_k u^{n-k}$, $t_i = \cos\theta_i$. We consider the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $(\theta_1, \dots, \theta_n) \mapsto (s_1, \dots, s_n)$. Its image Ω is the set of all n-tuples $X = (X_1, \dots, X_n)$ such that all roots of the polynomial $P_X(u) = u^n + \sum_{k=1}^n X_k u^{n-k}$ are real and belong to the interval $[-1, 1]$. Therefore Ω is bounded by the union of the hypersurface $\{X \mid \mathrm{discr}_u P_X(u) = 0\}$ and two hyperplanes $\{X \mid P_X(\pm 1) = 0\}$. When the point X moves from Ω crossing the discriminantal hypersurface, two real roots disappears. When it crosses the hyperplane $X = \pm 1$, one of the roots gets out from the interval $[-1, 1]$. One easily checks that Ω is the only bounded component of the complement (cf. Thm. 5.1(i$_5$), Figure 4, §5.3; for $n = 2$ see [3, §4.7]).

By linearity, $\mathbf{\Gamma}(s_k, s_m)$ is the coefficient of $u^{n-k}v^{n-m}$ in $\mathbf{\Gamma}(P(u), P(v))$. We have

$$\mathbf{\Gamma}(t_i, t_j) = \mathbf{\Gamma}(\cos\theta_i, \cos\theta_j) = \delta_{ij}\sin\theta_i \sin\theta_j = \delta_{ij}(1 - t_i^2).$$

Hence

$$\mathbf{\Gamma}\big(P(u), P(v)\big) = \sum_{i,j} \big(\partial_{t_i} P(u)\big)\big(\partial_{t_j} P(v)\big)\mathbf{\Gamma}(t_i, t_j) = \sum_i \frac{P(u)P(v)(1 - t_i^2)}{(u + t_i)(v + t_i)}$$

$$= \frac{P(u)P(v)}{v - u} \sum_i \Big(\frac{1 + ut_i}{u + t_i} - \frac{1 + vt_i}{v + t_i}\Big) = \frac{Q(u)P(v) - Q(v)P(u)}{v - u}$$

where

$$Q(u) = \sum_i \frac{P(u)(1 + ut_i)}{u + t_i} = \sum_i P(u)\Big(u + \frac{1 - u^2}{u + t_i}\Big)$$
$$= nuP(u) + (1 - u^2)P'(u) = \sum_k s_k\Big(ku^{n-k+1} + (n-k)u^{n-k-1}\Big),$$

thus $\mathbf{\Gamma}\big(P(u), P(v)\big)$ is equal to

$$\sum_{k,m} s_k s_m \Bigg(k\frac{u^{n-k+1}v^{n-m} - v^{n-k+1}u^{n-m}}{v - u} + (n-k)\frac{u^{n-k-1}v^{n-m} - v^{n-k-1}u^{n-m}}{v - u}\Bigg)$$
$$= \sum_{k,m} s_k s_m \Bigg\{ k\left(\sum_{l=1}^{k-m} u^{n-k+l}v^{n-m-l} - \sum_{l=0}^{m-k} u^{n-k-l}v^{n-m+l}\right)$$
$$+ (n-k)\left(\sum_{l=0}^{k-m} u^{n-k+l-1}v^{n-m-l-1} - \sum_{l=1}^{m-k-1} u^{n-k-l-1}v^{n-m+l-1}\right)\Bigg\}.$$

Hence, for $a \le b$, we have

$$\mathbf{\Gamma}(s_a, s_b) = (n - b + 1)s_{a-1}s_{b-1} - a s_a s_b + \sum_{l \ge 1}(b - a + 2l)(s_{a-l}s_{b+l} - s_{a-l-1}s_{b+l-1})$$

It is easy to see that $\mathbf{\Delta}(s_a) = -a s_a$.

6.10. Quotients of $\mathbb{R}^n$ by the affine Coxeter groups $\widetilde{B}_n$ and $\widetilde{D}_n$.

Let the notation be as in §6.9. The ring of invariant Fourier polynomials for the affine Coxeter group $\widetilde{B}_n$ is freely generated by $s_1,\dots,s_{n-1}$ and

$$\hat{s}_n = \prod_{i=1}^n \sqrt{2}\cos(\theta_i/2) = \prod_{i=1}^n \sqrt{1+\cos\theta_i} = P(1)^{1/2}.$$

$\Gamma(s_a,s_b)$ for $a\le b\le n-1$ are as in §6.9 with the substitution $s_n = \hat{s}_n^2 - \sum_{k=0}^{n-1} s_k$ (recall that $s_0=1$). Using the computations in §6.9 we obtain

$$2\mathbf{\Gamma}(P,\hat{s}_n) = \frac{\mathbf{\Gamma}(P(u),P(1))}{P(1)^{1/2}} = \frac{Q(u)P(1)-Q(1)P(u)}{P(1)^{1/2}(1-u)} = \hat{s}_n\big((1+u)P'-nP\big),$$

hence $2\mathbf{\Gamma}(s_a,\hat{s}_n) = ((n-a+1)s_{a-1}-as_a)\hat{s}_n$ and

$$4\mathbf{\Gamma}(\hat{s}_n,\hat{s}_n) = \frac{\mathbf{\Gamma}(P(1),P(1))}{P(1)} = \frac{1}{P(1)}\sum_i \frac{P(1)^2(1-t_i^2)}{(1+t_i)^2} = \sum_i\left(\frac{2P(1)}{1+t_i}-P(1)\right)$$
$$= 2P'(1)-nP(1) = -n\hat{s}_n^2 + \sum_{k=0}^{n-1}(n-k)s_k.$$

We have $\mathbf{\Delta}(s_a) = -as_a$ and $\mathbf{\Delta}(\hat{s}_n) = -\frac{1}{4}n\hat{s}_n$.

The image of $\mathbf{\Delta}$ through $\Phi:\mathbb{R}^n\to\mathbb{R}^n$, $(\theta_1,\dots,\theta_n)\mapsto(s_1,\dots,s_{n-1},\hat{s}_n)$ is not a solution of the DOP problem when $n\ge 3$ (and $\widetilde{B}_2$ is the same as $\widetilde{C}_2$). Indeed, $\mathbf{\Gamma}(s_2,s_{n-1})$ has monomial $(n-1)s_1\hat{s}_n^2$ of degree 3. However, $\Phi_*(\mathbf{\Delta})$ is a solution of the weighted DOP problem with weights $(1,\dots,1,\frac12)$.

For the affine group $\widetilde{D}_n$, all the computations are almost the same and we omit the details. The ring of invariant Fourier polynomials is generated by $s_1,\dots,s_{n-2}$, $\hat{s}_n$, and $\hat{s}_{n-1} = \prod_{i=1}^n\sqrt{2}\sin(\theta/2) = \sqrt{(-1)^nP(-1)}$. For $a\le b\le n-2$, $\mathbf{\Gamma}(s_a,s_b)$, $\mathbf{\Gamma}(s_a,\hat{s}_n)$, and $\mathbf{\Gamma}(\hat{s}_n,\hat{s}_n)$ are the same as above but with the substitutions

$$s_n = \tfrac12\big(\hat{s}_n^2+(-1)^n\hat{s}_{n-1}^2\big) - \sum_{k\ge1}s_{n-2k},\qquad s_{n-1} = \tfrac12\big(\hat{s}_n^2+(-1)^n\hat{s}_{n-1}^2\big) - \sum_{k\ge1}s_{n-2k-1}.$$

The values of $\mathbf{\Gamma}(\hat{s}_{n-1},*)$ are computed similarly and we arrive to the same conclusion as above: the quotient by $\widetilde{D}_n$, $n\ge4$, does not provide a solution of the DOP problem, but it provides a solution of the weighted DOP problem with weights $(1,\dots,1,\frac12,\frac12)$.

6.11. Quotient of $\mathbb{R}^{n-1}$ by the affine Coxeter group $\widetilde{A}_{n-1}$.

Let E be $\mathbb{R}^n$ with coordinates $\theta_1,\dots,\theta_n$ and $H=\{\theta_1+\cdots+\theta_n=0\}$. The affine Coxeter group $\widetilde{A}_{n-1}$ acting on H is generated by the orthogonal reflections in the hyperplanes $x_i=x_j$ and a suitable translation. The ring of invariant Fourier polynomials is freely generated by $s_1,\dots,s_{n-1}$ where $P(u)=(u+t_1)\dots(u+t_n) = \sum_{k=0}^n s_ku^{n-k}$, $t_i=\exp(\mathbf{i}\,\theta_i)$, $\mathbf{i}=\sqrt{-1}$. Notice that $\bar{s}_k|_H = s_{n-k}|_H$, in particular $s_n|_H=1$ and $s_{n/2}|_H$ is real when n is even. We consider the mapping $\Phi: H\to\mathbb{R}^{n-1}$, $(\theta_1,\dots,\theta_n)\mapsto s=(s_1,\dots,s_{\lfloor n/2\rfloor})$ where we identify $\mathbb{R}^{n-1}$ with $\mathbb{C}^{(n-1)/2}$

(which we define as $\mathbb{C}^{(n-2)/2} \times \mathbb{R}$ when n is even). Then $\Phi(H)$ is bounded by the hypersurface $\mathrm{discr}_u P(u, Z) = 0$, $Z = (Z_1, \dots, Z_{\lfloor n/2 \rfloor}) \in \mathbb{C}^{(n-1)/2}$,

$$P(u,Z) = \begin{cases} u^{2k} + Z_1 u^{2k-1} + \dots + Z_k u^k + \bar{Z}_{k-1} u^{k-1} + \dots + \bar{Z}_1 u + 1, & n = 2k, \\ u^{2k+1} + Z_1 u^{2k} + \dots + Z_k u^k + \bar{Z}_k u^{k+1} + \dots + \bar{Z}_1 u + 1, & n = 2k+1 \end{cases}$$

(cf. Thm. 5.1(vi) and Figure 3; for $n = 2$ see [3, §4.12]). Let $\mathbf{\Delta} = \mathbf{\Delta}_H$ and let $\mathbf{\Gamma}$ be the associated carré du champ. Then $\mathbf{\Gamma}(s_k, s_m)$ is the coefficient of $u^{n-k} v^{n-m}$ in $\mathbf{\Gamma}(P(u), P(v))$. For any functions f, g we have $\mathbf{\Gamma}(f,g) = \mathbf{\Gamma}_E(f,g) - \frac{1}{n}(\partial_0 f)(\partial_0 g)$ where $\partial_0 = \sum_{i=1}^n \frac{\partial}{\partial \theta_i}$ (see (14)). Denote $\frac{\partial}{\partial t_i}$ by ∂_i. We have

$$\partial_0 P(u) = \sum_i \mathbf{i}\, t_i \partial_i P = \mathbf{i} \sum_i \frac{t_i P}{u + t_i} = \mathbf{i} \sum_i \Big(1 - \frac{u}{u+t_i}\Big) P = \mathbf{i}\big(nP(u) - uP'(u)\big),$$

thus $\mathbf{\Gamma}\big(P(u), P(v)\big) = \mathbf{\Gamma}_E\big(P(u), P(v)\big) + \frac{1}{n}\big(nP(u) - uP'(u)\big)\big(nP(v) - vP'(v)\big)$. We also have $\mathbf{\Gamma}_E(t_i, t_j) = (dt_i/d\theta_i)(dt_j/d\theta_j)\Gamma_E(\theta_i, \theta_j) = -\delta_{ij} t_i^2$. Then

$$-\mathbf{\Gamma}_E\big(P(u), P(v)\big) = -\sum_{i,j} \big(\partial_i P(u)\big)\big(\partial_j P(v)\big) \mathbf{\Gamma}_E(t_i, t_j) = \sum_i \frac{P(u)P(v) t_i^2}{(u+t_i)(v+t_i)}$$

$$= \frac{P(u)P(v)}{v-u} \sum_i \Big(v - u + \frac{u^2}{u+t_i} - \frac{v^2}{v+t_i}\Big) = nP(u)P(v) + \frac{u^2 P'(u) P(v) - v^2 P'(v) P(u)}{v-u}$$

$$= nP(u)P(v) + \sum_{k,m} (n-k) s_k s_m \frac{u^{n-k+1} v^{n-m} - v^{n-k+1} v^{n-m}}{v-u}$$

$$= nP(u)P(v) + \sum_{k,m} (n-k) s_k s_m \left(\sum_{l=1}^{k-m-1} u^{n-k+l} v^{n-m-l} - \sum_{l=0}^{m-k} u^{n-k-l} v^{n-m+l} \right)$$

Hence, for $a \le b$, we have

$$\mathbf{\Gamma}(s_a, s_b) = \frac{a(b-n)}{n} s_a s_b + \sum_{l \ge 1} (b - a + 2l) s_{a-l} s_{b+l}.$$

Setting $s_k = x_k + \mathbf{i} y_k$, $\mathbf{\Gamma}(s_a, s_b) = A + \mathbf{i}B$, and $\mathbf{\Gamma}(s_a, \bar{s}_b) = C + \mathbf{i}D$, we obtain for $a \le b \le n/2$:

$$2\mathbf{\Gamma}(x_a, x_b) = A + C,\ 2\mathbf{\Gamma}(x_a, y_b) = B - D,\ 2\mathbf{\Gamma}(y_a, x_b) = B + D,\ 2\mathbf{\Gamma}(y_a, y_b) = C - A,$$

$$A = \frac{a(b-n)}{n}(x_a x_b - y_a y_b) + \sum_{l \ge 1} (b - a + 2l)(x_{a-l} x_{b+l} - y_{a-l} y_{b+l}),$$

$$B = \frac{a(b-n)}{n}(x_a y_b + y_a x_b) + \sum_{l \ge 1} (b - a + 2l)(x_{a-l} y_{b+l} + y_{a-l} x_{b+l}),$$

$$C = -\frac{ab}{n}(x_a x_b + y_a y_b) + \sum_{l \ge 1} (n - a - b + 2l)(x_{a-l} x_{b-l} + y_{a-l} y_{b-l}),$$

$$D = \frac{ab}{n}(x_a y_b - y_a x_b) - \sum_{l \ge 1} (n - a - b + 2l)(x_{a-l} y_{b-l} - y_{a-l} x_{b-l}).$$

For $a \le n/2$ we have $\mathbf{\Delta}(x_a) = \lambda_a x_a$, $\mathbf{\Delta}(y_a) = \lambda_a y_a$, $\lambda_a = a(a-n)/n$.

7. Conical surfaces

Theorem 7.1. *Let (Ω, g, ρ) be a solution of the SDOP problem in $\mathbb{C}^3$ such that $\partial\Omega$ contains a relatively open subset of an irreducible conical surface Σ, i.e. of a surface $\Sigma = \{\Gamma(x,y,z) = 0\}$ where Γ is an irreducible homogeneous polynomial. Then* $\deg\Gamma \le 2$.

Remark 7.2. There exist solutions of the DOP problem in bounded domains whose boundaries contain a piece of the quadratic cone. For example, the solutions (1b), (1e), (3e), (3i), (5g), (6f) in [3, §7.2] (see Figure 7).

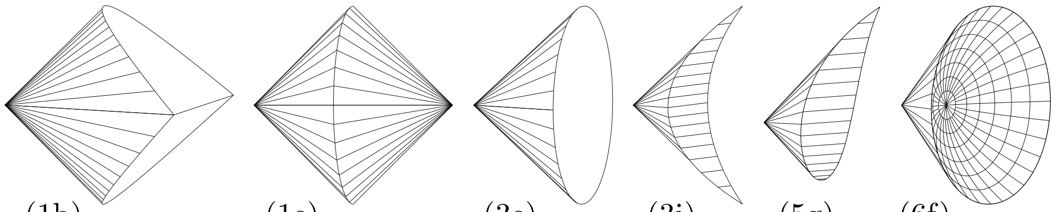

Figure 7. Bounded domains Ω from [3, §7.2] admitting solutions of the DOP problem, such that $\partial\Omega$ contains a piece of the quadratic cone.

Proposition 7.3. *Let (g, Γ) be a solution of the AlgDOP problem in $\mathbb{C}^3$ such that Γ is an irreducible homogeneous polynomial and* $\det g$ *is not a homogeneous polynomial of degree* 6. *Then* $\deg\Gamma \le 2$.

One can easily derive Theorem 7.1 from Proposition 7.3. Indeed, if $\det g$ (in the setting of Theorem 7.1) were a homogeneous polynomial of degree 6, then affine coordinates (x, y, z) could be chosen so that $\deg_x \det g = 6$ and Ω contains a half-cylinder $\{x > 0, y^2 + z^2 < 1\}$, which contradicts [3, Cor. 2.19].

The rest of this section is devoted to the proof of Proposition 7.3. Let Γ be as in Proposition 7.3. Let Σ be the surface in $\mathbb{C}^3$ defined by the equation $\Gamma = 0$, and let C be the curve in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ defined by the same equation. Any local branch γ of C has a parametrization of the form $t \mapsto (t^p, t^q + o(t^q))$, $1 \le p < q$, in some affine coordinates. We then say that γ is of type (p, q).

Lemma 7.4. *Any local branch of C is of type* $(1, 2)$ *or* $(2, 4)$.

Proof. The arguments are as in §3 but simpler. Let $\pi : \mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbb{P}^2$ be the quotient map (then $\Sigma = \pi^{-1}(C)$). Let γ be a local branch of C at $p \in \mathbb{CP}^2$ parametrized by $t \mapsto \gamma(t) = (\xi_1(t) : \xi_2(t) : \xi_3(t))$. Then Σ near the line $\pi^{-1}(p)$ is parametrized by $(t, u) \mapsto (u\xi_1(t), u\xi_2(t), u\xi_3(t))$. Similarly to §3, we rewrite the equations (3) in the form $E_1 = E_2 = E_3 = 0$ where

$$E_i = u \sum_{j=1}^{3} (\dot\xi_{j+1}\xi_{j-1} - \dot\xi_{j-1}\xi_{j+1}) g^{ij}(u\xi_1, u\xi_2, u\xi_3) = \sum_{\alpha=0}^{\infty} t^\alpha \sum_{\beta=1}^{3} E_{\alpha,\beta,i}\, u^\beta$$

(the indices $j \pm 1$ are considered mod 3) and the $E_{\alpha,\beta,i}$ are linear forms in g^{ij}_{klm} whose coefficients are polynomial functions of the coefficients of the ξ_i's.

We have $\deg C \le 5$ because otherwise $\det g$ would be homogeneous of degree 6. Hence C may have only local branches of type (p, q) with $q \le 5$. For each pair (p, q),

$1 \le p < q \le 5$, except $(1,2)$ and $(2,4)$ (thus for 8 pairs) we consider a branch γ of type (p,q) of the form $t \mapsto (1 : t^p : t^q + \sum_{k>q} a_k t^k)$ with indeterminate coefficients a_k and solve the maximal triangular subsystem of the system of equations $E_{\alpha,\beta,i} = 0$ for the unknowns g^{ij}_{klm}. This means that we find an equation implying that some unknown is zero, replace this unknown by zero in all other equations, and repeat this process as long as we can do. For all pairs (p,q) except $(1,4)$, $(1,5)$ we obtain that $\deg g$ is homogeneous of degree 6. In the two exceptional cases we obtain that z^2 divides $\det g$. Since $q \le \deg\Gamma$, this implies $\deg(z^2\Gamma) \ge 6$, hence $\det g = z^2\Gamma$ up to a scalar factor. This means that $\det g$ is homogeneous of degree 6. $\square$

Let d and $\mathbf{g}$ be the degree and the genus of C respectively. Let a_{2k}, $k \ge 2$, be the number of local branches of type $(2,4)$ which admit a parametrization $t \mapsto (t^2, t^{2k+1})$ in some local curvilinear coordinates (the A_{2k}-singularity). Let $\check{d}$ be the degree of the projectively dual curve $\check{C}$. Let $\mathbf{n} = \sum_\gamma \delta(\gamma) + \sum_{\gamma_1,\gamma_2}(\gamma_1 \cdot \gamma_2)$ where γ runs over all local branches of C, $\delta(\gamma)$ is the delta-invariant of γ, and (γ_1,γ_2) runs over all unordered pairs of local branches (see [3, §3.2] for more details). Due to Lemma 7.4, the Plücker-like equations [3, Eqs. (3.13)–(3.15)] take the form

$$\begin{aligned} &\mathbf{g} + \mathbf{n} + \sum k a_{2k} = (d-1)(d-2)/2, \\ &\check{d} = d(d-1) - 2\mathbf{n} - \sum (2k+1) a_{2k}, \\ &2 - 2\mathbf{g} = 2\check{d} - d - \sum a_{2k} \end{aligned}$$

(all the summations run over $k \ge 2$). One easily checks that these equations do not have any integer non-negative solution with $3 \le d \le 5$. Proposition 7.3 is proven.

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