CONSTRUCTION OF ARRANGEMENTS OF AN M-QUARTIC AND AN M-CUBIC WITH A MAXIMAL INTERSECTION OF AN OVAL AND THE ODD BRANCH[1]

S.Yu. Orevkov

Abstract. We construct 237 arrangements mentioned in the title. All the constructions consist in perturbing singular curves. Almost in all cases, we prove that all the curves with the given set of singularity types are considered under the condition that they could provide arrangements mentioned in the title. We prove that a certain mutual arrangement of a cubic and a quartic is realisable pseudo-holomorphically but unrealisable algebraically. The proof of the algebraic unrealisability is based on the cubic resolvent.

— Ça c'est vrai, dit le petit prince. Et qu'en fais-tu?
— Je les gère. Je les compte et je les recompte, dit le businessman. C'est difficile. Mais je suis un homme sérieux!

A. de Saint-Exupery. "Le petit prince"

In this paper, we construct 237 arrangements mentioned in the title (see §5). They include the arrangements which were constructed in [**2**, **3**, **11**]. All the constructions consist in perturbing singular curves. Almost in all cases, we prove that all the curves with the given set of singularity types are considered under the condition that they could provide arrangements mentioned in the title. In particular, we give in §1 a complete classification (up to isotopy) of arrangements of a quartic and a cubic with a maximal intersection of an oval and the odd branch (see Definition 0.2) which have irreducible double points with the total Milnor number equal to six. In §2, we give an analogous classification in two cases when the total Milnor number is four: symmetric arrangements with two A_2 (Sect. 2.1) and quartic with A_4 (Sect. 2.2). In §3, we give a classification of arrangements of a quartic and a cubic with an almost maximal intersection of an oval and the odd branch (see Definition 0.2) which have irreducible double points with the total Milnor number equal to six, with two exceptions: (1) the quartic has a singularity A_6 and (2) the quartic has A_4 and either the quartic or the cubic has A_2 so that the tangency at A_2 is not maximal. It is known a priori that these cases can add nothing to the list in §5 because the corresponding smoothings can be obtained as smoothings of curves from Sect. 2.2.

In Sections 1.3 and 6.3, we show (see. Remark 1.8 and Proposition 6.8) that a certain mutual arrangement of a cubic and a quartic is pseudoholomorphically realisable but algebraically unrealisable. Other examples of algebraically unrealisable mutual arrangements of two transversally intersecting non-degenerate real pseudoholomorphic curves have been

[1]English translation of the paper published in *Bulletin (vestnik) of Nizhni Novgorod State University (Lobaczewski University) "Mathematical modeling and optimal control", issue 2(24), 2002.*
Partially supported by the grant RFFI 01-01-00602.

known (see [**1**], [**8**]), however, Figure 13 is the first known to the author example of this kind such that both the construction and the proof of the algebraic non-realisability are obtained by simple arguments and do not require messy calculations.

I am grateful to G.M. Polotovskii for numerous discussions. This paper was written because of his insistence.

Definitions and notation. Recall that a curve has a *singularity of the type* A_n at a point p if it is defined by the equation $y^2 = \pm x^{n+1}$ in some local analytic coordinates centered at p. Such points are called *double*. A double point A_n is *reducible* when n is odd and *irreducible* when n is even. The integer n is its *Milnor number*. *A branch* of a real algebraic curve is by definition the image of a connected component of its normalisation (non-singular model). A branch of a curve in $\mathbf{R}P^2$ is called *even* (*odd*) if it realises a zero (non-zero) homology class in $H_1(\mathbf{R}P^2; \mathbf{Z}/2\mathbf{Z})$.

Definition 0.1. Suppose that one curve is non-singular at a point p and another curve has a singularity of the type A_n at this point. We shall say that the curves have a *maximal* (resp. *almost maximal*) tangency at p if the local multiplicity of the intersection is $n+1$ (resp. n).

Note, that if one curve is non-singular at a point p and another curve has a singularity of the type A_n at p then the intersection is maximal if and only if one of the curves curve is arranged from the both sides of the other one when restricted to any neighbourhood of p.

Definition 0.2. Let B_1 and B_2 be branches of algebraic curves C_1 and C_2 of degrees d_1 and d_2 respectively. Suppose that each of the curves C_1, C_2 has only irreducible double points as singularities. We shall say that the branches B_1 and B_2 are in *maximal mutual arrangement* if the following conditions hold:

(1) All the singularities of C_1, C_2 are located on the branches B_1, B_2;
(2) $C_1 \cap C_2 = B_1 \cap B_2$.
(3) C_1 and C_2 have no common singular points.
(4) The curves have a maximal intersection at each singular point.

Let us say that the branches B_1 and B_2 are in *almost maximal mutual arrangement* if the condition (4) is replaced by

($4'$) the intersection is almost maximal at one singular point and maximal at all the other singular points of each curve.

Notation 0.3. Let C be a curve in $\mathbf{R}P^2$ and p a point on C which is not a flex point. Let us choose coordinates $(x:y:z)$ so that $p=(0:1:0)$ and the line $z=0$ is the tangent to C at p. Let us choose a parameter a so that the conic $yz = ax^2$ intersects C at p with multiplicity ≥ 3. Let us denote by $f_{C,p}$ the birational quadratic transformation $(x:y:z) \mapsto (xz : yz - ax^2 : z^2)$ (the mapping $(X,Y) \mapsto (X, Y - aX^2)$ in the affine coordinates $X = x/z$, $Y = y/z$).

Notation 0.4. Let p and q be points in $\mathbf{R}P^2$ and let L be a line passing through q and not passing through p. Let us choose the coordinates $(x:y:z)$ so that $p=(0:1:0)$, $q = (0:0:1)$, $L = \{y=0\}$. Let us denote by $h_{p,q,L}$ the birational quadratic transformation $(x:y:z) \mapsto (x^2 : xy : yz)$. In the literature on the topology of real algebraic curves, this transformation is usually called the *hyperbolism* (O.Ya. Viro introduced this term referring to Newton).

§1. Maximal arrangements of a cubic and a quartic which have irreducible double points whose sum of Milnor numbers is equal to six

1.1. A smooth M-cubic and a quartic with a point A_6.

Lemma 1.1. *Let C be a non-singular M-cubic, E a conic, and L a line. Suppose that E meets the odd branch J of C at 6 points and let us denote one of these points by p. Suppose that L is tangent to E at p and also L is tangent to J at some point q. Then the arrangement of $C \cup E \cup L$ on $\mathbf{R}P^2$ is one of those depicted in Figures 1.1–1.5. Moreover, all these arrangements are realisable.*

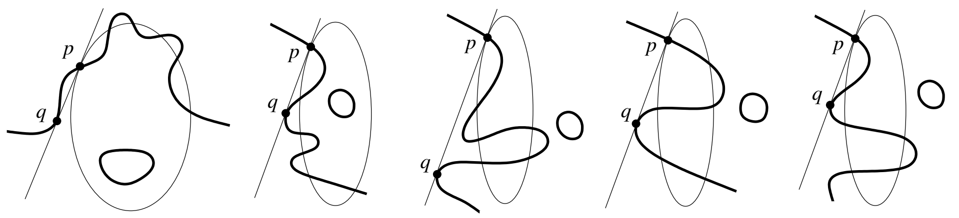


FIG. 1.1 FIG. 1.2 FIG. 1.3 FIG. 1.4 FIG. 1.5

Proof. Figure 1.1. Let x, y be affine coordinates. Let $E = \{x^2 + y^2 = 1\}$, $L = \{5y = 12x + 13\}$, $C = \{(x^2 + y^2 - 1)y + \alpha f(y) = 0\}$, $p = (-12/13, 5/13)$, and $q = (x_q, y_q)$ where $f(y) = (y - \frac{5}{13})(2y - 1)(4y - 3)$, $\alpha = \frac{91}{144} - \frac{13}{96}\sqrt{19} \approx 0.0416769$, $x_q = -(65 + \sqrt{19})/72$ and $y_q = (13 - \sqrt{19})/30$.

To realise the arrangements in Figures 1.2 – 1.5, let us fix C, $L = \{l = 0\}$, p, and q in the required way, and let us construct E.

Figure 1.2. Let $L_1 = \{l_1 = 0\}$ a line cutting J at three points which lie on the same arc pq. Let us set $E = \{l_1 l + \varepsilon f\}$ where $\{f = 0\}$ is the conic which is tangent to L at p and which has no other real intersections with L_1. Choose $|\varepsilon| \ll 1$.

Figure 1.3 – 1.5. Let $\{l_1 = 0\}$ be the line passing through p and cutting J at two more points. Set $E = \{l_1^2 + \varepsilon l_2 l\}$ where $\{l_2 = 0\}$ is a line close to L and $|\varepsilon| \ll 1$.

The fact that other arrangements are impossible easily follows from the classification due to Polotovskii [**10**; §3.1] of mutual arrangements of a conic and a cubic and from Bezout's theorem applied to an auxiliary line. □

Proposition 1.2. *Let C_3 be a non-singular M-cubic and C_4 a quartic which has a singularity of the type A_6 at p. Suppose that C_4 maximally meets the odd branch J of C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is one of those depicted in Figures 2.1 – 2.5. Moreover, all these arrangements are realisable.*

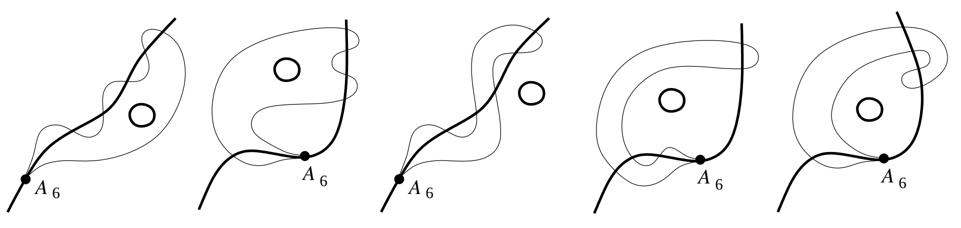


FIG. 2.1 FIG. 2.2 FIG. 2.3 FIG. 2.4 FIG. 2.5

Proof. Apply $f_{C,q}$ to the arrangements from 1.1. $\square$

1.2. A smooth M-cubic and a quartic with points A_4 and A_2.

Lemma 1.3. *Let p_1, p_2, q be points on the odd branch J of a non-singular M-cubic C. Let us denote the lines (p_1q), (p_2q), and (p_1p_2) by L_1, L_2, and L_3 respectively. Suppose that C is tangent to L_1 at q. Then C is arranged with respect to L_1, L_2, L_3 as in Figures 3.1– 3.5. Moreover, all these arrangements are realisable.*

Proof. The fact that there is no other arrangements is evident. To realise the arrangement in Figure 3.3, let us choose $p_1 \in J$, construct a tangent $L_1 = (p_1q)$ to J, and let L_3 be a line close to L_1.

To realise the arrangement in Figures 3.2 and 3.5, let us choose $p_1 \in J$ and construct two line L_1 and L_3' passing through p which are tangent to J. Let L_3 be a line close to L_3'.

To realise the remaining two arrangements, let us denote one of the inflection points of J by a. Let us construct successively the tangents to J as follows: ab (b is the tangency point), bc (is the tangency point), and cd (d is the tangency point on the arc ab). Let e be a point on the arc ad and f a point on the arc bd close to b. Then we obtain the arrangement in Figure 3.1 for $p_1 = b$, $p_2 = e$, $q = c$ and the arrangement in Figure 3.4 for $p_1 = c$, $p_2 = f$, $q = d$. $\square$

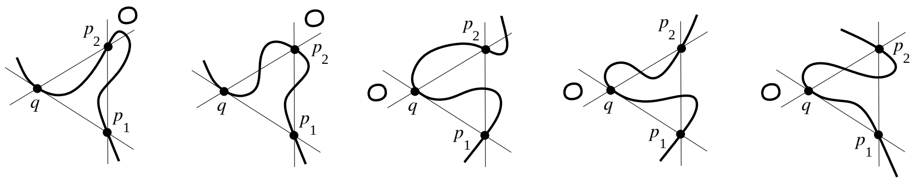


FIG. 3.1 FIG. 3.2 FIG. 3.3 FIG. 3.4 FIG. 3.5

Lemma 1.4. *Let C be a non-singular M-cubic, E a conic, and L_1, L_2 two lines. Suppose that E meets the odd branch J of C at 6 points. Let us denote two of these points by p_1 and p_2. Suppose that L_j is tangent to E at p_j, $j = 1, 2$, and L_1 is tangent to J at a point q. Suppose that L_2 also passes through q. Then the arrangement of $C \cup E \cup L_1 \cup L_2$ on $\mathbf{R}P^2$ is one of those depicted in Figures 4.1–4.8. Moreover, all these arrangements are realisable.*

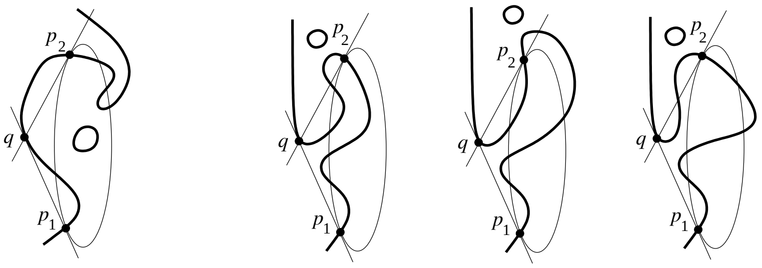


FIG. 4.1 FIG. 4.2 FIG. 4.3 FIG. 4.4

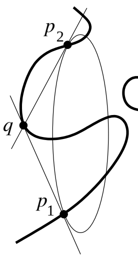


FIG. 4.5

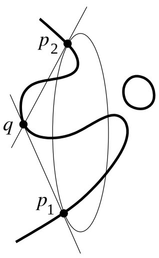


FIG. 4.6

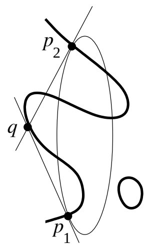


FIG. 4.7

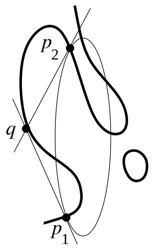


FIG. 4.8

Proof. All possible mutual arrangements of C, E and L_1 are described in Lemma 1.1. It is impossible to add the line L_2 to Figure 1.1 and the only way to add L_2 to Figure 1.2 (resp. 1.3; 1.4; 1.5) is as in Figure 4.1 (resp. 4.2–4.4; 4.5–4.6; 4.7–4.8). Let us show that all these arrangements are realizable.

The arrangements in Figures 4.3–4.7. In Lemma 1.3, let us set $E = \{l_3^2 + \varepsilon l_1 l_2 = 0\}$ where $L_j = \{l_j = 0\}$ and $|\varepsilon| \ll 1$.

The arrangements in Figure 4.1. Let an M-cubic C and two lines L and L_1 be arranged as in Figure 5. Consider the pencil of cubics $\{E(t)\}$, passing through p_1, r_1, and r_2 and touching L_1 at p_1. Let I be the segment of this pencil between $L \cup L_1$ and $(p_1 r_1) \cup (p_1 r_2)$ passing through the position depicted in Figure 5. Let $L_2(t)$ be the tangent to $E(t)$ passing through q and let $p_2(t)$ be the point of the tangency. When t runs through I, the point $p_2(t)$ moves continuously from $L \cap L_1$ to r_1. Hence, it crosses C at some moment.

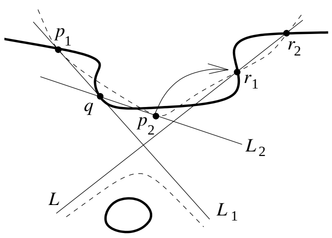


FIG. 5

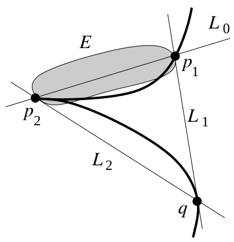


FIG. 6

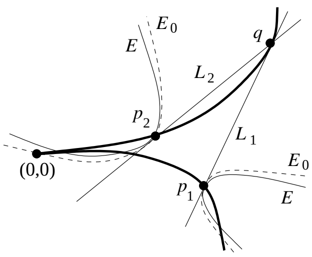


FIG. 7

The arrangements in Figure 4.2. Let us fix affine coordinates x, y and set $p_2 = (0,0)$, $L_0 = \{y = x\}$. Let $C(t)$, $t > 0$ be the M-cubic defined by $y^2 = x(x+t)(x+2t)$ and let $J(t)$ be its odd branch. Let $p_1(t)$ be the intersection point of $C(t)$ and L_0 which has the maximal x-coordinate. Let $L_1(t)$ be the tangent to $J(t)$ passing through $p_1(t)$ and let $q(t)$ be the point of the tangency. Let us denote the line through $p_2(t)$ and $q(t)$ by $L_2(t)$. The cubic $C(t)$ tends to $C_0 = \{y^2 = x^3\}$ as $t \to 0$. Let us choose the equations $l_j(t) = 0$ of the lines $L_j(t)$, $j = 0, 1, 2$, so that $l_j(t) \to l_j(0)$ as $t \to 0$. Let $E(t) = \{l_0^2 + \varepsilon l_1 l_2 = 0\}$. Let us fix ε such that the conic $E(0)$ is arranged as in Figure 6. Then for $0 < t \ll |\varepsilon|$ the curves $C(t)$, $E(t)$, and $L_j(t)$ are arranged as it is claimed.

The arrangements in Figure 4.8. Let us fix affine coordinates x, y and set $q = (1,1)$, $p_1 = (1/4, -1/8)$, $p_2 = (1/25, 1/125)$. These points lye on the curve $C_0 = \{y^2 = x^3\}$. Let $L_j = \{l_j = 0\}$ where $l_1 = 1 - 3x + 2y$, $l_2 = 1 - 31x + 30y$, $l_3 = 1 - 19x - 30y$, and let $E_0 = \{f_0 = 0\}$ where $f_0 = l_3^2 - l_1 l_2 = -4x - 92y + 270x^2 + 988xy + 840y^2$. Then $E = \{f_0 - \varepsilon l_1 l_2 = 0\}$ for $0 < \varepsilon \ll 1$ is arranged as in Figure 7. We define C as a small (with respect to $|\varepsilon|$) M-smoothing of C_0. $\square$

Proposition 1.5. *Let C_3 be a smooth M-cubic and C_4 a quartic which has two singular points of the types A_4 and A_2. Suppose that C_4 maximally intersects the odd branch J_3 of C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is one of those depicted in Figures 8.1–8.8. Moreover, all these arrangements are realisable.*

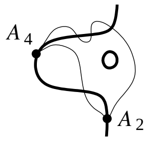


Fig. 8.1

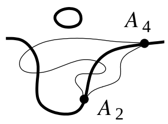


Fig. 8.2

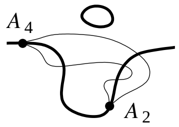


Fig. 8.3

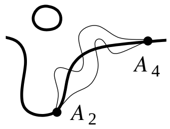


Fig. 8.4

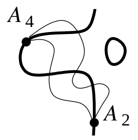


Fig. 8.5

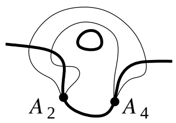


Fig. 8.6

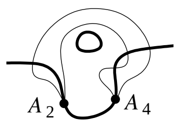


Fig. 8.7

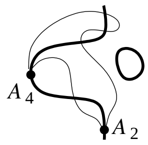


Fig. 8.8

Proof. We apply the hyperbolism h_{p,q,L_1} to the curves from Lemma 1.4 where p is the intersection point of L_2 and C which is different from p_2. $\square$

1.3. A smooth M-cubic and a quartic with three points A_2.

Lemma 1.6. *Let p_1, p_2, p_3 be points lying on the odd branch J of a smooth M-cubic C. Let us denote the lines (p_2p_3), (p_3p_1), (p_1p_2) by L_1, L_2, L_3 respectively. Let E be a conic touching L_1, L_2, L_3 at q_1, q_2, q_3 respectively. Suppose that J meets E at six points three of which are q_1, q_2, q_3. Then the arrangement of C with respect to E, L_1, L_2, L_3 is as in Figure 9.1 up to a permutation of p_1, p_2, p_3. Moreover, this arrangement is algebraically realisable.*

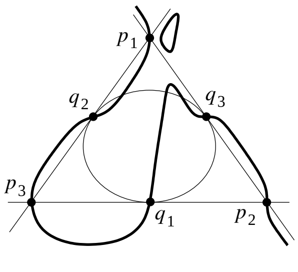


Fig. 9.1

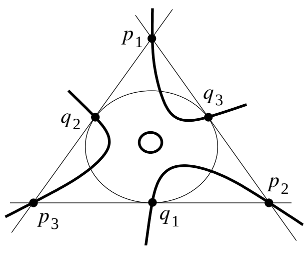


Fig. 9.2

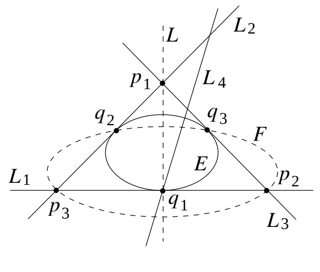


Fig. 10

Proof. Let us consider two conics E and F and five lines $L_1, \dots, L_4$, and L arranged as in Figure 10. Let $l_2 l_3 l_4 = 0$ and $lf = 0$ be the equations of $L_2 \cup L_3 \cup L_4$ and $L \cup F$ respectively. Then, for a suitable choice of the sign of a small parameter ε, the curve $C = \{l_2 l_3 l_4 + \varepsilon l f = 0\}$ is arranged with respect to L_1, L_2, L_3, and E as in Figure 9.1

All the arrangements different from Figure 9.1 and Figure 9.2 are impossible. This fact easily follows from Bezout's theorem for an auxiliary line and the classification due to Polotovskii [**10**] of mutual arrangements of a conic and a cubic. Let us show that the arrangement in Figure 9.2 is also impossible. Indeed, each of the lines $(p_1 q)$, where q runs the segment[2] $[q_1 p_2]$, meets J at three points. Hence, the oval O of C cannot intersect the triangle $T_1 = [q_1 p_1 p_2]$. Analogously, O cannot intersect the triangles $T_2 = [q_2 p_2 p_3]$ and $T_3 = [q_3 p_3 p_1]$. But since the lines $(p_1 q_1)$, $(p_2 q_2)$, and $(p_3 q_3)$ pass through the same point, the union of T_1, T_2, and T_3 coincides with the triangle $[p_1 p_2 p_3]$. $\square$

Proposition 1.7. *Let C_3 be a smooth M-cubic and C_4 a quartic which has three singular points of the type A_2. Suppose that C_4 maximally intersects the odd branch J_3 of C_3. Then $C_3 \cup C_4$ is arranged on $\mathbf{R}P^2$ as in Figure 11.1, moreover, this arrangement is realisable.*

Proof. Apply the quadratic transform $(x_1 : x_2 : x_3) \mapsto (x_1 x_2 : x_2 x_3 : x_3 x_1)$ to the curves from Lemma 1.6 where $L_i = \{x_i = 0\}$. $\square$

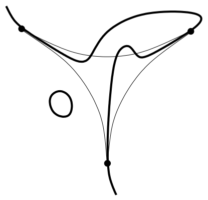


FIG. 11.1

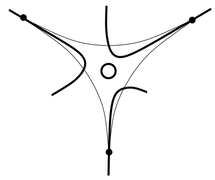


FIG. 11.2

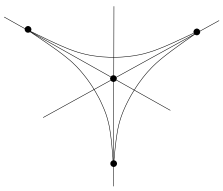


FIG. 12

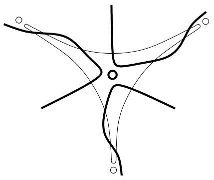


FIG. 13

Remark 1.8. The tangents at the singular points of a real tricuspidal quartic pass through the same point (see Figure 12). The unrealisability of Figure 11.2 means that the triple point in Figure 12 does not admit any M-smoothing preserving all the other singularities. It is evident that such a smoothing is realisable by real pseudoholomorphic curves (see a definition in [**1**,**6**,**8**]). After smoothing the other singularities of the quartic in Figure 11.2, one can obtain a pseudoholomorphic realisation of the arrangements of an M-cubic and an M-quartic depicted in Figure 13. Below, in Section 6.3 (Proposition 6.8), we shall prove that this arrangement is algebraically unrealisable.

This construction provides a new example of an algebraically unrealisable arrangement on $\mathbf{R}P^2$ of two smooth real pseudoholomorphic curves which meet each other transversally. Such examples can be found in [**1**], [**8**]. However, Figure 13 is the first known to the author example of this kind such that both the construction and the proof of algebraic unrealisability are elementary and do not require messy computations.

[2] When speaking of segments and triangles, we mean the affine chart corresponding to Figure 9.2.

1.4. A cuspidal cubic and a two-component quartic with a point A_4.

Lemma 1.9. *Let p_1, p_2, q be points on the odd branch J of a smooth M-cubic C. Let us denote the lines (p_1q), (p_2q), (p_1p_2) by L_1, L_2, L_3 respectively. Suppose that C touches L_1 at q and touches L_2 at p_2 Then C is arranged with respect to L_1, L_2, L_3 either as in Figure 14.1 or as in Figure 14.2. Moreover, the both arrangements are realisable.*

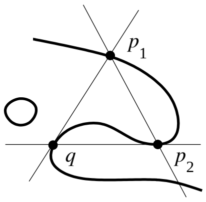


FIG. 14.1

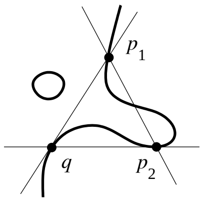
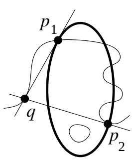


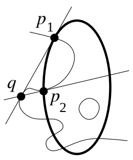
FIG. 14.2

Proof. The fact that there are no other arrangements is evident. Let us show that Figures 14.1 and 14.2 are realisable. Let a be the flex point of a smooth M-cubic, L the tangent at a, and L' the line passing through a and touching the odd branch at some other point. Let q be a point close to a. Choosing L_2 as a tangent close to L (resp. to L'), we obtain Figure 14.1 (resp. Figure 14.2). $\square$

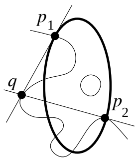
Lemma 1.10. *Let C be a smooth M-cubic, E a conic, and L_1, L_2 two lines. Suppose that E meets the odd branch J of C at 6 points and let us denote two of them by p_1 and p_2. Suppose that L_1 touches E at p_1 and touches J at q. Suppose also that L_2 touches C at p_2 and passes through q. Then the arrangement of $C \cup E \cup L_1 \cup L_2$ on $\mathbf{R}P^2$ is one of those depicted in Figures 15.1–15.13. Moreover, all these arrangements are realisable.*



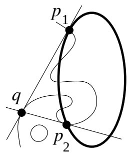
15.1



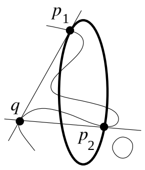
15.2



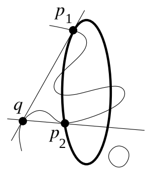
15.3



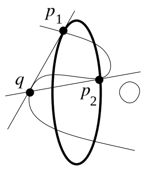
15.4



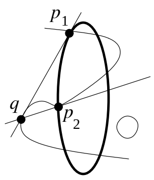
15.5



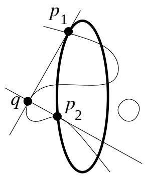
15.6



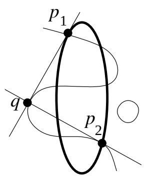
15.7



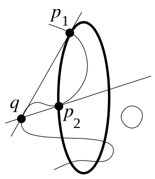
15.8



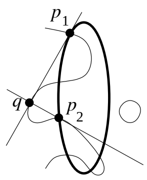
15.9



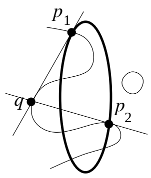
15.10



15.11



15.12



15.13

Proof. All possible mutual arrangements of C, E, and L_1 are described in Lemma 1.1. Figure 1.1 (resp. 1.2; 1.3; 1.4; 1.5) can provide only Figure 15.1 (resp. 15.2–15.3; 15.4–15.6; 15.7–15.10; 15.11–15.13). Let us show that all these arrangements are realisable.

The arrangements in Figures 15.5–15.10, 15.12, 15.13. Let us set in Lemma 1.9: $E = \{l_3^2 + \varepsilon l_1(l_3 + \delta l_2) = 0\}$, where $L_j = \{l_j = 0\}$ and $|\delta| \ll |\varepsilon| \ll 1$.

The arrangements in Figures 15.1 and 15.4 Let us consider two conics arranged with respect to the coordinate axes as in Figure 16.1 (C is obtained as a perturbation of the doubled line ab). Applying the quadratic transformation $(x : y : z) \mapsto (xy : yz : zx)$, we obtain Figure 16.2 whose perturbations yield Figures 15.1 and 15.4.

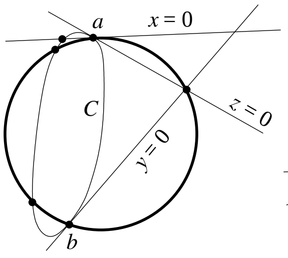


FIG. 16.1

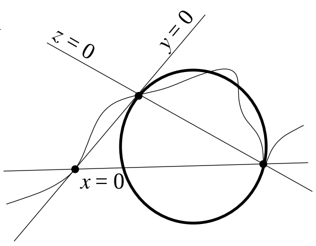


FIG. 16.2

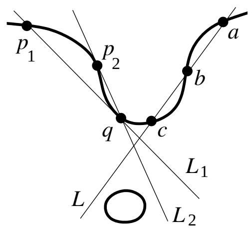
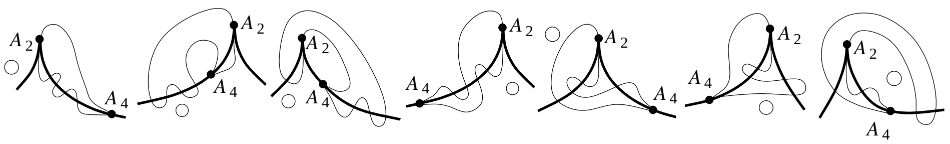


FIG. 17

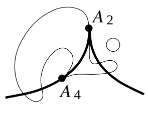
The arrangements in Figures 15.2 and 15.11. Let us consider an M-cubic C arranged with respect to three lines L, L_1, and L_2 as in Figure 17. Let E be the conic passing through the points p_1, p_2, a, b which is tangent to the line L_1 at p_1. Let p_0 be the flex point on the arc qp_2 and let L_0 be the tangent at this point. Let us fix C and L and let us move continuously the point q along the arc qp_0, changing continuously L_1, L_2, p_1, p_2, and E preserving the incidences and the tangencies. Then $E \to L_0 \cup L$ as $q \to p_0$, hence, after a certain moment, we obtain the arrangement in Figure 15.2. When q passes through the flex point, we get Figure 15.11 (when this happens, the order of the points along J becomes: a, b, c, p_1, p_2, q).

The arrangement in Figure 15.3. Let us fix a non-singular M-cubic C and a point q on its odd branch J. Let L_1 be the tangent at q and let p_1 be the point of its intersection with J. Let L_2 be the line through q touching at p_2 that arc qp_1 which contains two flex points. Let a be a point on that arc qp_2 which does not contain p_1. Let us denote the lines (p_1p_2), (p_1a), and (p_2a) by L_3, L_4, and L_5 respectively. Set $E = \{l_1l_5 + \varepsilon l_3l_4 = 0\}$ where $L_j = \{l_j = 0\}$ and $|\varepsilon| \ll 1$. $\square$

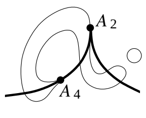
Proposition 1.11. *Let C_3 be a cuspidal cubic and let C_4 be a two-component quartic which has a singular point of the type A_4. Suppose that the odd branch of C_4 intersects maximally C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is one of those depicted in Figures 19.1–19.13. Moreover, all these arrangements are realisable.*



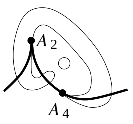
19.1 19.2 19.3 19.4 19.5 19.6 19.7



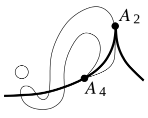
19.8



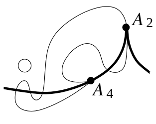
19.9



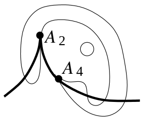
19.10



19.11



19.12



19.13

Proof. Apply the hyperbolism h_{p,q,L_1} to the curves from Lemma 1.10, where p is the intersection point of L_2 and E which is different from p_2. $\square$

1.5. A cuspidal cubic and a two-component quartic with two points A_2.

Lemma 1.12. *Let C be a non-singular M-cubic and L_1, L_2, L_3 lines. Let us denote $p_1 = L_2 \cap L_3$, $p_2 = L_3 \cap L_1$, $p_3 = L_1 \cap L_2$. Let q_1, q_2, q_3 be points on L_1, L_2, L_3 respectively which are different from p_1, p_2, p_3. Let E be the conic through p_1, q_2, q_3 which is tangent to L_1 at q_1. Suppose that E meets the odd branch J of C at 6 points including q_1, q_2, q_3. Suppose also that J passes through p_2, p_3 and touches L_2, L_3 at q_2, q_3. Then the arrangement of $C \cup E \cup L_1 \cup L_2 \cup L_3$ on $\mathbf{R}P^2$ is one of those depicted in Figures 20.1–20.3 up to swapping the indices 2 and 3. Moreover, all these arrangements are realisable.*

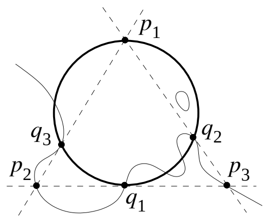


FIG. 20.1

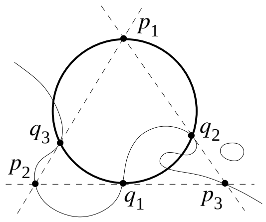


FIG. 20.2

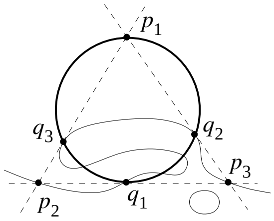


FIG. 20.3

Proof. The fact that other arrangements are impossible, easily follows from Polotovskii's classification [**10**] of mutual arrangements of a conic and a cubic and from Bezout's theorem applied to an auxiliary line. Let us show that the arrangements in Figures 20.1–20.3 are realisable.

The arrangements in Figures 20.1 and 20.2. Let us fix affine coordinates x, y. Then the points $p_1 = (10/7, 2/7)$, $p_2 = (1, -1)$, $p_3 = (1/25, 1/125)$, $q_1 = (1/16, -1/64)$, $q_2 = (0,0)$, $q_3 = (4,8)$, the cubic $C_0 = \{y^2 = x^3\}$, the conic $E = \{8y^2 - 18xy + 3x^2 + y = 0\}$, and the lines $L_1 = (p_2p_3) = \{21x + 20y = 1\}$, $L_2 = (p_3p_1) = \{x - 5y = 0\}$, $L_3 = (p_1p_2) = \{3x - y = -4\}$ are arranged as in Figure 21. Perturbing the singular point of C_0, we obtain Figures 20.1 and 20.2. The fact that the required perturbations exist, can be checked directly as follows. Namely, let us fix C_0, L_3, q_2, q_3, p_2 as above. Let us construct p_3, q_1, p_1 according to Figure 21, and let E be the conic through p_1, p_2, q_2 which is tangent to L_3 at q_3. Then one has $E = \{t(5t-1)x - (3t^2 - t - 1)y + \dots = 0\}$. Hence, for $t = 1/5$ we obtain Figure 21, for $0 < t < 1/5$ (resp. for $1/5 < t < 1/4$) a perturbation of C_0 yields Figure 20.1 (resp. Figure 20.2).

The arrangement in Figure 20.3. Let L_1, L_2, L_3, p_1, p_2, p_3, q_2, and q_3 be arranged as it is required. Let us fix an affine chart corresponding to Figure 20.3. Let $F = \{f = 0\}$ be the ellipse which is tangent to L_2, L_3 at q_2, q_3 and which cuts the segment $[p_2p_3]$ at two points Let us choose $q_1 \in [p_2p_3]$ so that $L_1 \cap F \subset [q_1p_2]$ and let us trace E in the required way. At last, we set $C = \{l_1f + \varepsilon l_2l_3l_4 = 0\}$ where $|\varepsilon| \ll 1$ and $\{l_4 = 0\}$ is the line through q_1 which does not cut F. $\square$

Proposition 1.13. *Let C_3 be a cuspidal cubic and let C_4 be a two-component quartic one of whose branches has two singular points of the type A_2. Suppose that the singular branch of C_4 maximally intersects C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is one of those depicted in Figures 22.1–22.3. Moreover, all these arrangements are realisable.*

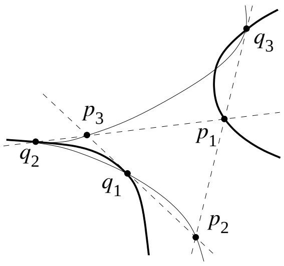


FIG. 21

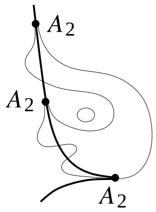


FIG. 22.1

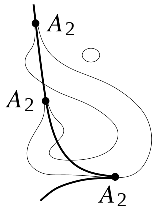


FIG. 22.2

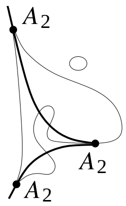


FIG. 22.3

Proof. Apply the quadratic transformation $(x : y : z) \mapsto (xy : yz : zx)$ to the curves from Lemma 1.12 (the coordinates are chosen so that the lines L_1, L_2, L_3 are the coordinate axes). $\square$

§2. Some maximal arrangements of a quartic and a cubic which have irreducible double points whose sum of Milnor numbers is equal to four

2.1. Symmetric arrangements of an M-cubic and a two-component quartic with two points A_2. In this section, we shall apply the method of construction used for the curve $A_2|$---$|A_2$ in the paper [**7**].

Lemma 2.1. *There exist mutual arrangements of a line L and three conics C, E, and H depicted in Figures 23.1–23.4.*

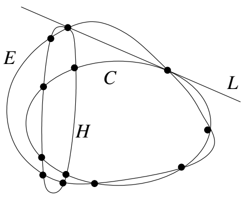


FIG. 23.1

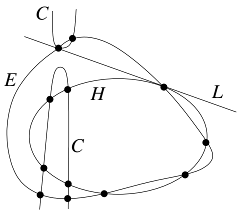


FIG. 23.2

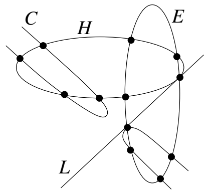


FIG. 23.3

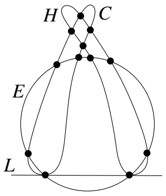


FIG. 23.4

Proof. The first two arrangements are constructed in the same way as in the paper [**7**]: In the case 23.1, one should change the sign of δ; in the case 23.2, one should swap C and H. The constructions of 23.3 and 23.4 are evident. $\square$

Proposition 2.2. *Let C_3 be an M-cubic and let C_4 be a two-component quartic which has two singular points of the type A_2 on the same branch and which maximally intersects C_3. Suppose that the both curves are symmetric with respect to the same axis L. Then the arrangement of $C_3 \cup C_4 \cup L$ on $\mathbf{R}P^2$ is one of those depicted in Figures 24.1–24.4. Moreover, all these arrangements are realisable.*

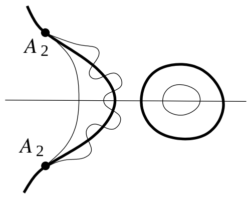


FIG. 24.1

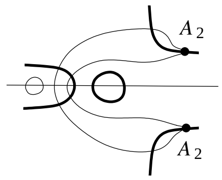


FIG. 24.2

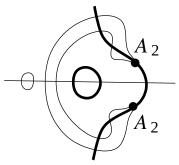


FIG. 24.3

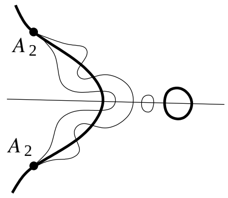


FIG. 24.4

Proof. Apply the construction from [**7**] to Figures 23.1–23.4. □

2.2. A smooth M-cubic and a two-component quartic with a point A_4.

Proposition 2.3. *Let C_3 be a non-singular M-cubic and let C_4 be a two-component quartic which has a singular point of the type A_4. Suppose that the singular branch of C_4 maximally intersects the odd branch J_3 of C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is either one of those depicted in Figures 25.1 and 25.2, or it is one of the* 31 *arrangements obtained from Figures 2.1–2.5, 8.1–8.8, 19.1–19.13 after modifications in Figure 26.*[3] *All these* $2 + 31 = 33$ *arrangements are realisable.*

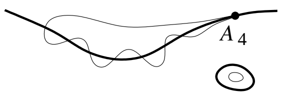


FIG. 25.1

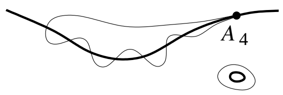
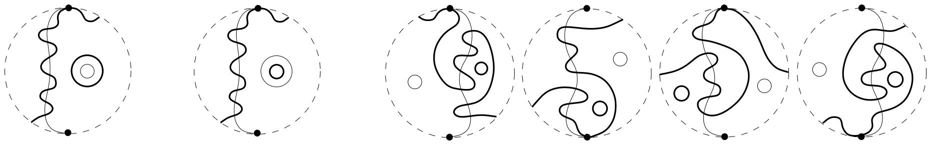


FIG. 25.2

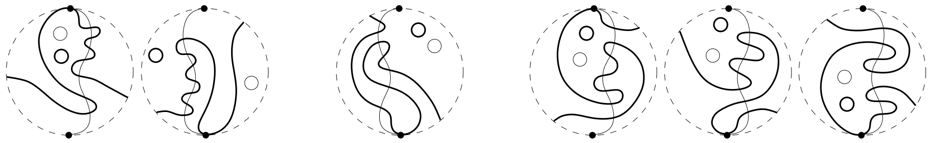
FIG. 26

Proposition 2.4. *Let C and C' be two non-singular M-cubics whose odd branches J and J' have a simple tangency at a point p and transversally cut each other at seven other points. Suppose that p is a flex point of C' and let L be the tangent to C' at p. Then the arrangement of $C \cup C' \cup L$ on $\mathbf{R}P^2$ is one of those depicted in Figures 27.1–27.33. Moreover, all these arrangements are realisable. In Figures 27.1–27.33, we depict* $\mathbf{R}P^2$

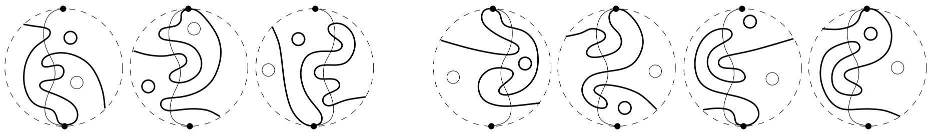
[3]Each of these modifications can be applied in two ways because of the reflection with respect to the non-singular branch.



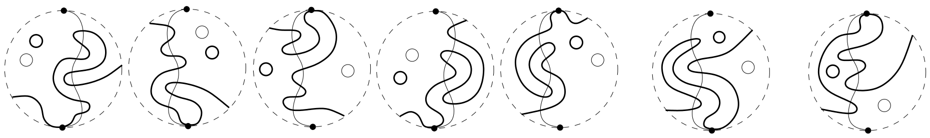
27.1 27.2 27.3 27.4 27.5 27.6



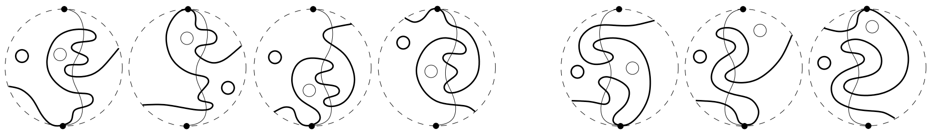
27.7 27.8 27.9 27.10 27.11 27.12



27.13 27.14 27.15 27.16 27.17 27.18 27.19



27.20 27.21 27.22 27.23 27.24 27.25 27.26

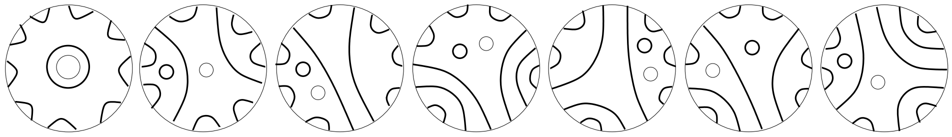


27.27 27.28 27.29 27.30 27.31 27.32 27.33

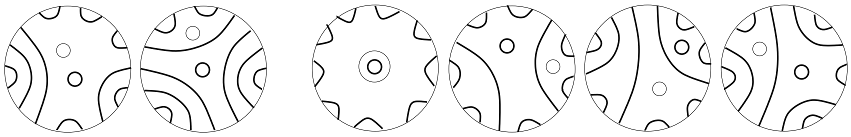
as a disc whose opposite boundary points are identified; the line L corresponds to the boundary of the disc.

Proof of Propositions 2.3 and 2.4. The transformation $f_{C,p}$ defines a one-to-one correspondence between arrangements from Propositions 2.4 and 2.3. Figure 27.1 and Figure 27.2 are transformed into Figure 25.1 and Figure 25.2. Thus, it is sufficient to realise only Figure 27.1 and 27.2. To this end, we fix C' and L satisfying the required conditions and set $C = \{f + \varepsilon l_1 l_2 l_3 = 0\}$, $|\varepsilon| \ll 1$ where $f = 0$ is the equation of C' and $L_i = \{l_i = 0\}$ $(i = 1, 2, 3)$ are lines each of which meets J' at three points and the lines L_1 and L_2 pass through p.

Let us prove that no other arrangement is possible under the hypothesis of Proposition 2.4. Polotovskii [**9**] classified all mutual arrangements of two M-cubics with maximally intersecting odd branches. Cutting $\mathbf{R}P^2$ along the odd branch of the first cubic, we obtain a disc. The second cubic and the oval of the first one are arranged on this disc. All such arrangements are presented in Figures 28.1–28.13 (swapping of the cubics defines the correspondence $1 \leftrightarrow 10$, $2 \leftrightarrow 11$, $3 \leftrightarrow 3$, $4 \leftrightarrow 4$, $5 \leftrightarrow 5$, $6 \leftrightarrow 12$, $7 \leftrightarrow 13$, $8 \leftrightarrow 8$, $9 \leftrightarrow 9$).



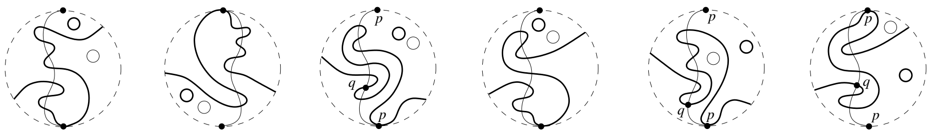
28.1 28.2 28.3 28.4 28.5 28.6 28.7



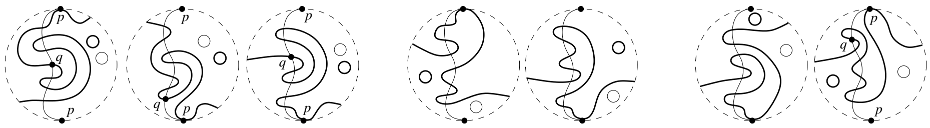
28.8 28.9 28.10 28.11 28.12 28.13

All the arrangements which could satisfy the hypothesis of Proposition 2.4, must be obtained from Figure 28.1–28.13 by degeneration of one of the digons into a simple tangency followed by adding a line L which has a 3rd order tangency with the boundary of the disc and which does not cut the ovals. Let us do it in all the possible ways so that L cuts the odd branch of each of the cubics at three points (counting the multiplicities). Immediately excluding the arrangements which contradicts Bezout's theorem for the auxiliary line passing through the tangency point and the oval of one of the cubics, we obtain Figures 27.1–27.33, and also Figures 29.1–29.13. The figures correspond to each other as follows:

28.1	$\to$	27.1		28.8	$\to$	27.20–27.24,	29.7–29.9
28.2	$\to$	27.3–27.6,	29.1	28.9	$\to$	27.25	
28.3	$\to$	27.7, 27.8,	29.2	28.10	$\to$	27.2	
28.4	$\to$	27.9,	29.3	28.11	$\to$	27.27–27.30,	29.10, 29.11
28.5	$\to$	27.10–27.12,	29.4	28.12	$\to$	27.26,	29.12, 29.13
28.6	$\to$	27.13–27.15,	29.5	28.13	$\to$	27.31–27.33,	29.6
28.7	$\to$	27.16–27.19					



29.1 29.2 29.3 29.4 29.5 29.6



29.7 29.8 29.9 29.10 29.11 29.12 29.13

The arrangement in Figures 29.3, 29.5–29.9, 29.13 contradicts Bezout's theorem for the auxiliary line pq.

The remaining 6 arrangements can be excluded using the method proposed in [**4**]. Let us choose a point q inside the oval of C'. Let $\mathcal{L}_q$ be the pencil of lines through q. The arrangements 29.1, 29.2, 29.4, and 29.10–29.12 determine the arrangements of C, C', L with respect to $\mathcal{L}_q$ ($\mathcal{L}_q$-arrangements) depicted in Figures 30.1–30.6 respectively (the lines of $\mathcal{L}_q$ correspond to the vertical lines in these figures). The braids determined by these $\mathcal{L}_q$-arrangements coincide with the braids determined by curves obtained by any of the modifications in Figures 31. None of these braids satisfies Murasugi-Tristram inequality (see details in [**4**]). $\square$

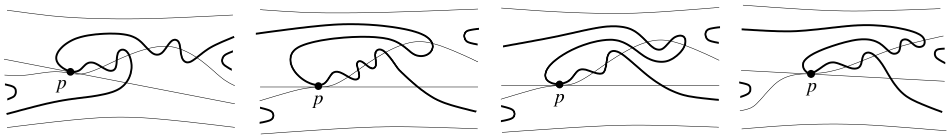


FIG. 30.1 FIG. 30.2 FIG. 30.3 FIG. 30.4

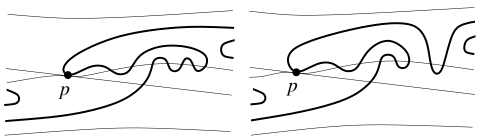


FIG. 30.5 FIG. 30.6

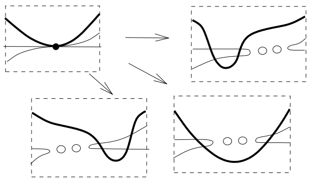


FIG. 31

§3. ALMOST MAXIMAL ARRANGEMENTS OF A QUARTIC AND A CUBIC WHICH HAVE IRREDUCIBLE DOUBLE POINTS WITH THE TOTAL MILNOR NUMBER EQUAL TO SIX

3.1. A smooth M-cubic and a quartic with points A_4 and A_2. (. . 1.2.)

Lemma 3.1. *Let C be a non-singular M-cubic, E a conic, L_1, L_2 tangents to E at p_1, p_2 respectively, and let $q = L_1 \cap L_2$. Suppose that the odd branch J of C meets E at six points including p_1. Suppose also that J is tangent to L_1 at q and cuts L_2 at two more points which are different from p_2. Then the arrangements of $C \cup E \cup L_1 \cup L_2$ on $\mathbf{R}P^2$ is either as in Figure 32, or it is obtained from Figure 4.1–4.8 by a perturbation of the cubic near the point p_2. Moreover, all these arrangements are realisable.*

Proof. Let J be the odd branch of an M-cubic C and let $L = (p_1 r) = \{l = 0\}$ be a line which is close to the tangent at a flex point and which cuts J at three points. Then it is

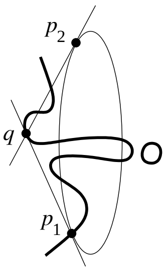


FIG. 32

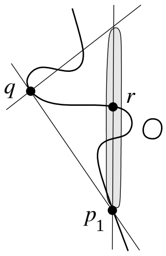


FIG. 33

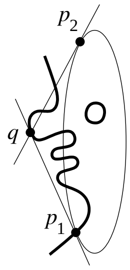


FIG. 34.1

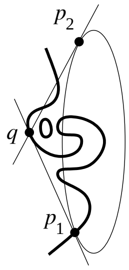


FIG. 34.2

not difficult to construct the lines $L_j = \{l_j = 0\}$, $j = 1, 2$, as in Figure 33. Adding the conic $E = \{l^2 = \varepsilon l_1 l_2\}$, $|\varepsilon| \ll 1$, we obtain Figure 32.

Combining Lemma 1.1 and Bezout's theorem for auxiliary lines, it is easy to exclude all the arrangements except Figure 32, perturbations of Figures 4.1–4.8, and also Figures 34.1 and 34.2. To exclude the two latter cases, we shall use Murasugi-Tristram inequality as it was done in the proof of Propositions 2.3 and 2.4. The arrangement of $C \cup E \cup L_1 \cup L_2$ with respect to the pencil of lines through a point inside the oval of the cubic has the form $\times_2^2 \times_5^5 (\times_5 \times_4 \times_5^2) \times_3^2 (\times_4^2 \times_3 \times_4)$ for Figure 34.1 and $\times_4^2 \supset_5 \subset_3 \times_2^3 \times_3^2 (\times_3 \times_4 \times_3^2) \times_5^2 (\times_4^2 \times_5 \times_4)$ for Figure 34.2 (see [**4**], [**6**], or [**7**] for the description of the encoding; the subwords in parentheses correspond to the points p_1 and q). The rest of the proof is as in [**4**] or [**7**]. $\square$

Proposition 3.2. *Let C_3 be a non-singular M-cubic and C_4 a quartic which has two singular points of the types A_4 and A_2. Suppose that C_4 almost maximally intersects the odd branch J_3 of C_3 so that it has a maximal tangency at A_2 and an almost maximal tangency at A_4. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is either as in Figures 35.1–35.8, or it is obtained from Figures 8.1–8.8 by a modification depicted in Figure 36. Moreover, all these arrangements are realisable.*

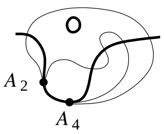


FIG. 35.1

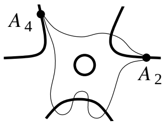


FIG. 35.2

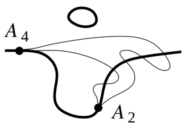


FIG. 35.3

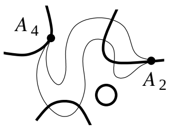


FIG. 35.4

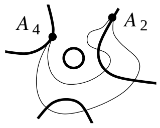


FIG. 35.5

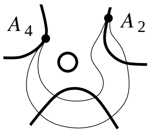


FIG. 35.6

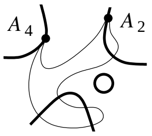


FIG. 35.7

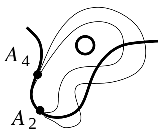


FIG. 35.8

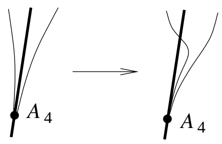


FIG. 36

Proof. Apply the hyperbolism h_{p,q,L_1} to the arrangements from Lemma 3.1, where p is an intersection point of L_2 and C which is different from q (let us denote the third intersection point by p'). In the case when the complement of $C \cup E \cup L_1 \cup L_2$ contains a curvilinear triangle adjacent to the segment $[p'p_2]$, we obviously obtain the perturbations of the curves in Figures 8.1–8.8.

In the other cases, the correspondence between the figures is following (prime denotes a perturbation) $4.1' \to 35.1$–2; $4.2' \to 8.2'$; $4.3' \to 8.3'$; $4.4' \to 35.3$; $4.5' \to 35.4$; $4.6' \to 35.5$; $4.7' \to 35.6$; $4.8' \to 35.7$; $32 \to 35.8$. $\square$

3.2. Almost maximal arrangements of a smooth M-cubic and a quartic with three points A_2. (Cp. Section 1.3.)

Lemma 3.3. *Let p_1, p_2, p_3 be three points on the odd branch J of a non-singular M-cubic C. Let us denote the lines (p_2p_3), (p_3p_1), (p_1p_2) by L_1, L_2, L_3 respectively. Let E be a conic touching L_1, L_2, L_3 at points q_1, q_2, q_3 respectively. Suppose that J does not pass through q_1 and meets E at six points two of which are q_2, q_3. Then (up to a renumbering of p_1, p_2, p_3) one of the following possibilities for the arrangement of C with respect to E, L_1, L_2, L_3 takes place: (a) it is as in Figures 37.1–37.4; (b) it is obtained from Figure 9.1 by perturbing the cubic near one of the points q_1, q_2, q_3; (c) it is obtained from Figure 9.2 by shifting the cubic to the right near q_1; (d) it is obtained from an arrangement corresponding to Cases (a)–(c) applying (maybe, successively) the modification depicted in Figure 38. Moreover, the arrangements corresponding to Cases (a)–(c) are realisable.*

Remark. We did not study the question of the realisability in Case (d).

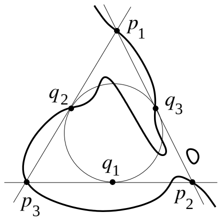


FIG. 37.1

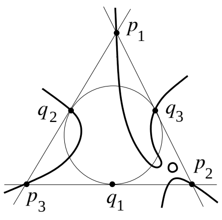


FIG. 37.2

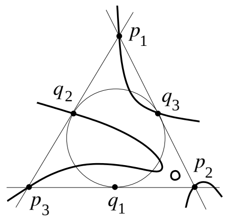


FIG. 37.3

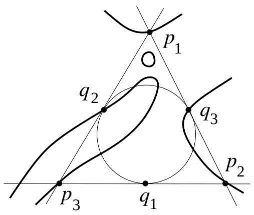


FIG. 37.4

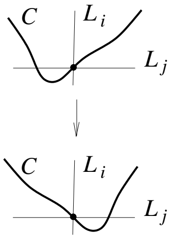


FIG. 38

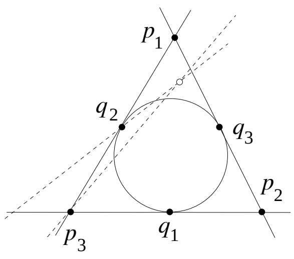


FIG. 39.2

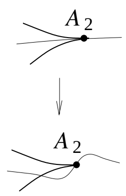


FIG. 40

Proof. Using Bezout's theorem for an auxiliary line and Polotovskii's classification [**10**] of mutual arrangements of a cubic and a conic, it is not difficult to check that all the other arrangement are impossible except, maybe, the one which is obtained from Figure 9.2 by shifting the cubic to the left near q_1. The latter arrangement can be excluded in the same way as in the proof of Lemma 1.6. Let us show that the arrangements in Figures 37.1–37.4 are realisable. Let $L_j = \{x_j = 0\}$, $j = 1,2,3$, and let $E = \{e = 0\}$ where $e = \sum x_j^2 - 2\sum_{j<k} x_j x_k$.

The arrangement in Figure 37.1. $C = \{fx_3 + \delta x_2^2(x_1 - x_3) = 0\}$ where $f = x_2x_3 - \varepsilon x_1(x_1 - \frac{1}{2}x_2 - x_3)$ and $0 \ll \delta \ll \varepsilon \ll 1$.

The arrangement in Figure 37.2. $C = \{(l_1 + \delta x_2)f = \eta x_1^2 l_2\}$ where $f = x_2x_3 + \varepsilon x_1 l_2$, $l_2 = x_1 - x_2 - x_3$, $\{l_1 = 0\}$ is a tangent to the conic $\{f = 0\}$ passing through p_1, and $|\eta| \ll |\delta| \ll \varepsilon \ll 1$.

The arrangement in Figure 37.3. $C = \{x_3(x_3 - x_1)(x_1 - \varepsilon x_2) = \delta x_2(x_3 + x_1)(x_2 - x_1)\}$ where $|\delta| \ll \varepsilon \ll 1$.

The arrangement in Figure 37.4. $C = \{x_3 l_1 l_2 = \varepsilon x_1 x_2(x_1 + x_2 - x_3)\}$ where $|\varepsilon| \ll 1$ and $l_1 = 0$, $l_2 = 0$ are the equations of the dashed lines in Figure 39.

Finally, let us show that the cubic in Figure 9.2 can be shifted to the right near q_1. Indeed, replace the cubic by a small perturbation of the union of the lines (p_1q_1'), (p_2q_2), and (p_3q_3) where $q_1' \in]q_1, p_2[$. □

Proposition 3.4. *Let C_3 be a non-singular M-cubic and let C_4 be a quartic which has three singular points of the type A_2. Suppose that C_4 almost maximally intersects the odd branch J_3 of C_3. Then the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is either as in Figures 41.1–41.5, or it is obtained from Figure 11.1 by a modification depicted in Figure 40. Moreover, all these arrangements are realisable.*

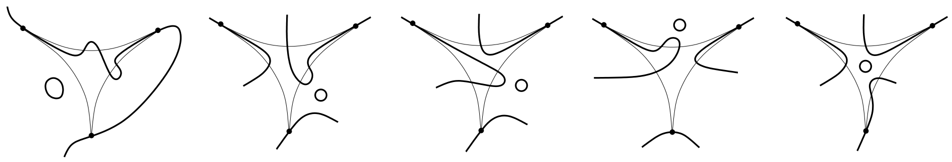


FIG. 41.1 FIG. 41.2 FIG. 41.3 FIG. 41.4 FIG. 41.5

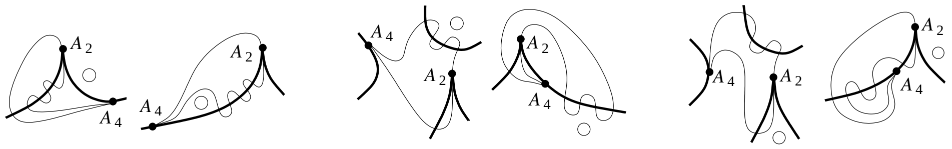
Proof. Apply the quadratic transformation $(x : y : z) \mapsto (xy : yz : zx)$ to the curves from Lemma 3.3. We choose he coordinates so that the lines L_1, L_2, L_3 are the coordinate axes. Then Figures 37.1–37.4 are transformed into Figure 41.1–41.4 respectively; a perturbation of Figure 9.2 (see Case (c) in Lemma 3.3) is transformed into Figure 41.5. It remains to note that the modification in Figure 38 does not change the isotopy type of the curve $C_3 \cup C_4$. □

3.3. Almost maximal arrangements of a cuspidal cubic and a two-components quartic with a point A_4. (Cp. Section 1.4.)

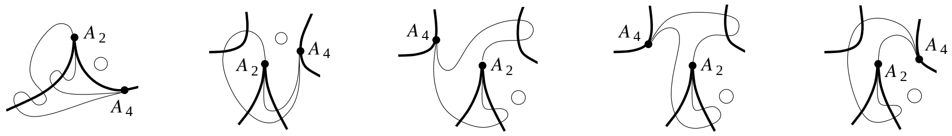
Lemma 3.5. *Let C be a non-singular M-cubic, E a conic, and L_1, L_2 two lines. Suppose that E meets the odd branch J of C at 6 points. Let us denote one of these points by p_1. Let p_2 be a point on J but not on E. Suppose that L_1 is tangent to E at p_1 and is tangent to J at q. Suppose also that L_2 is tangent to C at p_2, passes through q, and cuts E at two real points. Then either the arrangement of $C \cup E \cup L_1 \cup L_2$ on $\mathbf{R}P^2$ is obtained from Figure 32 by the rotation of L_2 clockwise around q till the first tangency with J, or it is obtained from Figures 15.1–15.13 by a perturbation of the conic near p_2. Moreover, all these arrangements are realisable.*

Proof. Combining Lemma 1.1 and Bezout's theorem for auxiliary lines, it is not difficult to exclude all the arrangements except those which are listed in Lemma 3.5 and those which would give Figures 34.1–34.2. by the rotation of L_2 around q. The unrealisability of the two latter cases is already proved in Lemma 3.1.

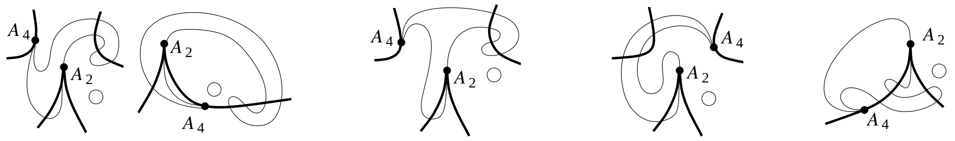
Proposition 3.6. *Let C_3 be a cuspidal cubic and C_4 a two-component quartic which has a singular point of the type A_4. Suppose that the singular branch of C_4 almost maximally intersects C_3 so that it has a maximal tangency at A_2 and an almost maximal tangency at A_4. Then either the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is as in Figures 42.1–42.16, or it is obtained from Figures 19.1–19.13 by a modification depicted in Figure 36. Moreover, all these arrangements are realisable.*



42.1 42.2 42.3 42.4 42.5 42.6



42.7 42.8 42.9 42.10 42.11



42.12 42.13 42.14 42.15 42.16

Proof. Apply the hyperbolism h_{p,q,L_1} to the curves from Lemma 3.5 where p is one of the intersection points of L_2 with E (let us denote the other one by p'). In the case when the complement of $C \cup E \cup L_1 \cup L_2$ contains o curvilinear triangle adjacent to the segment $[p'p_2]$, we evidently obtain the perturbations of the curve in Figures 19.1–19.13.

In the other cases, the correspondence between the figures is following (prime denotes a perturbation) $15.1' \to 42.1$–2; $15.2' \to 42.3$–4; $15.3' \to 42.5$–6; $15.4' \to 42.7$; $15.5' \to 19.5'$; $15.6' \to 19.6'$; $15.7' \to 42.8$; $15.8' \to 42.9$; $15.9' \to 42.10$; $15.10' \to 42.11$; $15.11' \to 42.12$–13; $15.12' \to 42.14$; $15.13' \to 42.15$; $32 \to 42.16$. $\square$

3.4. A cuspidal cubic and a two-component quartic with two points A_2: a non-maximal tangency at the cusp of the cubic. (Cp. Section 1.5.)

Lemma 3.7. *Let C -be a non-singular M-cubic and L_1, L_2, L_3 lines. Let us denote $p_1 = L_2 \cap L_3$, $p_2 = L_3 \cap L_1$, $p_3 = L_1 \cap L_2$, and let q_1, q_2, q_3 be points on L_1, L_2, L_3 respectively which differ from p_1, p_2, p_3. Let E be the conic passing through p_1, q_2, q_3 and touching L_1 at q_1. Suppose that E meets the odd branch J of C at 6 points including q_2, q_3. Suppose also that J passes through p_2, p_3, does not pass through q_1, and is tangent to L_2, L_3 at q_2, q_3. Then (up to swapping 2 and 3) either the arrangement of $C \cup E \cup L_1 \cup L_2 \cup L_3$ on $\mathbf{R}P^2$ is as in Figures 43.1–43.3, or it is obtained from Figures 20.1–20.3 by a perturbation of the cubic near q_1. Moreover, all these arrangements are realisable.*

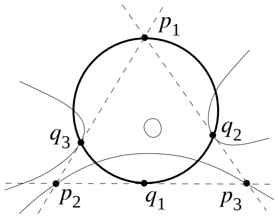


FIG. 43.1

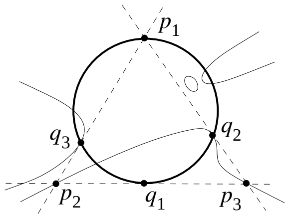


FIG. 43.2

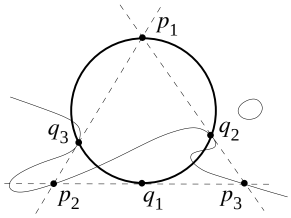


FIG. 43.3

Proof. Using Polotovskii's classification [**10**] of mutual arrangements of a conic and a cubic and Bezout's theorem applied to an auxiliary line, it is easy to check that all arrangements are impossible except those which are listed in this lemma and those which are depicted in Figure 44.1 and in Figure 44.2. To exclude the two latter cases, we shall apply the Murasugi-Tristram inequality as we did it in the proofs of Propositions 2.3, 2.4 and Lemma 3.1. The arrangement of $C \cup E \cup L_1 \cup L_2 \cup L_3$ with respect to the pencil of lines through a point inside the oval of the cubic has the form

$$(\times_5\times_6\times_5)(\times_5^2\times_6\times_5)\times_4^2(\times_2^2)\supset_3(\times_2\times_3\times_2)\subset_4\times_3\times_4(\times_5\times_6\times_5)\times_6(\times_5^2\times_6\times_5) \quad [\text{f.44.1}],$$
$$(\times_5\times_6\times_5)(\times_5^2\times_6\times_5)(\times_2^2)\supset_3(\times_2\times_3\times_2)(\times_3\times_4\times_3)\times_4\subset_4\times_5^3\times_4(\times_3^2\times_4\times_3) \quad [\text{f.44.2}].$$

the subwords in the parentheses correspond to the points $p_1, q_2, q_1, p_3, p_2, q_3$ in this order. The rest of the proof is as in [**4**] or [**7**].

Now let us show that the arrangements in Figures 43.1–43.3 are realisable.

The arrangement in Figure 43.1. Let us fix affine coordinates x, y and set $C = \{y^2 = x(x+1)(x+2)\}$ and $p_1 = (x_0, 0)$, $x_0 > 0$. Let L_2, L_3 be tangents to J passing through p_1. Let us define q_2, q_3, p_2, p_3, L_1 according to the conditions of the lemma. Let q_1 be the intersection point of L_1 and $\{y = 0\}$. Then the hyperbola E passing through p_1, q_2, q_3 and touching L_1 at q_1 is arranged in the required way.

The arrangement in Figure 43.2. One can check that the points $p_1 = (3{:}6{:}4)$, $p_2 = (36{:}75{:}64)$, $p_3 = (0{:}0{:}1)$, $q_1 = (25{:}12{:}108)$, $q_2 = (1{:}2{:}1)$, $q_3 = (2{:}5{:}8)$, the lines $L_1 = \{12y = 25x\}$, $L_2 = \{y = 2x\}$, $L_3 = \{16y = 28x + 3z\}$, the cubic $C = \{y^2z = x(x+1)^2\}$ and the conic $E = \{12332x^2 - 9336xy + 1584y^2 - 1121xz + 564yz - 3z^2 = 0\}$ are arranged in the required way (the cubic C has an ordinary double point with non-real tangents at $(-1{:}0{:}1)$; a perturbation of this point provides an oval).

The arrangement in Figure 43.3. Let us fix C, L_1, L_2, and L_3 as in Figure 43.3. Then, if we choose a point q_1 on the segment $[p_2p_3]$ sufficiently close to p_3, then the conic E, passing through p_1, q_2, q_3 and and touching L_1 at q_1 is arranged in the required way. $\square$

Proposition 3.8. *Let C_3 be a cuspidal cubic and C_4 a two-component quartic which has two singular points of the type A_2. Suppose that the singular branch of C_4 almost maximally intersects the curve C_3 so that it has a maximal tangency at the both cusps of C_4 and an almost maximal tangency at the cusp of C_3. Then either the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is as in Figure 45.1–45.2, or it is obtained from Figure 22.1–22.3 by a modification depicted in Figure 40. Moreover, all these arrangements are realisable.*

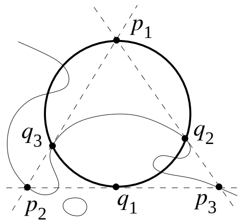


FIG. 44.1

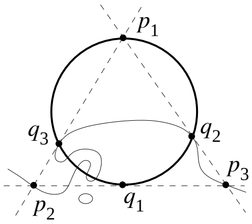


FIG. 44.2

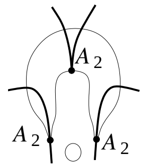


FIG. 45.1

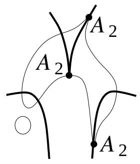


FIG. 45.2

Proof. Apply the quadratic transformation $(x : y : z) \mapsto (xy : yz : zx)$ to the curves from Lemma 3.7. The coordinates are chosen so that the lines L_1, L_2, L_3 are the coordinate axes. Then Figure 43.3 is transformed into Figure 22.2 modified as in Figure 40. $\square$

3.5. A cuspidal cubic and a two-component quartic with two points A_2: a non-maximal tangency at one of the cusps of the quartic.

(Cp. Section 1.5, 3.4.)

Lemma 3.9. *Let C be a non-singular M-cubic and L_1, L_2, L_3 lines. Let us denote $p_1 = L_2 \cap L_3$, $p_2 = L_3 \cap L_1$, $p_3 = L_1 \cap L_2$, and let q_1, q_2, q_3 be points on L_1, L_2, L_3 respectively which differ from p_1, p_2, p_3. Let E be the conic passing through p_1, q_2, q_3 and touching L_1 at q_1. Suppose that E meets the odd branch J of C at 6 points including q_1, q_2. Suppose also that J passes through p_2, p_3, is tangent to L_2 at q_2, and is tangent to L_3 at a point different from p_1, p_2, q_3. Then either the arrangement of $C \cup E \cup L_1 \cup L_2 \cup L_3$ on $\mathbf{R}P^2$ is as in Figure 46.1–46.4, or it is obtained from Figure 20.1–20.3 by a perturbation of the cubic near q_3 or q_2. Moreover, all these arrangements are realisable, except maybe Figures 46.3–46.4.*

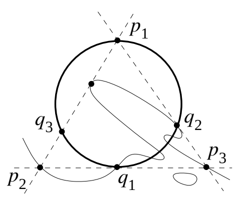


FIG. 46.1

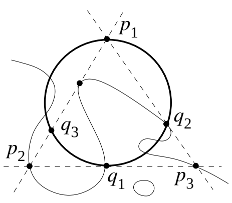


FIG. 46.2

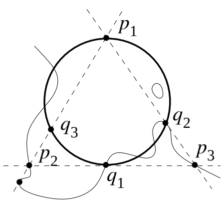


FIG. 46.3

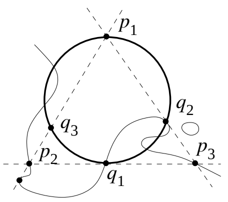


FIG. 46.4

Proof. Using Polotovskii's classification [**10**] of mutual arrangements of a conic and a cubic and Bezout's theorem applied to an auxiliary line, it is easy to check that all arrangements are impossible except those which are listed in this lemma and six more arrangements to exclude which we shall apply the Murasugi-Tristram inequality as we did it in the proofs of Propositions 2.3, 2.4, Lemma 3.1, and 3.7. The arrangements of $C \cup E \cup L_1 \cup L_2 \cup L_3$ with respect to the pencil of lines through a point inside the oval of the cubic have the form

$$(\times_5\times_6\times_5)\times_4(\times_5^2)\times_4^3\times_5(\times_5^2\times_4\times_5)(\times_6^2\times_5\times_6)(\times_4\times_5\times_4);$$
$$\times_5\times_4(\times_2\times_3\times_2)\times_4(\times_3^2\times_2\times_3)\times_3^2(\times_4^2)(\times_5\times_6\times_5)(\times_4^2\times_3\times_4)\supset_5(\times_2\times_3\times_2)\subset_4;$$
$$(\times_2\times_3\times_2)\times_4(\times_3^2\times_2\times_3)\times_4^2(\times_5\times_6\times_5)(\times_4^2\times_5\times_4)\times_4\times_3^2\times_4\supset_3(\times_2\times_3\times_2)\subset_3;$$
$$(\times_3\times_4\times_3)(\times_3^2\times_2\times_3)\times_3^4(\times_4^2\times_3\times_4)(\times_4\times_5\times_4)(\times_4^2)\times_3(\times_2\times_3\times_2);$$
$$(\times_5\times_6\times_5)\times_4(\times_3^2)(\times_2\times_3\times_2)(\times_4^2\times_3\times_4)\times_4^2(\times_5^2\times_6\times_5)\times_5^2\supset_4(\times_3\times_4\times_3)\subset_4;$$
$$(\times_5\times_6\times_5)\times_4(\times_3^2)(\times_2\times_3\times_2)(\times_4^2\times_3\times_4)(\times_5^2\times_6\times_5)\times_5\times_4^3\supset_5(\times_3\times_4\times_3)\subset_4.$$

The subwords in the parentheses correspond to the points p_1, p_2, p_3, q_1, q_2, and the tangency point of L_3 and C (not necessarily in this order). The rest of the proof is as in [**4**] or [**7**].

Now, let us show that the arrangements in Figures 46.1–46.2 are realisable.

The arrangement in Figure 46.1. Let us consider the conics E, F and the lines L_1, L_2, L_3, arranged as in Figure 47.1 ($p_1 = E \cap L_2 \cap L_3$; $p_2 = L_1 \cap L_3$; $p_3 = L_1 \cap L_2$; $q_2 = E \cap F \cap L_2$; E touches L_1 at q_1; F touches L_j at q_j for $j = 1, 2$). Let us set $C = \{l_1 f + \varepsilon l_3 l_4 l_5\}$, where $|\varepsilon| \ll 1$, $F = \{f = 0\}$, $L_j = \{l_j = 0\}$, and $L_3 = (q_1 q_2)$, $L_2 = (q_2 q_3)$.

The arrangement in Figure 46.2. See Figure 47.2. $\square$

Proposition 3.10. *Let C_3 be a cuspidal cubic and C_4 a two-component quartic which has two singular points of the types A_2. Suppose that the singular branch of C_4 almost maximally intersects C_3 so that it has a maximal tangency at the cusp of C_3 and at one of the cusps of C_4, and it has a non-maximal tangency at the other cusp of C_4. Then either the arrangement of $C_3 \cup C_4$ on $\mathbf{R}P^2$ is as in Figures 48.1–48.2, or it is obtained from Figures 22.1–22.3 by a modification depicted in Figure 40. Moreover, all these arrangements are realisable,.*

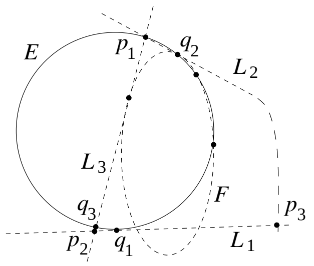


FIG. 47.1

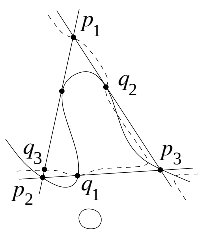


FIG. 47.2

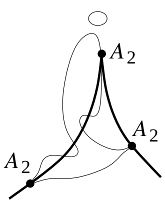


FIG. 48.1

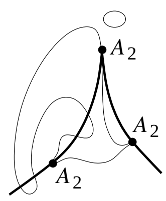


FIG. 48.2

Proof. Apply the quadratic transformation $(x : y : z) \mapsto (xy : yz : zx)$ to the curves from Lemma 3.9. The coordinates are chosen so that the lines L_1, L_2, L_3 are the coordinate axes. Then Figure s46.3–46.4 are transformed into Figures 22.1–22.2 modified as in Figure 40. $\square$

§4. Other constructions

4.1. A singular quartic, a conic, and a line.

Proposition 4.1. *(a). There exist a quartic with a singular point A_6 arranged with respect to a line L and a conic C as in Figure 49.*

(b). There exist a quartic with singular points A_4 and A_2 arranged with respect to a line and a conic as in Figure 50.

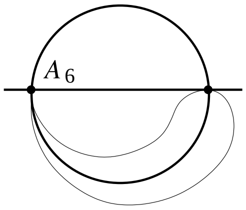


FIG. 49

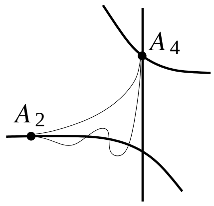


FIG. 50

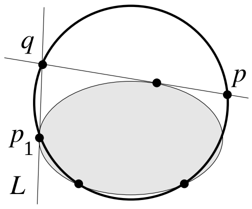


FIG. 51

Proof. (a). $f_{C,p}$ transforms the quartic into a circle and it transforms L and C into two tangents.

(b). $h_{p,q,L}$ transforms Figure 51 into Figure 50. □

4.2. M-cubic obtained by a perturbation of a simple and a double line, and an M-quartic. Let O_4 be an oval of an M-quartic C_4. Suppose that each of lines $L_1 = \{l_1 = 0\}$, $L_2 = \{l_2 = 0\}$ meets O_4 at four points. Up to isotopy, all such arrangements are listed in Figures 52.1–52.11 (this easily follows from Polotovskii's classification [**9**, **10**] of mutual arrangements of a quartic and a conic).

The first construction (see Figure 53). Let us fix a point $p \in L_1$ not on C_4. Let $\{l_3 = 0\}$ and $\{l_4 = 0\}$ cutting L_1 near p. Set $C_2 = \{c_2 = 0\}$ where $c_2 = l_1 l_2 + \varepsilon l_3 l_4$ and $|\varepsilon| \ll 1$, and let $C_3 = \{c_2 l_1 + \delta l_2^3\}$ where $|\delta| \ll |\varepsilon|$. According to a choice of the parameter ε, we obtain two a priori different arrangements of C_4 and C_3.

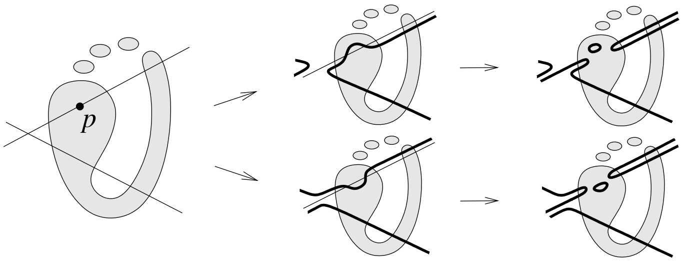


FIG. 53

The second construction (see Figure 54). Among the connected components of $\mathbf{R}P^2 \setminus (C_4 \cup L_1 \cup L_2)$, let us choose a digon D bounded by an arc of O_4 and a segment of L_1. It is easy to check that in all the cases, one can choose another intersection point of O_4 and L_1 so that a rotation of L_1 around this point makes D to degenerate into a tangency point (let us denote it by p) and all other intersections remain real.

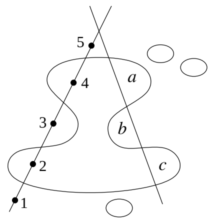


FIG. 52.1

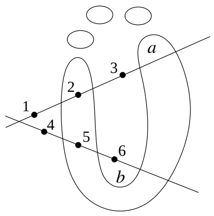


FIG. 52.2

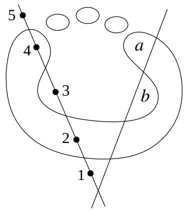


FIG. 52.3

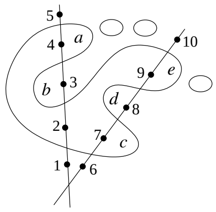


FIG. 52.4

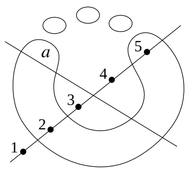


FIG. 52.5

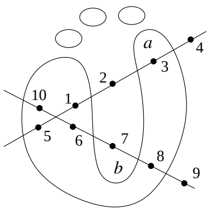


FIG. 52.6

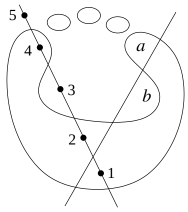


FIG. 52.7

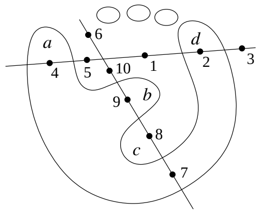


FIG. 52.8

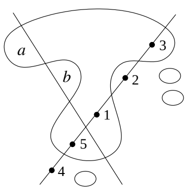


FIG. 52.9

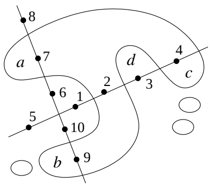


FIG. 52.10

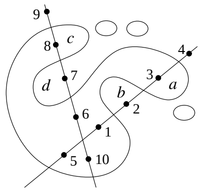


FIG. 52.11

Let $\{l_0 = 0\}$ a line cutting L_1 at p. Let $C_2 = \{c_2 = 0\}$ where $c_2 = l_1 l_2 + \varepsilon l_0^2$ and $|\varepsilon| \ll 1$, and let $C_3' = \{c_2 l_1 + \delta l_0^3\}$ where $|\delta| \ll |\varepsilon|$. According to choices of the signs of ε and δ, we obtain four a priori different arrangements of C_4 and C_3'. The curve C_3' has a singularity of the type A_2 (ordinary cusp) at p and it maximally intersects O_4. Let us perturb this singularity as in the right hand side of Figure 26. Let us denote the obtained M-cubic by C_3. One can apply this modification in two different ways because of the reflection with respect to O_4. We shall always choose that way when all the four new intersections of O_4 and J_3 lye on J_3 in the same order as on O_4 (the other way reduces to the first construction).

4.3. One more construction. Let us consider a conic C and three lines L_0, L_1, L_2 ($L_i = \{l_i = 0\}$) arranged with respect to the coordinate axes $x = 0$, $y = 0$, $z = 0$ as in Figure 55.1. Then for $|\delta| \ll |\varepsilon| \ll 1$, the conic $E = \{l_0^2 + \varepsilon l_1 l_2 + \delta l_1^2 = 0\}$ is arranged as it is depicted by a dashed line in Figure 55.1. Applying the transformation $(x : y : z) \mapsto (yz : zx : xy)$, we obtain Figure 55.2 which provides (by successive perturbations of singularities) Figure 55.3 and Figure 55.4.

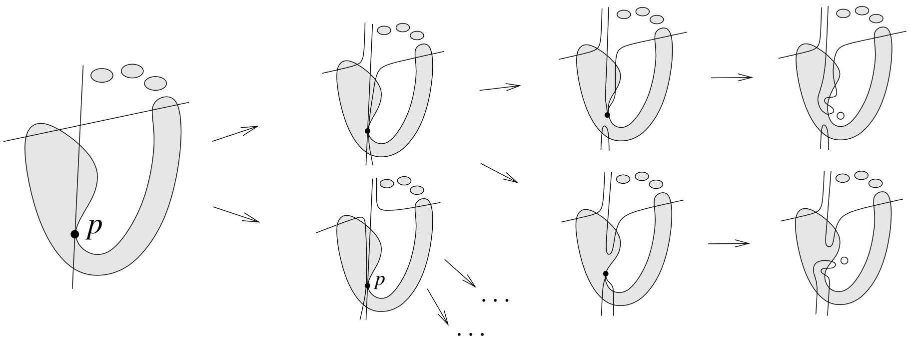
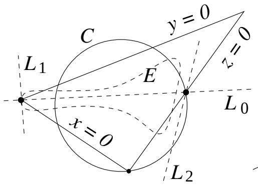
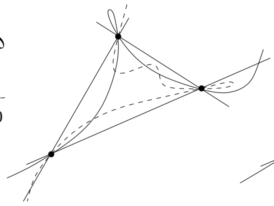
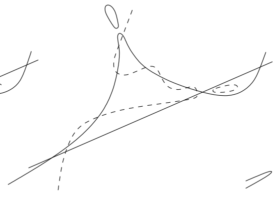


FIG. 54







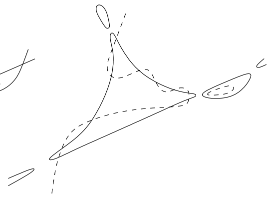


FIG. 55.1 FIG. 55.2 FIG. 55.3 FIG. 55.4

§5. The list of all the constructed arrangements of an M-cubic and an M-quartic with maximally intersecting an oval and the odd branch

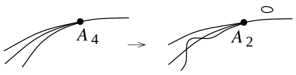
In this section, we present the list of all the mutual arrangements of an M-cubic and an M-quartic with maximally intersecting an oval and the odd branch which are constructed in Sections 4.2–4.3, and those which are obtained by perturbing singular curves constructed in §§1–3 and in Section 4.1. This list includes all the arrangements constructed by other methods in the papers [**11**], [**2**], [**3**] (note, that the arrangements no. 5, 10, and 11 in the paper [**3**] are depicted erroneously).

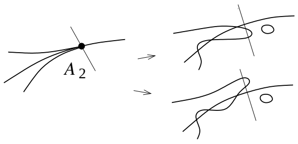
5.1. Applied perturbations. In the case of the arrangements in Figures 27.1–27.33, we apply the perturbations depicted in Figure 31.

In the case of a maximal tangency of a smooth branch with a branch having an irreducible double point, we apply (successively) the arrangements depicted in Figure 26 and Figure 56.1 (see details in [**7**]).

In the case of the arrangements in Figures 49 and 50 we apply the perturbations depicted in Figure 26 and Figure 56.1 followed by the perturbation in Figure 56.2.

In the case of a non-maximal tangency of a smooth branch with a branch having the singularity A_4, we apply the perturbations depicted in Figure 56.3.





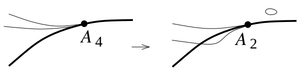


FIG. 56.1 FIG. 56.2 FIG. 56.3

5.2. Encoding of mutual arrangements of the intersecting branches. To denote the isotopy type of a mutual arrangement of intersecting branches J_3 and O_4 (the odd branch of the cubic and an oval of the quartic respectively) we use the encoding proposed by Polotovskii. Namely, let Γ_∞ be a pseudo-line (i.e. a simple closed curve in $\mathbf{R}P^2$ which is homologically nontrivial), disjoint from O_4 and cutting J_3 at a minimal possible number of points (in all the considered cases, this number is equal to 1 or 3). We shall call the points of $\Gamma_\infty \cap J_3$ *passages through the infinity.*

Let us number the points of $O_4 \cap J_3$ by digits[4] $1, \dots, 9, a, b, c$ in their order along O_4 so that the point 1 is an endpoint of a connected component of $J_3 \setminus (O_4 \cap J_3)$ crossing Γ_∞, but the point 2 is not. We shall encode the arrangement of $O_4 \cup J_3$ on $\mathbf{R}P^2$ by the word composed by digits $1, \dots, c$ in their order along J_3. Among all the words encoding the same isotopy type (if there are no symmetries, then the number of such words is twice the number of passages through the infinity), we shall always choose the word which is minimal in the lexicographic order. For the reader's convenience, we shall denote the passages through the infinity by "/".

First we list the arrangements which have one passage through the infinity and then those which have three passages. The both lists are ordered lexicographically (ignoring "/"). The arrangements with isotopic $J_3 \cup O_4$ are ordered arbitrarily. The points $1, \dots, c$ are not indicated in the pictures but we always assume that they are located clockwise along O_4, the point 1 being the leftmost. In the case of one passage through the infinity, we do not depict the free ovals "at the infinity" (i.e. in the connected component of the complement of $J_3 \cup O_4$ whose closure is non-orientable).

5.3. Encoding of the constructions. Under each arrangement, we refer to its construction(s). This is either a reference to the figure with the perturbed curve, or a reference to the paper where the curve is constructed,[5] or one of the expressions 2+3, 2+4, x^y whose meaning is as follows.

2+3. (see [**11**]). $C_4 = \{c_2^2 = \varepsilon f\}$ where $\{c_2 = 0\}$ is a conic cutting J_3 at six points.

2+4. The cubic C_3 is obtained as a small perturbation of $C_2 \cup L$ where C_2 is a conic meeting O_4 at eight points, and L is the line chosen as it was indicated in [**11**]. These constructions were done by G.M. Polotovskii.

x^y where $x = 1, \dots, 11$, $y = 1, 2, \dots$. The first construction from Section 4.2 where the point denoted in Figure 52.x by the number y is chosen as the point p. For example, 2^2 denotes the construction depicted in Figure 53.

x^y where $x = 1, \dots, 11$, $y = a, b, \dots$. The second construction from Section 4.2 where the digon denoted in Figure 52.x by the letter y is chosen as the digon D. For example, 8^c denotes the construction depicted in Figure 54.

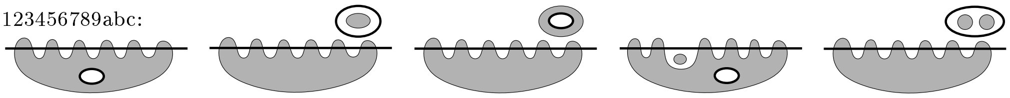
---

[4] In the computer programming, the characters a, b, c, d, e, f usually denote the hexadecimal digits $10, \dots, 15$.

[5] [**3**;n] Figure 5.n [**3**].

5.4. The list.

123456789abc:



2.1; 19.1

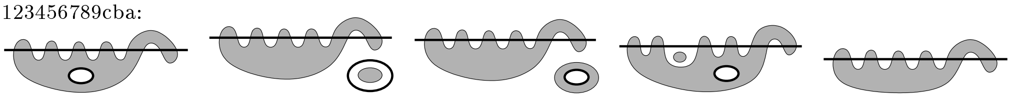
24.1; 25.1; 27.1 [**11**; 8.c]

25.2; 27.2

2.2; 8.1; 19.2,3; 22.1; 42.2

49; [**11**; 7.1]

123456789cba:



2.1; 19.1

24.1; 25.1

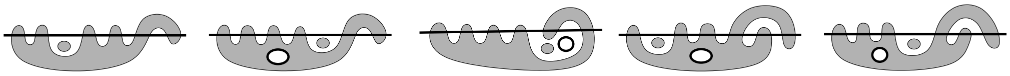
1^c; 25.2

1^a; 19.12

2.3; 19.1

12345678ba9c:

1234567abc98:



8.8; 19.2; 22.1 42.2

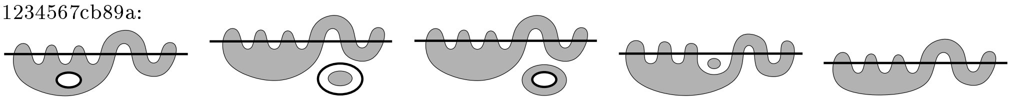
8.1; 19.11; 27.8; 22.1; 42.2

2.5; 19.3

1^2; 19.8 [**3**;3]

1^4; 19.9 [**3**;7]

1234567cb89a:



2.1; 19.1

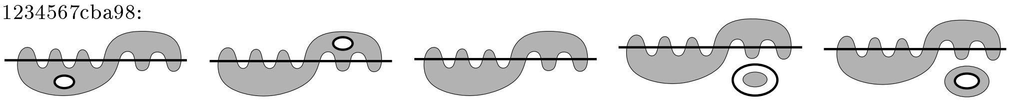
25.1; 27.1; 49

25.2; 27.2

8.5

2.3; 8.4; 19.4

1234567cba98:



2.1; 19.1

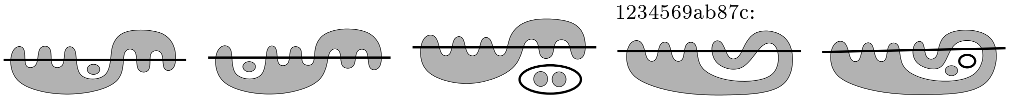
2.1; 19.4

8.4

25.1

25.2

1234569ab87c:



1^b; 8.5; 19.9

1^b; 19.8

50

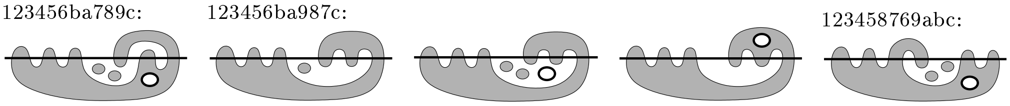
8.3; 19.5

8.6

123456ba789c:

123456ba987c:

123458769abc:



4^2; 19.10; [**11**; 7.2]

8.3

4^b; 8.6; 19.10

19.5

2.2; 8.1; 19.2,3 22.1

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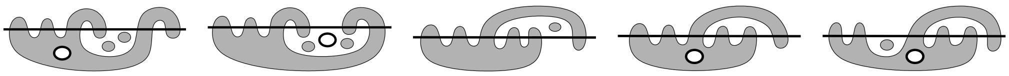
12345876ba9c:

1234589abc76:



4^c; 8.1; 19.11 22.1

2.5; 19.3

8.2; 19.8

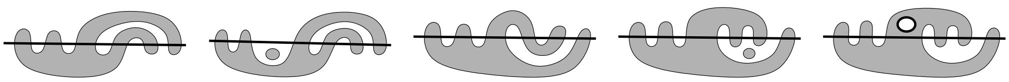
1^c9^a; 19.6

1^a; 42.16

1234589cba76:

12345a9678bc:

12345a9876bc:



1^19^4; 8.2; 19.6,8; 27.21; [**11**;10.1], [**2**;2]

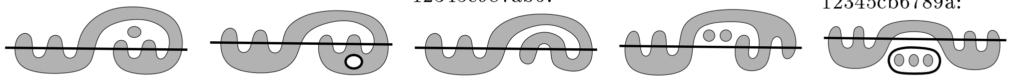
1^5; 42.16; [**2**;4]

8.2; 19.6; 41.1

8.2; 41.1

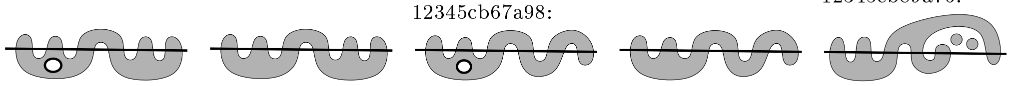
19.6

12345c789ab6: | 12345c987ab6: | 12345c9ab876: | 12345cb6789a:



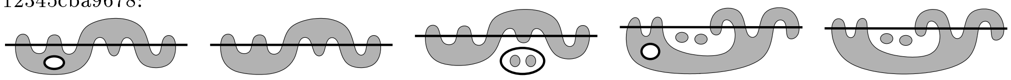
8.3,5; 19.9; | $4^a 11^c$; 19.5 | $4^5 11^4$; 8.3,5; 19.5,9; 27.11 | 4^d; 8.5; 19.9 | 49

12345cb67a98: | 12345cb89a76:



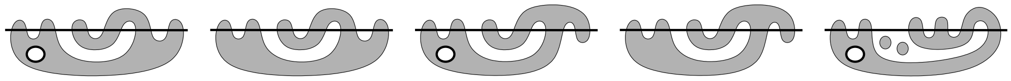
2.1; 19.4 | 8.4; 24.4 | 2^a; 2.1; 27.3; 19.4 | 2.3; 8.4; 24.4; 27.29; 19.4 | 4^d; 19.8

12345cba9678: | 123476589cba:



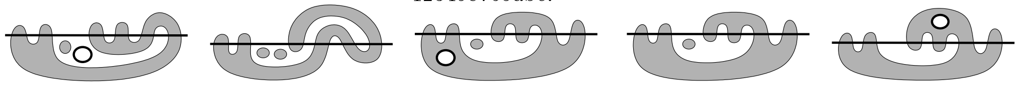
2.1; 19.4 | 2.3; 8.4; 19.1 | 49 | 4^e; 19.12 | 8.8; 19.2; 22.1

123478965abc: | 123478965cba: | 1234789ab65c:



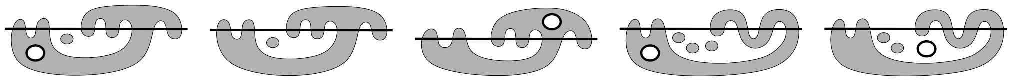
10^a; 2.2; 8.1; 19.2,3; 22.2 | 11.1; 22.3; 35.3; 42.7 | 10^b; 19.12; 27.26 | 7^5; 8.8; 11.1; 22.2,3; 19.2; 27.13,20 | 4^a; 2.2; 19.13

1234985cb67a: | 123498765abc:



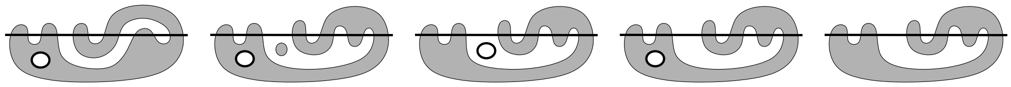
8.7; 35.8 | 4^1 | 2.2; 8.1; 19.2,3 22.2 | 11.1; 35.3 | 22.3; 42.7

123498765cba: | 1234987ab65c:



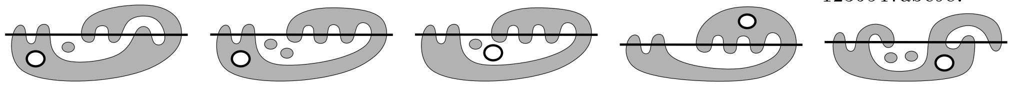
19.12 | 8.8;11.1;19.2; 22.2; 27.27 | 7^a; 22.3; 27.5 | 3^a; 2.2; 19.13 | 8.7; 35.8

12349ab8567c: | 12349ab8765c:



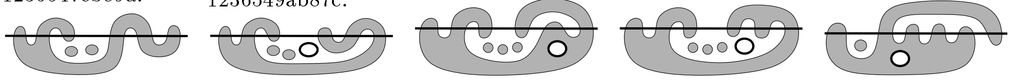
$2^5 8^a$; 27.22; 35.1; 42.4,6; [**11**; 7.4] | 2.2; 19.13; 27.28 | 2^b; 27.4; 8.7; 42.4 | 8^d | 42.1

1234ba98567c: | 1234ba98765c: | 1236547abc98:



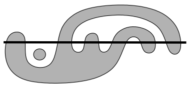
35.1;42.4,6;[**11**; 7.11] | 2.2; 19.13 | 8.7; 42.4 | 42.1 | 4^7; 19.9

1236547cb89a: | 1236549ab87c: | 123654ba789c: | 123654ba987c: | 1236789abc54:



8.5 | 8.6 | 3^2; 19.10; [**11**;7.12] | 3^b; 8.6; 19.10 | 27.8

1236789cba54:



27.15

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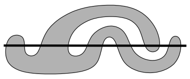
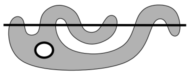
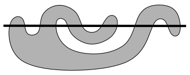


$6^9 8^6$; [**2**;1]

123874569cba:



8.1; 19.11
22.2



2+4; $10^8 11^9$; 11.1; 22.3;
27.23,24; 35.3; 42.7; [**3**;9]

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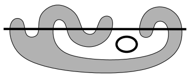
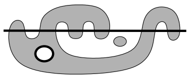
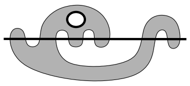


2.5; 19.3

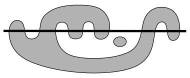
123876549cba:



8.1; 19.11
22.2



$10^b 11^a$; 27.3,6
22.3; 42.7



11.1; 27.29,30;
35.3

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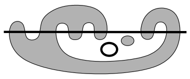
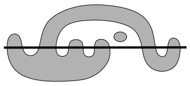
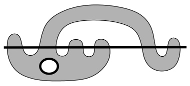


2.5; 19.3

12389abc7456:

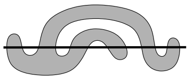


8.8; 11.1;
19.11,12; 22.2



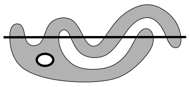
$9^a 10^{a,c} 11^c$
22.3;

12389cba7456:



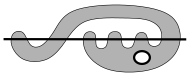
$9^4 10^{5,8} 11^4$; 8.8; 11.1;
19.11,12; 22.2,3; 27.9; [**3**;4]

123a9458bc76:



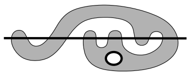
2^2; [**3**;12]

123c56789ab4:

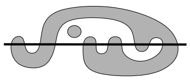


8^a; 27.8; 42.1

123c5678ba94:

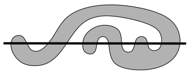


$6^a 8^d$



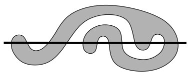
42.6

123c76589ab4:



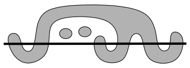
2+4; 8^3; 27.15

123c7658ba94:



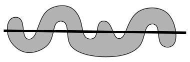
$6^4 8^3$; 27.17;
42.1,6; [**3**;2]

123c9ab87456:



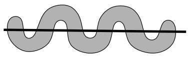
4^8; 8.8;
19.11,12; 22.1

123cb4567a98:



2.3; 8.4;
19.1,4

123cb45a9678:



2+3; 2^3; 2.3; 8.4;
19.4; 24.4; 27.23; 49

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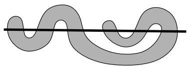
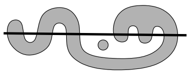
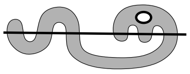


2+3; $3^5 5^1 7^5$; 8.3,5;
19.5; 27.18,24

123cb498765a:



8.3,5; 27.30;



$3^a 5^a 7^a$; 19.5; 27.6;

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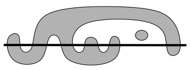
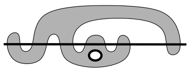


8.2; 27.29,30
41.1



19.6; 27.3,6

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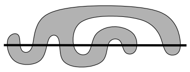

8.2; 19.6;
27.23,24; 41.1

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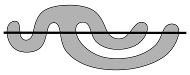

$4^{10} 11^9$; 8.2;
19.6; 27.20

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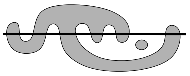
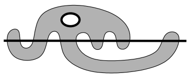


8.2; 27.27



$4^e 11^a$; 19.6; 27.5

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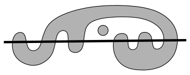
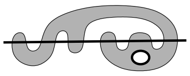


8.3; 27.27



19.5; 27.5

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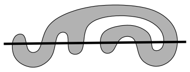

8.3; 19.5; 27.20

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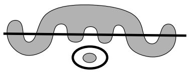
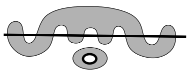
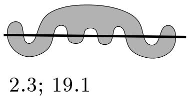


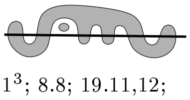
24.1; 49



55.4

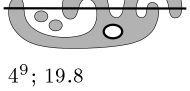


2.3; 19.1



1^3; 8.8; 19.11,12;
22.1; 27.15; 42.2

1254367abc98:



4^9; 19.8

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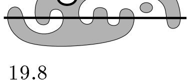

19.8

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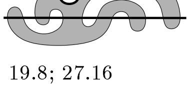
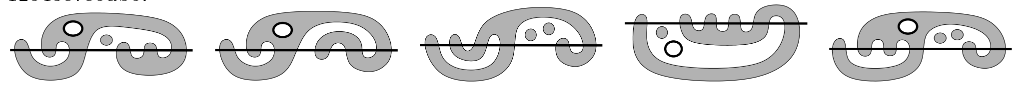


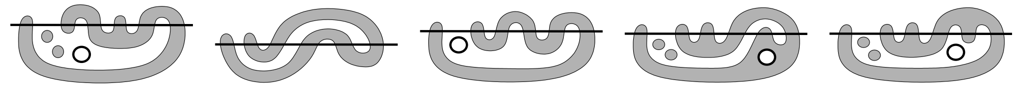
19.8; 27.16

12543c789ab6: | 12543c987ab6: | 1256743c9ab8: | 1256789ab43c: | 1276543c9ab8:



19.9 | 19.9; 27.32 | 4^6; 42.16 | 2.4 | 4^c; 42.16

1276589ab43c: | 1278963cb45a: | 1278965ab43c: | 12789ab6345c: | 12789ab6543c:



2.4 | 2+3; $2^1 2^4 5^1 6^4 6^9$; 27.25 | 2+3; 2.4; [**11**; 8.b] | 4^4; 19.7; [**11**; 7.3] | 4^b; 2.4; 19.7

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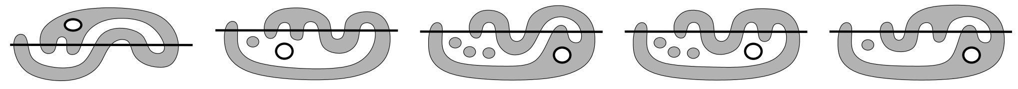
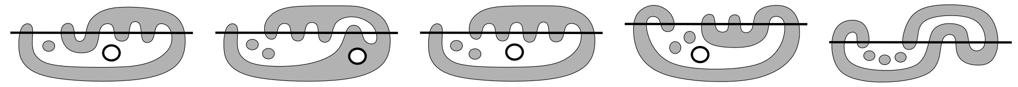


$2^a 5^a 6^a$ | 2.4 | 3^4; 19.7; [**11**; 7.15] | 3^b; 2.4; 19.7 | 19.7;27.14;[**11**;7.13]

129ab876543c: | 12ba9876345c: | 12ba9876543c: | 1432789ab65c: | 1432985cb67a:



2.4; 19.7; 27.7 | 19.7; [**11**;7.19] | 2.4; 19.7 | 4^3; 2.5; 8.7; 19.13; 35.8; [**11**;7.5] | 2+3; 3^1

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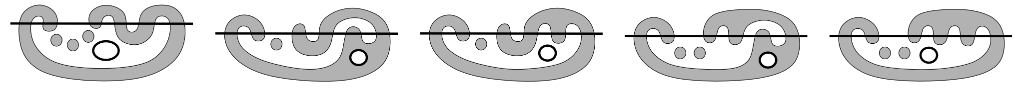
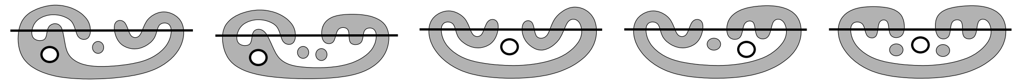


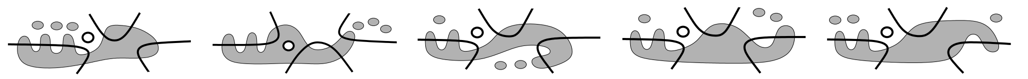
2+3; 3^3; 2.5; 8.7; 19.13; 35.8; [**11**;7.9] | 27.31; 35.1; 42.13; [**11**;7.10] | 2.5; 8.7; 19.13; 27.12; 42.13; [**11**;7.7] | 35.1; 42.13; [**11**;7.20] | 2.5; 8.7; 19.13; 42.13; [**11**;7.16]

1456329ab87c: | 145632ba987c: | 1652349ab87c: | 165234ba987c: | 165432ba987c:



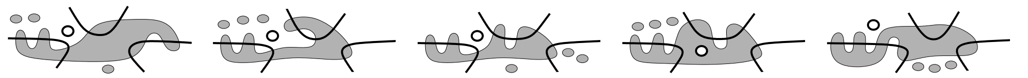
19.10; 27.33; [**11**;7.14] | 19.10; [**11**;7.18] | 2+3; 2^6; 8.6; 24.3; 27.19; [**11**;7.5] | 8.6; 24.3; 19.10; 27.10; [**11**;7.8] | 8.6; 24.3; 19.10; [**11**;7.17]

1234567c/98/ba: | 123456/b87c/9a: | 12345c/76/9ab8: | 12345c/76/b89a: | 12345c/76/ba98:



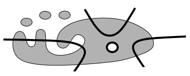
7^b, 27.7 | 8^b | $4^d 11^b$; 27.10,11; 35.4; 41.2; 42.10 | $4^d 11^b$; 27.7; 42.9 | $1^b 9^b$; 35.4; 42.9

12345c/9678/ba: | 12345c/9876/ba: | 12345cba/76/98:



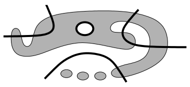
$1^b 9^b 10^d$; 41.2; 42.10; | 2+4; $7^3 3^b 7^b$; 35.5; 27.10–12; 41.4; 42.11 | $4^b 10^d 11^d$; 27.4 35.5; 41.4; 42.11; | 7^a; 27.28 | 8^c

1234765c/98/ba:



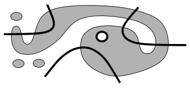
7^4; 27.14

1234/78/5abc96:



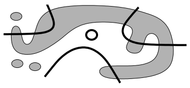
2+4; $8^9 8^a$

1234/78/5c9ab6:

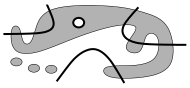


2+4;$4^4 11^3$;42.8;

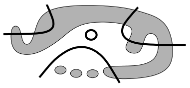
1234/78/5cba96:



$4^b 11^d$; 42.8;

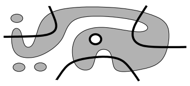


$6^a 8^d$

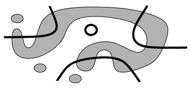


8^c

1234/789a/5cb6:

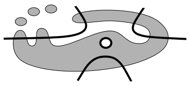


$4^a 11^c$; 42.15



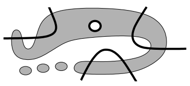
24.2; 35.6

123478/bc/965a:

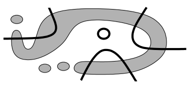


$5^2 7^1 8^b$

1234/7a98/5cb6:



$2^a 5^a 6^a$; 27.26; 35.6; 42.15



$4^3 11^2$; 27.13 42.15

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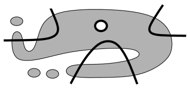
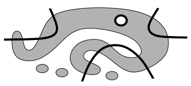
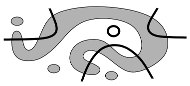


$4^2 10^2 11^1$; 27.22; 42.11; [**2**;6]

1234/7cb89a/56:

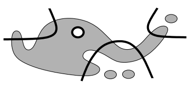


$10^c 11^c$



35.5; 41.4;

1234/965abc/78:



$9^a 10^a 10^c$

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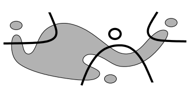
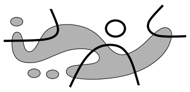
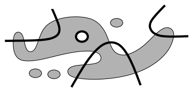


41.3; 45.2

1234/965cba/78:

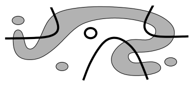


27.13; 41.3; 45.2



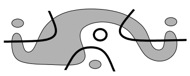
$10^b 11^a$; 27.26

1234/987a/5cb6:



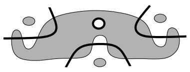
24.2

1234/9ab87c/56:



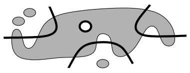
35.4, 41.2

1234/9abc/5678:

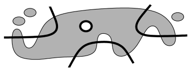


35.2; 11.5; 42.3,5

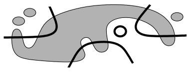
1234/9abc/5876:



1^a; 27.22; 35.2; 11.5; 42.3,5,12

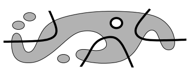


$1^c 9^a$; 42.14



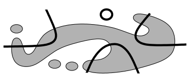
35.7; 42.3,5

1234/9cba/5876:

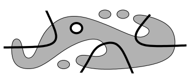


4^c; 27.31; 35.2; 11.5; 42.12

1234/9cba/7658:

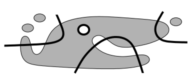


27.4; 35.7 42.3

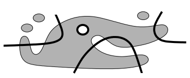


$4^e 11^a$; 27.28; 42.14

1234/ba789c/56:

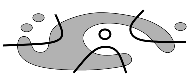


$1^4 9^1 10^2$; 42.10; [**2**;5]



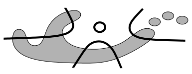
$1^2 9^3$; 42.9;

1234/ba9c/5876:



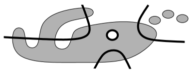
$1^3 9^2$; 27.17; 35.7; 42.5,12,14

123654/b87c/9a:



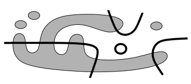
2+4; 8^8

12367854/bc/9a:



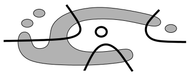
8^7

1236789c/54/ba:



11^b; 27.21; [**3**;8]

123874/ba9c/56:



$9^2 10^{3,6}$; 27.25; 41.3; 45.2

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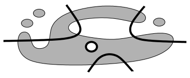
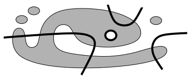


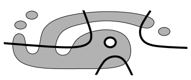
$10^7 11^{10}$; [**2**;3]

1238769c/54/ba:



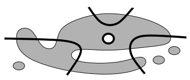
$10^9 11^8$; 27.16

12389a74/bc/56:



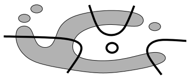
2+3; $9^3 10^4 10^7$; [**3**; 1]

12389c/54/ba76:



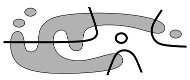
$9^5 10^9$; [**2**;10]

12389c/5674/ba:



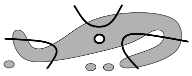
$10^{10} 11^7$; 27.9; 41.3; 45.2

123a9874/bc/56:



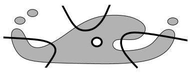
$9^b 10^d$

123c/54/78ba96:



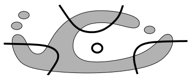
27.17

123c/54/b8769a:



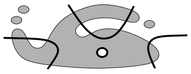
$4^9 11^8$; 27.14
42.9;

123c/5674/b89a:



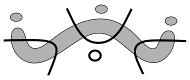
2+4;$4^8 11^7$;27.12;
35.7;42.12,14;

123c/569874/ba:



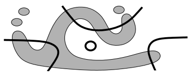
$4^7 11^6$; 27.32,33;
42.10; [**2**;7]

123c/5ab4/9678:



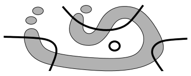
24.2

123c/765894/ba:



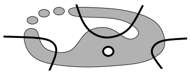
27.18,19; 35.4
41.2

123c/945876/ba:



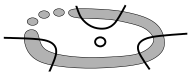
$10^3 11^2$; 27.18,19;
35.5; 41.4

123c/965478/ba:



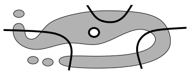
2+4; $3^2 7^2$;
27.31–33; 42.11;

123c/9678/b45a:



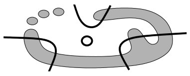
2+3;$3^3 2^3 5^4 6^2 7^3$;
$3^a 7^a$;27.9;35.6;42.15

123c98/54/7ab6:



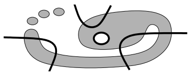
11^5; [**2**;8]

123c/98/b4567a:



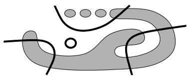
2+4; $6^2 8^1 3^b 7^b$;
27.21; 42.8;

123c/98/b4765a:



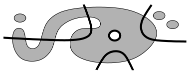
2+4; $3^4 5^5 7^4$;
27.16; 42.8;

123c/98/b6547a:



2+4; 8^5

123c9ab4/78/56:



2+4; $10^4 11^3$

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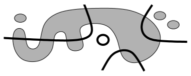
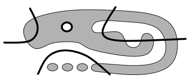


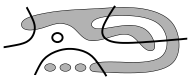
$10^d 11^d$

12/56/389abc74:



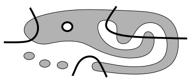
2+4; $8^4 8^b$

12/56/389cba74:



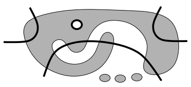
2+4; 8^{10}; [**3**;6]

12/56/3a98bc74:



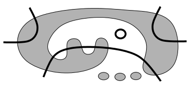
2+4; $6^3 8^2$

12/56ba789c/34:



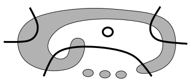
2+4; $6^8 8^7$

12/56ba987c/34:



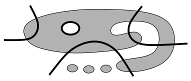
$6^b 8^c$

12/56ba9c/3874:



2+4; $6^7 8^8$

12/56bc/389a74:



2+4; $2^5 5^2 6^5 6^{6,8} 8^{4,9}$; [**2**;9]

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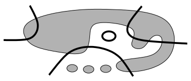
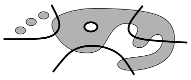
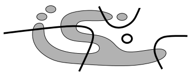


$2^b 6^b 8^c$



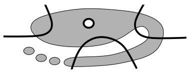
8^2

1258943c/76/ba:



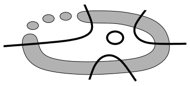
10^1; 27.25

12/58bc76/3a94:



2+4; $2^2 5^5 6^1 6^3 6^{10}$; [**2**;11]

1278/b43c/965a:



2+3; $2^6 5^3 6^7$

§6. Some restrictions

6.1. The case of nested free ovals. Recall that when speaking of a mutual arrangement of two curves, an oval of one of the curves is called *free* if it does not meet the other curve.

Proposition 6.1. *Suppose that the odd branch of an M-cubic meets an oval of an M-quartic at 12 points so that at least one free oval of one of the curves is contained inside a free oval of the other curve. Then the arrangements which are not listed in Section 5 are impossible.*

Proof. We shall apply the method proposed in [**6**; §3.3]. Let us consider the pencil of lines through a point inside the innermost of the nested free ovals. Then the arrangement of the union of the curves with respect to this pencil of lines has the form $\times^2_{i_1}\dots\times^2_{i_5}\times_3\supset_4 o_{j_1}o_{j_2}\subset_4\times_3$, where $i_1,\dots,i_5\in\{3,4\}$ and $j_1,j_2\in\{2,3,4,5\}$ (a description of the encoding can be found in [**4**], [**6**], or [**7**]). Computing the Alexander polynomial of the corresponding braid in each of the $2^5\cdot 4^2=512$ cases, we obtain a contradiction with the generalised Fox-Milnor theorem in all the cases not listed in Section 5. See details (including a computer program for computation of Alexander polynomial) in [**6**].

6.2. Oval of the cubic is outside the oval of the quartic but "not at infinity".

Proposition 6.2. *Suppose that the odd branch of an M-cubic C_3 meets the oval of an M-quartic C_4 at 12 points. Suppose also that there exists a connected component D of $\mathbf{R}P^2\setminus(J_3\cup C_4)$ whose closure is non-orientable. If the oval O_3 of C_3 is outside the ovals of the quartic and $O_3\not\subset D$, then the arrangement of $C_3\cup C_4$ is one of those listed in §5.*

Proof. We shall use the method from [**4**]. The arrangement of $C_3\cup C_4$ with respect to the pencil of lines centered inside O_3 has the form

$$\times_3\times^2_{i_1}\dots\times^2_{i_a}\times_3\supset_2 o_{j_1}o_{j_2}o_{j_3}\subset_2\times_3\times^2_{k_1}\dots\times^2_{k_b}\times_3,$$

where $a+b=4$, $i_1,\dots,i_a,k_1,\dots,k_b\in\{2,3\}$, $j_1,j_2,j_3\in\{2,3,5\}$. We shall consider all the $5\cdot 2^4\cdot 3^3=2160$ cases (by symmetry, the number of cases can be reduced). For each choice of $(a;i_1,\dots,i_4;j_1,j_2,j_3)$, we compute the braid corresponding to the pencil of lines centered inside O_3. The exponent sum of each of these braids is equal to 5. In all the cases not corresponding to the arrangements listed in Section 5, either Murasugi-Tristram inequality for the usual signature is not satisfied, or the Alexander polynomial is not identically zero.

6.3. Algebraic unrealisability of the flexible curve in Figure 13.

Definition 6.3. Let n be a positive integer and let $R(X,Z)=Z^3+b_1(X)Z^2+b_2(X)Z+b_3(X)$ where $a_k(X)$ is a polynomial in X of degree kn with real coefficients. Let us say that an interval $I=[X_1,X_2]$ is an *alternating interval for the polynomial R*, if the following conditions hold:

1. each of the polynomials $R(X_1,Z)$, $R(X_2,Z)$ has one simple root and one double root;
2. the polynomial $F(X_0,Z)$ has exactly one real root when $X_1<X_0<X_2$;
3. the double root is greater than the simple root for one of the polynomials $R(X_1,Z)$, $R(X_2,Z)$, and the simple root is greater than the double root for the other polynomial.

Definition 6.4. Let n be a positive integer and let $F(X,Y) = Y^4 + a_1(X)Y^3 + a_2(X)Y^2 + a_3(X)Y + a_4(X)$ where $a_k(X)$ is a polynomial in X of degree kn with real coefficients. Let us say that an interval $I = [X_1, X_2]$ is an *alternating interval for the polynomial* F, if the following conditions hold:

(1) each of the polynomials $F(X_1,Y)$, $F(X_2,Y)$ has one double root and two simple real roots;
(2) the polynomial $F(X_0,Y)$ has exactly two real roots when $X_1 < X_0 < X_2$;
(3) the double root is between the simple roots for one of the polynomials $F(X_1,Y)$, $F(X_2,Y)$ and the contrary for the other polynomial.

Lemma 6.5. *Let $R(Z) = Z^3 + b_2Z + b_3$ be a polynomial with real coefficients which has a simple root $Z = Z_1$ and a double root $Z = Z_2$. Then if $Z_1 < Z_2$ then $b_3 > 0$, and if $Z_2 < Z_1$ then $b_3 < 0$.*

Proof. We have $R(Z) = (Z - Z_1)(Z - Z_2)^2$. Then $b_3 = R(0) = -Z_1Z_2^2$, i.e. $\operatorname{sign} b_3 = -\operatorname{sign} Z_1$. It remains to note that $Z_1 + 2Z_2 = 0$, because the coefficient of Z^2 vanishes □

Lemma 6.6. *Let $R(X,Z)$ be as in Definition 6.3. Then it cannot have more than n alternating intervals.*

Proof. Performing if necessary the substitution $Z' = Z - b_1(X)$, we may assume that $b_1 = 0$. Let $D(X) = 4\,a_2^3 + 27\,a_3^2$ be the discriminant of R with respect to Z. Let $[X_1, X_2]$ be an alternating interval for R. Then the conditions (1)–(3) of Definition 6.3 and Lemma 6.5 imply that

(4) $D(X_1) = D(X_2) = 0$;
(5) $D(X) > 0$ for $X_1 < X < X_2$;
(6) $\operatorname{sign} b_3(X_1) = -\operatorname{sign} b_3(X_2)$.

The condition (6) implies that there exists $X_0 \in [X_1, X_2]$ such that $b_3(X_0) = 0$. Then, by (5) we have $4\,b_2(X_0)^3 = D(X_0) - 27\,b_3(X_0)^2 = D(X_0) > 0$, hence, $b_2(X_0) > 0$. Moreover, it follows from (4) that for $j = 1,2$ we have $4\,b_2(X_j)^3 = D(X_j) - 27\,b_3(X_j)^2 = -27\,b_3(X_j)^2 < 0$, hence $b_2(X_j) < 0$. Thus, the interval $[X_1, X_2]$ contains at least two roots of $b_2(X)$: one between X_1 and X_0, and another between X_0 and X_2. It remains to recall that $\deg b_2(X) = 2n$. □

Lemma 6.7. *Let $F(X,Y)$ be as in Definition 6.4. Then it cannot have more than $2n$ alternating intervals.*

Proof. Performing if necessary the substitution $Y' = Y - a_1(X)$, we may assume that $a_1 = 0$. Let $R(X,Y)$ be the cubic resolvent of $F(X,Y)$ with respect to Y. Let us recall its definition. For any fixed value of X, let us denote the roots of $F(X,Y)$ by $Y_1, \dots, Y_4$ and let us set

$$Z_1 = (Y_1 - Y_2)(Y_3 - Y_4), \quad Z_2 = (Y_1 - Y_3)(Y_2 - Y_4), \quad Z_3 = (Y_1 - Y_4)(Y_2 - Y_3),$$
$$R = (Z - Z_1)(Z - Z_2)(Z - Z_3) = Z^3 + b_1Z^2 + b_2Z + b_3.$$

The coefficients b_1, b_2, b_3 are symmetric polynomials in $Y_1, \dots, Y_4$, hence, they can be expressed polynomially via a_2, a_3, a_4 (see e.g. [**12**] for explicite formulas). Then b_k is a polynomial in X of degree $2kn$. Hence, by Lemma 6.6, R has at most $2n$ alternating intervals.

It remains to check that an interval is alternating for $R(X,Z)$ if and only if it is alternating for $F(X,Y)$. This can be easily proved using the definition of Z_1, Z_2, Z_3, and the relation $Y_1 + \dots + Y_4 = 0$. □

Proposition 6.8. *A real algebraic M-cubic C_3 cannot be arranged with respect to a real algebraic M-quartic C_4 as in Figure 13.*

Proof. Suppose that C_3 is arranged with respect to O_4 as in Figure 13. Let us introduce coordinates $(x : y : z)$ on $\mathbf{R}P^2$ so that the point $(0 : 1 : 0)$ is inside the oval of the cubic. Let $X = x/z$, $Y = y/z$ be the affine coordinates in the chart $z \neq 0$. The fact that the X-coordinate is monotone on all branches of the cubic implies that (under a suitable choice of the line at infinity) the curve C_4 is arranged as in Figure 57 with respect to some six vertical lines. Hence, there must be three alternating intervals for the polynomial $F(X,Y)$ which defines the curve C_4. However, by Lemma 6.7, $F(X,Y)$ cannot have more than two alternating intervals. $\square$

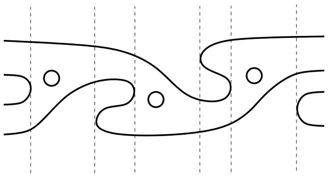


FIG. 57

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LAB. E. PICARD, UNIV. PAUL SABATIER, UFR MIG, 118 ROUTE DE NARBONNE, 31400, FRANCE *and* STEKLOV MATH. INST., GUBKINA 6, MOSCOW, RUSSIA