ON OSCULATING FRAMING OF REAL ALGEBRAIC LINKS

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Abstract. For a real algebraic link in $\mathbb{RP}^3$, we prove that its encomplexed writhe (an invariant introduced by Viro) is maximal for a given degree and genus if and only if its self-linking number with respect to the framing by the osculating planes is maximal for a given degree.

1. Introduction and statement of the main result

By *real algebraic curve* in $\mathbb{RP}^3$ we mean a complex curve in $\mathbb{CP}^3$ invariant under complex conjugation. We use the same notation for a real curve and the set of its complex points and, if it is denoted by A, then $\mathbb{R}A$ stands for the set of real points which is called a *real algebraic link* if it is non-empty and A is smooth. A real algebraic link is called *maximally writhed* or MW_λ*-link* if $|w_\lambda(L)|$ (a variation of Viro's invariant [7]) attains the maximal possible value $(d-1)(d-2)/2-g$ where d and g is the degree and genus of A respectively. We refer to [3] for a precise definition of w_λ.

In [3, Thm. 2] we proved that several topological and geometric invariants are maximized on MW_λ-links. In this paper we add one more item to this collection: we show that the self-linking number of L with respect to the osculating framing attains its maximal value (for links of a given degree) if and only if L is an MW_λ-link. The proof is very similar to that of the main theorem of [3]. Let us give precise definitions and statements.

Let L be an oriented link in a rational homology 3-sphere. A *framing* of L is a continuous 1-dimensional subbundle of the normal bundle of L or, equivalently, a continuous field (defined on L) of 2-dimensional planes tangent to L. Given a framed oriented link L, its *self-linking number* is defined as follows. Let F be an embedded annulus or Möbius band with core L, tangent to the framing. Then the self-linking number is $\frac{1}{2}\operatorname{lk}(L,\partial F)$ where the boundary ∂F of F is oriented so that $[\partial F]=2[L]$ in $H_1(F)$.

For an oriented link L in $\mathbb{RP}^3$, the *osculating framing* is the framing defined by the field of osculating planes. We denote the self-linking number of L with respect to this framing by $\operatorname{osc}(L)$. If L is a non-oriented link and O an orientation of L, we use the notation $\operatorname{osc}(L,O)$ which is self-explained.

Recall that a smooth irreducible real algebraic curve A is called an *M-curve* if $\mathbb{R}A$ has $g+1$ connected components where g is the genus of A. In this case $\mathbb{R}A$ divides A into two halves. The boundary orientation on $\mathbb{R}A$ induced by any of

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these halves is called a *complex orientation*. The main result of the paper is the following.

Theorem 1. *Let $L = \mathbb{R}A$ be an irreducible real algebraic link of degree $d \geq 3$ and O be an orientation of L. Then:*

(a) $|\operatorname{osc}(L, O)| \leq d(d-2)/2$.
(b) $|\operatorname{osc}(L, O)| = d(d-2)/2$ *if and only if L is an MW_λ-link* (by [3, Thm. 2], in this case A is an M-curve of genus at most $d-3$) *and O is its complex orientation.*

Remark. In the space of real algebraic links of a given degree and genus we can distinguish three kinds of "walls". The walls of the first kind correspond to curves with a double point with real local branches. When crossing such walls, both invariants $w_\lambda(L)$ and $\operatorname{osc}(L)$ are changed by ± 2. The walls of the second kind correspond to curves with a real double point with complex conjugate local branches. When crossing such walls, $w_\lambda(L)$ does change but $\operatorname{osc}(L)$ does not. The third kind of wall corresponds to curves which have a local branch parametrized by $t \mapsto (t, t^3 + o(t^3), t^4 + o(t^4))$ in some affine chart. When crossing such a wall, $w_\lambda(L)$ does not change but $\operatorname{osc}(L)$ does. So, in general, the invariants $w_\lambda(L)$ and $\operatorname{osc}(L)$ are more or less independent. Nevertheless, Theorem 1 implies that the chamber where they have maximal value is bounded only by the walls of the first kind – common for the both invariants.

2. A variant of Klein's formula for the number of real inflection points

Let $C \in \mathbb{P}^2$ be a nodal real irreducible algebraic curve. It may have three types of nodes: real nodes with real local branches of C, real nodes with imaginary local branches of C, or non-real nodes (coming in conjugate pairs). Denote the number of nodes of each type with h, e, and i respectively.

A *real flex* is a local real branch of C with the order of tangency ω to its tangent line greater than 1 (i.e. the local intersection number is $\omega + 1 \geq 3$). The multiplicity of a real flex is $\omega - 1$. In an affine chart of $\mathbb{P}^2$ a flex corresponds to a critical point of the Gauss map. It is easy to see that the multiplicity of a flex equals the multiplicity of the corresponding critical point. Thus a multiple flex can be thought of as $\omega - 1$ ordinary flexes collected at the same point. We denote with F the number of flexes counted with multiplicities.

A *solitary real bitangent* is a real line $L \subset \mathbb{P}^2$ which is tangent to C at a non-real point (and thus also at the complex conjugate point). The *multiplicity* of L is the sum of the orders ω over all local branches of $C \setminus \mathbb{RP}^2$ tangent to L. We denote with B the number of solitary real bitangents counted with multiplicities. Clearly, B is an even number.

Lemma 2.1. (Klein's formula [1] for nodal curves). *For a nodal real irreducible curve of degree d in $\mathbb{P}^2$ we have*

$$F + B = d(d-2) - 2h - 2i.$$

Proof. As in [6], we use additivity of the Euler characteristic χ to derive Klein's formula. Let $\nu : \tilde{C} \to C$ be the normalization. The space of all real lines in $\mathbb{P}^2$ is homeomorphic to $\mathbb{RP}^2$, and thus has the Euler characteristic 1. For a real line L the set $\nu^{-1}(L)$ consists of d distinct points unless L is tangent to C. Each tangency decreases the size of this set by ω.

Consider the space $X = \{(p, L) \mid p \in C,\ L \ni p\}$, where $L \subset \mathbb{RP}^2$ is a real line. From the observation above we deduce

$$\chi(X) + B + F + \chi(\mathbb{R}\tilde{C}) = d.$$

Note that $\chi(\mathbb{R}\tilde{C}) = 0$ and $\chi(X) = \chi(\nu^{-1}(C \setminus \mathbb{R}C)) = \chi(\tilde{C}) - 2e$, as each point of $\mathbb{R}C$ lifts to a circle in X while $\chi(S^1) = 0$. The lemma now follows from the adjunction formula $\chi(\tilde{C}) = 3d - d^2 + 2e + 2h + 2i$.

Remark 2.2. Lemma 2.1 can be also obtained as an almost immediate consequence from Schuh's generalization [5] of another Klein's formula

$$d - \sum_{x \in C \cap \mathbb{RP}^2} (m(x) - r(x)) = d^\vee - \sum_{x \in C^\vee \cap \mathbb{RP}^{2\vee}} (m^\vee(x) - r^\vee(x))$$

(see [6, Thm. 6.D] for a proof via Euler characteristics) combined with the class formula $d^\vee = d(d-1) - 2e - 2h - 2i$. Here $C^\vee$ is the dual curve, $d^\vee$ is its degree, $m(x)$ and $r(x)$ (resp. $m^\vee(x)$ and $r^\vee(x)$) are the multiplicity and the number of real local branches of C (resp. of $C^\vee$) at x.

3. Proof of the main theorem

Let $L = \mathbb{R}A$ be a smooth irreducible real algebraic link of degree d endowed with an orientation O. Let $\mathcal{U}$ be the set of points p in $\mathbb{RP}^3 \setminus L$ such that the projection of L from p is a nodal curve.

Fix a point $p \in \mathcal{U}$. Let $C_p = \pi_p(A)$ where $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$ is the linear projection from p. Consider the field of tangent planes to L passing through p, (so-called blackboard framing). Let $b_p(L)$ be the self-linking number with respect to it. We have

$$b_p(L) = \sum_q s(q), \qquad \text{thus} \qquad |b_p(L)| \le h(C_p) \tag{1}$$

where q runs over the hyperbolic (i. e., with real local branches) double points of C_p, $h(C_p)$ is the number of them, and $s(q)$ is the sign of the crossing at q in the sense of knot diagrams. The difference $|\operatorname{osc}(L) - b_p(L)|$ is bounded by one half of the number of those points where the osculating plane passes through p (each such point contributes $\pm 1/2$ or 0 to $\operatorname{osc}(L)$). This is the number of real flexes of C_p which we denote by $f(C_p)$. We have $f(C_p) \le d(d-2) - 2h(C_p)$ by Lemma 2.1. Thus

$$|\operatorname{osc}(L)| \le |\operatorname{osc}(L) - b_p(L)| + |b_p(L)| \le \tfrac{1}{2} f(C_p) + h(C_p) \le \tfrac{1}{2} d(d-2) \tag{2}$$

which is Part (a) of Theorem 1.

Now suppose that $|\operatorname{osc}(L)| = d(d-2)/2$. Then for any choice of $p \in \mathcal{U}$ we have the equality sign everywhere in (2), in particular, we have the equality sign in (1), i.e., all crossings are of the same sign, say, positive:

$$s(q) = +1 \qquad \text{for any hyperbolic crossing } q \text{ of } C_p. \tag{3}$$

By Lemma 2.1, the equality sign in the last inequality of (2) implies that all flexes of C_p are ordinary for any choice of $p \in \mathcal{U}$. This implies that L has non-zero torsion at each point. Indeed, otherwise there exists a real plane P which has tangency with L of order greater than 3. It is easy to check that $\mathcal{U}$ has non-empty intersection with any plane, thus we can choose a point $p \in \mathcal{U} \cap P$, and then C_p would have a k-flex with $k > 3$. Moreover, the positivity of all crossings for any generic projection implies that the torsion is everywhere positive (cf. the proof of [2, Prop. 1]).

Similarly to [2, Lem. 7] and [3, Lem. 4.10], we derive from these conditions that the real tangent surface TL (the union of all real lines in $\mathbb{RP}^3$ tangent to L) is a union of (non-smooth) embedded tori. Indeed, suppose that two tangent lines cross. Let P be the plane passing through them (any plane passing through them if they coincide) and let ℓ be the line passing through the two tangency points. Let p be a generic real point on ℓ. Then C_p has two real local branches at the same point such that each of them is either singular or tangent to the line $\pi_p(P)$. Since L has non-zero torsion, all singular branches of C_p are ordinary cusps. Then we can find a generic point close to p such that the projection from it does not satisfy (3).

Let $K_1, \dots, K_n$ be the connected components of L, and let TK_i be the connected component of TL that contains K_i (the union of real lines tangent to K_i). The same arguments as in [3, Lemma 4.12] show that, for some positive integers $a_1, \dots, a_n$, there exist real lines ℓ_i, ℓ'_i, $i = 1, \dots, n$, such that (for suitable choice of the orientations) the linking numbers of their real loci $l_i = \mathbb{R}\ell_i$ and $l'_i = \mathbb{R}\ell'_i$ with the components of L are:

$$2\,\mathrm{lk}(l_i, K_i) = a_i + 2, \qquad 2\,\mathrm{lk}(l'_i, K_i) = a_i. \tag{4}$$

Moreover, each TK_i splits $\mathbb{RP}^3$ into two solid tori U_i and V_i such that $l_i \subset U_i$, $l'_i \subset V_i$, the homology classes $[l_i]_U$ and $[l'_i]_V$ generate $H_1(U_i)$ and $H_1(V_i)$ respectively, and we have $[K_i]_U = a_i[l_i]_U$ and $[K_i]_V = (a_i + 2)[l'_i]_V$. It follows that

$$2\,\mathrm{osc}(K_i) = a_i(a_i + 2) \tag{5}$$

(the linking number of K_i with its small shift disjoint from TL). Indeed, if K_i is parametrized by $t \mapsto r(t)$ and the torsion is non-zero, then TK_i has a cuspidal edge along K_i and a small shift of K_i in the direction of the vector field $\ddot{r}$ is disjoint from TK_i (see Figure 1). A priori this argument proves (5) up to sign only. However the positivity of the torsion implies that $\mathrm{osc}(K_i)$ is positive.

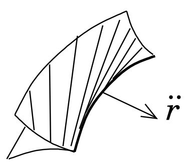

FIGURE 1

If L is connected (i. e., $n = 1$), it remains to note that then the condition $2\,\mathrm{osc}(K_1) = d(d-2)$ implies $(a_1 + 2)a_1 = d(d-2)$, hence $a_1 = d - 2$. Thus

L satisfies Condition (v) of [3, Thm. 1] which completes the proof that L is an MW-knot.

If L is not necessarily connected, we argue as follows. By Murasugi's result [4, Prop. 7.5] (see also [3, Prop. 1.2]), the number of crossings of any projection of K_i is at least $(a_i+2)(a_i-1)/2$. Hence, for $h = h(C_p)$, we have

$$2h \ge \sum_{i=1}^{n}(a_i+2)(a_i-1) + \sum_{i\neq j}|\operatorname{lk}(K_i,K_j)|. \qquad (6)$$

On the other hand, if we choose p on a line passing through a pair of complex conjugate points of A, then C_p has at least one elliptic double point (i. e., a real double point with complex conjugate local branches), whence by the genus formula we obtain

$$h \le (d-1)(d-2)/2 - g - 1 \le (d-1)(d-2)/2 - n \qquad (7)$$

(the second inequality in (7) is the Harnack's bound). Hence

$$\begin{aligned} d(d-2) = 2\operatorname{osc}(L) &= 2\sum_{i=1}^{n}\operatorname{osc}(K_i) + \sum_{i\neq j}\operatorname{lk}(K_i,K_j) \\ &\le \sum_{i=1}^{n} a_i(a_i+2) + 2h - \sum_{i=1}^{n}(a_i+2)(a_i-1) \qquad \text{by (5) and (6)} \\ &= 2h + 2n + \sum_{i=1}^{n} a_i \le (d-1)(d-2) + \sum_{i=1}^{n} a_i. \qquad \text{by (7)} \end{aligned}$$

Thus $\sum a_i \ge d-2$ and we conclude that L is an MW_λ-link. This fact follows from [3, Prop. 1.1] (which implies that $\operatorname{ps}(L) = \sum a_i$) combined with [3, Thm. 2] (which claims, in particular, that L is an MW_λ-link as soon as $\operatorname{ps}(L) \ge d-2$). Here we denote with $\operatorname{ps}(L)$ the *plane section number* of L. It is a topological invariant of a link in $\mathbb{RP}^3$ defined in [3] as the minimal number of intersection points with a generic plane where the minimum is taken over the isotopy class of the link.

Let us show that O is a complex orientation of L. It is easy to see that the plane section number is at most $d-2$ for any algebraic link of degree d. Indeed, it is enough to consider a small shift of a non-osculating tangent plane in a suitable direction. Thus the inequality in $\operatorname{ps}(L) = \sum a_i \ge d-2$ is in fact an equality. It follows that the equality is attained in all the inequalities used in the proof, in particular, we have $|\operatorname{lk}(K_i,K_j)| = \operatorname{lk}(K_i,K_j)$ for $i \neq j$. Since all components of an MW_λ-link endowed with a complex orientation are positively linked (see [3]), we are done. This completes the proof of the "only if" part of (b).

To prove the "if" part of (b), we notice that by [3, Thm. 3 and §4.4], any MW_λ-link L of degree d and genus g is a union of $g+1$ knots $K_0 \cup \dots \cup K_g$ and $\operatorname{lk}(K_i,K_j) = a_i a_j$, $i \neq j$, for some positive integers $a_0,\dots,a_g$ with $a_0+\dots+a_g = d-2$. Furthermore, the torsion of L is everywhere positive and each knot K_i is arranged on its tangent surface TK_i as described above, thus (5) holds for each i, and we obtain

$$\begin{aligned} 2\operatorname{osc}(L) &= \sum_{i=0}^{g}\operatorname{osc}(K_i) + \sum_{i\neq j}\operatorname{lk}(K_i,K_j) = \sum_{i=0}^{g} a_i(a_i+2) + \sum_{i\neq j} a_i a_j \\ &= \Big(\sum a_i\Big)^2 + 2\sum a_i = (d-2)^2 + 2(d-2) = d(d-2). \end{aligned}$$

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