COMPLEX ORIENTANTION FORMULAS FOR M-CURVES OF DEGREE $4d+1$ WITH 4 NESTS

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Abstract. We prove complex orientation formulas for M-curves in $\mathbb{RP}^2$ of degree $4d+1$ with 4 nests. They generalize the formulas of complex orientations for M-curves in $\mathbb{RP}^2$ with a deep nest. This is a step towards the isotopy classification of real M-curves of degree 9.

Introduction.

It was observed in [4], that in the case when a real algebraic M-curve of degree m in $\mathbb{RP}^2$ has a nest of depth $[m/2]-1$ (deep nest), a new complex orientation formula takes place. Here "new" means independent of Rohlin and Rohlin-Mishachev complex orientation formulas. The condition of deep nest is needed here for the existence of a pencil of lines such that each line *has at most two non-real intersection points with the curve* (satisfies *"≤2"-condition*). It is clear from the proof in [4] that similar formulas should take place for a real curve in a ruled surface if each fiber satisfies "≤2"-condition, in particular, if there is such a pencil of conics for a curve in $\mathbb{RP}^2$. In this paper we consider three cases when the latter situation occurs and prove a new complex orientation formula for each case (Propositions 1.1, 2.1, and 3.1). In Section 4, we discuss applications to the problem of classification of real M-curves of degree 9 up to isotopy.

Another proof of the complex orientation formula from [4] and a generalization for any (not necessarily relatively minimal) ruled surface is obtained by Welschinger [8, 9]. His general formula is not immediate to apply in a concrete situation, but, of course, the formulas from the present paper can be derived from it. Moreover, this is done in [9] for the formula of Proposition 1.1 (see below) as an example of application. Another specialization considered in [8, 9] (and not considered here) appeared to be very useful for the classification [5] of pseudoholomorphic M-curves of degree 8 in $\mathbb{RP}^2$ (it twice reduced the number of fiberwise arrangements to consider).

Here we give direct self-contained proofs using the same tool as in [4]: linking and 'self-linking' numbers of sublinks of $L = A \cap S^3$ where A is the complexification of the curve, and S^3 is the boundary of a neighbourhood of the union of the complexifications of real lines of a certain pencil.

To simplify the exposition, we formulate everything for real algebraic curves, but all statements hold for real pseudoholomorphic curves as well. The proofs also can be easily adopted for the pseudoholomorphic context.

I thank the referee for useful remarks.

Definitions and Notation. If A is a nonsingular real algebraic curve on $\mathbb{RP}^2$, then the set of its real points is denoted by $\mathbb{R}A$ and the set of its complex points is denoted just by A. A connected component of $\mathbb{R}A$ is called an *oval* if it is contractible in $\mathbb{RP}^2$ and it is called an *odd branch* otherwise. The complement of an oval V has two connected components: D (a disk) and M (a Möbius band). The component D is called the *interior* of V and it is denoted by $\operatorname{Int} V$. An oval of A is called *empty* if its interior does not contain other ovals. A *nest* of depth k of a curve A is a union of pairwise disjoint ovals $V_1, \dots, V_k$ such that $\operatorname{Int} V_k \subset \operatorname{Int} V_{k-1} \subset \dots \subset \operatorname{Int} V_1$. A nest of A is called *maximal* if it is not a subset of a bigger nest of A.

Throughout this paper, A is a nonsingular real algebraic M-curve in $\mathbb{RP}^2$ of degree $m = 4d+1$, $d \ge 2$. The odd branch of A is denoted by J. We suppose that A has four maximal nests $N_1, \dots, N_4$ of depths $d_1, \dots, d_4$ respectively. Let $p_1, \dots, p_4$ be generic points inside the innermost ovals of the nests $N_1, \dots, N_4$ respectively. Let $\mathcal{P}$ be the pencil of conics passing through $p_1, \dots, p_4$. We say $p_1, \dots, p_4$ are *in convex position with respect to* J if there exists a convex quadrangle Q with vertices at $p_1, \dots, p_4$ which does not intersect J. It is clear that if $p_1, \dots, p_4$ are not in convex position with respect to J then any conic from $\mathcal{P}$ meets J at least at 2 points.

We denote the ovals of N_i by $V_1^{(i)}, \dots, V_{d_i}^{(i)}$. We number them so that $V_{j+1}^{(i)} \subset \operatorname{Int} V_j^{(i)}$, in particular $V_1^{(i)}$ is the outermost oval of N_i. We call the ovals contained in the nests $N_1, \dots, N_4$ *big* and we call the other ovals *small*.

We are interested in situations when any conic from $\mathcal{P}$ must have at least $2m-2$ intersection points with the union of J and all big ovals of A (and hence, by Bezout's theorem, all small ovals are empty).

This is so in the following cases (see Figure 1 for $d = 3$):

(1) The nests $N_1, \dots, N_4$ are pairwise disjoint, and $d_1 = \dots = d_4 = d$. In this case the points $p_1, \dots, p_4$ are necessarily in convex position with respect to J, see Figure 1(1).

(2) The nests $N_1, \dots, N_4$ are pairwise disjoint, $d_1 = d_2 = d_3 = d$, and $d_4 = d-1$. The points $p_1, \dots, p_4$ are not in convex position, see Figure 1(2a)–(2b).

(3) The outermost ovals of N_2 and N_4 coincide (i.e., $V_1^{(2)} = V_1^{(4)}$), but the nests N_1, N_2', N_3, N_4' are pairwise disjoint where $N_j' = N_j \setminus V_1^{(j)}$. The depths are $d_1 = d_2 = d_3 = d_4 = d$. Moreover, the points $p_1, \dots, p_4$ are not in convex position, see Figure 1(3).

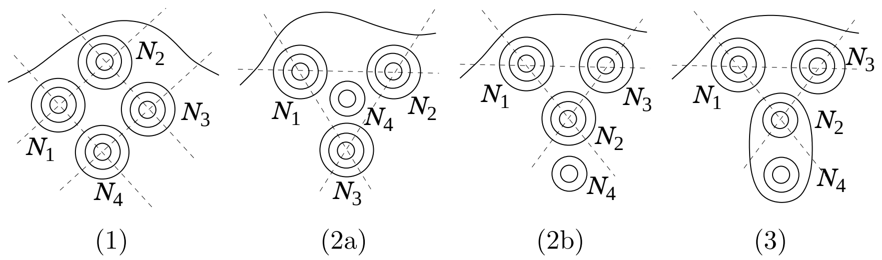


FIGURE 1

We fix a complex orientation on $\mathbb{R}A$. As usually, an oval V is caled *positive* (resp. *negative*) if $[V] = -2[J]$ (resp. $[V] = 2[J]$) in the homology group $H_1(\mathbb{RP}^2 \setminus \operatorname{Int} V)$.

For $S, s \in \{+,-\}$ and for $i = 1,\dots,4$, let $\pi_s^S(N_i)$ be the number of pairs of ovals (O, o) of the signs (S, s) respectively such that O is a non-empty oval contained in the nest N_i and o is an empty oval contained in $\operatorname{Int} O$. Similarly, $\Pi_s^S(N_i)$ will denote the number of pairs of ovals (O, o) of the signs (S, s) such that O is a big oval contained in N_i and o is a small oval contained in $\operatorname{Int} O$.

Let $K^S(N_i)$ be the number of ovals of the sign S in the nest N_i, and let $k^S(N_i)$ be the number of non-empty ovals among them.

1. Four disjoint nests of depth d in convex position.

In this section we assume that the nests $N_1,\dots,N_4$ are pairwise disjoint and $d_1 = \cdots = d_4 = d$. Then Bezout's theorem for auxiliary conics easily implies that all small ovals (i.e. those which are not involved in the nests $N_1,\dots,N_4$) are empty and the points $p_1,\dots,p_4$ are vertices of a convex quadrangle Q which does not meet J.

Let us number the points $p_1,\dots,p_4$ so that they are placed in this order along the boundary of Q. Let us set

$$\pi_i = \begin{cases} \pi_+^-(N_i) - \pi_-^-(N_i), & i = 1,3, \\ \pi_-^+(N_i) - \pi_+^+(N_i), & i = 2,4, \end{cases} \qquad k_i = \begin{cases} k^-(N_i), & i = 1,3, \\ k^+(N_i), & i = 2,4, \end{cases}$$

and let us define Π_i and K_i via $\Pi_s^S(N_i)$ and $K^S(N_i)$ in the same way.

Proposition 1.1. *One has*

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = k_1^2 + k_2^2 + k_3^2 + k_4^2 \tag{1}$$

and

$$\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = (K_1^2 - K_1) + \cdots + (K_4^2 - K_4). \tag{2}$$

Remark. The formula (2) is an equivalent version of (1). It does not provide any additional restriction on the complex scheme of $\mathbb{R}A$.

The rest of this section is devoted to the proof of Proposition 1.1. Let $\mathrm{cr} : \mathbb{RP}^2 \to \mathbb{RP}^2$ be the standard quadratic Cremona transformation centered at p_1, p_2, p_3. Let $\mathcal{L}$ be the pencil of lines through $\mathrm{cr}(p_4)$ (this is the image of the pencil of conics through $p_1,\dots,p_4$). Let the lines ℓ_1, ℓ_2, ℓ_3 be the transforms of the points p_1, p_2, p_3 respectively and let us denote the transforms of the lines p_2p_3, p_3p_1, p_1p_2 by q_1, q_2, q_3 respectively.

The arrangement of $\mathrm{cr}(\mathbb{R}A)$ with respect to $\mathcal{L}$ is as in Figure 2 up to zigzag removal (see [6; §5] for a discussion of zigzag removal). Here the pencil $\mathcal{L}$ is supposed to be the pencil of vertical lines. The dashed rectangle R is shown here because we shall refer to it in the proof of Lemma 1.2.

The image of the nest N_2 is shown in more detail in Figure 3.

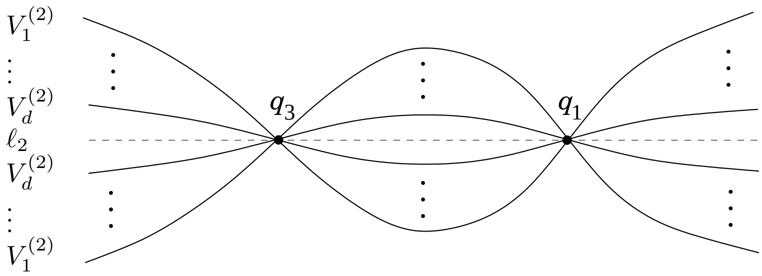


FIGURE 3. $\mathrm{cr}(N_2)$.

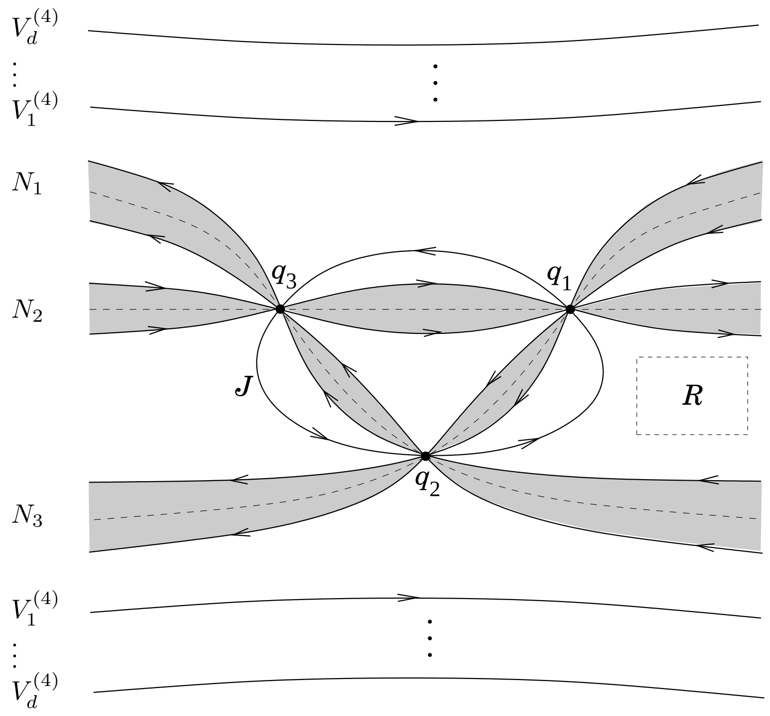


FIGURE 2. Orientations of $\mathrm{cr}(V_1^{(\nu)})$ for positive $V_1^{(\nu)}$

Let us fix the complex orientation on J as shown in Figure 2. Then the complex orientations of the ovals $\mathrm{cr}(V_1^{(\nu)})$ are depicted in Figure 2 under condition that all the ovals $V_1^{(\nu)}$ are positive.

Let b be the braid corresponding to the arrangement of $\mathrm{cr}(\mathbb{R}A)$ with respect to the pencil of lines $\mathcal{L}$ (see [4] or [5]). Let $L = \hat{b}$ be the link which is the braid closure of b. Let L_+ (resp. L_-) be the sublink of L composed of the strings of b oriented from the left to the right (resp. from the right to the left). When speaking of "left" and "right", we refer to Figure 2. Let $\tilde{L}$ be the link corresponding to the reducible curve $\mathrm{cr}(A)\cup\ell_1\cup\ell_2\cup\ell_3$ (then L is a sublink of $\tilde{L}$). We shall use the same notation ℓ_1, ℓ_2, ℓ_3 for the corresponding components of $\tilde{L}$.

Let b_+ be the braid corresponding to L_+, and let us denote the exponent sum (the algebraic length) of b_+ by $e(b_+)$.

Lemma 1.2. *One has*

$$e(b_+) + 2(\Pi_1 + \cdots + \Pi_4) = (2K_1^2 + K_1) + \cdots + (2K_4^2 + K_4). \tag{3}$$

Proof. Let us consider all real (not only algebraic) curves in $\mathbb{RP}^2 \setminus \{\mathrm{cr}(p_4)\}$ which are obtained from $\mathrm{cr}(\mathbb{R}A)$ by moving small ovals so that they remain to be disjoint from the set $A' = (\ell_1\cup\ell_2\cup\ell_3)\cup\mathrm{cr}(N_1\cup\cdots\cup N_4)$ but, maybe, they are distributed in other connected components of $\mathbb{RP}^2 \setminus A'$. When moving small ovals, we keep their order (from the left to the right) and their orientations coming from the complex orientation of $\mathbb{R}A$.

For any such curve B we can define the braid corresponding to $B\cup\ell_1\cup\ell_2\cup\ell_3$, the sublinks $L_\pm$ and ℓ_j of the link $\tilde L$, and all the quantities involved in the formula (3). Let us show that the quantity

$$\Phi(B)=e(b_+)+2(\Pi_1+\dots+\Pi_4)-4\sum_{j=1}^{3}K_j\,\mathrm{lk}(\ell_j,L_+)$$

does not depend on B (here lk stands for the linking number).

Indeed, if a small oval passes through a big oval which contributs to L_- (i.e., through an oval from N_j of the sign $(-1)^{j+1}$), then none of the terms of $\Phi(B)$ changes. If a small oval of sign s passes through a big oval $\mathrm{cr}(V_i^{(j)})$ which contributs to L_+ moving from $\mathbb{RP}^2\setminus\mathrm{cr}(\mathrm{Int}\,V_i^{(j)})$ to $\mathrm{cr}(\mathrm{Int}\,V_i^{(j)})$, then $e(b_+)$ changes by $2(-1)^js$ and $2\Pi_j$ changes by $-2(-1)^js$. If a small oval of sign s passes through ℓ_j, then its sign reverses and in this case both Π_j and $2K_j\,\mathrm{lk}(\ell,L_+)$ change by $(-1)^j2sK_j$.

Thus, to compute $\Phi(\mathrm{cr}(\mathbb{R}A))$ it is sufficient to compute $\Phi(B)$ when all small ovals of B are, say, in the rectangle R in Figure 2. Let us do it. For this curve B we have $\Pi_1=\dots=\Pi_4=0$ and

$$\begin{aligned}2\,\mathrm{lk}(\ell_1,L_+)=&-(2K_1+2K_2) &&\text{(contribution of } q_3)\\ &-(2K_1+2K_3+1) &&\text{(contribution of } q_2)\\ &+(2K_1+2K_2+2K_3+2K_4+1) &&\text{(contribution of } \Delta)\\ =&\;2K_4-2K_1.\end{aligned}$$

Similarly, $\mathrm{lk}(\ell_2,L_+)=K_4-K_2$ and $\mathrm{lk}(\ell_3,L_+)=K_4-K_3$. For the curve B we have also

$$\begin{aligned}e(b_+)=&-2K_2 &&\text{(contribution of } [q_1,q_3])\\ &-(K_1+K_2)(2K_1+2K_2-1) &&\text{(contribution of } q_3)\\ &-(K_1+K_3)(2K_1+2K_3+1) &&\text{(contribution of } q_2)\\ &-(K_2+K_3)(2K_2+2K_3-1) &&\text{(contribution of } q_1)\\ &+(K_1+\dots+K_4)(2K_2+\dots+2K_4+1) &&\text{(contribution of } \Delta)\end{aligned}$$

Summing up all these quantities, we see that $\Phi(B)$ is equal to the right hand side of (3). It remains to note that $\Phi(\mathrm{cr}(\mathbb{R}A))$ is the left hand side of (3) because $\mathrm{lk}(\ell_j,L_+)=0$ for it. This follows from the fact that all intersection points of $\mathrm{cr}(A)$ and ℓ_j are real, hence, the corresponding sublinks of $\tilde L$ bound disjoint embedded surfaces in the 4-ball (see [4] for details). ☐

Lemma 1.3. *One has* $e(b_+)=3(K_1+\dots+K_4)$.

Proof. Being an M-curve, $\mathbb{R}A$ has $(m-1)(m-2)/2$ ovals. Among them, there are $d_1+\dots+d_4=4d=m-1$ big ovals. Hence, $\mathbb{R}A$ has $(m-1)(m-4)/2$ small ovals. Hence, we have

$$\begin{aligned}e(b)=&-1-(m-1)(m-4)/2 &&\text{(contribution of } J \text{ and small ovals)}\\ &-3\times m(m-1)/2 &&\text{(contribution of } q_1,q_2,q_3)\\ &+m(2m-1) &&\text{(contribution of } \Delta)\\ =&\;12d.\end{aligned}$$

Let us denote the number of components of the links L and $L_\pm$ by $\mu(L)$ and $\mu(L_\pm)$ respectively. We have $\mu(L) = 4d + 2$ (each big oval contributes 1, and J together with the chain of small ovals contribute 2). Since the curve A is maximal, the link L bounds a surface F of genus zero in the 4-ball. Let $\mu(F)$ be the number of components of F. Since the genus of F is zero, we have $\chi(F) = 2\mu(F) - \mu(L)$. On the other hand, we have $\chi(F) = \deg \operatorname{cr}(A) - e(b) = (8d + 2) - 12d = 2 - 4d$ by Riemann-Hurwitz formula. Hence, $\mu(F) = (\chi(F)+\mu(L))/2 = (1-2d)+(2d+1) = 2$. This means that F is a union of two connected surfaces $F = F_+ \cup F_-$ such that $\partial F_\pm = L_\pm$.

Let m_+ be the number of strings of b_+. By the same arguments as above, we have $m_+ - e(b_+) = \chi(F_+) = 2 - \mu(L_+)$. Since $\mu(L_+) = K_1 + \cdots + K_4 + 1$ and $m_+ = 2(K_1+\cdots+K_4)+1$, this yields $e(b_+) = m_++\mu(L_+)-2 = 3(K_1+\cdots+K_4)$. $\square$

Proof of Proposition 1.1. The formula (2) is immediate from Lemmas 1.2 and 1.3. To deduce (1), let us show that $\pi_j - k_j^2 = \Pi_j - (K_j^2 - K_j)$. Indeed, if the innermost big oval from N_j is of the sign $(-1)^j$, then $\Pi_j = \pi_j + k_j$ and $K_j = k_j + 1$. Otherwise, $\Pi_j = \pi_j - k_j$ and $K_j = k_j$. $\square$

2. Four disjoint nests of depths $d, d, d, d-1$ in a non-convex position.

In this section we suppose that the nests $N_1, \dots, N_4$ are pairwise disjoint and $(d_1, \dots, d_4) = (d, d, d, d-1)$. We suppose also that the points $p_1, \dots, p_4$ are not in convex position with respect to J. This means that there is a triangle T whose vertices are three of these points, such that the fourth point is inside T, and $T \cap J = \varnothing$, i.e., the nests are arranged either as in Figure 1(2a) or as in Figure 1(2b) (the triangle T is not depicted in Figures 1(2a)–(2b)!). Let $V(T)$ be the set of vertices of T, i.e., $V(T) = \{p_1, p_2, p_3\}$ for Figure 1(2a) and $V(T) = \{p_1, p_3, p_4\}$ for Figure 1(2b). Let us set

$$\pi_i = \begin{cases} \pi_+^-(N_i) - \pi_-^-(N_i), & p_i \notin V(T), \\ \pi_-^+(N_i) - \pi_+^+(N_i), & p_i \in V(T), \end{cases} \qquad k_i = \begin{cases} k^-(N_i), & p_i \notin V(T), \\ k^+(N_i), & p_i \in V(T), \end{cases}$$

and let us define Π_i and K_i via $\Pi_s^S(N_i)$ and $K^S(N_i)$ in the same way.

Proposition 2.1. *The identities* (1) *and* (2) *hold in this situation* (*again,* (2) *is an equivalent version of* (1)).

Proof. The proof repeats almost word-by-word the proof of Proposition 1.1. We apply the Cremona transformation centered at p_1, p_2, p_3 (recall that $d_1 = d_2 = d_3 = d$ and $d_4 = d - 1$), and we consider the pencil of lines $\mathcal{L}$ through $\operatorname{cr}(p_4)$. We numerate the nest N_1, N_2, N_3 as in Figure 1(2a–b). The images of the nests are arranges with respect to $\mathcal{L}$ as in Figures 4(a–b) where, as in Figure 2, we have depicted the complex orientations of the big ovals under condition that all of them are positive.

The statement of Lemma 1.2 holds without changes. In its proof, if we place all small ovals of B into R so that the leftmost one is oriented clockwise (which means that it is positive for Figure 4(a) and negative for Figure 4(b)), then the values of $\operatorname{lk}(\ell_j, L_+)$ and $e(b_+)$ are as in the proof of Lemma 1.2 (though their computation is slightly different).

The statement of Lemma 1.3 also holds. Its proof must be modified as follows. This time there is one more small oval (because one big oval is missing), but since J does not contribute to $e(b)$, we still have $e(b) = 12d$. We still have $\mu(L) = 4d+2$

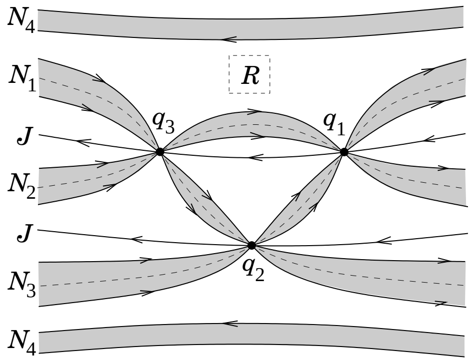


FIGURE 4(a)

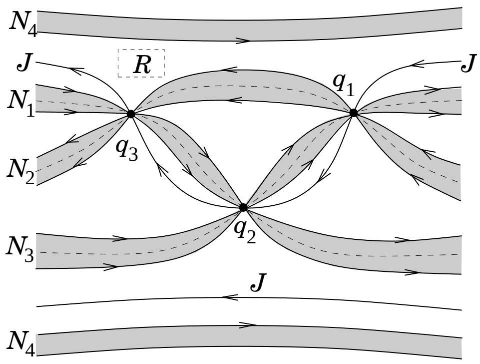


FIGURE 4(b)

(each of $4d-1$ big ovals and J contribute 1, and the chain of small ovals contibute 2). The rest of the proof repeats word-by-word. $\square$

3. Nests in a non-convex position, two outermost ovals coincide.

In this section we suppose that Case (3) takes place (see Figure 1(3)). Let us set $V = V_1^{(2)} = V_1^{(4)}$ (the common outermost oval of N_2 and N_4). Let T be the triangle with vertices p_1, p_2, p_3 such that $T \cap J = \varnothing$, and let $\operatorname{Int}^+ V = \operatorname{Int} V \setminus T$ and $\operatorname{Int}^- V = \operatorname{Int} V \cap T$. We set also $\operatorname{Int}^\pm V_i^{(j)} = \operatorname{Int} V_i^{(j)}$ for $V_i^{(j)} \neq V$.

For $S, s \in \{+,-\}$ and for $i = 1,\dots,4$, let $\tilde\Pi_s^S(N_i)$ be the number of pairs of ovals (O,o) of the signs (S,s) respectively such that $O \subset N_i$ is big and $o \subset \operatorname{Int}^S O$ is small. In particular, we have $\tilde\Pi_s^S(N_i) = \Pi_s^S(N_i)$ for $i = 1,3$.

Let us set

$$\Pi_i = \begin{cases} \tilde\Pi_-^+(N_i) - \tilde\Pi_+^+(N_i), & i = 1,3,4,\\ \tilde\Pi_+^-(N_i) - \tilde\Pi_-^-(N_i), & i = 2, \end{cases} \qquad K_i = \begin{cases} K^+(N_i), & i = 1,3,4,\\ K^-(N_i), & i = 2, \end{cases}$$

Proposition 3.1. *One has*

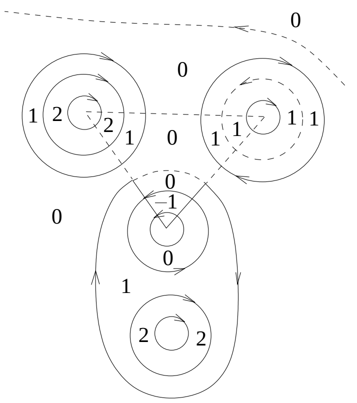
$$\sum_{i=1}^4 \Pi_i = \Big(\sum_{i=1}^4 (K_i^2 - K_i)\Big) - K_2 + \begin{cases} 0, & V \textit{ is positive}\\ 1, & V \textit{ is negative} \end{cases} \tag{4}$$

Remark. The left hand side of (4) can be rewritten as $-\sum_v \varphi_v \operatorname{sign} v$. Here the sum is taken over all small ovals v and φ_v is the value on v of a locally constant function φ defined on $\mathbb{RP}^2 \setminus \big(N_1^+ \cup N_2^- \cup N_3^+ \cup N_4^+ \cup \partial(T \cap \operatorname{Int} V)\big)$ where N_i^S is the union of big ovals of the sign S which are contained in N_i. The values of φ are given in Figures 5(a)–(b). The big ovals (arcs of them) where the function φ does not change the value are depicted by dashed lines.

As in the proofs of Propositions 1.1 and 2.1, let cr be the Cremona transformation centered at p_1, p_2, p_3, and let us introduce the same notation as above.

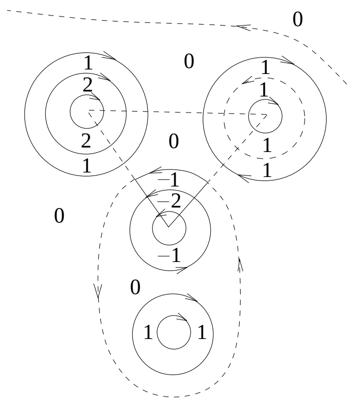
Lemma 3.2. *One has*

$$e(b_+) + 2\sum_{i=1}^4 \Pi_i = -2K_2 + \Big(\sum_{i=1}^4 (2K_i^2 + K_i)\Big) + \begin{cases} -2, & V \textit{ is positive,}\\ 0, & V \textit{ is negative.} \end{cases} \tag{5}$$



V is positive

FIGURE 5(A)



V is negative

FIGURE 5(B)

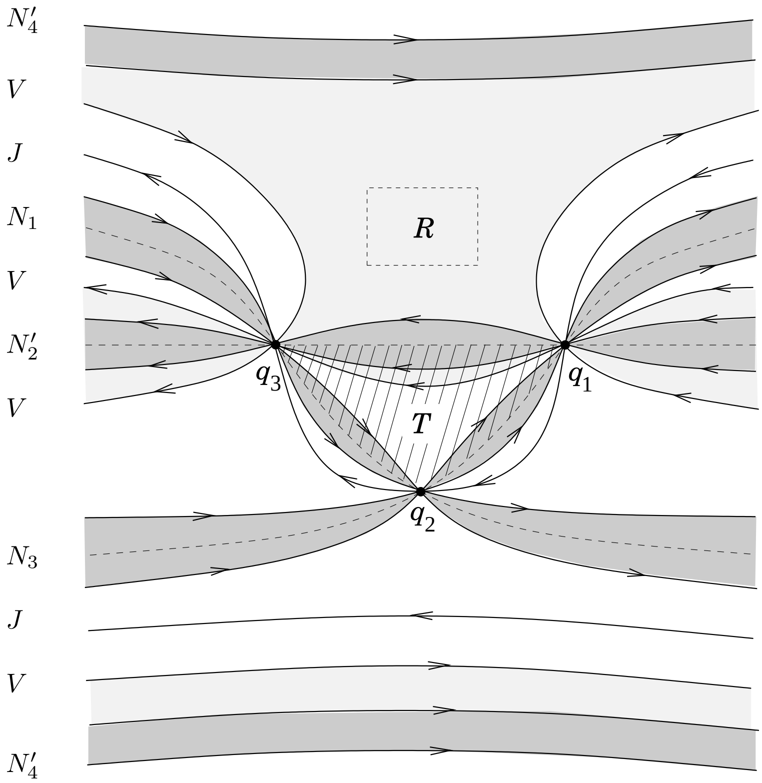


FIGURE 6. Orientations of $\mathrm{cr}(\mathbb{R}A)$ when big ovals are positive

Proof. Since the proof is similar to that of Lemma 1.2, we just sketch it. Again, we consider the set of curves obrained from $\mathrm{cr}(\mathbb{R}A)$ by vetrical moving of small ovals.

For such a curve B, we set

$$\begin{aligned}\Phi(B) =& e(b_+) + 2(\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4)\\ &- 4K_1 \operatorname{lk}(\ell_1, L_+) - (4K_2 - 2) \operatorname{lk}(\ell_2, L_+) - 4K_3 \operatorname{lk}(\ell_3, L_+).\end{aligned}$$

This amount does not change when small ovals move from one region to another. Indeed, suppose that a small oval v of sign s crosses ℓ_2 moving from from the bottom to the top according to Figure 6 (i.e., it leaves T). Then its contribution to $\operatorname{lk}(\ell_2, L_+)$ changes by $-s$. The sign of the small oval reverses. Hence, if V is positive, then the contribution of v into Π_2 (resp. to Π_4) switches from sK_2 to $-sK_2$ (resp. from 0 to s). If V is negative, then the contribution of v into Π_2 switches from sK_2 to $-s(K_2 - 1)$ and its contribution to Π_4 does not change. Thus, in the both cases, the contribution of v into $\Pi_2 + \Pi_4$ changes by $-s(2K_2 - 1)$, hence its contribution into $\Pi_2 + \Pi_4 - (2K_2 - 1) \operatorname{lk}(\ell_2, L_+)$ does not change. Other cases of moving of a small oval from one region to another are considered in the same way as in the proof of Lemma 1.2.

Let us compute $\Phi(B)$ for the curve B all whose small ovals are in the rectangle R (see Figure 6). Note, that the above discussion implies that $\Phi(B) = \Phi(\operatorname{cr}(\mathbb{R}A))$. For this curve B we have

$$\Pi_1 + \cdots + \Pi_4 = \begin{cases} -1, & V \text{ is positive,}\\ 0, & V \text{ is negative,}\end{cases}$$

$\operatorname{lk}(\ell_j, L_+) = K_4 - K_j$, $j = 1, 2, 3$, and

$$\begin{aligned} e(b_+) = &- (K_1 + K_2)(2K_1 + 2K_2 - 1) && \text{(contribution of } q_3)\\ &- (K_1 + K_3)(2K_1 + 2K_3 - 1) && \text{(contribution of } q_2)\\ &- (K_2 + K_3)(2K_2 + 2K_3 - 1) && \text{(contribution of } q_1)\\ &+ (K_1 + \cdots + K_4)(2K_2 + \cdots + 2K_4 - 1) && \text{(contribution of } \Delta)\end{aligned}$$

Thus, $\Phi(B)$ is equal to the right hand side of (5). The left hand side of (5) is equal to $\Phi(\operatorname{cr}(\mathbb{R}A))$. $\square$

Lemma 3.3. *One has* $e(b_+) = 3(K_1 + \cdots + K_4) - 2$.

Proof. The proof is similar to that of Lemma 1.3, but now we have one big oval less, i.e., one small oval more, hence $e(b) = 12d - 1$. We have $\mu(L) = 4d + 1$ (the contributions of $N_1, N_3, N_2', N_4', J, V \cup$ (small ovals) are $d, d, d - 1, d - 1, 1, 2$ respectively). Hence $\chi(F) = 2\mu(F) - \mu(L) = 3 - 4d$ and $\mu(F) = (\chi(F) + \mu(L))/2 = 2$. Therefore, as in Lemma 1.3, we have $e(b_+) = m_+ + \mu(L_+) - 2$. It remains to note that $m_+ = 2\mu(L_+) = 2(K_1 + \cdots + K_4)$. $\square$

Proposition 3.1 follows from Lemmas 3.2 and 3.3.

4. Towards a classification of M-curves of degree 9.

A preliminary study of M-curves of degree 9 was done by A.B. Korchagin. Analysing available examples, he formulated [3] the following conjectures about the parity of the numbers α_i in isotopy types of the form $J \sqcup \alpha_0 \sqcup 1\langle\alpha_1\rangle \sqcup \cdots \sqcup 1\langle\alpha_s\rangle$.

(1) If $s = 4$, then $\alpha_0 \equiv 0 \mod 4$ (proven in [7]);
(2) If $s = 4$, then all the numbers $\alpha_1, \ldots, \alpha_4$ are odd (proven in [1]);
(3) If $s = 3$, then at most one of the numbers α_1, α_2, α_3 is even (still open).

The proofs of Conjectures (1) and (2) use only the following tools:

(i) Kharlamov-Viro congruence mod 8 for the union of a 9th degree curve and three lines whose intersection points are in three different nests;
(ii) Bezout's theorem for auxiliary rational curves;
(iii) Rohlin-Mishachev formula for complex orientations;
(iv) Fiedler's rule of alternation of orientations in pencils of lines.

I expected that these tools combined with

(v) Propositions 1.1, 2.1, and 3.1 of this paper

would be enough to prove Conjecture (3). I suggested Severine Fiedler-Le Touzé to try to do it. Recently, using (ii)–(v), she proved a weaker version of Conjecture (3): *if* $s = 3$*, then one of the numbers* α_1*,* α_2*,* α_3 *is odd* (see [2]). Also she found a configuration of oriented embedded circles with respect to lines which contradicts Conjecture (3) but which does not seem to contradict the restrictions (i)–(v).

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