ALGEBRAIC CURVE IN THE UNIT BALL IN $\mathbb{C}^2$ PASSING THROUGH THE ORIGIN, ALL OF WHOSE BOUNDARY COMPONENTS ARE ARBITRARILY SHORT

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To the memory of Anatoliy Georgievich Vitushkin

1. Introduction

Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{C}^2$ centered at the origin. A.G. Vitushkin posed the following question (see [1], [2; Problem 5.3], [4]):

(1). Does there exist an absolute constant c such that for any complex algebraic curve A in $\mathbb{C}^2$ passing through the origin, there exists a connected component of the set $A \cap \mathbb{S}^3$ whose length is not less than c?
(2). Is it true that $c = 2\pi$?

In this paper, we give negative answers to the both questions.

Theorem 1.1. *a). Let Ω be a compact closed domain in an analytic surface, and let $M = \partial\Omega$ be its boundary. Let M_0 be the set of those points where M is a C^2-smooth strictly pseudoconvex real hypersurface. Suppose that some Riemannian metric is fixed on M. Let A be a complex analytic curve in Ω such that ∂A is contained in M_0 and realizes the zero homology class in $H_1(M_0;\mathbb{Z})$. Let P be any finite subset of A.*

Then for any 2-chain β in M_0 such that $\partial\beta = \alpha$ and for any $\varepsilon > 0$, there exists a complex analytic curve A' in Ω which is ε-close to $A \cup \operatorname{supp}\beta$ and such that the length of any of its boundary component is less than ε and $P \subset A'$.

b). If, moreover, $\Omega \subset \mathbb{C}^2$ and, for any point $p \in M_0$, the complex line T tangent to M at p does not meet Ω at other points and the restriction to T of the second fundamental form of M at p is positive definite (*by the strict pseudoconvexity of M the latter condition is equivalent to the positivity of its sectional curvature at p in the direction of T*)*, then one can choose A' to be an algebraic curve.*

This theorem follows immediately from Propositions 2.6 and 3.5. It is proved at the end of §3. The crucial role in the proof is played by the notion of a Legendrian net hanged on a transversal cycle in a contact 3-manifold introduced in §2.

A negative answer to Vitushkin's question is provided by applying Theorem 1.1 b) in the case when Ω is the unit ball, P is its center, and A is an arbitrary curve (for example, a line) passing through P.

Remark 1.2. The condition of the *strict* pseudoconvexity in Theorem 1.1 is important. Indeed, the answers to both Vitushkin's questions are positive if one considers the bidisk instead of the ball (see [1]).

Remark 1.3. We formulate Theorem 1.1 and Propositions 2.6 and 3.5 in "minimax generality", i.e., we try to give the most general statement that can be proved using exactly the same argument as in the simplest (known to us) proof for the case of an algebraic curve in the unit ball.

If one drops this principle, then Theorem 1.1 can be easily generalized as far as one's fantasy allows. For instance, the line T in Part b) could be replaced by an algebraic curve (but then the proof of the corresponding analogue of Lemma 3.1 would become more complicated), or one could take into consideration Shilov boundaries, polynomial convexity, etc.

Remark 1.4. It is shown in [1] that if one replaces algebraic curves by real surfaces which have open projections onto (complex) coordinate axes, then the answer to Question (1) is negative.

Remark 1.5. Recall some terminology from the link theory. A link $L \subset \mathbb{S}^3$ is called *indecomposable* (*into a disjoint sum*), if there does not exist two balls B_1, B_2 such that $B_1 \cup B_2 = \mathbb{S}^3$, $B_1 \cap B_2 = \partial B_1 = \partial B_2$, $L \cap B_1 \neq \varnothing \neq L \cap B_2$, and $L \cap B_1 \cap B_2 = \varnothing$. So, if one considers in Question (1) indecomposable components rather than connected components, then the answer will be *positive*. This follows from the fact (proved in [3]) that the boundary of a connected curve is an indecomposable link.

Remark 1.6. Apparently, Vitushkin's main motivation for asking this question was its relation with the problems about polynomial hulls of "bad" sets. Some links between these topics are discussed in the recent paper [4].

Remark 1.7. The answer to Question (2) (is it true that $c = 2\pi$) is negative even for curves of degree two. To see this, one can explicitly parametrize the real curve $\{\,(z,w) \in \mathbb{C}^2 \mid w^2 = az(z-1)\,\} \cap \mathbb{S}^3$, then check by numerical integration that its length is less than 4π for some values of a, and finally, observe that the perturbed curve $\{w^2 = az(z-1+\varepsilon)\} \cap \mathbb{S}^3$ for $0 < \varepsilon \ll 1$ consists of two equal halves whose total length is close to the length of the initial curve.

Thus, an absolute constant c does not exist. However if one fixes n — the number of connected components of $A \cap \mathbb{S}^3$, such a constant depending on n certainly does exist (it is clear that the total length of all the components is greater than 2π). Let $n(\varepsilon)$ denote the minimal number of connected components of $A \cap \mathbb{S}^3$ under the condition that A is an algebraic curve through the origin such that length of any connected component of $A \cap \mathbb{S}^3$ is less than ε.

It follows from the argument above that $n(\varepsilon) > 2\pi/\varepsilon$. It is not difficult to deduce from Stokes' formula that after the projection onto $\mathbb{C}P^1$, the sum of the oriented areas bounded by the projections of the components of $A \cap \mathbb{S}^3$ is greater than the area of the whole $\mathbb{C}P^1$, hence $n(\varepsilon) > \mathrm{const}/\varepsilon^2$ (see Proposition 4.9). On the other hand, a straightforward application of the construction provided by the proof of Theorem 1.1 yields an upper bound $n(\varepsilon) < \mathrm{const}/\varepsilon^4$.

A natural correction of Vitushkin's question suggests itself: is it true that the maximal length of components of $A \cap \mathbb{S}^3$ is essentially greater than the evident estimates? More precisely, what is the asymptotics of $n(\varepsilon)$ as $\varepsilon \to 0$? The same question can be asked about the quantity $d(\varepsilon)$ — the minimal degree of an algebraic curve satisfying the same condition. As we have seen, the order of growth of $n(\varepsilon)$ is between ε^{-2} and ε^{-4}. It seems plausible that it is ε^{-3}. In §6, we prove an upper bound for $n(\varepsilon)$ of the order ε^{-3}. In §5, we prove that this bound cannot be improved by the methods of this paper (i.e. using the construction based on a perturbation of a Legendrian net). In the end of §5, we propose a new question, a positive answer to which would imply a lower bound on $n(\varepsilon)$ of the order ε^{-3}.

2. Legendrian nets hanged on transverse cycles

All the statements of this section are almost obvious but we shall give their proofs anyhow.

We shall understand *chains, cycles*, and *boundaries* more or less in the sense of the theory of singular homologies, but we shall consider only piecewise smooth chains and we shall identify chains obtained from one another by subdivisions and reparametrizations. In particular, a 1-*chain* in a smooth manifold M is by definition an element of the quotient of the free abelian group generated by all piecewise smooth mappings $\alpha : I = [0,1] \to M$ modulo all relations of the form $\alpha = -(\alpha\circ\varphi)$ and $\alpha = (\alpha\circ\varphi_1) + (\alpha\circ\varphi_2)$ where φ is an orientation reversing piecewise smooth homeomorphism of the segment I onto itself, and φ_1, φ_2 are orientation preserving piecewise smooth homeomorphisms of the segment I onto the segments $[0,1/2]$ and $[1/2,1]$ respectively. For example, these relations imply that the constant mapping $I \to p \in M$ realizes the zero chain. A linear combination $\sum m_i\alpha_i$ representing a chain α will be called a *minimal realization of* α if $m_i \neq 0$ for all i and there do not exist indices i_1, i_2, segments $I_1, I_2 \subset I$, and a homeomorphism $\varphi : I_1 \to I_2$ such that $\alpha_{i_1}|_{I_1} = \alpha_{i_2}\circ\varphi$.

Let $\sum m_i\alpha_i$ be some minimal realization of a 1-chain α on a 3-manifold M. Then the set $\operatorname{supp}\alpha = \bigcup \alpha_i(I)$ is called the *support* of α. If, moreover, M is endowed with a Riemannian metric then the *length* of α is by definition $\operatorname{len}\alpha = \sum |m_i| \operatorname{len}\alpha_i$ where $\operatorname{len}\alpha_i$ is the length of the path α_i. A 1-chain α is called ε-*short* if $\operatorname{len}\alpha < \varepsilon$. Similarly, we define the support and the area of a 2-chain. In the sequel, we shall not distinguish between chains and their minimal realizations. A 1-cycle is called *generic* or *in general position* if it is a union of pairwise disjoint piecewise smoothly embedded oriented circles taken with multiplicity 1.

Recall that a *contact structure* on an oriented 3-manifold M is a smooth field of 2-planes which can be represented as $\ker\eta$ where η is a 1-form such that $\eta\wedge d\eta$ does not vanish. It is known that all contact structures are locally equivalent to each other.

A 1-chain on a contact 3-manifold (M,η) is called *Legendrian* if it is C^2-smooth and the restriction of η identically equals zero on its smooth pieces. A 1-chain α is called *positively transverse* if it can be represented as $\alpha = \sum m_i\alpha_i$ where $m_i > 0$ and $\alpha_i^*(\eta) > 0$ for all i (such a realization of α is automatically minimal).

Let us denote the standard coordinates in $\mathbb{R}^3$ by x, y, z and let us consider the contact structure defined by the 1-form $\eta = dz - y\,dx$. Let $\operatorname{pr} : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(x,y,z) \mapsto (x,y)$.

Lemma 2.1. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a C^2-smooth path starting at a point $p_0 = (x_0,y_0)$. Then for any $z_0 \in \mathbb{R}$ there exists a unique Legendrian path $\tilde\gamma : [0,1] \to \mathbb{R}^3$ starting at $\tilde p_0 = (x_0,y_0,z_0)$ such that $\gamma = \operatorname{pr}\tilde\gamma$. Moreover, the length of $\tilde\gamma$ is less than $L\sqrt{1+(|y_0|+L)^2}$ where L is the length of γ.*

The path $\tilde\gamma$ is called the *Legendrian lift* of γ starting at $\tilde p_0$.

Proof. Let $\gamma(t) = (x(t),y(t))$. Set $\tilde\gamma(t) = (x(t),y(t),z(t))$ where $z(t) = z_0 + \int_{\gamma([0,t])} y\,dx$. We have $|\tilde\gamma'|^2 = \dot x^2+\dot y^2+\dot z^2 = \dot x^2+\dot y^2+(y\,\dot x)^2 \le (1+y^2)|\gamma'|^2$. Hence, the length of $\tilde\gamma$ is less than $L\max\sqrt{1+y^2}$. It remains to note that $\max y \le |y_0|+L$. $\square$

Lemma 2.2. *Let $0<\varepsilon<1/2$ and let $\tilde p_0=(x_0,y_0,z_0)$ and $\tilde p_1=(x_1,y_1,z_1)$ be points in $\mathbb{R}^3$ such that $|y_0|<1$, $|y_1|<1$, and $\|\tilde p_1-\tilde p_0\|<\varepsilon^2$. Then there exists a piecewise smooth Legendrian path from $\tilde p_0$ to $\tilde p_1$ whose length is less than $c_1\varepsilon$ for some absolute constant c_1.*

Proof. Let γ_1 be a straight line segment connecting $p_0=\mathrm{pr}(\tilde p_0)$ to $p_1=\mathrm{pr}(\tilde p_1)$ and let $\tilde\gamma_1$ be the Legendrian lift of γ_1 starting at $\tilde p_0$. Let $\tilde p_1'=(x_1,y_1,z_1')$ be the end of $\tilde\gamma_1$. Let $\gamma_2=\mathrm{sign}(z_1-z_1')\partial D$ where D is a disk of area $|z_1-z_1'|$ such that $p_1\in\partial D$. Let $\tilde\gamma_2$ be the Legendrian lift of $\tilde\gamma_2$ starting at $\tilde p_1'$. Then the end of $\tilde\gamma_2$ coincides with p_1 because $\int_{\tilde\gamma_2}dz=\int_{\gamma_2}y\,dx=\pm\,\mathrm{Area}(D)$. The estimate for the length of $\tilde\gamma_1$ is obtained by a straightforward application of Lemma 2.1. $\square$

Lemma 2.3. *Let M be a contact C^2-smooth 3-manifold endowed with a Riemannian metric. Let α be a Legendrian zero-homologous 1-cycle on M. Then for any $\varepsilon>0$ there exist ε-short Legendrian 1-cycles $\alpha_1,\dots,\alpha_n$ on M such that $\sum\alpha_j=\alpha$.*

Proof. It is known that all contact structures are locally equivalent to each other. Hence, for any $p\in M$ there exist its neighbourhood U_p and a smooth embedding $\varphi_p:U_p\to\mathbb{R}^3$ taking the given contact structure on M to the contact structure on $\mathbb{R}^3$ defined by the form $\eta=dz-y\,dx$. Replacing U_p by a smaller neighbourhood, if necessary, we may assume that the set $\varphi_p(U_p)$ is convex, contained in the layer $\{|z|<1\}$, and there exists a constant $m_p>0$ such that $\|d\varphi_p(v)\|>m_p\|v\|$ for all $v\in TU_p$. In each U_p, let us choose an open subset V_p such that $p\in V_p$ and $\overline{V}_p\subset U_p$.

Let β be a 2-cycle in M whose boundary is α. Let us choose a finite subfamily $\mathcal{U}=\{(U_i,V_i,\varphi_i)\}_{i=1,\dots,k}\subset\{(U_p,V_p,\varphi_p)\}_{p\in M}$ such that the support of β is contained in $\bigcup_{i=1}^k V_i$, and let $m=\min_{(U_p,V_p,\varphi_p)\in\mathcal{U}}m_p$.

Let $\varepsilon_1=\min_i\mathrm{dist}\big(\varphi_i(\overline{V}_i),\,\mathbb{R}^3\setminus\varphi_i(U_i)\big)$ and let $\varepsilon_2=\min(\varepsilon_1,m\varepsilon/3)/c_1$ (here c_1 is the constant from Lemma 2.2). Let us represent β as a sum of simplices $\beta=\beta_1+\dots+\beta_n$ so that:

1. each β_j is contained in some V_{i_j} and $\mathrm{diam}\,\varphi_{i_j}(\beta_j)<\varepsilon_2^2$ far any $i=1,\dots,n$;
2. the lengths (with respect to the metric on M) of those edges of β_j's which contribute to $\alpha=\partial\beta$ are smaller than $\varepsilon/3$.

Let $\Gamma=\{\gamma\}$ be the set of those edges of the simplices β_j's which do not contribute to α (for each pair of edges which cancel against each other in $\partial\sum\beta_j$, we include only one of them in Γ). For each $\gamma\in\Gamma$, $\gamma\subset\partial\beta_j$, using Lemma 2.2, we can choose a piecewise Legendrian path γ' which joins the ends of $\varphi_{i_j}(\gamma)$ and which is shorter than $c_1\varepsilon_2$. Since $c_1\varepsilon_2\le\varepsilon_1$, we have $\gamma'\subset U_{i_j}$, hence $\gamma''=\varphi_{i_j}^{-1}(\gamma')$ is a Legendrian path in M shorter than $\varepsilon/3$. Let Γ'' be the set of all such γ''.

Finally, for each $j=1,\dots,n$, we define α_j as the cycle obtained from the boundary of β_j by replacing each of its edges $\gamma\in\Gamma$ with the corresponding path $\gamma''\in\Gamma''$. $\square$

Definition 2.4. Let α be a positively transverse 1-cycle in a contact 3-manifold M. A finite collection of 1-cycles $\mathcal{A}=\{\alpha_1,\dots,\alpha_n\}$ in M is called a *Legendrian net hanged on α* if

1. each α_i decomposes into the sum of two 1-chains $\alpha_i=\alpha_i^{\mathrm{pt}}+\alpha_i^{\mathrm{leg}}$ (each of them may be zero) where α_i^{pt} is positively transverse and α_i^{leg} is Legendrian;

(2) $\alpha_1 + \cdots + \alpha_n = \alpha_1^{\mathrm{pt}} + \cdots + \alpha_n^{\mathrm{pt}} = \alpha$;

The cycles $\alpha_1, \ldots, \alpha_n$ will be called the *cells* of $\mathcal{A}$, and the union of their supports will be called the *support* of $\mathcal{A}$.

Definition 2.5. Let α be a generic positively transverse cycle in M. A Legendrian net $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ hanged on α is called *generic* or *in general position* if there exists a piecewise smoothly embedded graph Γ with Legendrian edges such that

(1) the multiplicity (i.e. the number of incident edges) of any vertex of Γ is either 1 or 3;
(2) each end (i.e. vertex of multiplicity 1) of Γ is a smooth point of the support of α, and the tangents to Γ and to α at this point are distinct;
(3) $\Gamma \cap \operatorname{supp} \alpha$ coincides with the set of the ends of Γ;
(4) each chain α_i^{leg} is a sum of edges of Γ taken with the coefficients ± 1, every edge contributing to exactly two cells with the opposite signs.

Proposition 2.6. *Let M be a contact C^2-smooth 3-manifold endowed with a Riemannian metric. Let α be a positively transverse 1-cycle in M which is homologous to zero. Then for any ε there exists a Legendrian net $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ hanged on α all whose cells are ε-short. The support of $\mathcal{A}$ can be done arbitrarily close to the support of any given 2-chain β such that $\partial\beta = \alpha$.*

If, moreover, α is generic, then $\mathcal{A}$ can also be chosen generic.

The proof is the same as that of 2.3, and we omit it. To achieve the genericity of $\mathcal{A}$, one should apply the following statement.

Proposition 2.7. *Let M be a contact C^2-smooth 3-manifold endowed with a Riemannian metric. Let α be a generic positively transverse 1-cycle in M, and let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ be a Legendrian net hanged on α.*

Then for any $\delta > 0$ there exists a generic Legendrian net $\mathcal{A}' = \{\alpha_1', \ldots, \alpha_n'\}$ hanged on α such that for any $i = 1, \ldots, n$, the cycles α_i' and α_i are δ-close in Hausdorff metric and $|\operatorname{len}\alpha_i - \operatorname{len}\alpha_i'| < \delta$.

Proof. Step 1. Let us show that after an arbitrarily small perturbation of $\mathcal{A}$, one can find an embedded graph Γ with Legendrian edges such that Condition (4) of Definition 2.5 is satisfied.

By the definition of 1-chains, there exist piecewise smooth Legendrian paths $\gamma_1, \ldots, \gamma_k$ and integer coefficients m_{ij} such that $\alpha_i^{\mathrm{leg}} = \sum_j m_{ij}\gamma_j$. We must achieve $|m_{ij}| \le 1$ for all i, j. To this end we shall successively reduce the quantity $\sum_{ij} \max(0, |m_{ij}| - 1)$. Suppose that $m_{i_0,j_0} \ge 2$ for some i_0, j_0 (the case $m_{i_0,j_0} \le -2$ is analogous). Since γ_{j_0} does not contribute to $\sum_i \alpha_i$, we have $\sum_i m_{i,j_0} = 0$. Hence, there exists an index i_1 such that $m_{i_1,j_0} < 0$. Let γ' be a Legendrian perturbation of γ_{j_0} such that $\partial\gamma' = \partial\gamma_{j_0}$ and $(\operatorname{supp}\gamma') \cap (\operatorname{supp}\Gamma) = \operatorname{supp}(\partial\gamma')$. Let us replace α_{i_0} with $\alpha_{i_0} - \gamma_{j_0} + \gamma'$ and α_{i_1} with $\alpha_{i_1} + \gamma_{j_0} - \gamma'$. It easy to see that this reduces the quantity $\sum_{ij} \max(0, |m_{ij}| - 1)$ at least by one.

Step 2. Suppose that there exists an embedded graph Γ with Legendrian edges which satisfies Condition (4) of Definition 2.5, and let us show that it can be perturbed so that (1)–(3) are satisfied.

Let p be a vertex of $\hat\Gamma = \Gamma \cup (\operatorname{supp}\alpha)$ of multiplicity $k > 3$. Let us consider an auxiliary graph G_p defined as follows. Its vertices are the edges of $\hat\Gamma$ incident to

p. Two vertices γ and γ' of G_p (i.e. edges of $\hat\Gamma$) are connected by an edge in G_p when $\gamma\subset\operatorname{supp}\alpha_i$ and $\gamma'\subset\operatorname{supp}\alpha_i$ for some α_i. The condition $\sum_i\alpha_i=\alpha$ implies that after removing certain edges from G_p, one obtains a disjoint union of graphs $E_1\sqcup\dots\sqcup E_m$, moreover, the graphs $E_2,\dots,E_m$ are combinatorially equivalent to a circle, and E_1 is equivalent either to a circle (when $p\notin\operatorname{supp}\alpha$), or to a segment whose endpoints correspond to the edges of $\hat\Gamma$ lying on $\operatorname{supp}\alpha$. Denote the vertices of E_k by $\gamma_{k,1},\dots,\gamma_{k,c_k}$ so that $\gamma_{k,j}$ is connected to $\gamma_{k,j+1}$ by an edge in E_k.

Let U_p be a sufficiently small neighbourhood of p diffeomorphic to the ball, such that $\Gamma_p=U_p\cap\Gamma=\bigcup_{k,j}(\gamma_{k,j}\cap U_p)$ and each of $\gamma_{k,j}\cap U_p$ is an embedded segment transverse to ∂U_p. Set $\Gamma_{p,k}=\bigcup_{j=1}^{c_k}(\gamma_{k,j}\cap U_p)$ and $q_{k,j}=\gamma_{k,j}\cap\partial U_p$. Let $\Gamma'_{p,k}$, $k=1,\dots,m$, be an arbitrary plane tree embedded into a disk Δ such that all of its vertices have multiplicity 1 or 3, the number of ends (i.e. vertices of multiplicity 1) is equal to c_k, and all the ends lie on $\partial\Delta$. Let us denote the ends of $\Gamma'_{k,j}$ by $q'_{k,1},\dots,q'_{k,c_k}$ in the cyclic order along $\partial\Delta$. When $p\in\operatorname{supp}\alpha$, we shall also assume that there exists a vertex p' of $\Gamma'_{p,1}$ connected by edges to $q'_{1,1}$ and q'_{1,c_1}.

To perturb Γ as is required, we replace each tree $\Gamma_{p,k}$ by the image of $\Gamma'_{p,k}$ under an embedding into M which has the following properties. It takes $q'_{k,j}$ to $q_{k,j}$, it maps the union of the edges $[p',q'_{1,1}]\cup[p',q'_{1,c_1}]$ homeomorphically onto the arc $q_{1,1}q_{1,c_1}$ of α (the vertex p' being sent to a smooth point of this arc), and the images of all other edges of $\Gamma'_{p,k}$ are Legendrian. $\square$

3. Approximation of a Legendrian net by the union of boundaries of analytic disks

Let V be a complex analytic surface and M an oriented real hypersurface in V. Then the field of complex tangents is defined on M. It can be represented as $\ker\eta$ for some 1-form η. We shall call a curve $\gamma:[0,1]\to M$ *Legendrian* (resp. *positively transverse*) if $\gamma^*\eta=0$ (resp. $\gamma^*\eta>0$). In the case when M is strictly pseudoconvex, the field of complex tangents is a contact structure on M, hence these definitions are coherent with the definitions in §2.

Lemma 3.1. *Let U be an open subset in $\mathbb{C}^2$, and $M\subset U$ a real hypersurface defined by an equation $f=0$ where f is a real C^2-smooth function in U. Let $\gamma:[0,t_1]\to M$ be a Legendrian C^2-smooth path and let $p_0=\gamma(0)$. Let T be the complex tangent line to M at p. Suppose that the Hessian H at p_0 of the restriction $f|_T$ is positive definite.*

Let L_t be the complex line passing through the points p and $\gamma(t)$. Let S_t^+ and S_t^- denote the arcs into which the curve $L_t\cap M$ is divided by the points $\gamma(0)$ and $\gamma(t)$. Then we have

$$\lim_{t\to0}\frac{2\operatorname{len}(S_t^\pm)}{\operatorname{len}\big(\gamma([0,t])\big)}=\frac{\operatorname{len}(E)}{d(E,\gamma'(0))}<\pi\sqrt{K_1/K_2},\tag{1}$$

where E is the ellipse $\{H=1\}$, $d(E,v)$ is the length of its diameter in the direction of a vector v, and K_1, K_2 $(K_1\ge K_2)$ are the principal curvatures of M in the direction of T.

Proof. Let us denote the coordinates in $\mathbb{C}^2$ by (z,w). Without loss of generality we may assume that p_0 is the origin, T is the axis $w=0$, and $\gamma'(0)=(1,0)$. Then we have

$$f'_z(0,0)=f'_{\bar z}(0,0)=0\qquad\text{and}\qquad f'_w(0,0)=\overline{f'_{\bar w}(0,0)}=a\ne0.\tag{2}$$

Since f is twice differentiable, we have

$$f(z,w) = aw + \bar a\bar w + Az^2 + 2Bz\bar z + \bar A\bar z^2 + w\,g_1 + \bar w\,g_2 + (z\bar z + w\bar w)\,g_3, \qquad (3)$$

where

$$2A = f''_{zz}(0,0), \quad 2B = f''_{z\bar z}(0,0), \qquad \lim_{(z,w)\to(0,0)} g_{1,2,3}(z,w) = 0.$$

Let us set $\gamma(t) = (z(t), w(t))$. The condition that the path γ is Legendrian means that

$$f'_z(\gamma(t))\,z'(t) + f'_w(\gamma(t))\,w'(t) = 0, \qquad t \in [0, t_1]. \qquad (4)$$

For $t = 0$, by (2), this implies $w'(0) = 0$. Hence we have

$$z(t) = t(1+\alpha_1(t)), \quad w(t) = b\,t^2(1+\alpha_2(t)), \quad 2b = w''(0), \quad \lim_{t\to 0}\alpha_{1,2}(t) = 0. \quad (5)$$

Differentiating (4) at $t = 0$ and combining with (2), (3), and (5), we obtain

$$2ab + 2A + 2B = 0. \qquad (6)$$

Consider the parametrization of L_t given by $\varphi_t : \mathbb{C} \to \mathbb{C}^2$, $\zeta \mapsto \big(z(t)\zeta, w(t)\zeta\big)$. Let us denote the curve $\varphi_t^{-1}(L_t \cap M)$ by S_t. It is defined by $f(\varphi_t(\zeta)) = 0$. Using (3) and (5), one can rewrite the left hand side of this equation in the form

$$t^2 \cdot \Big(ab\zeta + \bar a\bar b\bar\zeta + A\zeta^2 + 2B\zeta\bar\zeta + \bar A\bar\zeta^2 + g(t,\zeta)\Big) \qquad (7)$$

where $g(t,\zeta)$ tends to zero as $t \to 0$ uniformly on any bounded subset of $\mathbb{C}$. Note, that the Hessian of the restriction $f|_T$ has the form $H(z) = (Az^2 + 2Bz\bar z + \bar A\bar z^2)/2$. Hence, combining (7) with (6) and dividing by $2t^2$, we obtain $S_t = \{\,\zeta \mid H(\zeta - 1/2) + g(t,\zeta) = H(1/2)\,\}$, and hence, $S_t \to E_{1/2}$ for $t \to 0$ where $E_{1/2} = \{\,\zeta \mid H(\zeta - 1/2) = H(1/2)\,\}$ is a translate of E. Since the second derivatives of f are continuous, $S_t \to E_{1/2}$ implies that $\mathrm{len}(S_t) \to \mathrm{len}(E)$ and $\mathrm{len}(S_t^{\pm}) \to \mathrm{len}(E)/2$. It remains to note that the length of $\varphi_t^{-1}(\gamma[0,t])$ tends to $d(E,1)$, because γ is twice differentiable. □

Remark 3.2. a). If we replace the condition that γ is Legendrian in Lemma 3.1 by the weaker condition that $\gamma'(0) \in T$, then S_t will still tend to some translate of E. However, it may then happen, that the center of the translated ellipse will not be on the real axis, and hence the arc $\varphi_t^{-1}(\gamma[0,t])$ will not tend to a diameter. Thus, the upper bound for the ratio of the length will fail.

b). The only place in the proof where the continuity of the second derivatives of f is used, is the implication $(S_t \to E_{1/2}) \Longrightarrow (\mathrm{len}(S_t) \to \mathrm{len}(E))$. Therefore, the assertion of the lemma remains true if we replace the condition that f is of class C^2 by the weaker condition that f is just twice differentiable but assume in addition that M is convex.

Corollary 3.3. *Let Ω be a domain in a complex analytic surface whose boundary $M = \partial\Omega$ is C^2-smooth. Suppose that M is endowed with a C^2-smooth Riemannian metric g, and let $p_0 \in M$. Suppose that M is strictly pseudoconvex in a neighbourhood of a point p_0. Let $\gamma : [0, t_1] \to M$, $\gamma(0) = p_0$, be a Legendrian C^2-smooth curve.*

Then, for any $\delta > 0$, there exists a family of analytic disks $\{D_t\}_{t\in[0,t_2]}$, $t_2 \le t_1$ such that $D_t \subset \Omega$, $\partial D_t \subset M$, $D_t \cap \gamma = \{p_0, \gamma(t)\}$, D_t is transverse to M, and

$$\lim_{t\to 0} \frac{\operatorname{len} S_t^{\pm}}{\operatorname{len}\gamma([0,t])}, < 1 + \delta \tag{8}$$

where S_t^+ and S_t^- are the arcs into which the curve ∂D_t is divided by the points $\gamma(0)$ and $\gamma(t)$.

Proof. Let us choose the coordinates (z, w) as in the proof of Lemma 3.1. Then the coordinate change $(z, w) \to (z, w + cz)$ transforms H into

$$(A + ac)z^2 + 2Bz\bar{z} + (\bar{A} + \bar{a}\bar{c})\bar{z}^2.$$

Let us choose c so that $A + ac = B - \delta_1$ when $\delta_1 \ll \delta$, and apply Lemma 3.1. $\square$

Definition 3.4. Let M be a smooth contact manifold. A *Positive Transverse Simple Crossing Curve* (PTSC-curve) in M is a union of piecewise smooth embedded positively transverse oriented closed curves $S = S_1 \cup \dots \cup S_m$ (called the *components of* S) which meet each other at most pairwise and so that if S_i and S_j intersect at p then each of these curves is smooth at p and the tangents to S_i and to S_j at p are distinct.

A *circuit* of a PTSC-curve S is an oriented piecewise smooth embedded circle γ which is a union of arcs of S such that

(1) on any smooth arc a of γ, the orientation induced from γ coincides with the orientation induced from S;
(2) if γ passes through the intersection point of two components of S then it switches from one component to the other one.

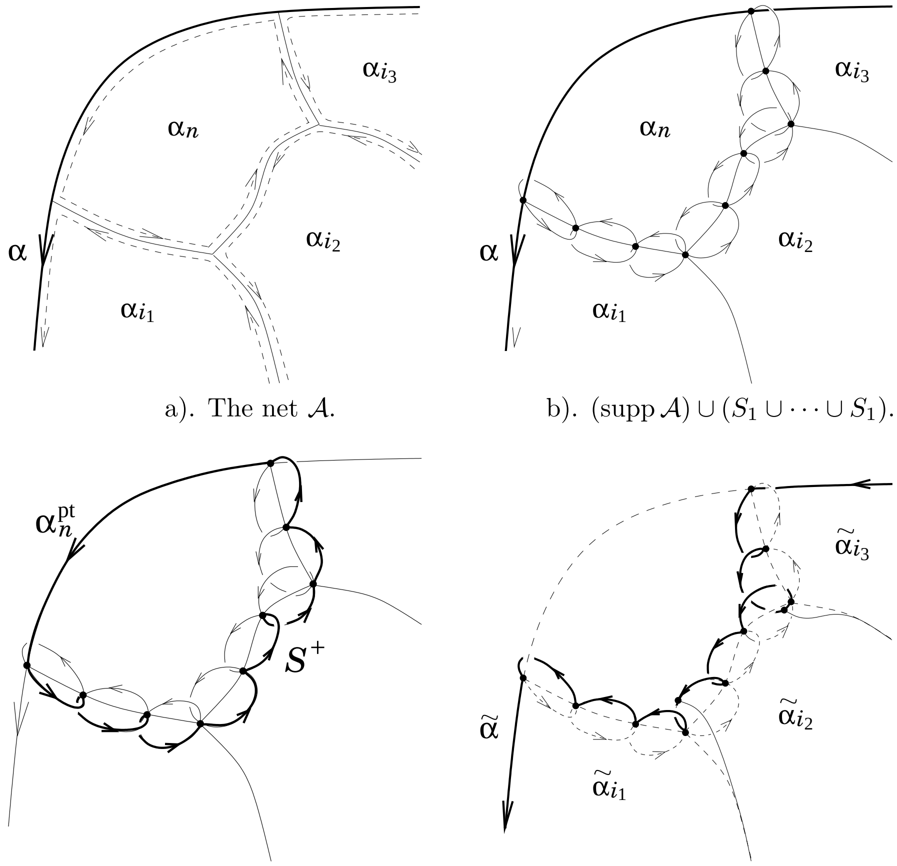
It is clear that any two circuits may intersect each other only at intersection points of components of S, and the sum of all circuits is S.

Proposition 3.5. *a). Let Ω be a domain in a complex analytic surface whose boundary $M = \partial\Omega$ is C^2-smooth. Suppose that M is endowed with a C^2-smooth Riemannian metric g. Let α be a positively transverse curve which is a union of disjoint piecewise C^2-smoothly embedded circles, and let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ be a generic Legendrian net hanged on α. Suppose that M is strictly pseudoconvex in a neighbourhood of* $\operatorname{supp}\mathcal{A}$.

Then, for any $\delta > 0$, there exists a PTSC-curve $S = S_1 + \dots + S_N$ such that:

(1) *each S_j is the boundary of an analytic disk D_j in Ω;*
(2) *$S + \alpha$ has exactly n circuits $\beta_1, \dots, \beta_n$;*
(3) *for any $i = 1, \dots, n$, the Hausdorff distance between β_i and α_i is less than δ, and $\operatorname{len}(\beta_i) < \operatorname{len}(\alpha_i) + \delta$.*

b). If, moreover, Ω is a domain in $\mathbb{C}^2$ and the sectional curvature of M in the direction of complex tangents does not vanish in some neighbourhood of M, then the disks $D_1, \dots, D_N$ can be chosen so that each of them is the intersection of Ω with some complex line, but in this case the estimate for the lengths should be replaced by $\operatorname{len}(\beta_i) < c_3 \operatorname{len}(\alpha_i) + \delta$ where c_3 is a constant depending on M, g, and $\mathcal{A}$ (when Ω is the unit ball and g is induced by the standard metric in $\mathbb{C}^2$, one has $c_3 = \pi/2$).



a). The net $\mathcal{A}$. b). $(\operatorname{supp}\mathcal{A}) \cup (S_1 \cup \cdots \cup S_1)$.

c). The circuit $\beta_n = \alpha_n^{\mathrm{pt}} + S^+$. d). The net $\tilde{\mathcal{A}}$.

FIG. 1

Proof. a). Induction by n. The case $n = 0$ is trivial. Suppose that we have proved the required statement for Legendrian nets having $n-1$ cells. Let us prove it for a Legendrian net $\mathcal{A}$ which has n cells. Let α_i^{pt} and α_i^{leg} be as in Definition 2.4. The construction described below is illustrated in Figures 1(a–d).

By Corollary 3.3, for any point $p \in \alpha_n^{\mathrm{leg}}$ there is a neighbourhood U_p such that for any $q \in U_p \cap \alpha_n$ there exists an analytic disk $D_{pq} \subset \Omega$ satisfying inequality (8) with an arbitrarily given number δ_1 instead of δ. Choosing a finite subcovering $\{U_p\}$, we can represent α_n^{leg} as the sum of arcs $\alpha_n^{\mathrm{leg}} = \gamma_1 + \cdots + \gamma_k$ so that for any $i = 1, \ldots, k$ there exists an analytic disk $D_i \subset \Omega$ such that $\partial D_i = S_i = S_i^+ + S_i^-$, $\partial S_i^\pm = \pm\partial\gamma_i$, and $\operatorname{len}(S_i^\pm)/\operatorname{len}(\gamma_i) < 1 + \delta_1$. We may also assume that the length of each arc γ_i is less than an arbitrarily given number, and that any edge of Γ (the graph from Definition 2.4) contributing to α_n^{leg} is the sum of a subset of arcs γ_i.

Perturbing the disks D_i, we can achieve that they are transverse to each other and hence, the curves S_i have distinct tangents at the intersection points. We may also assume that if an end of γ_i lies on α then the tangents at this point to α and to γ_i are distinct. Let us set $S^\pm = \sum_{i=1}^k S_i^\pm$. These are positively transverse chains such that $\partial S^+ = \partial\alpha_n^{\mathrm{leg}} = -\partial S^-$. Hence $\tilde\alpha = \alpha - \alpha_n^{\mathrm{leg}} + S^-$ is a generic positively transverse cycle.

Passing if necessary from the arcs γ_j to their subdivisions, we may assume that the collection of arcs $\gamma_1,\dots,\gamma_k$ can be completed up to $\gamma_1,\dots,\gamma_k,\gamma_{k+1},\dots,\gamma_m$, so that $\alpha_i=\sum_{j=1}^m a_{ij}\gamma_j$, $i=1,\dots,n$, for some matrix of integer coefficients a_{ij} such that $a_{ij}\in\{-1,0,1\}$ for any $i=1,\dots,n$, $j=1,\dots,m$. Some of $\gamma_{k+1},\dots,\gamma_m$ being positively transverse, the others being Legendrian.

Let us denote the set of ends of $\gamma_1,\dots,\gamma_k$ not belonging to α by P. In other words, $P=(S\cap\operatorname{supp}\alpha_n)\setminus\operatorname{supp}\alpha=(S^-\cap\operatorname{supp}\alpha_n)\setminus\operatorname{supp}\alpha$. For every $p\in P$, let us define a point $\tilde p$ as follows. Let γ_i, $1\le i\le k$, be an arc whose end is p (there are two such arcs but we choose any of them). Then we define $\tilde p$ as an interior point of S_i^- which is closer to p than to the other end of S_i^-. If p is the end of an arc γ_i and $p\notin P$, we set $\tilde p=p$.

For each $i=1,\dots,m$, let us define an arc $\tilde\gamma_i$ as follows. Let $\partial\gamma_i=q-p$, and let $\tilde p$ and $\tilde q$ be the points chosen as it is described above starting from p and q respectively. If $1\le i\le k$, we define $\tilde\gamma_i$ as the path on S^- connecting $\tilde p$ to $\tilde q$. If $k<i\le m$ and the arc γ_i is Legendrian, we define $\tilde\gamma_i$ as a Legendrian path from $\tilde p$ to $\tilde q$. If the arc γ_i is positively transverse, we set $\tilde\gamma_i=\gamma_i$. In all the cases, we orient $\tilde\gamma_i$ so that $\partial\tilde\gamma_i=\tilde q-\tilde p$. It follows from Lemma 2.2 that the arc $\tilde\gamma_i$ can be chosen arbitrarily close to γ_i.

Let us set $\tilde{\mathcal A}=\{\tilde\alpha_1,\dots,\tilde\alpha_{n-1}\}$ where $\tilde\alpha_i=\sum_{j=1}^m a_{ij}\tilde\gamma_j$, $i=1,\dots,n-1$. It is easy to check that this is a generic Legendrian net hanged on $\tilde\alpha$ (see Figure 1 d). Hence, by the induction hypothesis, we can find a PTSC-curve $\tilde S=S_{k+1}+\dots+S_N$ so that the statement of the lemma holds for $\tilde{\mathcal A}$ instead of $\mathcal A$ and for an arbitrarily chosen constant instead of δ. Then, for a suitable choice of the constants involved in the construction of $\tilde{\mathcal A}$, the curve $S=S_1+\dots+S_k+\tilde S$ will satisfy the conclusion of the lemma. Indeed, let us denote the circuits of $\tilde S$ by $\beta_1,\dots,\beta_{n-1}$. Then the curve S has n circuits, namely, $\beta_1,\dots,\beta_{n-1}$, and $\beta_n=\alpha_n^{\mathrm{pt}}+S^+$. By the induction hypothesis, the circuits $\beta_1,\dots,\beta_{n-1}$ are close to the circles $\tilde\alpha_1,\dots,\tilde\alpha_{n-1}$, hence also to the cycles $\alpha_1,\dots,\alpha_{n-1}$. The circuit β_n is close to the cycle α_n by construction (see Figure 1 c).

b). The proof is more or less the same as in Part a), but the manifold M should be replaced by a neighbourhood of the support of $\mathcal A$ where the quantity K_1/K_2 from (1) is bounded from below by some constant. $\square$

Let us denote $\mathbb R_+=\{x\in\mathbb R\,|\,x\ge0\}$ and $\mathbb R_-=\{x\in\mathbb R\,|\,x\le0\}$.

Lemma 3.6. *Let x,y,z be coordinates in $\mathbb R^3$ and let $f_1,f_2:\mathbb R^3\to\mathbb C^2$ be the mappings given by $f_1(x,y,z)=x+iz$, $f_2(x,y,z)=y+iz$. For any complex number c, let us denote the real curve $\{p\in\mathbb R^3\,|\,f_1(p)f_2(p)=c\}$ by S_c. Then, for $c\notin\mathbb R_-$, the curve S_c has exactly two branches (i.e. two connected components) S_c^+ and S_c^- such that $S_c^+\subset\{x+y>0\}$, $S_c^-\subset\{x+y<0\}$, and the restriction of the linear function $\mathbb R^3\to\mathbb R$, $(x,y,z)\mapsto x-y$, to each of the branches $S_c^\pm$ is a diffeomorphism.*

Moreover, $S_c^\pm$ tends in any reasonable sense to $S_0^\pm$ as $c\to0$, $c\notin\mathbb R_-$ where $S_0^+=\{z=xy=0,x+y\ge0\}$ and $S_0^-=\{z=xy,x+y\le0\}$.

Proof. Set $a=\operatorname{Re}c$, $b=\operatorname{Im}c$. Then the curve S_c is given by the system of simultaneous equations $xy-z^2=a$, $z(x+y)=b$. By the change of variables $x-y=2u$, $x+y=2v$ we transform this system to $v^2-u^2-z^2=a$, $2zv=b$.

If $b=0$ and $a>0$ then S_c is the hyperbola $v^2-u^2=a$ in the plane $z=0$.

If $b \neq 0$ then the intersection of S_c with the plane $u = u_0$ can be found by solving the system of simultaneous equations $v^2 - u^2 - z^2 = a$, $2zv = b$, $u = u_0$. Eliminating u, z, we obtain the equation $v^4 - (u_0^2 + a)v^2 - (b/2)^2 = 0$ with respect to the variable v. It is clear that for any value of u_0, this equation has exactly two roots one of them being positive and the other one being negative. $\square$

Remark. For $c \in \mathbb{R}_-$, the curve S_c is not smooth. It is the union of the hyperbola $v^2 - u^2 = c$ in the plane $z = 0$ and the circle $u^2 + z^2 = -c$ in the plane $v = 0$ which intersect at the two points $z = v = 0$, $u = \pm\sqrt{-c}$.

Lemma 3.7. *Let M be a C^2-smooth oriented real 3-manifold, and let f_1, f_2, h be C^2-smooth complex valued functions on M such that $f_1(p_0) = f_2(p_0) = 0$, $h(p_0) \neq 0$, and each of f_1, f_2 is a submersion in a neighbourhood of some point $p_0 \in M$. Let us denote the real curves $f_j^{-1}(0)$ by γ_j, $j = 1, 2$. On each γ_j near p_0, let us introduce the orientation induced by the submersion f_j. Suppose that the tangents to γ_1 and γ_2 at p_0 are distinct.*

Then there exist a number $\theta_0 \in [0, 2\pi[$ and a neighbourhood U of p_0 such that each of the curves $U \cap \gamma_j$, $j = 1, 2$, is diffeomorphic to an open interval and for any fixed $\theta \not\equiv \theta_0 \mod 2\pi$ there exists $r_0 = r_0(\theta) > 0$ such that for $0 < r < r_0$, the curve $S_{r,\theta} = \{\, p \in U \mid f_1(p) f_2(p) = re^{i\theta} h(p) \,\}$ consists of two smooth branches one of which tending to $\gamma_1^- \cup \gamma_2^+$ and the other one tending to $\gamma_2^- \cup \gamma_1^+$ as $r \to 0$ where $\gamma_j^\pm$ denotes the preimage of $\mathbb{R}_\pm$ under an orientation preserving embedding $(U \cap \gamma_j, p_0) \to (\mathbb{R}, 0)$.

Proof. It is clear that if the statement of the lemma holds for f_1, f_2, h, then it holds (with maybe another number θ_0) also for $c_1 f_1, c_2 f_2, c_3 h$ where c_1, c_2, c_3 are arbitrary nonzero complex numbers. Therefore, we may assume that $h(p_0) = 1$. Let us choose a local real coordinate z in a neighbourhood of p_0 so that the both curves γ_1, γ_2 lie on the surface $z = 0$. Multiplying f_1 and f_2 by suitable complex numbers, we may assume that $\partial(\operatorname{Re} f_j)/\partial z(p_0) = 0$, $j = 1, 2$. Let us set $x = \operatorname{Re} f_1$, $y = \operatorname{Re} f_2$. Then (x, y, z) is a local coordinate system where the functions f_1, f_2 have the form $f_1(x, y, z) = x + iz + O(x^2 + y^2 + z^2)$, $f_2(x, y, z) = y + iz + O(x^2 + y^2 + z^2)$. Therefore, the statement follows from Lemma 3.6 combined with the fact that the curve $H_r(S_{r,\theta})$ tends to the curve $\{(x + iz)(y + iz) = e^{i\theta}\}$ as $r \to 0$, $\theta = \text{const} \not\equiv \pi \mod 2\pi$ where H_r stands for the homothety $(x, y, z) \mapsto (x/\sqrt{r},\, y/\sqrt{r},\, z/\sqrt{r})$. $\square$

Proposition 3.8. *Let Ω be an arbitrary domain in $\mathbb{C}^2$ with a compact C^2-smooth boundary $M = \partial\Omega$, and let A be an algebraic curve in $\mathbb{C}^2$ given by $f = 0$. Suppose that $S = A \cap M$ is a PTSC-curve. Let h be a polynomial which does not vanish at the double points of S. Then there exists a finite set $\Theta \subset [0, 2\pi[$ such that for any $\theta \in [0, 2\pi[\,\setminus\Theta$, there is $r_0 = r_0(\theta) > 0$ such that the real curve $S_{r,\theta} = \{\, p \in M \mid f(p) = re^{i\theta} h(p) \,\}$ for $0 < r < r_0$ is smooth and its connected components converge to the circuits of S as $r \to 0$.*

Proof. Follows from Lemma 3.7. $\square$

Proof of Theorem 1.1. By Proposition 2.6, we can construct a Legendrian net hanged on ∂A all of whose cells are small. By Proposition 3.5, it can be approximated by the boundary of the union D of analytic (resp., linear) disks so that the circuits of $D \cap M$ are arbitrarily small. Using Proposition 3.8 in the algebraic case, and the standard techniques of analytic sheaves on open Riemann surfaces in the analytic case, we can perturb D so that all of its boundary components become close to circuits of S. Moreover, this perturbation can be chosen so that the points of P do not move (in the algebraic case, we just choose h in Proposition 3.8 so that it vanishes on P). $\square$

4. Some elementary facts about the standard contact structure on $\mathbb{S}^3 \subset \mathbb{C}^2$.

For the reader's convenience, in this section we shall give some well-known facts about curves on $\mathbb{S}^3$ and their projections to $\mathbb{P}^2$, and we shall deduce from them a lower bound for $n(\varepsilon)$ of the order $1/\varepsilon^2$.

Let $z = x+iy$ and $w = u+iv$ be the standard coordinates in $\mathbb{C}^2$. Let us denote:

$$\rho = \rho(z,w) = |z|^2 + |w|^2 = x^2 + y^2 + u^2 + v^2,$$

$$\eta = 1/2\, d^c\rho = i/2\,(z\,d\bar z - \bar z\,dz + w\,d\bar w - \bar w\,dw) = x\,dy - y\,dx + u\,dv - v\,du,$$

$$\omega = 1/2\, d\eta = i/2\,(dz\wedge d\bar z + dw \wedge d\bar w) = dx\wedge dy + du \wedge dv.$$

Let $\mathbb{B}^4 = \{\rho \le 1\}$, $\mathbb{S}^3 = \partial\mathbb{B}^4 = \{\rho = 1\}$, $\mathbb{P}^1 = \big(\mathbb{C}^2 \setminus \{(0,0)\}\big)/_{(z,w)\sim(\lambda z,\lambda w)}$ and let $\mathrm{pr} : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ and $\mathrm{pr}_{\mathbb{S}^3} : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{S}^3$ be the standard projections. The field of real 2-planes $\ker \eta|_{\mathbb{S}^3}$ is the field of complex tangents to $\mathbb{S}^3$. It defines the standard (tight) complex structure on $\mathbb{S}^3$.

Let $\|\cdot\|_{\mathbb{P}^1}$ and $\omega_{\mathbb{P}^1}$ be the Riemannian Fubini-Studi metric on $\mathbb{P}^1$ and the corresponding volume form which are defined by

$$\|d\zeta\|_{\mathbb{P}^1} = \frac{|d\zeta|^2}{(1+|\zeta|^2)^2}, \qquad \omega_{\mathbb{P}^1} = \frac{i}{2}\,\frac{d\zeta\wedge d\bar\zeta}{(1+|\zeta|^2)^2}, \qquad \zeta = z/w.$$

$\mathbb{P}^1$ equipped with this metric is isometric to the standard 2-sphere of radius $1/2$, in particular, we have

$$\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = \pi.$$

Let

$$\eta^* = \mathrm{pr}^*_{\mathbb{S}^3}(\eta|_{\mathbb{S}^3}) \qquad \text{and} \qquad \omega^* = \mathrm{pr}^*(\omega_{\mathbb{P}^1}).$$

It is easy to check that

$$\eta^* = \frac{\eta}{\rho} = \frac{1}{2}\, d^c \log\rho \qquad \text{and} \qquad d\eta^* = \frac{2\omega}{\rho} - \frac{d\rho\wedge\eta}{\rho^2} = 2\omega^*. \tag{10}$$

Lemma 4.1. *Let F be a 2-chain in $\mathbb{S}^3$. Then*

$$\int_{\partial F}\eta = 2\int_{\mathrm{pr}_* F}\omega_{\mathbb{P}^1}.$$

Proof. Follows from Stokes' theorem and from (10). $\square$

Let $\|\cdot\|_{\mathbb{S}^3}$ be the Riemannian metric on $\mathbb{S}^3$ induced by the standard metric in $\mathbb{C}^2$. It is easy to check that

$$\|v\|^2_{\mathbb{S}^3} = |\eta(v)|^2 + \|\,\mathrm{pr}_* v\|^2_{\mathbb{P}^1}, \qquad v \in T\mathbb{S}^3. \tag{12}$$

In particular, if D is the disk which is cut by $\mathbb{B}^4$ on a complex line passing through the origin, then the circle ∂D is orthogonal to the contact structure and

$$\int_{\partial D}\eta = 2\pi. \tag{13}$$

Lemma 4.2. *Let A be a smooth complex algebraic curve in $\mathbb{C}^2$ passing through the origin and having there a non-degenerate tangency with a complex line L. Let F be the closure of $\mathrm{pr}_{\mathbb{S}^3}(A\cap\mathbb{B}^3\setminus 0)$. Then $\partial F=\partial(A\cap\mathbb{B}^4)-\partial(L\cap\mathbb{B}^4)$. In particular,*

$$\int_{\partial(A\cap\mathbb{B}^4)}\eta = 2\pi + 2\int_{\mathrm{pr}_*F}\omega_{\mathbb{P}^2}\ge 2\pi. \tag{14}$$

Proof. Apply the *real* blowup of the origin (identifying $\mathbb{C}^2$ with $\mathbb{R}^4$). $\square$

Definition 4.3. An n-chain β with piecewise smooth boundary on an oriented n-manifold M is called *positive* (resp., *strictly positive*) if each connected component of the complement of $\partial\beta$ contributes to β with a *nonnegative* (resp., *positive*) multiplicity. We shall write in this case $\beta\ge 0$ (resp., $\beta>0$).

Every n-chain β on M can be represented in a unique way as $\beta=\beta^+-\beta^-$ so that $\beta^+\ge 0$, $\beta^-\ge 0$, and $(\operatorname{supp}\beta^+)\cap(\operatorname{supp}\beta^-)=(\operatorname{supp}\partial\beta^+)\cap(\operatorname{supp}\partial\beta^-)$. The chains $\beta^\pm$ are called the *positive* and the *negative parts* of β.

If U is a domain in M which has piecewise smooth boundary and if β is an n-chain, then the *restriction* of β to U is the n-chain $\beta|_U=\sum m_i(\beta_i\cap U)$ where $\beta=\sum m_i\beta_i$ is the representation of β as a linear combination of domains with piecewise smooth boundaries.

Remark 4.4. Let M be an oriented n-manifold. We shall identify n-chains on M having piecewise smooth boundaries with integer-valued functions that are linear combinations of characteristic functions of domains. Namely, if $\beta_1,\dots,\beta_k$ are domains in M having piecewise smooth boundaries, then the chain $\beta=\sum m_i\beta_i$, $m_i\in\mathbb{Z}$, will be identified with the function $\chi_\beta=\sum m_i\chi_{\beta_i}$ where χ_{β_i} is the characteristic function of the domain β_i (i.e. $\chi_{\beta_i}|_{\beta_i}=1$, $\chi_{\beta_i}|_{M\setminus\beta_i}=0$).

The integral of a 2-form ξ corresponds under this identification to $\int_M\chi_\beta\xi$. Taking the restriction of β to U corresponds to the multiplication by χ_U, etc.

Lemma 4.5. (*Isoperimetric inequality for 2-chains on $\mathbb{S}^2$.*) *Let $\mathbb{S}^2$ be the sphere of radius R in $\mathbb{R}^3$ endowed with the standard Riemannian metric and the standard area form dS. Let β be a 2-chain on $\mathbb{S}^2$ which has a piecewise smooth boundary whose length (taking into account the multiplicities if there are multiple segments) is equal to a, and let $b=\int_\beta dS$ be the oriented area of β. Let β^+ (resp. β^-) be the positive (resp. negative) part of β, and let $b^\pm=\int_{\beta^\pm}dS$.*

Suppose that $|b|<2\pi R^2$ $a<2\pi R$. Then

$$|b|\le b^++b^-\le S_R(a)\qquad\text{where}\quad S_R(a)=2\pi R^2\big(1-\sqrt{1-a^2/(2\pi R)^2}\,\big) \tag{15}$$

and if, moreover, the set $\operatorname{supp}\partial\beta$ is connected then

$$\operatorname{diam}_{\mathbb{S}^2}\operatorname{supp}\beta\le a/2. \tag{16}$$

Proof. If β is a domain on the sphere, then (15) is the classical isoperimetric inequality.

In the general case, the boundary of β can be represented as a disjoint union of closed curves whose lengths we denote by $a_1,\dots,a_k$. Each of these curves is the common boundary of two domains in the sphere, and the area of at least one of them does not exceed $2\pi R^2$. Choosing in a suitable way the signs of these domains, we obtain a 2-chain whose boundary coincides with $\partial\beta$. Adding if necessary several times $\pm[\mathbb{S}^2]$, we obtain a 2-chain β' such that $\beta-\beta'$ is a zero-homologous cycle, $\partial\beta'=\partial\beta$, and β' has the form $\beta'=m[\mathbb{S}^2]+s_1\beta_1+\dots+s_k\beta_k$ where $m\in\mathbb{Z}$, $s_i=\pm1$, and β_i is a domain of area $b_i\le 2\pi R^2$. For each of these domains, we have $b_i\le S_R(a_i)$. Since the function S_R is convex and $S_R(0)=0$, it follows that $S_R(a)=S_R(a_1+\dots+a_k)\ge S_R(a_1)+\dots+S_R(a_k)$. Hence,

$$|b-4\pi R^2 m|=\Big|\sum s_ib_i\Big|\le\sum b_i\le\sum S_R(a_i)\le S_R(a).$$

Let us show that $m=0$. Indeed, recall that $|b|<2\pi R^2$. Combining this inequality with $S_R(a)<2\pi R^2$, we obtain $4\pi R^2|m|\le|b|+|b-4m\pi R^2|<2\pi R^2+S_R(a)<4\pi R^2$, i.e. $|m|<1$. But $m\in\mathbb{Z}$, hence $m=0$.

Let us set $\hat\beta^\pm=\sum_{s_i=\pm1}\beta_i$, $\hat b^\pm=\sum_{s_i=\pm1}b_i$. We have proven that $\hat b^++\hat b^-\le S_R(a)$ and for deducing (15), it remains to note that $b^\pm\le\hat b^\pm$. The latter fact is evident because the decomposition $\beta^+-\beta^-$ can be obtained from $\hat\beta^+-\hat\beta^-$ by successive cancellation of connected components of $\operatorname{supp}\partial\beta$ contributing simultaneously to β^+ and β^-.

Now let us suppose that the set $\operatorname{supp}\partial\beta$ is connected and prove (16). First, let us show that $\operatorname{supp}\beta$ does not contain pair of antipodal points. Indeed, let us denote the central symmetry by $\sigma:\mathbb{S}^2\to\mathbb{S}^2$. The estimate $\operatorname{len}\partial\beta<2\pi R$ yields $\sigma(\operatorname{supp}\partial\beta)\cap\operatorname{supp}\partial\beta=\varnothing$. Since $\operatorname{supp}\beta$ is connected, this implies that $\sigma(\operatorname{supp}\beta)$ is contained in a single connected component of the complement of $\operatorname{supp}\partial\beta$. This component cannot be contained in $\operatorname{supp}\beta$ because its area is greater than the area of $\sigma(\operatorname{supp}\beta)$, and hence greater than the area of $\operatorname{supp}\beta$. Therefore, we have $\sigma(\operatorname{supp}\beta)\cap\operatorname{supp}\beta=\varnothing$.

Let $p,q\in\operatorname{supp}\beta$. Let us denote the shortest geodesic from p to $\sigma(q)$ (resp., from q to $\sigma(p)$) by γ_p (resp., by γ_q). Since the points $\sigma(p)$ and $\sigma(q)$ do not belong to $\operatorname{supp}\beta$, there exist points $p'\in\gamma_p\cap\operatorname{supp}\partial\beta$ and $q'\in\gamma_q\cap\operatorname{supp}\partial\beta$. Therefore, we have $\operatorname{dist}_{\mathbb{S}^2}(p,q)\le\operatorname{dist}_{\mathbb{S}^2}(p',q')\le(\operatorname{len}\partial\beta)/2=a/2$. $\square$

Remark 4.6. The classical isoperimetric inequality (the inequality (15) for a single domain in the sphere) can be equivalently reformulated as $4\pi b-b^2/R^2\le a^2$. In this form, it holds without the assumptions $b<2\pi R^2$ and $a<2\pi R$. An analogue of this inequality for 2-chains is

$$\max_{m\in\mathbb{Z}}\Big(|b-4m\pi R^2|-(b-4m\pi R^2)/R^2\Big)\le a^2.$$

It holds also without the assumptions $|b|<2\pi R^2$ and $a<2\pi R$. The graph of the left hand side of the latter inequality (considered as a function of b) is the union of the upper halves of ellipses centered at the points $\big((2+4m)\pi R^2,0\big)$, $m\in\mathbb{Z}$. The ellipses touch each other at the points $\big(4m\pi R^2,0\big)$.

Lemma 4.7. *Let γ be a positively transverse curve on $\mathbb{S}^3$ (e.g. a connected component of the intersection of a complex analytic curve with $\mathbb{S}^3$). Let us denote:*

$$a=\operatorname{len}_{\mathbb{P}^1}(\operatorname{pr}\gamma),\qquad b=\int_\gamma\eta,\qquad \ell=\operatorname{len}_{\mathbb{S}^2}(\gamma).$$

Then we have

$$\max(a,b) \le \ell \le a+b \tag{17}$$

and if, moreover, $\ell < \pi/2$, then we have

$$b \le S_{1/2}(a) = \frac{\pi}{2}\left(1-\sqrt{1-\frac{a^2}{\pi^2}}\,\right) = \frac{a^2}{4\pi} + \frac{a^4}{16\pi^3} + \dots \tag{18}$$

Proof. The inequalities (17) follow from (12) combined with the fact that the form η is positive on γ.

To prove (18), let us consider a 2-chain in $\mathbb{S}^3$ whose boundary is the cycle γ and let us denote its projection to $\mathbb{P}^1$ by β. Recall that $\mathbb{P}^1$ is isometric to the sphere of the radius $R = 1/2$. Hence, $\ell < \pi/\sqrt{2}$ combined with (17) implies $a < \ell < \pi/2 < \pi = 2\pi R$ and $b < \ell < \pi/2 = 2\pi R^2$, and the result follows from Lemma 4.5. $\square$

Corollary 4.8. *Let γ be a positively transverse curve in $\mathbb{S}^3$ and let ℓ and b be as in Lemma 4.7. If $\ell < \pi/2$ then $b \le S_{1/2}(\ell)$.*

Proof. Follows from (17), (18), and the monotonicity of $S_{1/2}$ $\square$

Combining all the above facts, it is easy to obtain the quadratic estimate for $n(\varepsilon)$ announced in the Introduction:

Proposition 4.9. *If $\varepsilon < \pi/2$, then $n(\varepsilon) > 2\pi/S_{1/2}(\varepsilon) = 8\pi^2/\varepsilon^2 - 2 + O(\varepsilon^2)$.*

Proof. Let A be a complex algebraic curve in $\mathbb{C}^2$ passing through the origin such that all connected components $\gamma_1,\dots,\gamma_n$ of $A\cap\mathbb{S}^3$ are shorter than ε. Perturbing A, we may assume that the conditions of Lemma 4.2 are satisfied. Thus, by (14) and Corollary 4.8, we have

$$2\pi \le \int_{\partial(A\cap\mathbb{B}^4)} \eta = \sum_{i=1}^n \int_{\gamma_i} \eta \le nS_{1/2}(\varepsilon) = \frac{n\varepsilon^2}{4\pi}\cdot\big(1+\frac{\varepsilon^2}{4\pi^2}+O(\varepsilon^4)\big). \quad \square$$

5. A lower bound of the order ε^{-3} for the number of cells of a Legendrian net

In this section, we shall prove the following result which means that any upper bound obtained by the method of §§2–3 cannot be better than $n(\varepsilon) = O(\varepsilon^{-3})$. More precisely, we shall prove the following result.

Proposition 5.1. *Let A be an algebraic curve in $\mathbb{C}^2$ passing through the origin, and let $\Gamma = A\cap\mathbb{S}^3$. Let $\mathcal{A} = \{\alpha_1,\dots,\alpha_n\}$ be a Legendrian net hanged on Γ (see Definition 2.4). Suppose that every cell of $\mathcal{A}$ is shorter than ε. Then*

$$n > \frac{2c_0}{\varepsilon S_{1/2}(\varepsilon)} = \frac{8c_0\pi}{\varepsilon^3} - \frac{2c_0}{\pi\varepsilon} + O(\varepsilon)$$

where c_0 is a constant depending only on A. In the case when A is a complex line, one can set $c_0 = \pi^2/4$.

Remark. It seems that a similar statement should hold for any contact 3-manifold.

Proof. We shall use the notation introduced in §4. We shall assume that $\varepsilon < \pi/2$. Let us define a function $f : \mathbb{P}^1 \to \mathbb{R}_+$ by setting $f(p) = \mathrm{dist}_{\mathbb{P}^1}(p, \mathrm{pr}\,\Gamma)$. For each cell α_i, let us consider a 2-chain $\tilde\beta_i$ in $\mathbb{S}^3$ such that $\partial\tilde\beta_i = \alpha_i$, and let us set $\beta_i = \mathrm{pr}_*\tilde\beta_i$. Let $\beta_i = \beta_i^+ - \beta_i^-$ be the decomposition of β_i into the positive and negative part (see Definition 4.3). Let us denote

$$b_i = \int_{\beta_i} \omega_{\mathbb{P}^1}, \quad b_i^+ = \int_{\beta_i^+} \omega_{\mathbb{P}^1}, \quad b_i^- = \int_{\beta_i^-} \omega_{\mathbb{P}^1}, \qquad c_0 = \int_{\mathbb{P}^1} f\,\omega_{\mathbb{P}^1}.$$

In the case when A is a complex line, it is not difficult to compute in spherical coordinates that $c_0 = \pi^2/4$. It follows from (18) that $b < \pi/2$, hence, by Lemma 4.5 we have

$$\mathrm{diam}_{\mathbb{P}^1}(\mathrm{supp}\,\beta_i) < \varepsilon/2, \qquad i = 1, \dots, n. \tag{19}$$

Perturbing A if necessary, we may assume that it is non-degenerate. Let F and L be as in Lemma 4.2. Let $\tilde\beta_0$ be a 2-chain in $\mathbb{S}^3$ such that $\partial\tilde\beta_0 = \partial(L \cap \mathbb{B}^4)$. Then, according to (14), we have

$$\sum_{i=1}^n \partial\tilde\beta_i = \sum_{i=1}^n \alpha_i = \Gamma = \partial F + \partial(L \cap \mathbb{B}^4) = \partial F + \partial\tilde\beta_0.$$

It follows from (13) and from $\mathrm{pr}_*\,\partial\tilde\beta_0 = 0$, that $\mathrm{pr}_*\,\tilde\beta_0 = [\mathbb{P}^1]$. Hence $\sum_{i=1}^n \beta_i = \mathrm{pr}_* F + \mathrm{pr}_*\,\tilde\beta_0 = \mathrm{pr}_* F + [\mathbb{P}^1]$. Thus,

$$c_0 = \int_{\mathbb{P}^1} f\,\omega_{\mathbb{P}^1} \le \int_{\mathrm{pr}_* F} f\,\omega_{\mathbb{P}^1} + \int_{\mathbb{P}^1} f\,\omega_{\mathbb{P}^1} = \sum_{i=1}^n \int_{\beta_i} f\,\omega_{\mathbb{P}^1}. \tag{20}$$

Let us set $m_i^+ = \max_{\mathrm{supp}\,\beta_i^+} f$ and $m_i^- = \min_{\mathrm{supp}\,\beta_i^-} f$. Then

$$\int_{\beta_i} f\,\omega_{\mathbb{P}^1} = \int_{\beta_i^+} f\,\omega_{\mathbb{P}^1} - \int_{\beta_i^-} f\,\omega_{\mathbb{P}^1} \le b_i^+ m_i^+ - b_i^- m_i^-$$

$$= b_i^+(m_i^+ - m_i^-) + (b_i^+ - b_i^-)m_i^- = b_i^+(m_i^+ - m_i^-) + b_i m_i^-. \tag{21}$$

By Lemma 4.5, we have $b_i^+ \le S_{1/2}(\varepsilon)$. Since $|f(p) - f(q)| \le \mathrm{dist}_{\mathbb{P}^1}(p, q)$, it follows from (19) that $m_i^+ - m_i^- < \mathrm{diam}_{\mathbb{P}^1}\,\mathrm{supp}\,\beta_i < \varepsilon/2$, hence

$$b_i^+(m_i^+ - m_i^-) \le S_{1/2}(\varepsilon)\varepsilon/2. \tag{22}$$

Let us show that

$$b_i m_i^- = 0. \tag{23}$$

Indeed, let $\alpha_i = \alpha_i^{\mathrm{leg}} + \alpha_i^{\mathrm{pt}}$ be the decomposition from Definition 2.4. Let us consider two cases: $\alpha_i^{\mathrm{pt}} = 0$ and $\alpha_i^{\mathrm{pt}} \ne 0$. In the former case, the cycle α_i is Legendrian, hence

$$b_i = \int_{\beta_i} \omega_{\mathbb{P}^1} = \int_{\tilde\beta_i} \omega^* = \int_{\alpha_i} \eta^* = 0.$$

In the latter case, $\operatorname{supp}\tilde\beta_i$ has a non-empty intersection with Γ, hence f vanishes in $\operatorname{supp}\beta_i$ which implies $m_i^- = 0$. Equality (23) is proved. Combining (20) – (23), we obtain $c_0 \le nS_{1/2}(\varepsilon)\varepsilon/2 = O(\varepsilon^3)$. $\square$

Remark. In the case when A is a complex line passing through the origin, the quantity $\int_{\beta_i} f\omega_{\mathbb{P}^1}$ playing the central role in the proof can be interpreted as the moment of β_i (considered as a measure on $\mathbb{P}^1$) with respect to the point $\operatorname{pr} A$. So, the proof reduces to the following argument: the measure $\omega_{\mathbb{P}^1}$ whose moment is equal to an absolute constant $\pi^2/4$ is represented as the sum of measures β_i whose moments are of the order ε^3.

Finally, let us formulate an open question, an affirmative answer to which would imply a lower bound for $n(\varepsilon)$ of the order ε^{-3} (by the same method as in the proof of Proposition 5.1).

Let $\mathcal{L}$ be the set of positive functions on $\mathbb{P}^1$ satisfying the Lipschitz condition with the constant 1, i.e. functions $f : \mathbb{P}^1 \to \mathbb{R}_+$ such that $|f(p)-p(q)| \le \operatorname{dist}_{\mathbb{P}^1}(p,q)$ for all $p, q \in \mathbb{P}^1$.

Does there exist an absolute constant c such that the inequality

$$\max_{f\in\mathcal{L}}\Big(\int_{m[\mathbb{P}^1]+\operatorname{pr}_* F} f\omega_{\mathbb{P}^1} - \int_{A\cap\mathbb{S}^3}(f\circ\operatorname{pr})\cdot\eta\Big) > c, \qquad F = \overline{\operatorname{pr}_{\mathbb{S}^3}(A\cap\mathbb{B}^4\setminus\{0\})},$$

holds for any algebraic curve $A \subset \mathbb{C}^2$ whose multiplicity at the origin is m? (As in (14) and (20), here pr_* denotes the homomorphism between the groups of 2-chains induced by $\operatorname{pr} : \mathbb{S}^3 \to \mathbb{P}^1$; under the identification of 2-chains in $\mathbb{P}^1$ with integer-valued functions discussed in Remark 4.4, the 2-chain $m[\mathbb{P}^1] + \operatorname{pr}_* F$ corresponds to the function whose value at the line L through 0 is the number of intersection points of L and $A \cap \mathbb{B}^4$ counted with multiplicity).

6. Construction of a Legendrian net in $\mathbb{S}^3$ providing an upper bound for $n(\varepsilon)$ of the order $1/\varepsilon^3$

Let us denote the coordinate axis $\{w = 0\}$ by L. Let $\Gamma = L\cap\mathbb{S}^3$. For an integer n, we denote the rotation $(z,w) \mapsto (e^{2\pi i/n}z, w)$ by $\tilde R_n : \mathbb{C}^2 \to \mathbb{C}^2$, and let $R_n : \mathbb{P}^1 \to \mathbb{P}^1$ be the correspondent rotation $(z : w) \mapsto (e^{2\pi i/n}z : w) = (z : e^{-2\pi i/n}w)$. Let us set $p_0 = (0 : 1)$, $p_\infty = (1 : 0)$. These are the fixed points of R_n.

Let us fix a small number $\varepsilon > 0$, and let $m = [10\pi/\varepsilon] + 1$. Let us set

$$r_k = \frac{k\pi}{2m}, \qquad \Delta_k = \{\, q \in \mathbb{P}^1 \mid \operatorname{dist}_{\mathbb{P}^1}(p_0, q) \le r_k \,\}, \qquad k = 0,\dots,m.$$

Recall that $\mathbb{P}^1$ is isometric to the sphere of radius 1/2, hence

$$\{p_0\} = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_m = \mathbb{P}^1.$$

Let us denote the closure of $\Delta_k \setminus \Delta_{k-1}$ by A_k, and let us set $a_k = \operatorname{Area}(A_k)$, $s_k = \operatorname{Area}(\Delta_k) = a_1 + \cdots + a_k$, $k = 1,\dots,m$. Let $\ell_k = (\operatorname{len}\partial\Delta_k + \operatorname{len}\partial\Delta_{k-1})/2$. For each $k = 1,\dots,m$, we set $n_k = 2^{\nu_k}$ where ν_k is chosen so that

$$\frac{\varepsilon}{40} < \ell_k^+ \le \frac{\varepsilon}{20}, \qquad \ell_k^+ = \frac{s_k\ell_k}{n_k a_k}. \tag{24}$$

It is clear that ν_k is uniquely determined by this condition. Indeed, (24) implies that $\nu_k = [\log_2(20 s_k \ell_k/(\varepsilon a_k))]$.

By definition, we have

$$\ell_k = \frac{\pi}{2}\Big(\sin\frac{k\pi}{m} + \sin\frac{(k-1)\pi}{m}\Big),\quad s_k = \frac{\pi}{2}\Big(1-\cos\frac{k\pi}{m}\Big),\quad a_k = s_k - s_{k-1}. \tag{25}$$

It follows easily that

$$\nu_k \le \nu_{k+1} \le \nu_k + 2, \qquad k = 1,\dots,m-1. \tag{26}$$

Let us denote

$$\beta^+_{k,0} = A_k \cap \Big\{\, |\operatorname{Arg}\zeta - \theta_k| \le \frac{\pi\ell^+_k}{\ell_k}\Big\},\qquad b^+_k = \operatorname{Area}(\beta^+_{k,0}),\qquad k=1,\dots,m,$$

where $\zeta = z/w$ is a standard complex coordinate on $\mathbb{P}^1$, and the numbers $\theta_1,\dots,\theta_m$ will be chosen later. In other words, the angular width of the domain $\beta^+_{k,0}$ is equal to $2\pi\ell^+_k/\ell_k = (2\pi s_k)/(n_k a_k)$. This implies

$$b^+_k = \frac{a_k\ell^+_k}{\ell_k} = \frac{s_k}{n_k}. \tag{27}$$

Let us set

$$\beta^+_{k,j} = R^j_{n_k}(\beta^+_{k,0}),\qquad \beta^-_{k,j} = \beta^+_{k,j}\cap\beta^+_{k,j+1},\qquad k=1,\dots,m,\quad j=1,\dots,n_k$$

(here and below, when using the double index (k,j), we assume that j is a residue mod n_k).

The angular width of $\beta^+_{k,j}$ is equal to $(2\pi s_k)/(n_k a_k)$ which is not less than the angle of the rotation R_{n_k} (because $s_k/a_k \ge 1$). Hence $\beta^-_{k,j}\ne\varnothing$ and

$$b^-_k := \operatorname{Area}(\beta^-_{k,j}) = b^+_k - \frac{a_k}{n_k} = \frac{s_k}{n_k} - \frac{a_k}{n_k} = \frac{s_{k-1}}{n_k}. \tag{28}$$

Let $p_{k,j}$ be the midpoint of the arc $(\partial\Delta_k)\cap\beta^-_{k,j}$, and let $q_{k,j}$ be the midpoint of the arc $(\partial\Delta_{k-1})\cap\beta^+_{k,j}$. Now let us choose the numbers θ_k used in the definition of the domains $\beta^+_{k,0}$ so that $p_{k,0} = q_{k+1,0}$ for all k. Since $p_{k,j} = R^j_{n_k}(p_{k,0})$ and $q_{k,j} = R^j_{n_k}(q_{k,0})$, we have

$$\begin{aligned}
&\{p_{k,j}\,|\,0\le j<n_k\} = \{q_{k+1,j}\,|\,0\le j<n_{k+1}\} &&\text{for } n_k = n_{k+1},\\
&\{p_{k,j}\,|\,0\le j<n_k\} \subset \{q_{k+1,j}\,|\,0\le j<n_{k+1}\} &&\text{for } n_k < n_{k+1}.
\end{aligned}$$

Moreover, for all k,j we have

$$p_{k,j} = q_{k+1,\mu_k j},\qquad\text{where}\quad \mu_k = \frac{n_{k+1}}{n_k} = 2^{\nu_{k+1}-\nu_k}.$$

Note that by definition we also have

$$p_{m,1} = p_{m,2} = \cdots = p_{m,n_m} = p_\infty,\qquad q_{1,1} = q_{1,2} = \cdots = q_{1,n_1} = p_0.$$

Let $\alpha^+_{k,j}$, $k=1,\dots,m$, $j=1,\dots,n_k$, be the path going from $p_{k,j}$ to $p_{k,j-1}$ along the boundary of $\beta^+_{k,j}$ in the positive direction which passes through any point of $(\partial\beta^+_{k,j})\setminus\{p_{k,j}\}$ at most once. In the case $k=m$ this definition is ambiguous (because $p_{m,j}=p_{m,j-1}=p_\infty$), but we assume that $\alpha^+_{m,j}$ is the complete loop around $\beta^+_{k,j}$ in the positive direction starting and finishing at p_∞. Let $\gamma^{(0)}_{k,j}$ (resp. $\gamma^{(1)}_{k,j}$) be the half of $\alpha^+_{k,j}$ going from $p_{k,j}$ to $q_{k,j}$ (resp. from $q_{k,j}$ to $p_{k,j-1}$). Finally, let us set (see Figures 2 and 3)

$$\alpha^-_{k,j}=\gamma^{(1)}_{k,j+1}+\gamma^{(0)}_{k,j},\qquad k=1,\dots,m,\quad j=1,\dots,n_k,$$

$$\alpha_{m,1}=\alpha^+_{m,1},\qquad \alpha_{k,1}=\alpha^+_{k,1}-\sum_{j=0}^{\mu_k-1}\alpha^-_{k+1,j},\quad k=1,\dots,m-1,$$

$$\alpha_{k,j+1}=R_{n_k}(\alpha_{k,j}),\qquad k=1,\dots,m,\quad j=1,\dots,n_k-1.$$

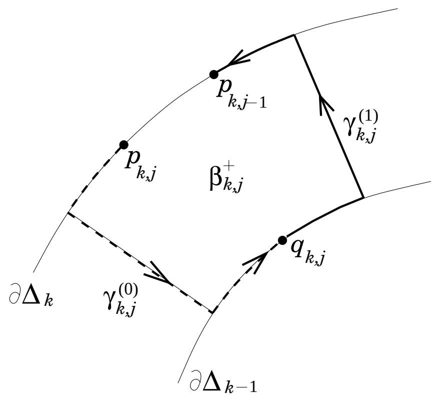


FIG. 2. $\alpha^+_{k,j}=\gamma^{(0)}_{k,j}+\gamma^{(1)}_{k,j}$.

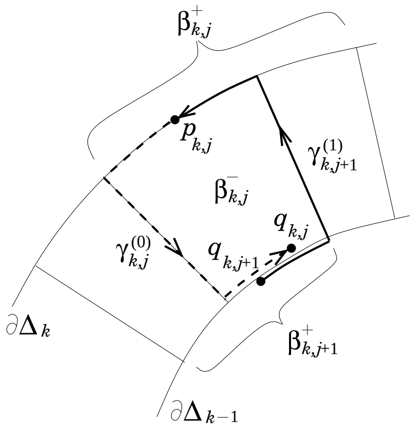


FIG. 3. $\alpha^-_{k,j}=\gamma^{(0)}_{k,j}+\gamma^{(1)}_{k,j+1}$.

Let $\gamma:[0,1]\to\mathbb{P}^1$ be a piecewise smooth path and $\tilde p$ a point in $\mathbb{S}^3$ such that $\mathrm{pr}(\tilde p)=\gamma(0)$ (as in §4, here pr denotes the standard projection $\mathbb{S}^3\to\mathbb{P}^1$). Then there exists a unique Legendrian path $\tilde\gamma:[0,1]\to\mathbb{S}^3$ such that $\tilde\gamma(0)=\tilde p$ and $\mathrm{pr}\circ\tilde\gamma=\gamma$. This follows from the fact that the fibers of $\mathrm{pr}:\mathbb{S}^3\to\mathbb{P}^1$ are transverse to the field of complex tangents $\ker(\eta|_{\mathbb{S}^3})$. The path $\tilde\gamma$ is called the *Legendrian lift* of γ starting at $\tilde p$.

We shall construct Legendrian lifts $\tilde\alpha^\pm_{k,j}$ and $\tilde\alpha_{k,j}$ of $\alpha^\pm_{k,j}$ and $\alpha_{k,j}$ and we shall show that $\{\tilde\alpha_{k,j}\}$ is the required Legendrian net.

Let us set

$$\tilde p_{m,j}=(e^{2\pi ij/n_m},0)\in\mathbb{C}^2,\qquad j=1,\dots,n_m.$$

The points $p_{m,j}$ belong to Γ, and we have $\tilde R_{n_m}(p_{m,j})=p_{m,j+1}$. Let $\tilde\gamma_{m,j}$, $j=1,\dots,n_m$, be the path $[(j-1)/n_m,j/n_m]\to\Gamma$, $t\mapsto(e^{it},0)$. It goes along Γ from $\tilde p_{m,j-1}$ to $\tilde p_{m,j}$. By (27), we have $\mathrm{Area}(\beta^+_{m,j})=b^+_m=s_m/n_m=\pi/n_m$. Thus,

$$\int_{\tilde\gamma_{m,j}}\eta=\frac{1}{n_m}\int_\Gamma\eta=\frac{2\pi}{n_m}=2b^+_m.$$

The subsequent construction will be recurrent (successively for $k = m, m-1, \dots, 1$). Suppose that for some $k \le m$, we have constructed points $\tilde p_{k,j}$ and paths $\tilde\gamma_{k,j}$ in $\mathbb{S}^3$ such that for any $j = 1, \dots, n_k$ the following conditions hold (as we have seen above, they do hold for $k = m$).

(i) $\partial\tilde\gamma_{k,j} = \tilde p_{k,j} - \tilde p_{k,j-1}$;

(ii) $\mathrm{pr}(\tilde p_{k,j}) = p_{k,j}$ and $\mathrm{pr}(\tilde\gamma_{k,j})$ is the arc of $\partial\Delta_k$ going in the positive direction from $p_{k,j-1}$ to $p_{k,j}$ and passing any point of $\partial\Delta_k$ at most once.

(iii) $\int_{\tilde\gamma_{k,j}} \eta = 2b_k^+$;

(iv) $\tilde R_{n_k}(\tilde p_{k,j}) = \tilde p_{k,j+1}$ and $\tilde R_{n_k}(\tilde\gamma_{k,j}) = \tilde\gamma_{k,j+1}$.

Let $\tilde\alpha^+_{k,j}$ be the Legendrian lift of $\alpha^+_{k,j}$ starting at $\tilde p_{k,j}$. Let us show that the end of $\tilde\alpha^+_{k,j}$ is $\tilde p_{k,j-1}$. Indeed, let us denote the end of $\tilde\alpha^+_{k,j}$ by $\tilde p$, and let $[\tilde p, \tilde p_{k,j-1}]$ be the arc of the circle $\mathrm{pr}^{-1}(p_{k,j-1})$ from $\tilde p$ to $\tilde p_{k,j-1}$ chosen so that the cycle $\tilde\alpha^+_{k,j} + \tilde\gamma_{k,j} + [\tilde p, \tilde p_{k,j-1}]$ is zero-homologous in $\mathbb{S}^3$. It is clear that the projection of this cycle to $\mathbb{P}^1$ coincides with $\partial\beta^+_{k,j}$. Hence, Lemma 4.1 implies

$$2b_k^+ = 2\,\mathrm{Area}(\beta^+_{k,j}) = \int_{\tilde\alpha^+_{k,j}} \eta + \int_{\tilde\gamma_{k,j}} \eta + \int_{[\tilde p, \tilde p_{k,j-1}]} \eta = 0 + 2b_k^+ + \int_{[\tilde p, \tilde p_{k,j-1}]} \eta.$$

Therefore, $\int_{[\tilde p, \tilde p_{k,j-1}]} \eta = 0$ and thus, $\tilde p = \tilde p_{k,j-1}$.

Let us show that $\tilde R_{n_k}(\tilde\alpha^+_{k,j}) = \tilde\alpha^+_{k,j+1}$. Indeed, let $\mathcal{F}_{k,j}$ be the field of real tangent lines on the torus $T_{k,j} = \mathrm{pr}^{-1}(\alpha^+_{k,j})$ which is cut by the field of complex tangents $\ker\eta|_{\mathbb{S}^3}$. Then $\tilde\alpha^+_{k,j}$ is the integral curve of $\mathcal{F}_{k,j}$ passing through $\tilde p_{k,j}$. It remains to note that $\tilde R_{n_k}$ takes $p_{k,j}$ and $T_{k,j}$ into $p_{k,j+1}$ and $T_{k,j+1}$ respectively. Since, moreover, $\tilde R^*_{n_k}(\eta) = \eta$, it takes $\mathcal{F}_{k,j}$ into $\mathcal{F}_{k,j+1}$.

Let $\tilde q_{k,j}$, $j = 1, \dots, n_k$, be the point on $\tilde\alpha^+_{k,j}$ such that $\mathrm{pr}(\tilde q_{k,j}) = q_{k,j}$. Let us set $\tilde\alpha_k = \sum_{j=1}^{n_k} \tilde\alpha^+_{k,j}$. This is a closed spiral-like Legendrian curve on $\mathbb{S}^3$ passing through the points $\tilde p_{k,j}$ and $\tilde q_{k,j}$ which is invariant under the rotation $\tilde R_{n_k}$. Let $\tilde\alpha^-_{k,j}$ be the Legendrian lift of $\alpha^-_{k,j}$ starting at $\tilde q_{k,j+1}$. Then $\tilde\alpha_k = \sum_{j=1}^{n_k} \tilde\alpha^-_{k,j}$. Moreover, the curve $\tilde\alpha_k$ is divided by the points $\tilde p_{k,j}$ into the arcs $\tilde\alpha^+_{k,j}$, and it is divided by the points $\tilde q_{k,j}$ into the arcs $\tilde\alpha^-_{k,j}$.

Let $\gamma^-_{k,0}$ be the arc of $\partial\Delta_{k-1}$ going in the positive direction from $q_{k,0}$ to $q_{k,1}$ and passing any point of $\partial\Delta_{k-1}$ at most once. Let us choose a (non-Legendrian) lift $\tilde\gamma^-_{k,0}$ of $\gamma^-_{k,0}$ from $\tilde q_{k,0}$ to $\tilde q_{k,1}$ so that the cycle $\tilde\alpha^-_{k,0} + \tilde\gamma^-_{k,0}$ is zero-homologous in $\mathbb{S}^3$. Its projection to $\mathbb{P}^1$ coincides with $\partial\beta^-_{k,0}$, hence, by Lemma 4.1 we have

$$2b_k^- = 2\,\mathrm{Area}(\beta^-_{k,0}) = \int_{\tilde\alpha^-_{k,0}} \eta + \int_{\tilde\gamma^-_{k,0}} \eta = 0 + \int_{\tilde\gamma^-_{k,0}} \eta. \tag{29}$$

Let us set $\tilde\gamma^-_{k,j} = \tilde R^j_{n_k}(\tilde\gamma^-_{k,0})$, $j = 1, \dots, n_k$, and let

$$\tilde p_{k-1,0} = \tilde q_{k,0}, \qquad \tilde\gamma_{k-1,1} = \sum_{j=0}^{\mu_{k-1}-1} \tilde\gamma^-_{k,j},$$

$$\tilde p_{k-1,j+1} = \tilde R_{n_{k-1}}(\tilde p_{k-1,j}) \quad \text{and} \quad \tilde\gamma_{k-1,j+1} = \tilde R_{n_{k-1}}(\tilde\gamma_{k-1,j}).$$

To complete the recurrent construction, it remains to check that Conditions (i)–(iv) are satisfied for the points $\tilde{p}_{k-1,j}$ and for the paths $\tilde{\gamma}_{k-1,j}$. Indeed, by (29) we have

$$\int_{\tilde{\gamma}_{k-1,1}} \eta = \sum_{j=0}^{\mu_{k-1}-1} \int_{\tilde{\gamma}^-_{k,j}} \eta = 2\mu_{k-1} b_k^-.$$

Combining this with (28) and $\mu_{k-1} = n_k/n_{k-1}$, we obtain (iii) for $\tilde{\gamma}_{k-1,j}$. The other conditions are obvious.

Finally, for $k = m$, let us set

$$\tilde{\alpha}_{m,j} = \tilde{\alpha}^{\mathrm{leg}}_{m,j} + \tilde{\alpha}^{\mathrm{pt}}_{m,j}, \qquad \text{where} \qquad \tilde{\alpha}^{\mathrm{leg}}_{m,j} = \tilde{\alpha}^+_{m,j}, \ \ \tilde{\alpha}^{\mathrm{pt}}_{m,j} = \tilde{\gamma}_{m,j}, \quad j = 1, \dots, n_m,$$

and for $k < m$, let us set

$$\tilde{\alpha}_{k,1} = \tilde{\alpha}^{\mathrm{leg}}_{k,1} = \tilde{\alpha}^+_{k,1} - \sum_{j=0}^{\mu_k - 1} \tilde{\alpha}^-_{k+1,j},$$

$$\tilde{\alpha}_{k,j+1} = \tilde{\alpha}^{\mathrm{leg}}_{k,j+1} = \tilde{R}_{n_k}(\tilde{\alpha}_{k,j}), \quad j = 1, \dots, n_k - 1.$$

$$\tilde{\alpha}^{\mathrm{pt}}_{k,j} = 0, \qquad j = 1, \dots, n_k.$$

(see Figure 4).

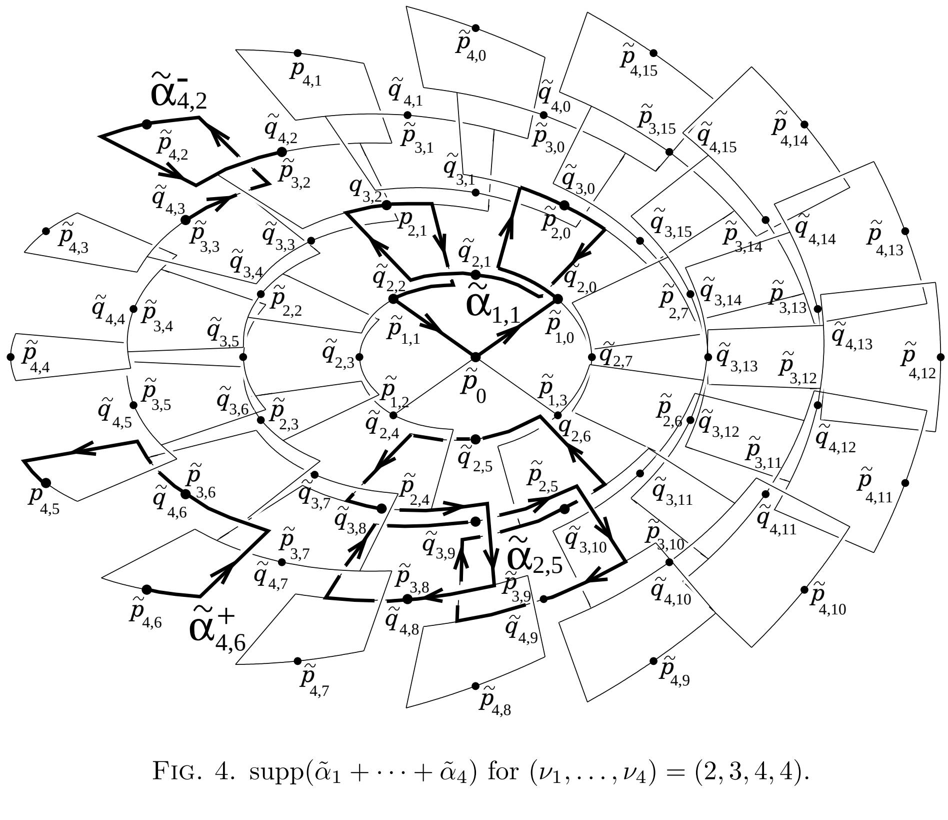


FIG. 4. $\operatorname{supp}(\tilde{\alpha}_1 + \dots + \tilde{\alpha}_4)$ for $(\nu_1, \dots, \nu_4) = (2, 3, 4, 4)$.

Let us denote $\mathcal{A}_\varepsilon = \{\tilde\alpha_{k,j} \,|\, 1 \le k \le m,\, 1 \le j \le n_k\}$.

Proposition 6.1. *$\mathcal{A}_\varepsilon$ is a Legendrian net hanged on Γ.*

Proof. It is easy to see that the chains $\tilde\alpha_{k,j}$ are cycles. Since the chains $\tilde\alpha^{\rm leg}_{k,j}$ (resp. $\tilde\alpha^{\rm pt}_{k,j}$) are Legendrian (resp. positively transverse) by construction, it remains to check that $\sum \tilde\alpha_{k,j} = \Gamma$. Indeed, we have

$$\sum_{j=1}^{n_m} \tilde\alpha_{m,j} = \sum_{j=1}^{n_m} \tilde\alpha^+_{m,j} + \sum_{j=1}^{n_m} \tilde\gamma_{m,j} = \tilde\alpha_m + \Gamma,$$

and for $k < m$, since $\tilde R_{n_k} = \tilde R^{\mu_k}_{n_{k+1}}$, we have

$$\tilde R^j_{n_k}(\tilde\alpha^-_{k+1,j'}) = \tilde R^{\mu_k j}_{n_{k+1}}(\tilde\alpha^-_{k+1,j'}) = \tilde\alpha^-_{k+1,j'+\mu_k j},$$

hence,

$$\sum_{j=1}^{n_k} \tilde\alpha_{k,j} = \sum_{j=1}^{n_k} \tilde\alpha^+_{k,j} - \sum_{j=1}^{n_k}\sum_{j'=0}^{\mu_k-1} \tilde R^j_{n_k}(\tilde\alpha^-_{k+1,j'}) = \sum_{j=1}^{n_k} \tilde\alpha^+_{k,j} - \sum_{j=1}^{n_{k+1}} \tilde\alpha^-_{k+1,j} = \tilde\alpha_k - \tilde\alpha_{k+1}.$$

Therefore,

$$\sum_{k,j} \tilde\alpha_{k,j} = (\tilde\alpha_1 - \tilde\alpha_2) + (\tilde\alpha_2 - \tilde\alpha_3) + \dots + (\tilde\alpha_{m-1} - \tilde\alpha_m) + (\tilde\alpha_m + \Gamma) = \tilde\alpha_1 + \Gamma.$$

It remains to note that $\tilde\alpha_1 = 0$ (see Figure 4) because $\beta^-_{1,j}$ is a segment of a geodesic between $p_{0,j} = p_0$ and $p_{1,j}$ and hence $\tilde\alpha^-_{1,j} = 0$ for all $j = 1, \dots, n_1$. $\square$

Proposition 6.2. $\operatorname{len}\tilde\alpha_{k,j} < \varepsilon$ *for all* k, j.

Proof. It follows from (12) that the length of a path on $\mathbb{P}^1$ is equal to the length of its Legendrian lift to $\mathbb{S}^3$. Hence,

$$\operatorname{len}\tilde\alpha_{k,j} = \begin{cases} \operatorname{len}\partial\beta^+_{m,j} + \operatorname{len}\tilde\gamma_{m,j}, & k = m,\\ \operatorname{len}\partial\beta^+_{k,j} + \mu_k \operatorname{len}\partial\beta^-_{k+1,j'} & k < m.\end{cases}$$

It is clear that

$$\operatorname{len}\partial\beta^\pm_{k,j} \le 2\ell^\pm_k + 2\cdot(\text{width of } A_k) \le 2\ell^\pm_k + \frac{\pi}{m} \le 2\cdot\frac{\varepsilon}{20} + \frac{\varepsilon}{10} = \frac{\varepsilon}{5},$$

and (26) implies that $\mu_k \le 4$ for all k. Hence, for $k < m$, we have

$$\operatorname{len}\tilde\alpha_{k,j} \le (1 + \mu_k)\frac{\varepsilon}{5} \le \varepsilon.$$

It follows from (24) and (25) that

$$\frac{1}{n_m} \le \frac{\varepsilon a_m}{20 s_m \ell_m} = \frac{\varepsilon(1 - \cos(\pi/m))}{20\pi\sin(\pi/m)} = \varepsilon\cdot O(1/m) = O(\varepsilon^2).$$

Thus, $\operatorname{len}\tilde\alpha_{m,j} \le \operatorname{len}\partial\beta^+_{m,j} + O(\varepsilon^2) \le (\varepsilon/5) + O(\varepsilon^2) \le \varepsilon$. $\square$

Corollary 6.3. *An upper bound* $n(\varepsilon) \leq \operatorname{Card} \mathcal{A}_\varepsilon = O(\varepsilon^{-3})$ *holds.*

Proof. It follows from (26) that $n_1 \leq n_2 \leq \cdots \leq n_m$, hence $\operatorname{Card} \mathcal{A}_\varepsilon \leq mn_m$. It is clear that $m = O(\varepsilon)$, and it easily follows from (25) that $n_m = O(\varepsilon^{-2})$. Therefore, $\operatorname{Card} \mathcal{A}_\varepsilon = O(\varepsilon^{-3})$.

The estimate $n(\varepsilon) \leq \operatorname{Card} \mathcal{A}_\varepsilon$ follows from the construction given in §3. □

Remark 6.4. By Proposition 2.7, it is not important for us whether the net $\mathcal{A}_\varepsilon$ is generic or not. However, it is generic everywhere except at the point p_0 (see Figure 4). If one changes slightly the parameters of the construction of $\mathcal{A}_\varepsilon$, it is not difficult to achieve $n_1 \leq 3$. In this case, $\mathcal{A}_\varepsilon$ will be generic everywhere, including p_0.

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