AGNIHOTRI-WOODWARD-BELKALE POLYTOPE AND THE INTERSECTION OF KLYACHKO CONES

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Abstract. Agnihotri-Woodward-Belkale polytope Δ (resp. Klyachko cone $\mathcal{K}$) is the set of solutions of the multiplicative (resp. additive) Horn's problem, i.e., the set of triples of spectra of special unitary (resp. traceless Hermitian) $n \times n$ matrices satisfying $AB = C$ (resp. $A + B = C$). $\mathcal{K}$ is the tangent cone of Δ at the origin. The group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ acts naturally on Δ.

In this note, we report on a computer calculation which shows that Δ coincides with the intersection of $g\mathcal{K}$, $g \in G$, for $n \leq 14$ but does not coincide for $n = 15$.

Our motivation was an attempt to understand how to solve the multiplicative Horn problem in practice for given conjugacy classes in $SU(n)$.

Introduction

For a special unitary matrix $A \in SU(n)$, let us denote by $\lambda(A)$ its *spectrum* i.e. the vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ which is uniquely defined by the conditions

$$\lambda_1 + \dots + \lambda_n = 0, \qquad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n \geq \lambda_1 - 1, \tag{1}$$

and $\exp(2\pi i\lambda_1), \dots, \exp(2\pi i\lambda_n)$ are the eigenvalues of A. The mapping $\lambda : SU(n) \to \mathbb{R}^n$ identifies conjugacy classes of $SU(n)$ with points of the $(n-1)$-simplex $\mathfrak{A}$ given by (1). This simplex is called *Weyl alcove* of $SU(n)$. It sits in the *Weyl chamber* $\mathfrak{t}_+$ defined by (1) with the last inequality excluded.

Agnihotri-Woodward [1] and Belkale [2] obtained necessary and sufficient conditions on three vectors α, β, γ for the existence of matrices $A, B, C \in SU(n)$ such that $\alpha = \lambda(A)$, $\beta = \lambda(B)$, $\gamma = \lambda(C)$, and $AB = C$. They have shown that the image of the mapping $\Lambda : SU(n)^2 \to \mathfrak{A}^3$, $(A, B) \mapsto (\lambda(A), \lambda(B), \lambda(AB))$, is a polytope $\Delta \subset \mathfrak{A}^3$ explicitely described in terms of quantum Schubert calculus (see §1).

This result is a generalization of Klyachko's solution [8] of Horn's problem (see an excellent survey [7]). Namely, Klyachko have described the set $\mathcal{K}$ of all triples of n-vectors realizable by spectra of traceless Hermitian matrices A, B, and $A + B$. Of course, $\mathcal{K}$ is the tangent cone of Δ at the origin O of $\mathbb{R}^{3n}$. Its facets are described in terms of the classical Schubert calculus. We call Δ and $\mathcal{K}$ the *Agnihotri-Woodward-Belkale polytope* and the *Klyachko cone* respectively.

How to apply these results in practice? Suppose, we have a concrete unitary matrix C and we ask if it is realizable as a product of matrices of given conjugacy classes (see [12] for an example of applications). Though there exists a finite set of inequalities to check, the number of them grows exponentially and, for matrices, say, 30×30 there is no chance to generate all inequalities in a reasonable time. This concerns both additive and multiplicative problems. However, for the additive

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problem, the honeycomb description of $\mathcal{K}$ due to Knutson and Tao [9, 10] allows to reduce the problem to the existence of a solution of $3\binom{n}{2}$ simultaneous linear inequalities on $\binom{n-1}{2}$ variables (see also [6]). The same problem for k matrices is reduced to the existence of a solution of $3(k-2)\binom{n}{2}$ inequalities on $(k-2)\binom{n-1}{2} + (k-3)(n-1)$ variables. For the multiplicative problem, such a reduction seems to be unknown (see [14]). This is why we study when the multiplicative problem can be reduced to the additive problem.

Let I be the identity $n \times n$ matrix and let $\omega = \exp(2\pi i/n)$. Easy to see that $\mathfrak{A}$ is the convex hull of $\lambda(Z)$ where $Z = \{\omega^k I\}$ is the center of $SU(n)$. Moreover, the action of Z on $SU(n)$ by multiplication induces affine linear actions of Z on $\mathfrak{A}$ and of $G = Z \times Z$ on Δ. This yields natural lower and upper estimates for Δ:

$$\operatorname{conv}(GO) \subset \Delta \subset \Delta_{\mathcal{K}} \qquad \text{where} \quad \Delta_{\mathcal{K}} = \bigcap_{g \in G} g\mathcal{K} \tag{2}$$

("conv" stands for the convex hull; $GO = \{gO \mid g \in G\} = \Lambda(G)$). Are these estimates sharp? For the lower estimate, this question was asked and answered in [2; Sect. 7]. The answer is: the equality $\operatorname{conv}(GO) = \Delta$ holds for $n \le 3$ and it fails for $n = 4$. For the upper estimate, we ask and answer this question in this note. The answer is: the equality $\Delta = \Delta_{\mathcal{K}}$ holds for $n \le 14$ and it fails for $n = 15$.

Agnihotri and Woodward [1; Sect. 8] discuss the action of G on Δ and its relation with a hidden symmetry of Gromov-Witten invariants. In particular, they point out that the action of G allows to reduce degree d invariants to degree zero invariants for small values of n (which implies $\Delta = \Delta_{\mathcal{K}}$ for those n), but, for $n = 10$, they give an example where the reduction is impossible. In terms of the polyhedra Δ and $\Delta_{\mathcal{K}}$, this means that for $n = 10$, among the inequalities which are used in [1] for Δ, there is at least one which cannot be reduced by G to a homogeneous inequality. However, this inequality does not belong to a smaller system of inequalities for Δ which is given by Belkale in [2]. For the smaller system of inequalities, such examples occur only for $n \ge 15$.

1. Description of Δ and $\mathcal{K}$ in terms of (quantum) Schubert calculus

Fix positive integers r, k such that $r + k = n$. Let $QH^*(G_r(\mathbb{C}^n))$ be the quantum cohomology ring of the Grassmanian of r-planes in $\mathbb{C}^n$. It is an algebra over $\mathbb{Z}[q]$ (q is an indeterminate) which is generated as a $\mathbb{Z}[q]$-module by the elements $\{\sigma_a\}$ where a runs over the set of partitions $\mathcal{P}_{r,k} = \{\,(a_1, \dots, a_r) \mid k \ge a_1 \ge \dots \ge a_r \ge 0\,\}$. Let $N_{ab}^c(r,k) = \sum_{d=0}^{\infty} N_{ab}^{c,d}(r,k)\, q^d$ be the structure constants of this algebra, i.e.

$$\sigma_a \cdot \sigma_b = \sum_{c \in \mathcal{P}_{r,k}} \sum_{d=0}^{\infty} N_{ab}^{c,d}(r,k)\, \sigma_c q^d.$$

The quantum multiplication is homogeneous if we set $\deg q = n$, $\deg \sigma_a = |a| := a_1 + \dots + a_r$, i.e. $N_{ab}^{c;d}(r,k)$ is nonzero only if $nd = |a| + |b| - |c|$. If $d = 0$ then $N_{ab}^{c;d}(r,k)$ coincides with the classical Littlewood-Richardson coefficient N_{ab}^c (in particular, it does not depend on r and k). An algorithm of computation of $N_{ab}^{c;d}(r,k)$ is given in [4]. It is implemented in [5].

Let $\bar{\mathcal{I}} = \{\,(r,k;a,b,c;d) \;\mid\; r + k = n, \;\; (a,b,c) \in \mathcal{P}_{r,k}^3\,\}$. For $t = (r,k;a,b,c;d) \in \mathcal{I}$ we also denote $N_{ab}^{c;d}(r,k)$ by N_t. Let $\mathcal{I} = \{t \in \bar{\mathcal{I}} \mid N_t = 1\}$ and let $\mathcal{I}_0$ be the subset of $\mathcal{I}$ defined by $d = 0$.

For $t = (r,k;a,b,c;d) \in \mathcal{I}$, we define $H_t = H_{ab}^{c,d}(r,k)$ as the half-space of $\mathbb{R}^{3n}$ given by the inequality $h_t(\alpha,\beta,\gamma) \ge 0$ where

$$h_t(\alpha,\beta,\gamma) = d + \sum_{i=1}^{r} \gamma_{k+i-c_i} - \sum_{i=1}^{r} \alpha_{k+i-a_i} - \sum_{i=1}^{r} \beta_{k+i-b_i},$$

As usually, we regard an elements $a = (a_1,\dots,a_r)$ of $\mathcal{P}_{r,k}$ as a *Young diagram* inscribed in the rectangle $r \times k$ which has r rows (numbered from the top to the bottom) and k columns (numbered from the left to the right). The Young diagram of a is the union of a_1 leftmost squares of the first row, a_2 leftmost squares of the second row etc. So, its area is $|a|$.

Let $\lambda : SU(n) \to \mathfrak{A}$, $\Lambda : SU(n)^2 \to \Delta$, $\mathfrak{t}^+$, and $\mathcal{K}$ be as in Introduction. The results of Klyachko and Agnihotri-Woodward-Belkale discussed in Introduction state that

$$\mathcal{K} = \mathfrak{t}_+^3 \cap \Big(\bigcap_{t\in\mathcal{I}_0} H_t \Big), \qquad \Delta = \mathfrak{A}^3 \cap \Big(\bigcap_{t\in\mathcal{I}} H_t \Big).$$

2. The action of G

Recall that $G = Z \times Z$ where Z is the center of $SU(n)$. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be the affine linear mapping defined by

$$\Omega(x_1,\dots,x_n) = (x_2,x_3,\dots,x_n,\, x_1-1) + (1/n,\,\dots,\,1/n).$$

It is easy to check that $\mathfrak{A} = \bigcap_{j=0}^{n-1} \Omega^j(\mathfrak{t}_+) = \operatorname{conv}\{\, \Omega^j(\vec{0}) \,|\, 0 \le j < n\}$. The action of G we spoke about in Introduction, is

$$g(\alpha,\beta,\gamma) = \big(\Omega^i(\alpha), \Omega^j(\beta), \Omega^{i+j}(\gamma)\big) \qquad \text{for} \quad g = (\omega^i,\omega^j).$$

Obviously, Δ is invariant under G. Indeed, $AB = C$ iff $(\omega^i A)(\omega^j B) = \omega^{i+j} C$.

It is shown in [1; Sect. 8] that the action of G on Δ corresponds to the action of G on $\mathcal{I}$ defined as follows. Quantum Pierri formula (see [3]) implies that

$$\sigma_k \sigma_a = \begin{cases} \sigma_{k,a_1,\dots,a_{r-1}}, & a_r = 0, \\ q\sigma_{a_1-1,\dots,a_r-1}, & a_r \ne 0. \end{cases}$$

Hence, for any $a,b,c \in \mathcal{P}_{r,k}$ and $i,j \in \mathbb{Z}$, we have

$$\sigma_k^i \sigma_a = \sigma_{a'} q^{d'_a}, \quad \sigma_k^j \sigma_b = \sigma_{b'} q^{d'_b}, \quad \sigma_k^{i+j} \sigma_c = \sigma_{c'} q^{d'_c}$$

for certain $d'_a, d'_b, d'_c \ge 0$ and $a',b',c' \in \mathcal{P}_{r,k}$ uniquely determined by a,b,c and i,j. Thus, for $t = (r,k;a,b,c;d)$ and $t' = (r,k;a',b',c';d')$ where $d' = d + d'_c - d'_a - d'_b$, we have $N_t = N_{t'}$. In other words, the quantum Littlewood-Richardson coefficients are symmetric under the action of G on $\bar{\mathcal{I}}$ defined by $gt = t'$ for $g = (\omega^i,\omega^j)$ (this action is well defined because $\sigma_k^n = q^k$). In particular, $\mathcal{I}$ is invariant under the action of G. It is shown in [1] that the two actions of G (on $\mathcal{I}$ and on Δ) are coherent, i.e., $gH_t = H_{gt}$.

3. Symmetries of Δ leaving $\mathcal{K}$ invariant

The the action of any non-trivial $g \in G$ is such that $g\mathcal{K} \neq \mathcal{K}$. There are also evident symmetries which are common for Δ and $\mathcal{K}$.

Since $AB = C \Leftrightarrow B^{-1}A^{-1} = C^{-1}$ and $A + B = C \Leftrightarrow (-A) + (-B) = -C$, both Δ and $\mathcal{K}$ are invariant under the involution $(\alpha, \beta, \gamma) \mapsto (\alpha^*, \beta^*, \gamma^*)$ where $(\alpha_1, \dots, \alpha_n)^* = (-\alpha_n, \dots, -\alpha_1)$. For $t \in \bar{\mathcal{I}}$, we denote the image of H_t under the mapping $(\alpha, \beta, \gamma) \mapsto (\alpha^*, \beta^*, \gamma^*)$ by H_t^*. This action on the facets of Δ corresponds to the following symmetry of (quantum) Littlewood-Richardson coefficients provided by the isomorphism between the Grassmanians $G_r(\mathbb{C}^n)$ and $G_k(\mathbb{C}^n)$.

Let $t \mapsto t^*$ be the involution $\bar{\mathcal{I}} \to \bar{\mathcal{I}}$ defined by

$$(r, k;\ a, b, c;\ d)^* = (k, r; a^*, b^*, c^*; d)$$

where

$$(a_1, \dots, a_r)^* = (a_1^*, \dots, a_k^*), \qquad a_i^* = \max\{j \mid a_j \geq i\}$$

(the Young diagrams of a and a^* are symmetric with respect to the main diagonal). For any $t \in \bar{\mathcal{I}}$, we have $N_t = N_{t^*}$ and $H_t^* = H_{t^*}$.

For $a = (a_1, \dots, a_r) \in \mathbb{P}_{r,k}$, let $\bar{a} = (k - a_r, \dots, k - a_1)$. Then $\bar{\mathcal{I}}$ is invariant under the mappings $t \mapsto (r, k;\ b, a, c;\ d)$ and $t \mapsto (r, k;\ b, \bar{c}, \bar{a};\ d)$ (commutativity and the Poincaré duality). These symmetries correspond to the evident symmetries of Δ provided by $AB \sim BA$ and $AB = C \Leftrightarrow BC^{-1} = A^{-1}$.

Let G_0 be the group of linear transformations of $\mathbb{R}^{3n}$ generated by all symmetries discussed in this section and let $\tilde{G}$ be the group of affine transformations generated by G and G_0. It is clear that $|G_0| = 12$ and $|\tilde{G}| = 12n^2$.

4. Statement of the results

Proposition 1. *If $n \leq 14$, then $\mathcal{I} = G\mathcal{I}_0 = \bigcup_G g\mathcal{I}_0$. In particular, $\Delta = \Delta_{\mathcal{K}}$.*

Proposition 2. *If $n = 15$, then $\mathcal{I} = G\mathcal{I}_0 \cup \tilde{G}t_0$ and $t_0 \notin G\mathcal{I}_0$ where*

$$t_0 = (6, 9;\ 663300, 663300, 666300;\ 1).$$

In particular, $\Delta = \Delta_{\mathcal{K}} \cap \big(\bigcap_{\tilde{G}} gH_{t_0}\big)$.

Moreover, $\Delta \neq \Delta_{\mathcal{K}}$, in particular $p = (\alpha, \beta, \gamma) \in (\Delta_{\mathcal{K}} \setminus H_{t_0}) \subset (\Delta_{\mathcal{K}} \setminus \Delta)$ for

$$\alpha = \beta = \tfrac{1}{17}\,(6, 6, 6, 6, 6,\ 0, 0, 0, 0, 0,\ -6, -6, -6, -6, -6),$$

$$\gamma = \tfrac{1}{17}\,(8, 8, 8,\ 1, 1, 1, 1, 1, 1,\ -3, -3, -3, -3,\ -9, -9).$$

The minimum of h_{t_0} on $\Delta_{\mathcal{K}}$ is attained at p.

Corollary 3. *If $n \leq 15$, then the set of inequalities $h_t \geq 0$, $t \in \mathcal{I}$, defining Δ is minimal, i.e., for any $t' \in \mathcal{I}$, we have*

$$\mathfrak{A}^3 \cap \Big(\bigcap_{t \in \mathcal{I} \setminus \{t'\}} H_t \Big) \neq \Delta$$

Proof. Follows from Propositions 1 and 2 combined with Knutson-Tao-Woodward's result [11] on the minimality of the system of inequalities $\{h_t \geq 0\}_{t \in \mathcal{I}_0}$ defining $\mathcal{K}$. $\square$

Proposition 4. *If* $n = 16$*, then* $\mathcal{I} = G\mathcal{I}_0 \cup \tilde{G}\mathcal{I}'$ *where*

$$\begin{aligned}\mathcal{I}' = \{\, &(6, 10;\ 553300, 663300, 663300;\ 1),\\ &(7, 9;\ 5533000, 6633000, 6633000;\ 1),\ (7, 9;\ 5533300, 6633000, 6633300;\ 1),\\ &(7, 9;\ 5553000, 6443000, 6553000;\ 1),\ (7, 9;\ 6433300, 6633000, 6633300;\ 1),\\ &(8, 8;\ 44431000, 54441100, 54441100;\ 1),\ (8, 8;\ 44431000, 54442000, 54442000;\ 1),\\ &(8, 8;\ 44440000, 55441100, 55441100;\ 1),\ (8, 8;\ 44441000, 54441100, 55441100;\ 1),\\ &(8, 8;\ 44441100, 54431000, 54441100;\ 1),\ (8, 8;\ 44441100, 54441000, 55441100;\ 1),\\ &(8, 8;\ 44441100, 54441100, 55441110;\ 1)\,\}.\end{aligned}$$

We do not know any human readable proof of Propositions 1, 2, and 4. They are obtained on a computer. To check Proposition 1, we used the program `lrcalc` written by Buch [5]. Namely, using this program, for each $n \le 14$ we generated all elements $t \in \mathcal{I}$ and, for each t, we checked that its orbit Gt (see §2) contains an element with $d = 0$.

Doing the same computations for $n = 15$, we found the element t_0 whose orbit is disjoint from $\mathcal{I}_0$. The only orbit of $\tilde{G}$ disjoint from $\mathcal{I}_0$ is $\tilde{G}t_0$.

To check that the inequality $h_{t_0} \ge 0$ is independent of the others, we minimized (using [13]) h_{t_0} under the constraints (1) and $h_t \circ g \ge 0$, $t \in \mathcal{I}_0$, $g \in G$. These are 3 135 129 030 constraints on 30 variables (since $a = b$ in t_0, we may set $\alpha = \beta$). The capacity of available computers was not enough for a straight forward solution of this problem. In rest of this section we explain the trick we used to reduce the number of constraints up to 148 295.

Consider the following 15×15 diagonal matrices:

$$\begin{aligned}A &= \mathrm{diag}(1, 1, 1, 1, 1,\ -1, \dots, -1),\\ B_1 &= \mathrm{diag}(1, 1, 1,\ -1, \dots, -1,\ 1, 1),\\ B_2 &= \mathrm{diag}(1, 1,\ -1, \dots, -1,\ 1, 1, 1).\end{aligned}$$

Then

$$\lambda(A) = \lambda(B_1) = \lambda(B_2) = \tfrac{1}{2}(-1, -1, -1, -1, -1,\ 0, 0, 0, 0, 0,\ 1, 1, 1, 1, 1),$$

$$\lambda(AB_1) = \tfrac{1}{2}(-1, -1,\ 0, \dots, 0,\ 1, 1), \text{ and } \lambda(AB_2) = \tfrac{1}{2}(-1, -1, -1,\ 0, \dots, 0,\ 1, 1, 1).$$

Let $p_i = \Lambda(A, B_i)$, $i = 1, 2$. By definition, we have $p_1, p_2 \in \Delta$. We have also $h_{t_0}(p_1) = h_{t_0}(p_2) = 0$. Let

$$\Delta' = \cap_{t \in \mathcal{I}'} H_t, \qquad \mathcal{I}' = \{t \in \mathcal{I} \mid t \ne t_0,\ h_t(p_1) = h_t(p_2) = 0\}$$

Lemma. *If* $\min_{\Delta'} h_{t_0} < 0$*, then* $\Delta \ne \Delta_{\mathcal{K}}$*.*

Proof. Suppose that $\min_{\Delta'} h_{t_0} < 0$. Let $p' \in \Delta'$ be such that $h_{t_0}(p') < 0$. Set $p_0 = (p_1 + p_2)/2$. Let $\mathcal{I}'' = \mathcal{I} \setminus (\mathcal{I}' \cup \{t_0\})$ and $Q = [p', p_0] \cap \Big(\bigcap_{t \in \mathcal{I}''} \{h_t = 0\}\Big)$. For any $t \in \mathcal{I}''$, the values of h_t at p_1 and p_2 are non-negative and at least one of them is positive, hence $h_t(p_0) > 0$. Therefore, $p_0 \notin Q$. Let q be the point of Q closest to p_0. Then $h_t(q) \ge 0$ for any $t \in \mathcal{I}''$. On the other hand, $h_t(q) \ge 0$ for any $t \in \mathcal{I}'$ because $q \subset \Delta'$. Thus, $q \in \Delta_{\mathcal{K}} \setminus \Delta$. $\square$

Thus, to show that $\Delta \neq \Delta_{\mathcal{K}}$, it is enough to find the minimum of h_{t_0} under the constraints only from $\mathcal{I}'$ and this set is more than 30 000 times smaller than $\mathcal{I}$. The minimum is attained at the point $p = (\alpha, \beta, \gamma)$ presented in Proposition 2 and we have $h_{t_0}(p) = -\frac{1}{17} < 0$. It follows from Lemma that $\Delta \neq \Delta_{\mathcal{K}}$. However, we cannot conclude that $\min_{\Delta_{\mathcal{K}}} h_{t_0} = \min_{\Delta} h_{t_0}$. To prove this fact and to ensure that floating point computations in [13] did not affect the result, we checked that $p \in \Delta_{\mathcal{K}}$ (using `lrcalc`, this takes few minutes). Namely, we checked that $h_t(p) < 0$ only for $t = t_0$ and $h_t(p) = 0$ only in the following cases (up to swapping a and b):

(1) $r = 3$, $a = b = (8, 4, 0)$, $c = (9, 0, 0)$;
(2) $r = 5$, $a = (7, 7, 3, 0, 0)$, $b = (7, 7, 4, 4, 0)$, $c = (8, 8, 6, 1, 1)$;
(3) $r = 7$, $a = (5, 5, 2, 2, 0, 0, 0)$, $b = (6, 6, 6, 3, 3, 0, 0)$, $c = (5, 5, 5, 5, 3, 0, 0)$;
(4) $r = 7$, $a = (5, 5, 3, 3, 3, 0, 0)$, $b = (5, 5, 3, 3, 3, 0, 0)$, $c = (5, 5, 5, 5, 3, 0, 0)$;
(5) $r = 13$, $a^* = b^* = (9, 5)$, $c^* = (11, 2)$;

in all these cases we have $d = 1$.

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