IRREDUCIBILITY OF LEMNISCATES

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Abstract. We prove that lemniscates (i.e., sets of the form $|P(z)| = 1$ where P is a complex polynomial) are irreducible real algebraic curves.

A *lemniscate* (or *polynomial lemniscate*) is a real curve in $\mathbb{C}$ given by the equation $|P(z)| = 1$ where $P(z)$ is a non-constant polynomial with complex coefficients.

We say that a subset of $\mathbb{R}^2$ is an *irreducible real algebraic curve* if it is the zero set of an irreducible over $\mathbb{C}$ real polynomial in two variables. We prove in this note that

any lemniscate is an irreducible real algebraic curve in $\mathbb{C}$

(under the standard identification of $\mathbb{R}^2$ and $\mathbb{C}$). This fact is an immediate consequence of Corollary 1 below. (indeed, since $\{[P| = 1\} = \{|P^d| = 1\}$, any lemniscate can be defined via a polynomal which is not a power of another one).

Theorem 1. *Let P and Q be two polynomials in one variable with complex coefficients. Then the polynomial $P(z)Q(w) - 1$ is reducible if and only if*

$$P(z) = P_1(z)^d \qquad \text{and} \qquad Q(w) = Q_1(w)^d$$

for $d > 1$ and some polynomials $P_1(z)$ and $Q_1(w)$.

Corollary 1. *Let $P(z)$ be a polynomial in one variable with complex coefficients and $f(x, y)$ be the real polynomial given by*

$$f(x, y) = P(x + iy)\bar{P}(x - iy) - 1$$

where $\bar{P}$ is the polynomial whose coefficients are conjugates to those of P. Then $f(x, y)$ is reducible over $\mathbb{C}$ if and only if $P(z) = P_1(z)^d$ for $d > 1$ and a polynomial $P_1(z)$.

Proof. It is enough to apply Theorem 1 to P and $\bar{P}$ after the linear change of variables $z = x + iy$, $w = x - iy$ in $\mathbb{C}^2$. $\square$

Remark 1. (F. B. Pakovich). If $P(z)$ and $Q(w)$ are arbitrary rational functions, then the problem of reducibility of the algebraic curve $P(z)Q(w) = 1$ seems to be very hard. For example, in another particular case (in a sense, opposite to the ours), when $P(z)$ and $1/Q(w)$ are polynomials, this problem is solved in [2] (up to finite number of cases) and in [1] (completely) but the solution relies on the classification of simple finite groups.

The rest of the paper is devoted to the proof of Theorem 1. Let

$$P(z)=\prod_{j=1}^{k}(z-z_j)^{p_j},\ \ \deg P=p \quad\text{and}\quad Q(w)=\prod_{j=1}^{l}(w-w_j)^{q_j},\ \ \deg Q=q$$

where $z_1,\dots,z_k$ are pairwise distinct as well as $w_1,\dots,w_l$. Let C be the closure in $\mathbb{P}^1\times\mathbb{P}^1$ of the affine algebraic curve in $\mathbb{C}^2$ given by $P(z)Q(w)=1$ (here we represent $\mathbb{P}^1$ as $\mathbb{C}\cup\{\infty\}$).

Suppose that $P(z)Q(w)-1=f'(z,w)f''(z,w)$ with non-constant polynomials f' and f''. Let C' and C'' be the corresponding subsets of C. We denote their local intersection numbers with the infinite lines $L_1=\mathbb{P}^1\times\{\infty\}$ and $L_2=\{\infty\}\times\mathbb{P}^1$ as follows:

$$\begin{aligned}&(C'\cdot L_1)_{(z_j,\infty)}=p'_j, &&(C''\cdot L_1)_{(z_j,\infty)}=p''_j, &&(j=1,\dots,k),\\ &(C'\cdot L_2)_{(\infty,w_j)}=q'_j, &&(C''\cdot L_2)_{(\infty,w_j)}=q''_j, &&(j=1,\dots,l).\end{aligned}$$

Let (p',q') and (p'',q'') be the bidegree of C' and C'' respectively. Then

$$p=p'+p'',\quad q=q'+q'',\quad p_j=p'_j+p''_j,\quad q_j=q'_j+q''_j.$$

The germ of C at (z_j,∞) has equation $u^q=v^{p_j}$ in some local analytic coordinates (u,v). So, it has $\gcd(q,p_j)$ local branches which are distributed in some proportion between C' and C''. By comparing the the degree of the projections of the germs of C' and C'' onto the coordinate axes, we conclude that $p'_j/p''_j=q'/q''$. Similarly, $q'_j/q''_j=p'/p''$. We denote these quotients by α and β respectively. Since $\sum p'_j=p'$ and $\sum p''_j=p''$, we obtain

$$\beta p''=p'=p'_1+\dots+p'_k=\alpha p''_1+\dots+\alpha p''_k=\alpha p''$$

whence $\alpha=\beta$. Let $\alpha=d'/d''$ with coprime d' and d''. Since

$$\frac{p'_1}{p''_1}=\dots=\frac{p'_k}{p''_k}=\frac{q'_1}{q''_1}=\dots=\frac{q'_l}{q''_l}=\frac{d'}{d''},$$

we obtain $p'_j=a_jd'$, $p''_j=a_jd''$ and $q'_j=b_jd'$, $q''_j=b_jd''$ for some integers $a_1,\dots,a_k$ and $b_1,\dots,b_l$. Hence $p_j=p'_j+p''_j=a_jd$ and $q_j=b_jd$ for $d=d'+d''$, and we finally obtain $P(z)=P_1(z)^d$ and $Q(w)=Q_1(w)^d$ with

$$P_1(z)=\prod_{j=1}^{k}(z-z_j)^{a_j}\quad\text{and}\quad Q_1(w)=\prod_{j=1}^{l}(w-w_j)^{b_j}.$$

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References

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