MAXIMALLY WRITHED REAL ALGEBRAIC LINKS

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ABSTRACT. Oleg Viro introduced an invariant of rigid isotopy for real algebraic knots and links in $\mathbb{RP}^3$ which is not a topological isotopy invariant. In this paper we study real algebraic links of degree d with the maximal value of this invariant. We show that these links admit entirely topological description. In particular, these links are characterized by the property that any of their planar diagram has at least $(d-1)(d-2)/2 - g - 1$ crossing points where g is the genus of the complexification. Also we show that these links are characterized by the property that any generic plane intersects them in at least $d - 2$ real points. In addition we give a complete topological classification of these links.

1. INTRODUCTION. PLANAR AND SPATIAL REAL ALGEBRAIC CURVES

Topological study of real projective algebraic curves can be traced back at least to the works of A. Harnack [7] and F. Klein [9]. Given two natural numbers d, n there are only finitely many homeomorphism types of pairs $(\mathbb{RP}^n, \mathbb{R}A)$, where $\mathbb{R}A$ is a smooth real algebraic curve $\mathbb{R}A \subset \mathbb{RP}^n$ of degree d. These homeomorphism types are known as *topological arrangements* of real algebraic curves, and a natural question is to list them all. Clearly, this question is not too interesting for $n > 3$ as topologically $\mathbb{R}A$ is a disjoint union of embedded circles, and they cannot be knotted or linked in high dimensions. However for $n = 2$ (when one oval of $\mathbb{R}A$ may sit either inside or outside of another oval) and for $n = 3$ (when the topology of $\mathbb{R}A$ becomes the subject of Knot Theory) the question is far from being trivial.

The case of $n = 2$ was popularized by D. Hilbert[1], and the corresponding problem became known as the 16th Hilbert problem[2]. This problem remains one of the few open problems in Hilbert's famous problem list. As it was shown in [7], a planar real algebraic curve may have up to $(d-1)(d-2)/2 + 1$ components. The curves with exactly $(d-1)(d-2)/2 + 1$ components are known as M-curves. (Note that $g = (d-1)(d-2)/2$ is the genus of a smooth curve of degree d in the plane by the adjunction formula.)

Currently the complete topological classification is known only for curves of degree ≤ 7 [20] (and of degree ≤ 8 [17] for M-curves in the pseudoholomorphic version of the problem). The number of topological arrangements grows roughly as the exponent of the square of degree (see [8]). Even with many known restrictions and a powerful *patchworking* technique for construction (see e.g. [21] for a survey) it

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[1]with a particular focus on the case of $d = 6$, and an even more particular attention to a certain curve proven not to exist by Petrovsky [18].

[2]more specifically, its first part.

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is not currently clear how one could approach the question for arbitrary d apart from trying to construct or prove non-existence of each suitable topological arrangement individually.

Meanwhile, there is a distinguished class of real smooth algebraic curves in $\mathbb{RP}^2$ (or, rather, $\mathbb{RP}^2$ with three coordinate lines fixed) with a unique topological arrangement for every degree d. This class of curves is known as *simple Harnack curves*, it can be described in several equivalent ways, see [11]. One of its possible characterization is through the so-called *quantum index* κ, a half-integer number that can be associated with some real algebraic curves in the plane (of type I and intersecting the coordinate axes only at real points), see [12]. Namely, if $|\kappa|$ attains its maximal possible value $d^2/2$, then the curve is a simple Harnack curve, and its topological arrangement is described by [11].

The case of $n = 3$ corresponds to the study of topological arrangements of spatial algebraic curves. The genus of a spatial curve is not determined by the degree, so for a classification we fix the genus g in addition to the degree d. The classification of topological arrangements of spatial curves in the case $d \leq 5$ and $g = 0$ was obtained in [1]. It was advanced to $d \leq 6$ and $g \leq 1$ in [13].

These classifications used the so-called *encomplexed writhe* w introduced by Viro in [22] as well as its modification w_λ for curves with $g > 0$. The number w_λ is integer and subject to the inequality

$$|w_\lambda| \leq (d - 1)(d - 2)/2 - g.$$

It is not a topological invariant, but is preserved under deformations in the class of smooth real algebraic curves in $\mathbb{RP}^3$. Indeed, according to the experimental data from $d \leq 6$ classifications, the same topological arrangement may correspond to distinct w_λ in the case of $|w_\lambda| < (d - 1)(d - 2)/2 - g$, but not in the case of $|w_\lambda| = (d - 1)(d - 2)/2 - g$. We refer to curves with $|w_\lambda| = (d - 1)(d - 2)/2 - g$ as *maximally writhed*, or MW_λ-curves.

In this paper we show that MW_λ-curves are topologically classified by partitions of the number $d - 2$ into $g + 1$ positive integer numbers. In particular, for $g = 0$ their topological arrangement is unique for any d (which was proved in [14]). In addition, we provide several characteristic properties of MW_λ-curves that may be used as their alternative definition. In particular, we show that MW_λ-curves are the only spatial curves of degree d such that any of its planar diagram has at least $(d - 1)(d - 2)/2 - g - 1$ crossings. We also show that MW_λ-curves are the only spatial curves of degree d such that any generic plane intersects them at least in $d - 2$ real points. The curves for $d = 6$ and $g = 1$ corresponding to the partitions $4 = 2 + 2$ and $4 = 3 + 1$ are depicted at Figure 1.
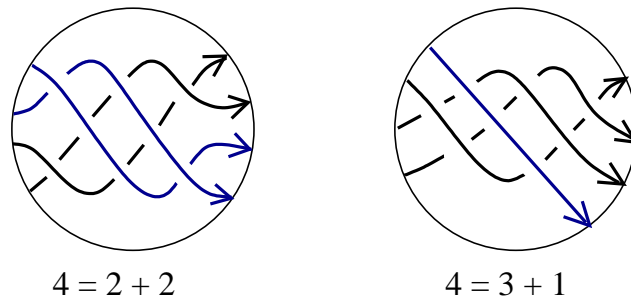

$$4 = 2 + 2 \qquad\qquad 4 = 3 + 1$$

FIGURE 1. MW_λ-links of degree 6 and genus 1

2. STATEMENT OF MAIN RESULTS

2.1. Rational curves and real algebraic knots. We use the same terminology as in [14]. Namely, a *real algebraic curve* in $\mathbb{P}^3$ is a (complex) one-dimensional subvariety A in $\mathbb{P}^3 = \mathbb{CP}^3$ invariant under the involution of complex conjugation conj $: \mathbb{P}^3 \to \mathbb{P}^3$, $(x_0 : x_1 : x_2 : x_3) \mapsto (\bar{x}_0 : \bar{x}_1 : \bar{x}_2 : \bar{x}_3)$. The conj-invariance is equivalent to the fact that A can be defined by a system of homogeneous polynomial equations with real coefficients. The degree of A is defined as its homological degree, i. e. the number d such that $[A] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^3) \cong \mathbb{Z}$. A curve of degree d intersects a generic complex plane in d points.

We denote the real locus (i.e., the set of real points) of A by $\mathbb{R}A$. We say that a real curve A is *smooth* if it is a smooth complex submanifold of $\mathbb{P}^3$. In this case, $\mathbb{R}A$ is a smooth real submanifold of $\mathbb{RP}^3$ and if each of its irreducible components is non-empty, we call it a *real algebraic link* or, more specifically, a *real algebraic knot* in the case when $\mathbb{R}A$ is connected.

According to our definition, if $L = \mathbb{R}A$ is an algebraic link, then A is uniquely determined by L. In this case we call it the *complexification of L* and we write $A = \mathbb{C}L$. We emphasize again that, by definition, the complexification of a real algebraic link is a *smooth* complex subvariety of $\mathbb{CP}^3$. We say that a real algebraic link is *irreducible* if its complexification is connected. All links considered in this paper are irreducible.

Two real algebraic links are called *rigidly isotopic* if their complexifications belong to the same connected component of the space of real algebraic links of the same degree. A rigid isotopy classification of real algebraic knots of genus 0 in $\mathbb{P}^3$ is obtained in [1] up to degree 5 and in [13] up to degree 6. Also we gave in [13] a rigid isotopy classification for genus one knots and links up to degree 6 (here we speak of the genus of the complex curve A rather than the minimal genus of a Seifert surface of $\mathbb{R}A$).

It happens that in all the above-mentioned cases, a rigid isotopy class is completely determined by the usual (topological) isotopy class, the complex orientation (for genus one links), and the invariant of rigid isotopy w introduced by Viro [22] (called in [22] *encomplexed writhe*). This invariant is defined as the sum of signs of crossings of a generic projection but the real crossings with non-real branches are also counted with appropriate signs, while the crossings between different link components are not counted; see details in [22] (the definition of w is also reproduced in [13]). So, w changes by 2 each time when the sign of a crossing is switched from -1 to $+1$, i.e. it can be viewed as a first order Vassiliev invariant.

Let $T(p,q) = \{(z,w) \mid z^p = w^q\} \cap \mathbb{S}^3$, $p \geq q \geq 0$, be the (p,q)-torus link in the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ of radius $\sqrt{2}$. If $p \equiv q \mod 2$, we define the *projective torus link* $T_{\mathrm{proj}}(p,q) = T(p,q)/(-1) \subset \mathbb{S}^3/(-1) = (\mathbb{C}^2 \setminus 0)/\mathbb{R}^\times = \mathbb{RP}^3$ (here we refer to an identification of $\mathbb{R}^4$ with $\mathbb{C}^2$). Note that the link $T_{\mathrm{proj}}(p,q)$ sits in the torus $\{|z|^2 = |w|^2 = 1\}/(-1) \subset \mathbb{S}^3/(-1)$ which corresponds to a hyperboloid under the identification of $\mathbb{S}^3/(-1)$ with $\mathbb{RP}^3$. The link $T_{\mathrm{proj}}(p,q)$ is an algebraic curve of bidegree $(a,b) = ((p+q)/2, (p-q)/2)$ on this hyperboloid, see [13; §3] and Section 5.1. This is why in [13] we called this link *hyperboloidal link $h_{a,b}$*. The curve $T_{\mathrm{proj}}(p,q) \subset \mathbb{P}^3$ is a real algebraic curve of degree p. If $\gcd(a,b) = 1$ then it is irreducible, and, furthermore, rational. If $b = 1$ then this curve is smooth (otherwise it has a pair of complex conjugate singularities of type $u^a = v^b$), thus $T_{\mathrm{proj}}(d, d-2)$ is an algebraic knot.

For a link L in $\mathbb{RP}^3$ (a differentiable real one-dimensional submanifold), we define the *crossing number* $\mathrm{cr}(L)$ in the same way as for links in $\mathbb{S}^3$. It is the minimal number of crossings of a generic plane projection where the minimum is taken over the isotopy class of L. Similarly we define the *plane section number* $\mathrm{ps}(L)$ as the minimal number of intersections with a generic plane where again the minimum is taken over the isotopy class of L. The plane section number of a link in $\mathbb{RP}^3$ can be equivalently defined as the smallest number of immersed arcs of any diagram of the link (in the sense of [2]; see also Section 5.3).

Let $N_d = (d-1)(d-2)/2$. This is the maximal number of nodes for an irreducible planar curve of degree d and thus also the maximal possible value of w for real algebraic knots of degree d. It is not difficult to check that $w(T_{\mathrm{proj}}(d, d-2)) = N_d$. The main result for genus 0 is the following.

Theorem 1. *Let K be a real algebraic knot of genus 0 and of degree $d \geq 3$. Then the following conditions are equivalent:*[3]

  (i) $|w(K)| = N_d$;
  (ii) *any real plane tangent to K has only real intersections with $\mathbb{C}K$;*
 (iii$^{\mathrm{a}}$) *any generic real plane cuts K at d or $d-2$ real points;*
 (iii$^{\mathrm{t}}$) $\mathrm{ps}(K) = d-2$;
 (iv$^{\mathrm{a}}$) *any generic projection of K has at least $N_d - 1$ hyperbolic double points;*
 (iv$^{\mathrm{t}}$) $\mathrm{cr}(K) = N_d - 1$;
  (v) *K or its mirror image is isotopic to $T_{\mathrm{proj}}(d-2, d) = h_{d-1,1}$.*

In [14] we have defined *MW-knots* as real algebraic knots satisfying (i). In the same paper we have proved the implication (i) $\Rightarrow$ (v).

As it is seen from the formulation, the implications (v)$\Rightarrow$(iii$^{\mathrm{t}}$) and (v)$\Rightarrow$(iv$^{\mathrm{t}}$) are purely topological facts. They are specializations for $(p, q) = (d, d-2)$ of the following results.

Proposition 2.1. *Let $1 \leq q \leq p$ and $p \equiv q \mod 2$. Then $\mathrm{ps}(T_{\mathrm{proj}}(p, q)) = q$.*

This proposition is proven in §5.2.

Proposition 2.2. (Murasugi [15]). *Let $1 \leq q \leq p$ and $p \equiv q \mod 2$. Then $\mathrm{cr}(T_{\mathrm{proj}}(p, q)) = p(q-1)/2$.*

Proof. By [15; Proposition 7.5], $\mathrm{cr}(T(p, q)) = p(q-1)$. It remains to observe that the double covering of any projection of $T_{\mathrm{proj}}(p, q)$ is a projection of $T(p, q)$. $\square$

Conjecture 2.3. For $1 \leq q \leq p$, any diagram of $T(p, q)$ with $p(q-1)$ crossings is positive (i.e. all its crossings are positive).

Conjecture 2.4. All *MW*-knots of degree d are rigidly isotopic to $T_{\mathrm{proj}}(d, d-2)$.

2.2. Generalization for arbitrary genus.

Theorem 1 admits a generalization for algebraic links of any genus. Given an irreducible algebraic link in s connected components $L = L_1 \sqcup \cdots \sqcup L_s = \mathbb{R}A$ where A is a *dividing* (or *Type I*) real algebraic curve (this means that $A \setminus \mathbb{R}A$ is not connected; see [19]), it is convenient to introduce another invariant (see [13]):

$$w_\lambda(L) = w(L) + 2\sum_{i<j} \mathrm{lk}(L_i, L_j), \tag{1}$$

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[3] The superscripts $^{\mathrm{a}}$ and $^{\mathrm{t}}$ in "(iii$^{\mathrm{a}}$)", "(iii$^{\mathrm{t}}$)", etc. come from 'algebraic' and 'topological'.

where the linking numbers are considered with respect to a *complex orientation* of $\mathbb{R}A$, that is the boundary orientation induced from one of the two halves into which $\mathbb{R}A$ divides A. In other words, w_λ is defined in the same way as w, but all the real crossings are counted, not only those where the both crossing branches belong to the same link component.

For an arbitrary (not necessarily dividing) real algebraic link $L = L_1 \sqcup \cdots \sqcup L_s = \mathbb{R}A$ we may also define the invariant

$$w_{|\lambda|}(L) = |w(L)| + 2 \sum_{i<j} |\operatorname{lk}(L_i, L_j)|.$$

It is clear that the absolute values of all the three invariants $w(L)$, $w_\lambda(L)$, and $w_{|\lambda|}(L)$ are bounded by $N_d - g$ where g is the genus of A. Indeed, $N_d - g$ is the maximal possible number of double points of an irreducible plane projective curve of degree d and genus g.

We recall that a non-singular real algebraic curve A is called an M-curve if the number of connected components of $\mathbb{R}A$ attains its maximal possible value $\operatorname{genus}(A) + 1$. It is well-known (and immediately seen from Euler characteristics considerations) that M-curves are dividing.

Theorem 2 generalizes Theorem 1 to the case of arbitrary genus. In particular, it provides a description of the isotopy type of links with the maximal (for given degree and genus) value of w_λ and $w_{|\lambda|}$.

For positive integers $a_0, \ldots, a_g$ we define $W_g(a_0, \ldots, a_g)$ as the (topological) link

$$\left(\mathbb{S}^3 \cap \bigcup_{i=0}^{g} \left\{ (w - c_i z)^{a_i} = \varepsilon z^{a_i+2} \right\} \right)/(-1) \qquad \text{in} \quad \mathbb{S}^3/(-1) \cong \mathbb{RP}^3$$

where $c_0, \ldots, c_g$ are distinct complex numbers and $0 < \varepsilon \ll 1$. We orient it as the boundary of the complex curve in the 4-ball. This is a $(g+1)$-component link $K_0 \cup \cdots \cup K_g$, each component K_i being isotopic to $T_{\text{proj}}(a_i + 2, a_i)$ and placed in a tubular neighborhood U_i of the line $l_i = \{w = c_i z\}/(-1)$ so that $[K_i] = a_i[l_i]$ in $H_1(U_i)$ (we orient l_i also as the boundary of the complex disk in the 4-ball). We assume that ε is so small that the U_i's can be chosen pairwise disjoint.

We call $l_0 \cup \cdots \cup l_g$ the *Hopf link* in $\mathbb{RP}^3$. It is indeed a union of fibers of the Hopf fibration $\eta : \mathbb{RP}^3 = (\mathbb{C}^2 \setminus 0)/\mathbb{R}^\times \to (\mathbb{C}^2 \setminus 0)/\mathbb{C}^\times = \mathbb{CP}^1$. So, the Hopf link is a union of oriented lines in $\mathbb{RP}^3$ such that any two lines are linked positively (the linking number is $\frac{1}{2}$). We show in Proposition 3.1 that it is uniquely determined by this condition.

Theorem 2. *Let L be an irreducible real algebraic link of degree $d \geq 3$ and genus g which is not contained in a plane. Then the following conditions are equivalent:*

   (i) $w_{|\lambda|}(L) = N_d - g$;
   (i$'$) $\mathbb{C}L$ *is dividing and* $|w_\lambda(L)| = N_d - g$;
   (ii) *any real plane tangent to L has only real intersections with $\mathbb{C}L$;*
  (iii^a) *any generic real plane cuts L at d or $d-2$ real points;*
  (iii^t) $\operatorname{ps}(L) = d - 2$;
  (iv^a) *any linear projection of L from a generic point of $\mathbb{RP}^3$ has at least $N_d - g - 1$ hyperbolic double points;*
  (iv^t) $\operatorname{cr}(L) = N_d - g - 1$;

(v) L or its mirror image is isotopic to $W_g(a_0, \ldots, a_g)$ for some positive integers $a_0, \ldots, a_g$ such that $a_0 + \cdots + a_g = d - 2$ (in particular, $g \le d - 3$);

(v′) Condition (v) holds in the sense of oriented links for a complex orientation of L (note that (v) implies that $\mathbb{C}L$ is an M-curve, so it has complex orientation);

Remark 2.5. (1) Given any curve A (not necessarily dividing) and any orientation O of $L = \mathbb{R}A$, the right hand side of (1) is well-defined and we denote the resulting invariant of (L, O) by $w_\lambda(L, O)$. Then Condition (i) in Theorem 2 can be replaced (without any change in the proof) by the condition that $|w_\lambda(L, O)| = N_d - g$ for some orientation O of L.

(2) It is highly likely that Condition (ii) can be replaced by a weaker condition that A is hyperbolic with respect to any real tangent line (see Section 4.3 for definition of hyperbolic spatial curves). The equivalence of these conditions would be consequence of the following conjecture: for any smooth non-planar spatial curve, almost all tangent lines intersect the curve at the tangency point only.

We call irreducible real algebraic links satisfying any of the conditions (i)–(v′) of Theorem 2 MW_λ-*links*. The following corollary shows that, in a sense, MW-links of nonzero genus do not exist.

Corollary 2.6. *Under the hypothesis of Theorem 2, if $|w(L)|$ attains the upper bound $N_d - g$, then $g = 0$.*

Proof. We have $|w| \le w_{|\lambda|} \le N_d - g$. Thus $|w| = N_d - g$ implies (i) and hence (v′). However, if $g > 0$, then the linking number of any two components of $W_g(a_0, \ldots, a_g)$ is positive, which contradicts the fact that $|w| = w_{|\lambda|}$. $\square$

It is interesting to study what is the maximal possible value of w for links of degree d and genus g. The classification obtained in $[13]^4$ shows that for $g = 1$ and $d = 4, 5, 6$, the maximal value is attained on $W_1(d - 3, 1)$ and thus is equal to $(N_d - 1) - (d - 3)$. It is natural to expect that for any $g \ge 0$ and any $d \ge g + 3$, the maximum of w is attained on $W_g(d - g - 2, 1, \ldots, 1)$ and thus is equal to $(N_d - g) - g(d - g - 2) - g(g - 1)/2$.

The following theorem claims realizability of all isotopy types allowed by Theorem 2 thus finishing the topological classification of MW_λ-links of arbitrary degree and genus.

Theorem 3. *For any $d \ge 3$ and any positive integers $a_0, \ldots, a_g$ such that $a_0 + \cdots + a_g = d - 2$, there exists an irreducible real algebraic link of degree d and genus g, isotopic to $W_g(a_0, \ldots, a_g)$.*

2.3. Organization of the paper.

We prove Theorem 2 in §4 (Theorem 1 is, evidently, a partial case of Theorem 2). The longest part of the proof concerns the implication (ii) $\Rightarrow$ (v′) (§4.4). The main idea is to show that if L satisfies (ii), then the tangent surface of L (that is the union of all tangent lines) is topologically isotopic to the boundary of a tubular

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[4]For $d = 4$, $g = 1$, contrary to an erroneous statement from [9, §7.1], there exists a non-planar link, namely $W_1(1, 1)$. Using methods of [13] it is easy to show that there are exactly 4 rigid isotopy classes: $\pm W_1(1, 1)$ $(w_\lambda = \pm 2)$, $\bigcirc\bigcirc$ $(w_\lambda = 0)$, and $\bigcirc$ $(w = 0)$.

neighborhood of the Hops link (the isotopy cannot be smooth because the tangent surface is not: it has a cuspidal edge along L; see Figures 5 and 6).

In §3 we prove that any configuration of positively linked lines is rigidly isotopic to the projective Hopf link. This fact is used in the proof of Theorem 2 but in our opinion it is interesting by itself. In §5 we discuss some topological properties of the projective links $T_{\mathrm{proj}}(p, q)$. The subsections 5.3–5.6 are not used in the rest of the paper. In §6 we prove Theorem 3.

3. Uniqueness of a configuration of positively linked lines in $\mathbb{RP}^3$ up to rigid isotopy

In this section a real line is understood in the usual sense, as a subset of $\mathbb{RP}^3$ rather than a conj-invariant complex line in $\mathbb{CP}^3$. The following fact is certainly well known for specialists but we did not find any reference.

Proposition 3.1. *Let $L = l_0 \cup \cdots \cup l_n$ and $L' = l'_0 \cup \cdots \cup l'_n$ be two unions of n pairwise disjoint oriented lines in $\mathbb{RP}^3$ such that $\mathrm{lk}(l_i, l_j) = \mathrm{lk}(l'_i, l'_j) = \frac{1}{2}$ for any $i < j$. Then L is rigidly isotopic to L'.*

(When speaking of rigid isotopy, we consider L and L' as reducible real algebraic links.)

Proof. It is enough to show that L is rigidly isotopic to some standard arrangement of lines. As a standard arrangement we choose a collection of lines lying on the same hyperboloid. Note that such an arrangement is a particular case of a Hopf link (see the introduction) because $\eta^{-1}(C)$ is a hyperboloid when C is a circle on $\mathbb{CP}^1$; recall that $\eta : \mathbb{RP}^3 \to \mathbb{CP}^1$ is the Hopf fibration.

Let us fix an affine chart (x, y, z) so that l_0 is the line at infinity in the plane $x = 0$. Then (for an appropriate orientation of l_0) the condition $\mathrm{lk}(l_0, l_i) > 0$ for $i > 0$ means that the x-coordinate increases along each of $l_1, \ldots, l_n$, i.e. all the lines are oriented from the left to the right (see Figure 2(a)). Let us consider the parallel projection $\pi : (x, y, z) \mapsto (x, y)$. We assume that the lines are numbered in the descending order according to the slope of $\pi(l_i)$. Then the positivity of the linking numbers means that each l_i passes over $l_1, \ldots, l_{i-1}$ at the crossings (see Figure 2(a)).
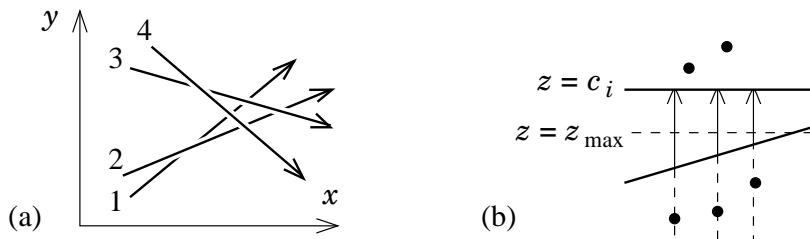

FIGURE 2. (a) $\pi(L \setminus l_0)$; (b) moving l_i inside $\pi^{-1}(l_i)$

Let $z_{\max}$ be the maximum of the z-coordinates of all the preimages of the crossing points, and let $y = c_i x + b_i$ be the equation of $\pi(l_i)$, $i = 1, \ldots, n$. Our assumption about the slopes means that $c_1 > c_2 > \cdots > c_n$. By shifting vertically all the configuration, we may achieve that $c_n > z_{\max}$.

At the first step we move each line l_i (one by one starting with l_n) inside the plane $\pi^{-1}(l_i)$ to place it in the horizontal plane $z = c_i$; see Figure 2(b) where the

lines l_j for $j \neq i$ are represented by their intersection points with the plane $\pi^{-1}(l_i)$. The condition $c_n > z_{\max}$ guarantees that l_i does not cross the other lines during this motion; see Figure 2(b). Indeed, let $l_{ij} = \pi^{-1}(\pi(l_i) \cap \pi(l_j))$. Then, during the motion of l_i, the final position of $l_i \cap l_{ij}$ ($j \neq i$) is over its initial position (because $c_i > z_{\max}$), and $l_j \cap l_{ij}$ cannot be between them.

At the second step we parallelly translate each line (staying in the horizontal plane) to make it cross the axis $x = y = 0$. Thus the final position of the i-th line is given by the system of two equations $z = c_i$, $y = c_i x$. This means that all the lines (including l_0 !) sit on the hyperboloid $y = zx$. $\square$

4. Classification of MW_λ-links (proof Theorem 2)

4.1. Non-vanishing of the torsion.

Recall that the sign of the (differential geometric) torsion of a curve $t \mapsto r(t) \in \mathbb{R}^3$, $t \in \mathbb{R}$, coincides with the sign of $\det(r', r'', r''')$ and it does not depend on the parametrization if $r' \neq 0$. The sign of the torsion of a curve in $\mathbb{RP}^3$ does not depend on a choice of positively oriented affine chart.

Lemma 4.1. *Let $L = \mathbb{R}A$ be an irreducible real algebraic link of degree d which is not contained in any plane. If Condition (ii) of Theorem 2 holds, then the torsion of L nowhere vanishes.*

Proof. Let $p \in L$. We can always choose affine coordinates (x_1, x_2, x_3) centered at p such that L near p admits a parametrization

$$t \mapsto (x_1(t), x_2(t), x_3(t)), \quad 1 = \text{ord}_t\, x_1 < \text{ord}_t\, x_2 < \text{ord}_t\, x_3.$$

Suppose that the torsion at p is zero, i.e. $a = \text{ord}_t\, x_3 \geq 4$, and let us prove that then (ii) does not hold.

Consider the projection $\pi : (x_1, x_2, x_3) \mapsto (x_1, x_3)$. Then $\pi(L)$ is parametrized by $t \mapsto \gamma(t) = (t, ct^a + o(t^a))$, $c \neq 0$. Let ℓ_t be the line on the plane (x_1, x_3) tangent to $\pi(L)$ at $\gamma(t)$. If $0 < |t| \ll 1$, then ℓ_t has at least two non-real intersections with $\pi(A)$ near the origin, thus the plane $\pi^{-1}(\ell_t)$ does not satisfy (ii). $\square$

4.2. Hyperbolic planar curves.

Let C be a real algebraic curve of degree d in $\mathbb{RP}^2$ (maybe, singular). Recall that C is called *hyperbolic* with respect to a point $q \in \mathbb{RP}^2$ (which may or may not belong to C), if any real line through q intersects C at real points only. We denote:

$$\text{hyp}(C) = \{q \mid C \text{ is hyperbolic with respect to } q\}. \tag{2}$$

It is easy to check that $\text{hyp}(C)$ is either empty or a convex closed set. It is possible that $\text{hyp}(C)$ contains only one point. In this case, the point should be singular. For example, if C is a cuspidal cubic, then $\text{hyp}(C)$ consists of the cusp only.

Lemma 4.2. *Let C be a real plane curve, $q \in \text{hyp}(C)$, and $q_1 \in \mathbb{R}C \setminus \{q\}$. Then each local branch of C at q_1 is smooth, real, and transverse to the line (qq_1). The projection from q defines a covering $\mathbb{R}\tilde{C} \to \mathbb{RP}^1$ where $\tilde{C}$ is the normalization of C.*

Proof. Let (x, y) be an affine chart such that $q_1 = (0, 0)$ and q is the infinite point of the y-axis. Let $t \mapsto (x(t), y(t))$, $0 \mapsto 0$, be a real analytic parametrization of a local branch of C at q_1. Let $x(t) = c\, t^a + o(t^a)$, $c \neq 0$, $a \geq 1$. If $a > 1$, then the

line $\{x = \varepsilon\}$, $0 < |\varepsilon| \ll 1$, $\varepsilon c < 0$, has non-real intersections with C near q_1 which contradicts the hyperbolicity condition. Thus $a = 1$ and the result follows $\quad\square$

A real algebraic curve is called *dividing* if its normalization is divided by the real locus into two halves. The following lemma is due to Rokhlin [19] in the case of smooth curves, the same proof also holds in the singular case.

Lemma 4.3. *Let C be a real algebraic curve in $\mathbb{P}^2$, and $p \in \mathrm{hyp}(C)$. Then C is dividing and a complex orientation of $\mathbb{R}C$ is the pull-back of an orientation of $\mathbb{R}\mathbb{P}^1$ under the central projection $\mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ restricted to $\mathbb{R}C$.* $\quad\square$

4.3. Hyperbolic spatial curves.

Let $L = \mathbb{R}A$ be an irreducible real algebraic link in $\mathbb{R}\mathbb{P}^3$ of degree d. The line (in $\mathbb{P}^3$) tangent to A at a point p will be denoted by T_p.

Given a point $p \in \mathbb{R}\mathbb{P}^3$, let $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$ be the linear projection from a point p and let $\hat{\pi}_p : A \to \mathbb{P}^2$ be the restriction of π_p to A. If $p \in L$, then we extend $\hat{\pi}_p$ to p by continuity, thus $\pi_p^{-1}(\hat{\pi}_p(p)) = T_p \setminus \{p\}$. Given a point $p \in \mathbb{R}\mathbb{P}^3$, we denote $\hat{\pi}_p(A)$ by C_p.

We say that A is *hyperbolic* with respect to a real line ℓ if, for any real plane P passing through ℓ, each intersection point of A and $P \setminus \ell$ is real. The following property of hyperbolic spatial curves is immediate from the definition:

Lemma 4.4. *Let ℓ be a real line in $\mathbb{P}^3$ and $p \in \mathbb{R}\ell$. Then A is hyperbolic with respect to ℓ if and only if $\pi_p(\ell) \in \mathrm{hyp}(C_p)$.* $\quad\square$

Lemma 4.5. *Suppose that A is hyperbolic with respect to a real line ℓ and let $p \in (\mathbb{R}A) \setminus \ell$. Then $\ell \cap T_p = \varnothing$.*

Proof. Combine Lemma 4.2 and Lemma 4.4. Namely, let $p_1 \in \mathbb{R}\ell$ and $q = \pi_{p_1}(\ell)$. Then $q \in \mathrm{hyp}(C_{p_1})$ by Lemma 4.4. Hence the line $\pi_{p_1}(T_p)$ cannot pass through q by Lemma 4.2 because otherwise it would be a line through q non-transverse to the image of the germ (A, p) under the projection π_{p_1}. $\quad\square$

Lemma 4.6. *Consider a crossing of a generic projection of $\mathbb{R}A$. Let $p, q \in \mathbb{R}A$ be the preimages of the crossing point under this projection. Suppose that A is hyperbolic with respect to the tangent line T_pA and the torsion of $\mathbb{R}A$ at p is positive. Then the considered crossing is also positive.*

Projection along T_pA

Projection along a direction close to T_pA

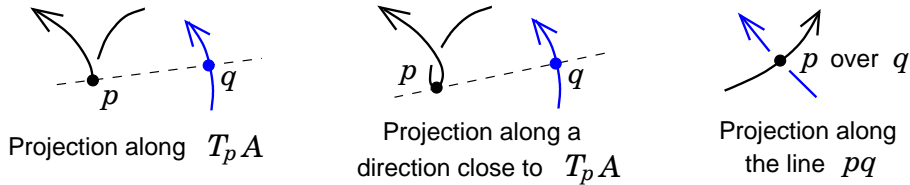Projection along the line pq

p over q

FIGURE 3. Positive crossing and positive torsion at p

Proof. Briefly speaking, the statement follows from Lemma 4.3 and Figure 3. Indeed, since the torsion at p is positive, the curve near p looks as in Figure 3 (left) because a small perturbation of this projection should produce a positive crossing near p as shown in Figure 3 (middle). Since the curve is hyperbolic with respect to T_pA, by Lemma 4.3 the mutual orientations at p and q should be as in Figure 3 (left), i.e., when looking from p to q, the rotation from $\vec{v}_p$ to $\vec{v}_q$ is counterclockwise

where $\vec{v}_p$ and $\vec{v}_q$ are positive tangent vectors at p and q respectively. This means exactly that the crossing is positive; see Figure 3 (right). $\quad\square$

Lemma 4.7. *Let $\mathbb{R}A$ be an irreducible real algebraic link in $\mathbb{RP}^3$ which is hyperbolic with respect to any real tangent line (and thus dividing by Lemma 4.3 and Lemma 4.4). Suppose that the torsion of $\mathbb{R}A$ is nonzero at any point. Then the torsion of $\mathbb{R}A$ at all its points has the same sign. Furthermore all non-isolated crossings of any projection have the same sign as the torsion of $\mathbb{R}A$ once we endow $\mathbb{R}A$ with a complex orientation.*

Proof. For any two generic points $p, q \in \mathbb{R}A$, the torsion at these points has the same sign due to Lemma 4.6 applied to the projection along the line pq. Hence the torsion is positive everywhere. Then the positivity at any crossing of any projection follows again from Lemma 4.6. $\quad\square$

4.4. Tangent surface. In this subsection we prove the implication (ii) $\Rightarrow$ (v′) in Theorem 2. Let the notation be as introduced in the beginning of §4.3 and assume that Condition (ii) of Theorem 2 holds, i.e., any real tangent plane has only real intersections with A, in particular, A is hyperbolic with respect to T_p for any $p \in L = \mathbb{R}A$. By Lemma 4.3 combined with Lemma 4.4, the curve A is dividing, so we fix a complex orientation on L and we orient the real locus of each tangent line $\mathbb{R}T_p$, $p \in L$, accordingly.

The torsion of L does not vanish by Lemma 4.1, thus Lemma 4.7 implies that the torsion of L has the same sign everywhere. We assume that it is positive (otherwise we replace L by its mirror image).

Lemma 4.8. *Let $p \in L$. Then $A \cap T_p = \{p\}$.*

Proof. Suppose that there exists a point $q \in (A \cap T_p) \setminus \{p\}$. Then q is real by (ii) applied to a tangent plane at p, and A is hyperbolic with respect to T_q by (ii) applied to a tangent plane at q. If $T_p \neq T_q$, then $p \notin T_q$, and we obtain a contradiction with Lemma 4.5 for $\ell = T_q$. Hence $T_p = T_q$.

Let π be a generic projection. Then $\pi(A)$ is tangent to $\pi(T_p)$ at $\pi(p)$ and $\pi(q)$. Let ℓ be a line tangent to $\pi(A)$ at a point close to $\pi(p)$. By Lemma 4.1 we may assume that the curvature of $\pi(A)$ at $\pi(p)$ and $\pi(q)$ is non-zero, hence ℓ can be chosen so that it has two non-real intersections with $\pi(A)$ near $\pi(q)$. Then the plane $\pi^{-1}(\ell)$ contradicts (ii). $\quad\square$

Lemma 4.9. *Let $p \in L$, $q \in A \setminus L$, and let ℓ be the line $(q\bar{q})$. Then $T_p \cap \ell = \varnothing$.*

Proof. If $T_p \cap \ell \neq \varnothing$, then (ii) does not hold for the plane containing these two lines. $\quad\square$

Lemma 4.10. *Let p and q be two distinct points on L. Then $T_p \cap T_q = \varnothing$ and $\mathrm{lk}(\mathbb{R}T_p, \mathbb{R}T_q) = 1/2$.*

Proof. Let $\ell = T_q$. Then A is hyperbolic with respect to ℓ by Condition (ii), and $p \notin \ell$ by Lemma 4.8. Thus $T_p \cap \ell = \varnothing$ by Lemma 4.5. The positivity of the linking number follows from Lemma 4.7. $\quad\square$

Lemma 4.11. *For any $p \in L$ there exist two real lines ℓ and ℓ' arbitrarily close to T_p such that A is hyperbolic with respect to each of them, $\ell \cap A = \varnothing$, and ℓ' crosses A without tangency at a pair of complex conjugated points.*

Proof. Let $p_0 \in \mathbb{R}T_p \setminus \{p\}$. The curve C_{p_0} has an ordinary cusp at $q_0 = \pi_{p_0}(p)$ because L has non-zero torsion at p. Let p_1 and p_2 be generic points close to p_0 which are chosen on different sides of the osculating plane of L at p. Then C_{p_1} and C_{p_2} are obtained from C_{p_0} by a perturbation of the cusp as shown in Figure 4 where q_2 is a solitary node of C_{p_2} (a point where two conjugate non-real local branches cross).
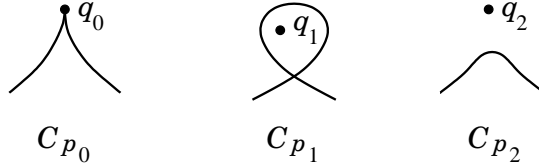

FIGURE 4. Two perturbations of a cusp in the proof of Lemma 4.11

Let $\ell = \pi_{p_1}^{-1}(q_1)$ and $\ell' = \pi_{p_2}^{-1}(q_2)$ where the points q_1 and q_2 are chosen as in Figure 4. The fact that A is hyperbolic with respect to T_p implies $q_0 \in \mathrm{hyp}(C_{p_0})$ by Lemma 4.4. We see in Figure 4 that then $q_j \in \mathrm{hyp}(C_{p_j})$ for $j = 1, 2$. $\square$

Let $K_0, \ldots, K_n$ be the connected components of L. For each $i = 0, \ldots, n$, let ℓ_i and ℓ'_i be the lines given by Lemma 4.11 close to T_p for some $p \in K_i$. To simplify the notation, we denote $\mathbb{R}\ell_i$ and $\mathbb{R}\ell'_i$ by l_i and l'_i. We endow l_i and l'_i with the orientation inherited from $\mathbb{R}T_p$ and we set

$$a_i = 2\,\mathrm{lk}(l'_i, K_i).$$

Let

$$T = TK_0 \cup \cdots \cup TK_n \quad \text{where} \quad TK_i = \bigcup_{p \in K_i} \mathbb{R}T_p\,.$$

By Lemma 4.10, T is a disjoint union of a continuous family of real projective lines (topologically, circles) parametrized by L. We are going to show that T is topologically isotopic in $\mathbb{R}\mathbb{P}^3$ to a union of hyperboloids. Note that T is not smooth. It has a cuspidal edge along L (see Figures 5–7 and Remark 4.15).

Lemma 4.12. *For any i we have $a_i > 0$ and*

$$
\begin{aligned}
&2\,\mathrm{lk}(l'_i, L) = d - 2, && 2\,\mathrm{lk}(l_i, L) = d, \\
&2\,\mathrm{lk}(l'_i, K_i) = a_i, && 2\,\mathrm{lk}(l_i, K_i) = a_i + 2, \\
&2\,\mathrm{lk}(l'_j, K_i) = a_i, && 2\,\mathrm{lk}(l_j, K_i) = a_i && \textit{for } j \neq i.
\end{aligned}
$$

Proof. The positivity of a_i follows from the hyperbolicity of L with respect to ℓ'_i. The first two rows of identities in Lemma 4.12 follow from hyperbolicity and the definition of a_i. We also have $\mathrm{lk}(l_j, K_i) = \mathrm{lk}(l'_j, K_i)$ for $j \neq i$ because both l_j, l'_j are close to T_p for some $p \in K_j$, and T_p is disjoint from K_i.

So, it remains to show that $|\mathrm{lk}(l'_i, K_i)| = |\mathrm{lk}(l'_j, K_i)|$. Indeed, each of ℓ'_i, ℓ'_j passes through a conjugate pair of points of A. Since $A \setminus L$ has two connected components, there exists a continuous path $p : [0, 1] \to A \setminus L$ such that $l'_i = l(0)$ and $l'_j = l(1)$ where $l(t)$ is the real locus of the line $\bigl(p(t)\,\overline{p(t)}\bigr)$. Each line $l(t)$ is disjoint from L by Lemma 4.9, hence $|\mathrm{lk}(l(t), K_i)|$ is constant. $\square$

By Lemma 4.12 we have

$$a_0 + \cdots + a_n = \sum_{i=0}^{n} \mathrm{lk}(l'_1, K_i) = \mathrm{lk}(l'_1, L) = d - 2. \tag{3}$$

Let $i \in \{0, \ldots, n\}$. The line l'_i is disjoint from T by Lemma 4.5. Let P be a real plane through l'_i. Again by Lemma 4.5, P crosses each line T_p, $p \in K_i$, at a single point. Let us denote this point by $\xi_P(p)$. Then $\xi_P : K_i \to \mathbb{R}P$ is a continuous mapping. It is injective by Lemma 4.10 and its image (which is $TK_i \cap \mathbb{R}P$) is disjoint from l'_i. Hence $TK_i \cap \mathbb{R}P$ is a Jordan curve in the affine real plane $\mathbb{R}P \setminus l'_i$. Let D_P be the disk bounded by this Jordan curve, and let $U_i = \bigcup_P D_P$ where P runs over all the real planes passing through l'_i. Then U_i is fibered by disks over a circle which parametrizes the pencil of planes through l'_i. Since $\mathbb{R}P^3$ is orientable, this fibration is trivial, thus U_i is a solid torus and $\partial U_i = TK_i$. Each P transversally crosses K_i at a_i real points, thus K_i sits in TK_i and it realizes the homology class $a_i \alpha_i$ where α_i is a generator of $H_1(U_i)$.

The same arguments applied to the line l_i show that TK_i bounds a solid torus V_i such that K_i realizes the homology class $(a_i + 2)\beta_i$ where β_i is a generator of $H_1(V_i)$. Easy to see that $\alpha_i = [l_i] \in H_1(U_i)$ and $\beta_i = [l'_i] \in H_1(V_i)$. Using Proposition 5.1 we deduce that K_i realizes the knot $T_{\mathrm{proj}}(a_i, a_i + 2)$ sitting in the torus TK_i.

By comparing the linking numbers given by Lemma 4.12 (or using the fact that all the l'_i are homotopic to each other in the complement of T; see the proof of Lemma 4.12), we see that the solid tori $U_0, \ldots, U_n$ are pairwise disjoint, so their union can be considered as a tubular neighborhood of the line arrangement $\mathcal{L} = l_0 \cup \cdots \cup l_n$ (see Figure 7). All the linking numbers $\mathrm{lk}(l_i, l_j)$, $i \neq j$, are positive by Lemma 4.7, hence $\mathcal{L}$ is the Hopf link by Proposition 3.1.
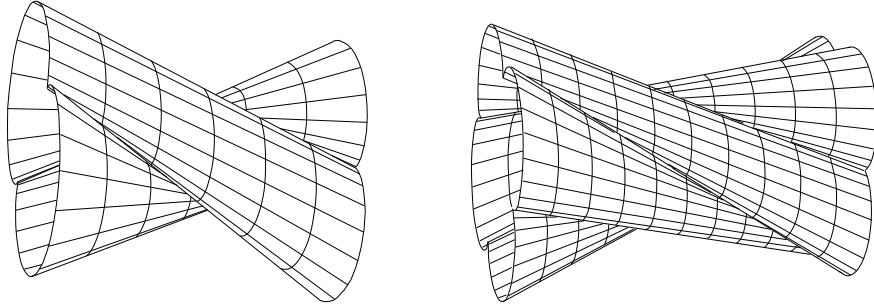

FIGURE 5. Tangent surface of $T_{\mathrm{proj}}(4, 2)$ and $T_{\mathrm{proj}}(6, 4)$ (view from V)
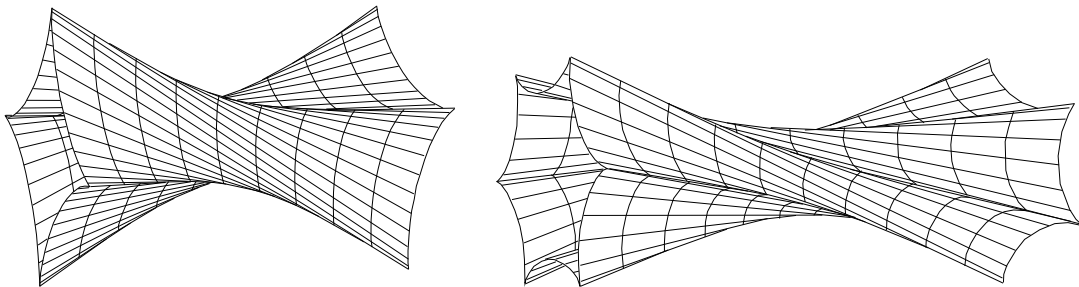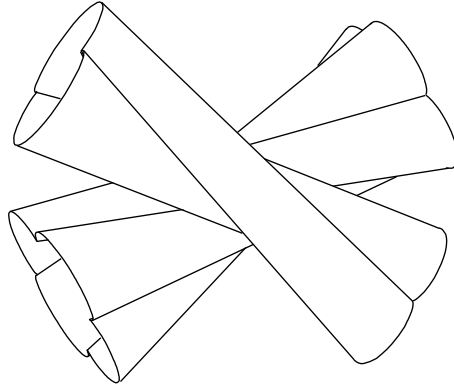

FIGURE 6. Tangent surface of $T_{\mathrm{proj}}(4, 2)$ and $T_{\mathrm{proj}}(6, 4)$ (view from U)

FIGURE 7. Mutual position of TK_0 and TK_1 for $W_1(4, 2)$

It remains to show that $n = g = \mathrm{genus}(A)$. Indeed, we have

$$\mathrm{cr}(L) \geq \sum_{i=0}^{n} \mathrm{cr}(K_i) + 2 \sum_{i<j} \mathrm{lk}(K_i, K_j)$$

$$= \tfrac{1}{2} \sum_{i=0}^{n} (a_i + 2)(a_i - 1) + \sum_{i<j} a_i a_j \qquad \text{by Proposition 2.2}$$

$$= \tfrac{1}{2} \Big(\sum_{i=0}^{n} a_i \Big)^2 + \tfrac{1}{2} \sum_{i=0}^{n} a_i - (n+1)$$

$$= N_d - n - 1 \qquad\qquad\qquad \text{by (3)}$$

(note that "$\geq$" can be replaced here by "$=$" because the minimal number of crossings for each K_i can be realized inside U_i). On the other hand, the number of double points of any projection is bounded by $N_d - g$ by genus formula, and there exists a projection with at least one solitary double point (it is enough to project from a real point on a line $(p\bar{p})$ for $p \in A \setminus \mathbb{R}A$), hence $\mathrm{cr}(L) \leq N_d - g - 1$. Thus $n \geq g$. Since $n \leq g$ by Harnack inequality, we conclude that $n = g$, hence $L \cong W_g(a_0, \ldots, a_g)$. The implication (ii) $\Rightarrow$ (v$'$) is proven.

The above computation of $\mathrm{cr}(L)$ also shows that

$$\mathrm{cr}(W_g(a_0, \ldots, a_g)) = N_d - g - 1 \qquad\qquad (4)$$

for any $g \geq 0$ and any positive a_i's satisfying $a_0 + \cdots + a_g = d - 2$.

4.5. End of the proof of Theorem 2. In Figure 8 we show the implications which are either evident or already proven. Here we prove the others.

(i) or (iv$^\mathrm{a}$) $\Rightarrow$ (iii$^\mathrm{a}$). Suppose that (iii$^\mathrm{a}$) is not satisfied. Let P be a generic real plane which cuts $L = \mathbb{R}A$ at less than $d - 2$ real points. Then there are at least two conjugate pairs of non-real points $p, \bar{p}$ and $q, \bar{q}$ in $P \cap A$. Since P is generic, we may assume that it does not contain any trisecant of A. Then the lines pq and $\bar{p}\bar{q}$ are distinct. Since they are contained in P, they cross each other. Then the projection from their intersection point has at least two non-real double points. This contradicts each of the conditions (i) and (iv$^\mathrm{a}$).

(ii) & (iv$^\mathrm{a}$) $\Rightarrow$ (i$'$). Recall that a double point (node) of a real curve in the plane is called *hyperbolic* if it is an intersection point of two real branches. A projection
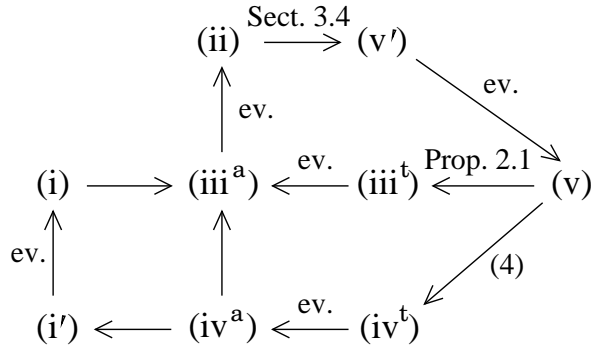
FIGURE 8. Scheme of implications ("ev." means "evident")

of A from a smooth point p on the tangent surface T has a cusp. If we push p from T then the cusp in the resulting projection gets replaced with a double point that is either hyperbolic or solitary depending on the choice of the side for moving p (see Figure 4). By (iv$^{\mathrm{a}}$) the number of hyperbolic nodes is at least $N_g - g - 1$. Thus this number is $N_g - g$ for one of the choices. By Lemma 4.7 (using the condition (ii) here as well as for deducing that A is dividing) all these nodes must correspond to positive crossing points of the diagram.

4.6. Concluding remarks for Theorem 2.

We see that T cuts $\mathbb{RP}^3$ into a union of solid tori $U = U_0 \cup \cdots \cup U_g$ and its complement $V = V_0 \cap \cdots \cap V_g$.

Proposition 4.13. (Compare with Condition (iv$^{\mathrm{a}}$)). *Let p be a generic point of $\mathbb{RP}^3$. Then C_p has only real double points. If $p \in U$, then all the double points are hyperbolic and the interior of $\mathrm{hyp}(C_p)$ is non-empty. If $p \in V$, then one double point q is solitary (i.e. has a conjugate pair of local branches), all the other double points are hyperbolic, and $\mathrm{hyp}(C_p) = \{q\}$.*

Proof. Let us consider a generic path $p(t)$ which connects the given point to a point on T. It defines a continuous deformation of the knot diagram which is a sequence of Reidemeister moves (R1) – (R3). However, (R2) is impossible because it involves a negative crossing, and (R1) may occur only when $p(t)$ passes through T. Thus the number and the nature of double points does not change during the deformation. The projection from a point of T is cuspidal and it is hyperbolic with respect to the cusp, so all the double points are hyperbolic by Lemma 4.2.

Non-emptiness of the interior of $\mathrm{hyp}(C_p)$ in case $p \in U$, follows from the fact that $\mathrm{hyp}(C_p)$ can disappear only by a move (R3). This is however impossible because all crossings are positive and the boundary orientation on $\partial(\mathrm{hyp}(C_p))$ agrees with an orientation of C_p due to Lemma 4.2 (see Figure 9).

The hyperbolicity of $C_{p(t)}$ with respect to the solitary node cannot fail during the deformation due to Lemma 4.9. $\square$
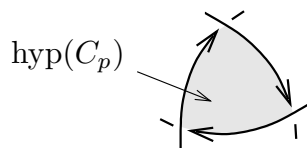

FIGURE 9. Impossibility of a move (R3) which eliminates $\mathrm{hyp}(C_p)$

The same arguments prove the following fact.

Proposition 4.14. *The family of real lines $(p\bar{p})$, $p \in A \setminus L$, defines a fibration of V over the quotient space $(A \setminus L)/\operatorname{conj}$.* $\square$
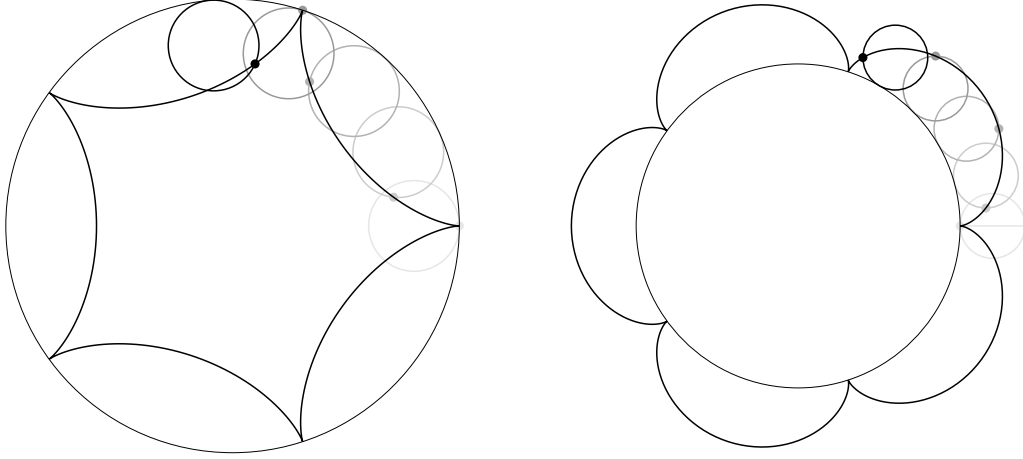

FIGURE 10. n-hypocycloid (left) and n-epicycloid (right) for $n = 5$

Remark 4.15. It is a nice and easy exercise to check that, in the case of the most symmetric (rational) MW-knot, the Jordan curves $T \cap P$ (discussed after Lemma 4.12) are d-hypocycloids (for $P \supset l_1'$) or $(d-2)$-epicycloids (for $P \supset l_1$). These are curves defined as trajectories of a point of a smaller circle which rolls without slipping around a bigger one (see Figure 10). By "the most symmetric MW-knot" we mean $K = \{z^d = w^{d-2}\}$ in $\mathbb{RP}^3 = \mathbb{S}^3/(-1)$, and $l_1' = \{z = 0\}$, $l_1 = \{w = 0\}$. In Figures 5 and 6 we show the true shape of the tangent surface T for this knot for $d = 4$ and $d = 6$. The grid in these figures is composed of the tangent lines T_p, $p \in K$, and the plane sections which are hypo- or epicycloids.

5. Some topological properties and characterizations of $T_{\operatorname{proj}}(d, d-2)$

5.1. Equivalent definitions of $T_{\operatorname{proj}}(p, q) = h_{a,b}$.

Let H be a hyperboloid in $\mathbb{RP}^3$. It cuts $\mathbb{RP}^3$ into two solid tori U and V. Let u and v be core circles of U and V respectively. We orient them so that $\operatorname{lk}(u, v) = 1/2$. The surface H is doubly ruled. The two families of projective lines on H give a diffeomorphism $H \approx \mathbb{RP}^1 \times \mathbb{RP}^1$ as well as a basis (α, β) of $H_1(H; \mathbb{Z})$ (defined up to transposition of basis vectors and changing their sign). We choose it so that any two lines representing α are linked positively while $\operatorname{lk}(\alpha, u) = \frac{1}{2}$, $\operatorname{lk}(\beta, u) = \frac{1}{2}$, $\operatorname{lk}(\alpha, v) = \frac{1}{2}$, $\operatorname{lk}(\beta, v) = -\frac{1}{2}$. Note that the definition of the link $T_{\operatorname{proj}}(p, q)$ given in the introduction extends to the case when p and q are of arbitrary signs (but, however, $(p, q) \neq (0, 0)$) by replacing z^p with $\bar{z}^{|p|}$ (resp. w^q with $\bar{w}^{|q|}$) if $p < 0$ (resp. if $q < 0$). We have isotopies: $T_{\operatorname{proj}}(p, q) \sim T_{\operatorname{proj}}(-p, -q) \sim T_{\operatorname{proj}}(q, p)$. The following fact is very easy and we omit its proof (see some more details in [13; §3]).

Proposition 5.1. *Let L be a smooth oriented one-dimensional compact submanifold of H. Let $p = a + b$, $q = a - b$ where $a, b \in \mathbb{Z}$, $(a, b) \neq (0, 0)$. We assume that any non-empty one-dimensional submanifold of L is non-trivial in $H_1(H; \mathbb{Z})$ (i.e., any connected component of L is homologically non-trivial and no pair of parallel*

components have opposite orientations). *If any of the following conditions holds,
then L is isotopic to $T_{\mathrm{proj}}(p,q) = h_{a,b}$:*

   (i) $[L] = a\alpha + b\beta$ *in* $H_1(H;\mathbb{Z})$;
   (ii) $\mathrm{lk}(L,u) = p/2$ *and* $\mathrm{lk}(L,v) = q/2$;
   (iii) $[L] = q[u]$ *in* $H_1(U;\mathbb{Z})$ *and* $[L] = p[v]$ *in* $H_1(V;\mathbb{Z})$.   □

Any of the properties (i)–(iii) can be chosen for a definition of $T_{\mathrm{proj}}(p,q)$ considered up to isotopy.

5.2. Plane section number of a projective torus link in $\mathbb{RP}^3$. The following is an equivalent statement of Proposition 2.1.

Proposition 5.2. *The plane section number of $T_{\mathrm{proj}}(p,q)$ is equal to* $\min(|p|,|q|)$.

Proof. Let the notation be as in Section 5.1 and let L be as in Proposition 5.1. Let P be a generic surface in $\mathbb{RP}^3$ isotopic to a plane $\mathbb{RP}^2 \subset \mathbb{RP}^3$. Then either $P \cap U$ or $P \cap V$ is orientable (since each non-orientable component of $P \setminus H$ contains a Möbius band whose complement is a disk), let it be $P \cap U$. Let $C = P \cap H$ oriented as the boundary of $P \cap U$. Let $j_* : H_1(H) \to H_1(U)$ be the homomorphism induced by the inclusion. Then $[C] \in \ker j_*$ and $\ker j_*$ is generated by $\alpha + \varepsilon\beta$, $\varepsilon = \pm 1$, thus $[C] = k(\alpha + \varepsilon\beta)$, $k \in \mathbb{Z}$.

Note that k must be odd since the $\mathbb{Z}_2$-homology class of a plane section of $H \subset \mathbb{RP}^3$ is defined invariantly of P and thus $[C] \equiv \alpha + \beta \mod 2$. In particular, $k \neq 0$. By Proposition 5.1 we have $[L] = \frac{p+q}{2}\alpha + \frac{p-q}{2}\beta$. Therefore $|[L].[C]| = |k(\frac{p+q}{2} \pm \frac{p-q}{2})| \geq \min(|p|,|q|)$.   □

5.3. Diagrams of projective links and projective braid closures. Similarly to links in $\mathbb{S}^3$, a link in $\mathbb{RP}^3$ can be represented by a diagram in a (round) disk $\mathbb{D}$. It is a union of immersed circles and/or arcs with endpoints on $\partial\mathbb{D}$ transverse to each other. The crossings are presented by over- and underpasses in the same way as for the usual links. The endpoints are placed on $\partial\mathbb{D}$ symmetrically with respect to the center. The link in $\mathbb{RP}^3$ represented by a diagram is defined as follows. First we consider a tangle in the 3-ball represented by the diagram assuming that the endpoints of the tangle are placed on the equator. Then we identify the opposite points of the 3-ball's boundary, see [2] for more details.

We define the braid closure in $\mathbb{S}^3$ in the usual way and we define the braid closure in $\mathbb{RP}^3$ (the *projective braid closure*) as the link in $\mathbb{RP}^3$ represented by a diagram of the braid placed in a disk with the endpoints of the strands on the boundary.
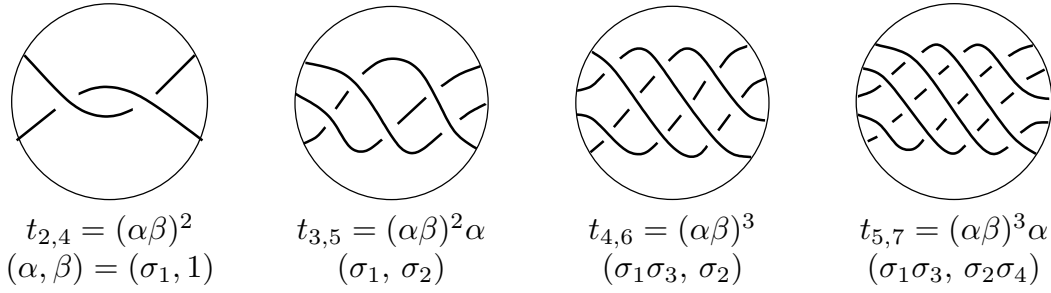
Let p and q be positive integers of the same parity. Then $T(p,q)$ is the closure in $\mathbb{S}^3$ of the p-braid $(\alpha\beta)^q$ where $\alpha = \sigma_1\sigma_3\ldots$ and $\beta = \sigma_2\sigma_4\ldots$. Similarly, $T_{\mathrm{proj}}(p,q)$ is the closure in $\mathbb{RP}^3$ of the braid represented by the first half of the word $(\alpha\beta)^q$, i. e., the braid

$$t_{p,q} = \begin{cases} (\alpha\beta)^{q/2} & \text{if } p \text{ and } q \text{ are even,} \\ (\alpha\beta)^{(q-1)/2}\alpha & \text{if } p \text{ and } q \text{ are odd,} \end{cases} \tag{5}$$

see Figure 11. We obtain this braid by projecting $T_{\mathrm{proj}}(p,q)$ (defined by the formulas in the introduction) from a point on a suitable coordinate axis.

5.4. Permutation braids. Let B_n be the braid group on n strands. Let $\mu = \mu_n : B_n \to S_n$ be its standard homomorphism to the symmetric group (which takes each σ_i to the transposition $(i, i+1)$) and let $e = e_n : B_n \to \mathbb{Z}$ be the *exponent sum* homomorphism (which takes each σ_i to 1).

$$t_{2,4} = (\alpha\beta)^2 \qquad t_{3,5} = (\alpha\beta)^2\alpha \qquad t_{4,6} = (\alpha\beta)^3 \qquad t_{5,7} = (\alpha\beta)^3\alpha$$
$$(\alpha,\beta) = (\sigma_1, 1) \qquad (\sigma_1, \sigma_2) \qquad (\sigma_1\sigma_3, \sigma_2) \qquad (\sigma_1\sigma_3, \sigma_2\sigma_4)$$

FIGURE 11. Knots $T_{\mathrm{proj}}(d, d-2)$ as projective braid closures

Permutation braids are introduced by El Rifai and Morton in [3]. They can be defined as braids represented by diagrams whose any two strands cross at most once and all crossings are positive. One can choose such a diagram so that each strand is represented by a straight line segment. Note that μ_n establishes a bijection of the set of permutation n-braids onto S_n; see more details in [3; Section 2].

The permutation n-braid corresponding to the permutation $i \mapsto n+1-i$, $i = 1, \ldots, n$, is a positive half-twist of all the strands $\Delta = \Delta_n = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i}\sigma_j$ called also the *Garside element* of the braid group B_n. It is a unique permutation braid whose exponent sum attains the maximal possible value $n(n-1)/2$.

5.5. Links in $\mathbb{RP}^3$ and their lifts in $\mathbb{S}^3$. Let L be a link in $\mathbb{RP}^3$ and let $\tilde{L}$ be its inverse image under the covering $\mathbb{S}^3 \to \mathbb{RP}^3$.

<div align="center">

Does $\tilde{L}$ determine L?

</div>

Surprisingly, this question seems to be open. The same question can be posed about links in lens spaces $L(p, q)$ and their lifts to $\mathbb{S}^3$ (note that $L(2, 1)$ is $\mathbb{RP}^3$) but, as shown in [10], the answer is negative for some knots in $L(4, 1)$ and in $L\left(p, \frac{p\pm 1}{2}\right)$ for odd $p \geq 5$. So, the question is not as simple as one could expect.

However, due to a result by González-Meneses [4], the answer in a sense is positive if we speak of braids instead of links. Namely, let $\tau : B_n \to B_n$ be the inner automorphism $X \mapsto \Delta X \Delta^{-1}$. We have $\tau(\sigma_i) = \sigma_{n-i}$, $1 \leq i < n$, thus $\tau(X)$ is X rotated by $180°$. We write $X \sim Y$ if the braids X and Y are conjugate.

Proposition 5.3. *(a). The conjugacy class of $X\Delta$ determines the projective braid closure of a braid X.*

  (b). The lift of the projective braid closure of X is the braid closure of $X\tau(X)$.

  (c). The conjugacy class of $X\tau(X)$ determines the closure of X in $\mathbb{RP}^3$.

Proof. (a,b). Evident.

  (c). Since Δ^2 belongs to the center of B_n, we have

$$X\tau(X) \sim Y\tau(Y) \iff (X\Delta)^2 \sim (Y\Delta)^2,$$

thus the result follows from González-Meneses' theorem [4] which states that the m-th root of a braid (for any m) is unique up to conjugacy. □

5.6. Characterizations of $T_{\mathrm{proj}}(d, d-2)$ via braids. A braid is called *positive* if it can be represented by a diagram with positive crossings only (i.e. if it is a product of the standard generators in positive powers). We denote the monoid of positive n-braids by B_n^+. Recall that $N_d = (d-1)(d-2)/2$.

Theorem 5.4. *(a). Let X be a permutation d-braid such that $e(X) = N_d$ and the projective closure of X is a knot* (the last condition is equivalent to the fact that $\mu_d(X\Delta_d)$ is a d-cycle). *Then the projective closure of X is $T_{\mathrm{proj}}(d, d-2)$.*

(b). Let X be a $(d-2)$-braid such that $e(X) = N_d - 1$, $X = \Delta A$ for some $A \in B_{d-2}^+$, and the projective closure of X is a knot. Then the projective closure of X is $T_{\mathrm{proj}}(d, d-2)$.

Remark 5.5. Using the algorithm in [3], for any given braid X it is very easy to check if the condition $X = \Delta A$, $A \in B_{d-2}^+$, in Theorem 5.4(b) is fulfilled or not.

Remark 5.6. A positive d-braid whose projective closure is $T_{\mathrm{proj}}(d, d-2)$ need not be a permutation braid. Example: $\sigma_{d-2}^{-1}\Delta_{d-1}\sigma_2$ viewed as a d-braid, $d \geq 4$.

Lemma 5.7. *Let $x_1, \ldots, x_n$ be elements of a group G such that $x_i x_j = x_j x_i$ for $j - i \geq 2$. Let $(y_1, \ldots, y_n)$ be a permutation of $(x_1, \ldots, x_n)$. Then there exists an element u of the subgroup generated by $x_1, \ldots, x_{n-1}$ such that $u(y_1 \ldots y_n)u^{-1} = x_1 \ldots x_n$.*

Proof. Induction on n. If $n = 1$, the statement is obvious. Suppose that $n \geq 2$. Let $x = x_1 \ldots x_n$, $y = y_1 \ldots y_n$ and let G_k be the subgroup generated by $x_1, \ldots, x_k$. We have $y = Ax_{n-1}Bx_nC$ or $y = Ax_nBx_{n-1}C$ with $A, B, C \in G_{n-2}$. In the former case we have $y = Ax_{n-1}x_nBC$. In the latter case, for $u_1 = x_{n-1}C \in G_{n-1}$, we have $u_1yu_1^{-1} = x_{n-1}CAx_nB = x_{n-1}x_nCAB$. Thus, in the both cases, there exists $u_1 \in G_{n-1}$ such that $u_1yu_1^{-1} = z_1 \ldots z_{n-1}$ where $(z_1, \ldots, z_{n-1})$ is a permutation of $(x_1, \ldots, x_{n-2}, x_{n-1}')$ for $x_{n-1}' = x_{n-1}x_n$. Since x_{n-1}' commutes with $x_1, \ldots, x_{n-3}$, by the induction hypothesis there exists $u_2 \in G_{n-2}$ such that $u_2(z_1 \ldots z_{n-1})u_2^{-1} = x_1 \ldots x_{n-2}x_{n-1}' = x$ and hence $uyu^{-1} = x$ for $u = u_2u_1$. $\square$

Proof of Theorem 5.4. (a). First, note that $T_{\mathrm{proj}}(d, d-2)$ can be represented by a braid which satisfies these conditions. For example, one can choose the braid (5).

Thus, it is enough to show that if Y is another braid satisfying the same conditions, then the projective closure of X and Y coincide as knots in $\mathbb{RP}^3$. To this end we show that the braids $X' = \Delta X^{-1}$ and $Y' = \Delta Y^{-1}$ are conjugate in this case. Indeed, X' and Y' are permutation braids by [3; Theorem 2.6] and

$$e(X') = e(Y') = e(\Delta) - e(X) = \frac{d(d-1)}{2} - \frac{(d-1)(d-2)}{2} = d - 1.$$

Thus each of X' and Y' is a product of $d-1$ generators. The connectedness of the projective braid closure implies that each σ_i, $1 \leq i < d$, occurs in this product and we conclude that X' and Y' are conjugate by Lemma 5.7.

(b). Proof is similar but simpler (we do not use [3; Theorem 2.6] and we apply Lemma 5.7 directly to A). $\square$

6. Construction of MW_λ-links of arbitrary genus (proof of Theorem 3)

Theorem 3 is an immediate consequence of Lemma 6.4 and Lemma 6.5 below. We say that a smooth irreducible algebraic curve C in $\mathbb{CP}^n$ is *special* if the the hyperplane section divisor D on C is special, i.e., $h^1(D) > 0$ where, as usual, $h^i(D)$ is an abbreviation for $\dim H^i(C, \mathcal{O}_C(D))$.

Lemma 6.1. *Let A be a smooth irreducible real algebraic non-special curve of genus g and degree d in $\mathbb{RP}^3$ and let ℓ be a real line which crosses A at a single point p without tangency. Then there exists a smooth irreducible real algebraic non-special curve A' of degree $d+1$ and genus g arbitrarily close to $A \cup \ell$ whose real locus is isotopic to the link obtained from $\mathbb{R}A \cup \mathbb{R}\ell$ by any of the two local modifications at p shown in Figure 12.*

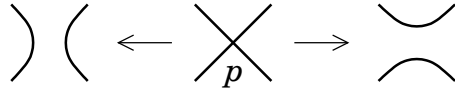*Moreover, the curve A' can be chosen isomorphic to A as an abstract real algebraic curve.*

FIGURE 12. Local smoothings in Lemma 6.1

Remark 6.2.

(1) For $g = 0$ this result follows from [1; Theorem 2.4].

(2) A simple example showing that Lemma 6.1 may fail for special curves can be found for $(d, g) = (4, 3)$ (then A is contained in a plane). In this case $A \cup \ell$ cannot be smoothed out into a 5-th degree curve of genus 3 because smooth spatial curves of degree 5 and genus 3 do not exist. Indeed, such a curve should sit either in a plane or in an irreducible quadric by [13; Proposition 1]. However the maximal genus of a quintic curve in a nonsingular quadric is $\max_{a+b=5}(a-1)(b-1) = 2$ and the genus of any smooth quintic curve in a plane (resp. in a quadratic cone) is 6 (resp. 2).

(3) Lemma 6.1 also admits an easy generalization to the case of two non-special irreducible real algebraic curves A and B intersecting at a single point p without tangency. In such case there exists a smooth curve C of degree equal to the sum of degrees, and of genus equal to the sum of genera such that its real locus is isotopic to the link obtained from $\mathbb{R}A \cup \mathbb{R}B$ by any of the two local modifications at p shown in Figure 8. However for the purposes of this paper the special case given by Lemma 6.1 suffices.

Proof. Let us choose homogeneous coordinates $(x : y : z : w)$ so that p is not on the coordinate planes and ℓ passes through the point $(0 : 0 : 0 : 1)$. Up to rescaling, the embedding of A into $\mathbb{P}^3$ is determined by the effective divisors D_x, D_y, D_z, D_w which are cut on A by the coordinate planes. We are going to define A' as the image of an embedding $f : A \to \mathbb{P}^3$ defined by new effective divisors D'_x, D'_y, D'_z, D'_w. To achieve the result, it is enough to choose these divisors so that they all belong to the same linear system and

$$D'_x = D_x + p, \quad D'_y = D_y + p, \quad D'_z = D_z + p, \quad D'_w = \widetilde{D}_w + \tilde{p}$$

where $\widetilde{D}_w$ is close to D_w and $\tilde{p}$ is a real point on A close to p and chosen on a required side of p (the choice between the two local smoothings in Figure 12 is determined by the side from which $\tilde{p}$ approaches to p; see Figure 13).

For a positive integer n, let $s_n : S^n(A) \to \mathcal{J}_A$ be the Abel-Jacobi mapping where $S^n(A)$ is the n-th symmetric power of A and $\mathcal{J}_A$ is the Jacobean of A (to define
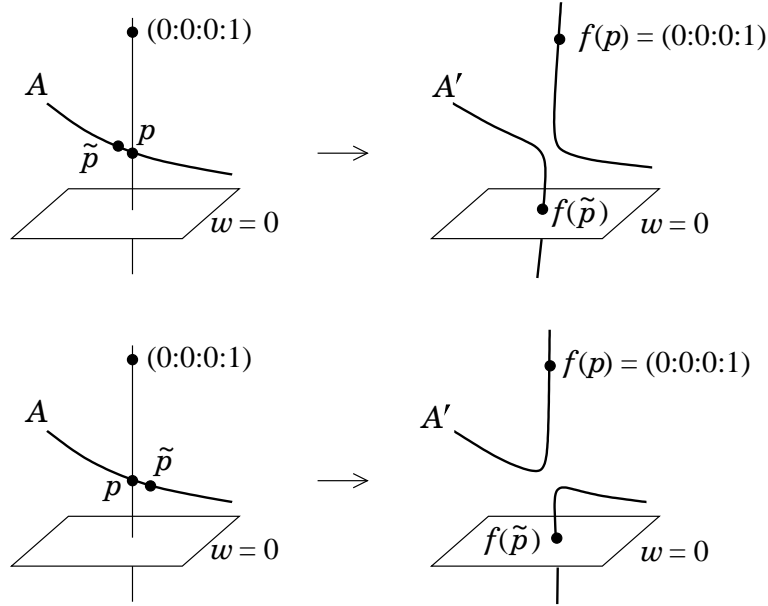
FIGURE 13. Global smoothings in Lemma 6.1

s_n we need to fix a point p_0 on A). Note that the both varieties admit natural real structure (an antiholomorphic involution Conj) and s_n is a Conj-equivariant regular morphism (we assume here that p_0 is real). The real locus $\mathbb{R}\mathcal{J}_A$ is a subgroup of $\mathcal{J}_A$. It is a union of 2^{l-1} real g-dimensional tori where $l = b_0(\mathbb{R}A)$ (see [6], [16; §8]).

It is clear that $D'_x \sim D'_y \sim D'_z$. The condition $D'_x \sim D'_w$ reads $s_{d+1}(D_x + p) = s_{d+1}(\widetilde{D}_w + \tilde{p})$. Since $D_x \sim D_w$, this is equivalent to

$$s_d(D_w) - s_d(\widetilde{D}_w) = s_1(\tilde{p}) - s_1(p).$$

Thus we need to prove that for any sufficiently small $h \in \mathbb{R}\mathcal{J}_A$ there exists $\widetilde{D}_w \in \mathbb{R}S_d(A)$ close to D_w such that $s_d(D_w) - s_d(\widetilde{D}_w) = h$. This follows from the surjectivity of the differential $\mathrm{d}s_d$ at non-special divisors D.

It remains to check that A' is non-special. Indeed, the corresponding plane section divisor on A is $D'_x = D_x + p$. We have $h^0(K_A - D_x) = 0$ because A is non-special. Hence $h^0(K_A - D'_x) = 0$. $\square$

Lemma 6.3. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let C be a nonsingular algebraic curve on X of bidegree $(2, g+1)$, $g \geq 0$. Let D be a divisor on C which is cut by a generic curve of bidegree $(1,1)$. Then $h^1(D) = 0$.*

Proof. Let A and B be algebraic curves on X of bidegree $(1,0)$ and $(0,1)$ respectively, and let Let $A|_C$ and $B|_C$ be the divisors in C which are cut by them. By Serre's duality we have $h^1(D) = h^0(K_C - D)$. Let us show that $|K_C - D|$ is empty. Suppose it is not. Let $D_1 \in |K_C - D|$, thus

$$K_C \sim D_1 + D = D_1 + A|_C + B|_C. \tag{6}$$

The curve C is hyperelliptic of genus g, and the hyperelliptic projection $\pi : C \to \mathbb{P}^1$ is the restriction of the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor (this projection contracts B to a single point). Without loss of generality we may assume that A is transverse to C, thus $\pi(\mathrm{supp}(A|_C))$ consists of $A \cdot C = A \cdot (2A + (g+1)B) = g+1$

distinct points. Since all summands in the right hand side of (6) are effective divisors, we conclude that the support of $D_1 + D$ contains that of $A|_C$ whence $\pi(\mathrm{supp}(D_1 + D))$ has at least $g + 1$ points.

On the other hand, $D_1 + D \in |K_C|$ (see (6)) and it is well known (see, e.g., [5; Ch. 3, §3.5]) that any element of $|K_C|$ is supported by a union of at most $g - 1$ fibers of π. The obtained contradiction completes the proof. $\square$

Lemma 6.4. *For any $g \geq 0$ there exists an irreducible non-special real algebraic link of genus g and degree $d = g + 3$, isotopic to $W_g(1, \ldots, 1)$.*

Proof. Let $L_0, \ldots, L_g$ be real lines belonging to one ruling of a hyperboloid H, and let L and $\bar{L}$ be a pair of complex conjugate lines from the other ruling. We smooth out (remaining on H) the union of all these $d = g + 3$ lines. The genus of the resulting curve is g. It is non-special by Lemma 6.3, and its real locus is isotopic to $\mathbb{R}L_0 \cup \cdots \cup \mathbb{R}L_g$ which is a projective Hopf link. $\square$

Lemma 6.5. *Let $a_0, \ldots, a_g$ be positive integers and $d = a_0 + \cdots + a_g + 2$. Suppose that $W_g(a_0, \ldots, a_g)$ is realizable by an irreducible non-special real algebraic link of degree d and genus g. Then $W_g(a_0 + 1, a_1, \ldots, a_g)$ is realizable by an irreducible non-special real algebraic link of degree $d + 1$ and genus g.* $\square$

Proof. Let $\mathbb{R}A$ be a non-special real algebraic link of type $W_g(a_0, \ldots, a_g)$. We shall use for it the notation introduced in §4.3 and §4.4. Let TK_0 be the component of the tangent surface of A corresponding to the a_0-component K_0 of $\mathbb{R}A$. For a point p close to TK_0 the curve $C_p \subset \mathbb{P}^2$ (obtained as the image of A under the projection $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$) is a hyperbolic curve.

Since p is close to TK_0, the curve C_p is obtained by a perturbation of a cuspidal hyperbolic curve $C_{p'}$, $p' \in TK_0$, with a cusp on the projection of K_0. Thus we may choose $q \in \mathrm{hyp}(C_p)$ such that $\ell = \pi_p^{-1}(q) \cap K_0 \neq \varnothing$ so that A and ℓ intersect at a single point without tangency.

The two local modifications of $\mathbb{R}A \cup \mathbb{R}\ell$ provided by Lemma 5.1 produce non-special real algebraic links with the invariant w_λ equal to $w_\lambda(\mathbb{R}A) \pm (d - 1)$. Denote by $\mathbb{R}B$ the one corresponding to the plus sign so that $w_\lambda(\mathbb{R}B) = N_d - g + (d - 1) = N_{d+1}$.

By Theorem 2 the isotopy type of $\mathbb{R}B$ is $W_g(b_0, \ldots, b_g)$ where the numbers b_j are determined by Lemma 4.12. We get $b_j = a_j$ for $j > 0$ and $b_0 = a_0 + 1$ once we keep the numeration of the components of $\mathbb{R}B$ and $\mathbb{R}A$ consistent. $\square$

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