CONGRUENCE MODULO 8 FOR REAL ALGEBRAIC CURVES OF DEGREE 9

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1. Introduction and statement of the result. Let A be a non-singular real algebraic curve of degree m in $\mathbf{RP}^2$. Its connected components are embedded circles. Those of them whose complement in $\mathbf{RP}^2$ is not connected are called *ovals.* One says that an oval u lies *inside* an oval v if u is contained is the orientable component of the complement of v. A union of d ovals $v_1,\dots,v_d$ such that v_i is inside v_{i+1}, $1\le i<d$, is called a *nest of the depth* d. An oval is called *exterior* if it does not lye inside any other oval; an oval is called *empty* if there is no other ovals inside it. An oval is called *even* if it is contained inside an even number of other ovals, and *odd* otherwise. Denote by p and n the number of even and odd ovals respectively. One says that A is an M*-curve* if it has the maximal possible number of connected components which equals $M(m)=(m-1)(m-2)/2+1$. If A has $M(m)-i$ connected components then it is called an $(M-i)$*-curve* Let $\mathbf{C}A$ be the complexification of A. If $\mathbf{C}A\setminus A$ is not connected, A is a curve *of type I*; if $\mathbf{C}A\setminus A$ is connected then A is a curve *of type II.*

For curves of an even degree $m=2k$, in some cases, the difference $p-n$ satisfies congruences. For example,
Gudkov-Rohlin congruence $p-n\equiv k^2 \mod 8$ for M-curves,
Gudkov-Krahnov-Kharlamov congruence $p-n\equiv k^2\pm1 \mod 8$ for $(M-1)$-curves,
Kharlamov-Marin congruence $p-n\not\equiv k^2+4 \mod 8$ for M-curves of type II, and
Arnold congruence $p-n\equiv k^2 \mod 4$ for curves of type I.
These statements do not extend to curves of *odd* degrees. So, for an M-curve of any odd degree $2k+1$ with $k\ge3$, the residue $p-n \mod 8$ may take any values congruent to $k \mod 2$. As far as we know, the following theorem is the first result of this kind.

Theorem 1. *Let A be a curve of degree $m=4d+1$ which has 4 paiwise distinct nests of the depth d. Then*

if A is an M-curve then	$p-n\equiv-2d$	$\mod 8$;	(1)
if A is an $(M-1)$-curve then	$p-n\equiv-2d\pm1$	$\mod 8$;	
if A is an $(M-2)$-curve of type II then	$p-n\not\equiv-2d+4$	$\mod 8$;	
if A is a curve of type I then	$p-n\equiv-2d$	$\mod 4$;	

It is clear that (1) for $d=2$ is equivalent to the fact that the number of exterior empty ovals of an M-curve of degree 9 with 4 nests is divisible by 4. This was conjectured by Korchagin [2]. Theorem 1 is obtained below (see Sect. 4) as a consequence of Kharlamov-Viro congruence [1] which generalizes the classical congruences to the case of singular curves of even degrees.

2. Brown - van der Blij invariant. By a *quadratic space* we mean a triple $(V, \circ, q)$ composed of a vector space V over the field $\mathbf{Z}_2$, a bilinear form $V \times V \to \mathbf{Z}_2$, $(x, y) \mapsto x \circ y$, and a function $q : V \mapsto \mathbf{Z}_4$ which is quadratic with respect to $\circ$ in the sense that $q(x+y) = q(x) + q(y) + 2x \circ y$. A quadratic space is determined by its *Gram matrix* with respect to a base $e_1, \dots, e_n$ of V, i.e. the matrix $Q = (q_{ij})$ where $q_{ii} = q(e_i)$ and $q_{ij} = e_i \circ e_j$ for $i \neq j$ (the diagonal entries are defined mod 4, the others mod 2; note that $q(x) \equiv x \circ x \mod 2$). It is easy to see that by elementary changes of the base, one can put the Gram matrix to the block-diagonal form $\operatorname{diag}(d_1, \dots, d_t) \oplus Q_1 \oplus \dots \oplus Q_s$ where each block Q_i is either $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. If all $d_i \neq 2$, we say that the form q is *informative* and in this case we define its *Brown - van der Blij invariant* $B(q) = \sum B(d_i) + \sum B(Q_i) \mod 8$ where $B(0) = 0$, $B(1) = 1$, $B(-1) = -1$, $B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$, and $B\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4$.

3. Kharlamov-Viro congruence for nodal curves. Let A be a curve in $\mathbf{RP}^2$ of degree $2k$ defined by $f = 0$ and let each of its singular points be the point of transverse intersection of two smooth real local branches. A is called an *M-curve* (a curve *of type I*) if the normalization of any its irreducible component is an M-curve (a curve of type I). A curve which is not of type I, is of type II. Let $x_1, \dots, x_s$ be the singular points and Γ_A be the union of the connected components of A passing through them. Let $b = 0$ if $\mathbf{RP}^2_+ = \{f \geq 0\}$ is contractable in $\mathbf{RP}^2$ and $b = (-1)^k$ otherwise.

Suppose that Γ_A is connected. Let us define a quadratic space $(V, \circ, q)$ as follows. Let $C_1, \dots, C_r$ be the oriented components of $\mathbf{RP}^2 \setminus \Gamma_A$ on which $f > 0$ near Γ_A. Let $(V_0, \circ, q_0)$ be the quadratic space with the orthogonal base $e_1, \dots, e_s$ such that $q_0(e_1) = \dots = q_0(e_s) = -1$. Set $c_i = \sum_{j \in \alpha_i} e_j$ where $\{x_j\}_{j \in \alpha_i}$ are the singular points through which ∂C_i passes only once. In the cases when either Γ_A is contractible in $\mathbf{RP}^2$ or, as in Sect. 4, there is a *branch of* Γ_A (i.e. a smoothly immersed circle) which is non-contractible in $\mathbf{RP}^2$, we define $V \subset V_0$ as the subset generated by $c_1, \dots, c_r$ and we set $q = q_0|_V$.

In the case when Γ_A is not contractible in $\mathbf{RP}^2$ but all its branches are, let us choose a simple closed curve in Γ_A which is not contractible in $\mathbf{RP}^2$. Let $(V_0', \circ, q_0')$ be the quadratic space with the base $(e_0, \dots, e_s)$ which contains V_0 as a quadratic subspace ($q_0'|_{V_0} = q_0$) and let $q_0'(e_0) = (-1)^k$, $e_0 \circ e_j = 0$ iff $L \sim 0$ in $H_1(RP^2_+, RP^2_+ \setminus x_j)$. Let $V \subset V_0'$ be the subspace generated by $c_1, \dots, c_r$, and $e_0 + \sum_{j \in \alpha_0} e_j$ where $\alpha_0 = \{j \mid L \not\sim 0 \text{ in } H_1(RP^2_-, RP^2_- \setminus x_j)\}$, and let $q = q_0'|_V$.

If Γ_A is not connected, we define $(V, \circ, q)$ as the direct sum of quadratic spaces associated as above to each connected component of Γ_A.

Theorem 2. *Suppose that each branch of A which is contractible in $\mathbf{RP}^2$ cuts other branches at $n \equiv 0 \mod 4$ singular points and each branch which is not contractible in $\mathbf{RP}^2$, at $n \equiv (-1)^{k+1} \mod 4$ singular points. If A is an M-curve then $\chi(\mathbf{RP}^2_+) \equiv k^2 + B(q) + b \mod 8$ and also the corresponding analogues of Gudkov-Krahnov-Kharlamov, Kharlamov-Marin, and Arnold congruences take place.*

Theorem 2 is a corollary of Theorem (3.B) on curves with arbitrary singularities from the paper by Kharlamov and Viro [1]. Theorem 2 is formulated here because there are mistakes in [1] in the discussion of the corresponding particular case (4.I), (4.J) of Theorem (3.B).

4 Proof of Theorem 1. Let us choose any three pairwise distinct nests of the depth d and a point inside the innermost oval of each of them. Theorem 1 follows from Theorem 2 applied to the union of A and the three straight lines passing through the three chosen points. Indeed, the union of the three chosen lines and the non-contractible branch of A divides $\mathbf{RP}^2$ into 4 triangles and 3 quadrangles (curvilinear). Let us choose the sign of f so that near the non-contractible branch of A one $f \geq 0$ on the three quadrangles and $f \leq 0$ on the four triangles. All ovals not belonging to the three chosen nests lye in the quadrangles (otherwise would exist a conic having too many intersections with A). Therefore, one has

$$\chi(\mathbf{RP}^2_+) = \chi(\bigcup \overline{C}_j) - p' + n' \tag{1}$$

where p' and n' are the numbers of even and odd ovals, not belonging to the three chosen nests. $B(q)$ can be computed according to Sect. 2 as follows.

We have $r = 3 + 6d$ and $C_1, \dots, C_r$ are curvilinear quadrangles. Let C_1, C_2, C_3 be those of them which are adjacent to the non-contractible branch of A. For the others, we change the notation and denote them by C^ν_{ij} where $\nu = 1, 2, 3$ is the number of the nest, $i = 1, \dots, d$ is the depth in the nest, and $j = 1, 2$ (see Figure 4). Let us denote by c_1, c_2, c_3, and c^ν_{ij} the corresponding vectors in the quadratic space V.

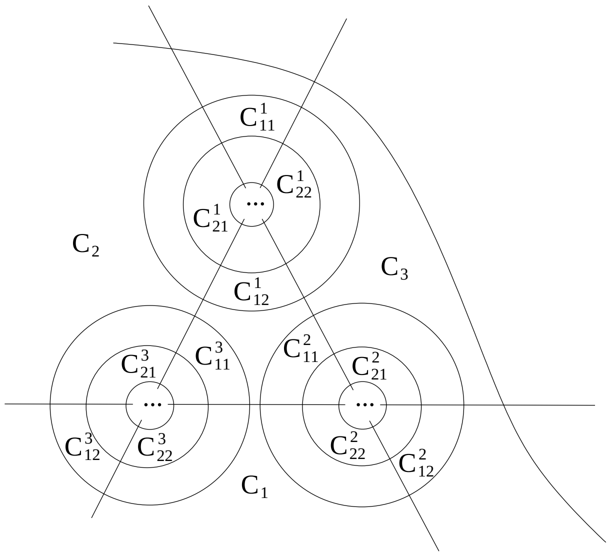

FIG. 3

Then we have:

$$q(c_1) = q(c_2) = q(c_3) = 2, \quad q(c^\nu_{ij}) = \begin{cases} 1 \text{ if } i = d, \\ 0 \text{ if } i < d; \end{cases}$$

$$c_\nu \circ c_{\nu\pm 1} = 1, \qquad c_{\nu\pm 1} \circ c^\nu_{1,j} = 1, \qquad c^\nu_{ij} \circ c^\nu_{i+1,k} = 1, \qquad c^\nu_{d\,j} \circ c^\nu_{dk} = 1$$

for all $j, k = 1, 2$, $i = 1, \dots, d-1$, and $\nu = 1, 2, 3$ (here $\nu \pm 1$ means $\nu \pm 1 \mod 3$) and the product $\circ$ is zero for any other pair of the base vectors.

Let us replace

$$c^\nu_{i,2} \longrightarrow c^\nu_{i,1} + c^\nu_{i,2}, \qquad c^\nu_{i,1} \longrightarrow b^\nu_i = \sum_{j=i}^{d} c_{i,1}, \qquad c_\nu \longrightarrow b_\nu = c_\nu + b^{\nu+1}_1 + b^{\nu-1}_1.$$

This is a triangular change of the base. It gives a decomposition of the quadratic space $(V, \circ, q)$ into the direct sum of a subspace where q and $\circ$ vanish (it is generated by $\{c^\nu_{i,1} + c^\nu_{i,2}\}$ and by b_1, b_2, b_3), and three isomorphic d-dimensional subspaces V_1, V_2, V_3 where V_ν is generated by $b^\nu_1, \dots, b^\nu_d$. The Gram matrix of $q|_{V_\nu}$ is $\operatorname{diag}\big(1, -1, \dots, (-1)^d\big)$. Hence,

$$B(q) \equiv 3\alpha \mod 8 \qquad \text{where} \quad \alpha = \begin{cases} 1, & \text{if } d \text{ is odd} \\ 0, & \text{if } d \text{ is even} \end{cases} \tag{2}$$

Easy to see that

$$p - n = p' - n' + 3\alpha \qquad k = 2d + 2, \qquad b = (-1)^k = 1, \tag{3}$$

$$\chi(\bigcup \overline{C}_j) = \chi(\mathbf{RP}^2) - 4 - 3 \cdot 2d = -3 - 6d. \tag{4}$$

Putting (1) – (4) into the congruence of Theorem 2 for M-curves, we get

$$\chi(\bigcup \overline{C}_j) - p' + n' \equiv (2d+2)^2 + B(q) + 1 \mod 8, \quad \text{hence,}$$

$$-3 - 6d - p + n + 3\alpha \equiv 4(d+1)^2 + 3\alpha + 1 \mod 8, \quad \text{hence,}$$

$$p - n \equiv -4 - 6d - 4(d^2 + 2d + 1) \equiv 4d^2 - 6d \equiv -2d \mod 8.$$

The other congruences (for $(M-1)$-curves etc.) are obtained the same way.

References

1. V.M. Kharlamov, O.Ya. Viro, *Extension of the Gudkov-Rohlin congruence*, Lect. Notes. Math. **1346** (1988), 357–406.
2. A.B. Korchagin, *Construction of new M-curves of 9th degree*, Lect. Notes. Math. **1524** (1991), 407–426.

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