AUTOMORPHISM GROUP OF THE COMMUTATOR SUBGROUP OF THE BRAID GROUP

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Introduction

Let $\mathbf{B}_n$ be the braid group with n strings. It is generated by $\sigma_1,\dots,\sigma_{n-1}$ (called *standard* or *Artin* generators) subject to the relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i \text{ for } |i-j|>1; \qquad \sigma_i\sigma_j\sigma_i=\sigma_j\sigma_i\sigma_j \text{ for } |i-j|=1.$$

Let $\mathbf{B}'_n$ be the commutator subgroup of $\mathbf{B}_n$. Vladimir Lin [16] posed a problem to compute the group of automorphisms of $\mathbf{B}'_n$. In this paper we solve this problem.

Theorem 1. *If $n\ge 4$, then the restriction mapping $\operatorname{Aut}(\mathbf{B}_n)\to\operatorname{Aut}(\mathbf{B}'_n)$ is an isomorphism.*

Dyer and Grossman [5] proved that $\operatorname{Out}(\mathbf{B}_n)\cong\mathbb{Z}_2$ for any n. The only nontrivial element of $\operatorname{Out}(\mathbf{B}_n)$ corresponds to the automorphism Λ defined by $\sigma_i\mapsto\sigma_i^{-1}$ for all $i=1,\dots,n-1$. The center of $\mathbf{B}_n$ is generated by Δ^2 where $\Delta=\Delta_n=\prod_{i=1}^{n-1}\prod_{j=1}^{n-i}\sigma_j$ is Garside's half-twist. Thus $\operatorname{Aut}(\mathbf{B}_n)\cong(\mathbf{B}_n/\langle\Delta^2\rangle)\rtimes\mathbb{Z}_2$. For an element g of a group, we denote the inner automorphism $x\mapsto gxg^{-1}$ by $\tilde g$.

Corollary 1. *If $n\ge 4$, then $\operatorname{Out}(\mathbf{B}'_n)$ is isomorphic to the dihedral group $\mathbf{D}_{n(n-1)}=\mathbb{Z}_{n(n-1)}\rtimes\mathbb{Z}_2$. It is generated by Λ and $\tilde\sigma_1$ subject to the defining relations $\Lambda^2=\tilde\sigma_1^{n(n-1)}=\Lambda\tilde\sigma_1\Lambda\tilde\sigma_1=\mathrm{id}$.*

For $n=3$, the situation is different. It is proven in [11] that $\mathbf{B}'_3$ is a free group of rank two generated by $u=\sigma_2\sigma_1^{-1}$ and $t=\sigma_1^{-1}\sigma_2$ (in fact, the free base of $\mathbf{B}'_3$ considered in [11] is u,v with $v=t^{-1}u$). So, its automorphism group is well-known (see [18; §3.5, Theorem N4]). In particular (see [18; Corollary N4]), there is an exact sequence

$$1\longrightarrow\mathbf{B}'_3\xrightarrow{\ \iota\ }\operatorname{Aut}(\mathbf{B}'_3)\xrightarrow{\ \pi\ }\mathrm{GL}(2,\mathbb{Z})\longrightarrow 1$$

where $\iota(x)=\tilde x$ and π takes each automorphism of $\mathbf{B}'_3$ to the induced automorphism of the abelianization of $\mathbf{B}'_3$ (which we identify with $\mathbb{Z}^2$ by choosing the images of u and t as a base). We have $\tilde\sigma_1(u)=t^{-1}u$, $\tilde\sigma_2(u)=ut^{-1}$, $\tilde\sigma_1(t)=\tilde\sigma_2(t)=u$, $\Lambda(u)=t^{-1}$, $\Lambda(t)=u^{-1}$ whence

$$\pi(\tilde\sigma_1)=\pi(\tilde\sigma_2)=\begin{pmatrix}1&1\\-1&0\end{pmatrix},\qquad \pi(\Lambda)=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}.$$

Thus, again (as in the case $n\ge 4$) the image of $\operatorname{Aut}(\mathbf{B}_3)$ in $\operatorname{Out}(\mathbf{B}'_3)\cong\mathrm{GL}(2,\mathbb{Z})$ is isomorphic to $\mathbf{D}_6$ but this time it is not the whole group $\operatorname{Out}(\mathbf{B}'_3)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let $\mathbf{S}_n$ be the symmetric group and $\mathbf{A}_n$ its alternating subgroup. Let $\mu = \mu_n : \mathbf{B}_n \to \mathbf{S}_n$ be the homomorphism which takes σ_i to the transposition $(i, i+1)$ and let μ' be the restriction of μ to $\mathbf{B}'_n$. Then $\mathbf{P}_n = \ker \mu_n$ is the group of *pure braids*. Let $\mathbf{J}_n = \mathbf{B}'_n \cap \mathbf{P}_n = \ker \mu'$. Note that the image of μ' is $\mathbf{A}_n$. The following diagram commutes where the rows are exact sequences and all the unlabeled arrows (except "$\to 1$") are inclusions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{J}_n & \longrightarrow & \mathbf{B}'_n & \xrightarrow{\mu'} & \mathbf{A}_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{P}_n & \longrightarrow & \mathbf{B}_n & \xrightarrow{\mu} & \mathbf{S}_n & \longrightarrow & 1 \end{array} \tag{1}$$

Recall that a subgroup of a group G is called *characteristic* if it is invariant under each automorphism of G. Lin proved in [15; Theorem D] that $\mathbf{J}_n$ is a characteristic subgroup of $\mathbf{B}'_n$ for $n \ge 5$ (note that this fact is used in our proof of Theorem 1 for $n \ge 5$). By Theorem 1, this result extends to the case $n = 4$.

Corollary 2. *$\mathbf{J}_4$ is a characteristic subgroup of $\mathbf{B}'_4$.*

Note that $\mathbf{J}_3$ is not a characteristic subgroup of $\mathbf{B}'_3$. Indeed, let $\varphi \in \mathrm{Aut}(\mathbf{B}'_3)$ be defined by $u \mapsto u$, $t \mapsto ut$. Then $ut \in \mathbf{J}_3$ whereas $\varphi^{-1}(ut) = t \notin \mathbf{J}_3$.

1. Preliminaries

Let $e : \mathbf{B}_n \to \mathbb{Z}$ be the homomorphism defined by $e(\sigma_i) = 1$ for all $i = 1, \dots, n$. Then we have $\mathbf{B}'_n = \ker e$.

1.1. Groups. For a group G, we denote its unit element by 1, the center by $Z(G)$, the commutator subgroup by G', the second commutator subgroup $(G')'$ by G'', and the abelianization G/G' by $G^{\mathfrak{ab}}$. We denote $x^{-1}yx$ by y^x (thus $\tilde{x}(y^x) = y$) and we denote the commutator $xyx^{-1}y^{-1}$ by $[x, y]$. For $g \in G$, we denote the centralizer of g in G by $Z(g, G)$. If H is a subgroup of G, then, evidently, $Z(g, H) = Z(g, G) \cap H$.

Lemma 1.1. *Let G be a group generated by a set $\mathcal{A}$. Assume that there exists a homomorphism $e : G \to \mathbb{Z}$ such that $e(\mathcal{A}) = \{1\}$. Let $\bar{e}$ be the induced homomorphism $G^{\mathfrak{ab}} \to \mathbb{Z}$. Let Γ be the graph such that the set of vertices is $\mathcal{A}$ and two vertices a and b are connected by an edge when $[a, b] = 1$.*

If the graph Γ is connected, then $(\ker e)^{\mathfrak{ab}} \cong \ker \bar{e}$.

Proof. Let $K = \ker e$. Let us show that $G' \subset K'$. Since K' is normal in G, and G' is the normal closure of the subgroup generated by $[a, b]$, $a, b \in \mathcal{A}$, it is enough to show that $[a, b] \in K'$ for any $a, b \in \mathcal{A}$.

We define a relation $\sim$ on $\mathcal{A}$ by setting $a \sim b$ if $[a, b] \in K'$. Since Γ is connected, it remains to note that this relation is transitive. Indeed, if $a \sim b \sim c$, then $[a, c] = [a, b][bab^{-2}, bcb^{-2}][b, c] \in K'$.

Thus $G' \subset K'$ whence $G' = K'$ and we obtain $K^{\mathfrak{ab}} = K/K' = K/G' = \ker \bar{e}$. $\square$

Remark 1.2. The fact that $\mathbf{B}''_n = \mathbf{B}'_n$ for $n \ge 5$ proven by Gorin and Lin [11] (see also [15; Remark 1.10]) is an immediate corollary of Lemma 1.1. Indeed, if we set $G = \mathbf{B}_n$ and $\mathcal{A} = \{\sigma_i\}_{i=1}^{n-1}$, then Γ is connected whence $\mathbf{B}'_n/\mathbf{B}''_n = (\ker e)^{\mathfrak{ab}} \cong \ker \bar{e} = \{1\}$. In the same way we obtain $G'' = G'$ when G is an Artin group of type D_n $(n \ge 5)$, E_6, E_7, E_8, F_4, or H_4.

1.2. Pure braids. Recall that $\mathbf{P}_n$ is generated by the braids σ_{ij}^2, $1 \le i < j \le n$, where $\sigma_{ij} = \sigma_{ji} = \sigma_{j-1} \dots \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$. For a pure braid X, let us denote the linking number of the i-th and j-th strings by $\mathrm{lk}_{ij}(X)$. If X is presented by a diagram with under- and over-crossings, then $\mathrm{lk}_{ij}(X)$ is the half-sum of the signs of those crossings where the i-th and j-th strings cross. Let A_{ij} be the image of σ_{ij}^2 in $\mathbf{P}_n^{\mathfrak{ab}}$. We have, evidently,

$$\mathrm{lk}_{\gamma(i),\gamma(j)}(X) = \mathrm{lk}_{i,j}(X^\gamma), \qquad \text{for any } X \in \mathbf{P}_n,\ \gamma \in \mathbf{B}_n \tag{2}$$

(here $\gamma(i) = \mu(\gamma)(i)$ which is coherent with the interpretation of $\mathbf{B}_n$ with a mapping class group; see §3.1).

It is well known that $\mathbf{P}_n^{\mathfrak{ab}}$ is freely generated by $\{A_{ij}\}_{1\le i<j\le n}$. This fact is usually derived from Artin's presentation of $\mathbf{P}_n$ (see [1; Theorem 18]) but it also admits a very simple self-contained proof based on the linking numbers. Namely, let L be the free abelian group with a free base $\{a_{ij}\}_{1\le i<j\le n}$. Then it is immediate to check that the mapping $\mathbf{P}_n \to L$, $X \mapsto \sum_{i<j} \mathrm{lk}_{i,j}(X) a_{ij}$ is a homomorphism and that the induced homomorphism $\mathbf{P}_n^{\mathfrak{ab}} \to L$ is the inverse of $L \to \mathbf{P}_n^{\mathfrak{ab}}$, $a_{ij} \mapsto A_{ij}$. In particular, we see that the quotient map $\mathbf{P}_n \to \mathbf{P}_n^{\mathfrak{ab}}$ is given by $X \mapsto \sum_{i<j} \mathrm{lk}_{i,j}(X) A_{ij}$.

Lemma 1.3. *If $n \ge 5$, then the mapping $\mathbf{J}_n \to \mathbf{P}_n^{\mathfrak{ab}}$, $X \mapsto \sum \mathrm{lk}_{ij}(X) A_{ij}$ defines an isomorphism $\mathbf{J}_n^{\mathfrak{ab}} \cong \{\sum x_{ij} A_{ij} \mid \sum x_{ij} = 0\} \subset \mathbf{P}_n^{\mathfrak{ab}}$.*

Proof. Follows from Lemma 1.1 with $\mathbf{P}_n$, $e|_{\mathbf{P}_n}$, and $\{\sigma_{ij}^2\}_{1\le i<j\le n}$ standing for G, e, and $\mathcal{A}$ respectively. □

So, when $n \ge 5$, we identify $\mathbf{J}_n^{\mathfrak{ab}}$ with its image in $\mathbf{P}_n^{\mathfrak{ab}}$. The following proposition will not be used in the proof of Theorem 1.

Proposition 1.4. *(a). $\mathbf{J}_n^{\mathfrak{ab}}$ is a free abelian group and*

$$\operatorname{rk} \mathbf{J}_n^{\mathfrak{ab}} = \binom{n}{2} + \begin{cases} 1, & n \in \{3,4\}, \\ -1, & \text{otherwise.} \end{cases}$$

(b). $\mathcal{E}_3 = \{\bar u \bar t, \bar t \bar u, \bar u^3, \bar t^3\}$ and $\mathcal{E}_4 = \mathcal{E}_3 \cup \{\bar c^2, \bar w^2, (\bar c \bar w)^2\}$ are free bases of $\mathbf{J}_3^{\mathfrak{ab}}$ and $\mathbf{J}_4^{\mathfrak{ab}}$ respectively; u, t, w, c are defined in the beginning of Section 5.

(Here and below $\bar x$ stands for the image of x under the quotient map $\mathbf{J}_n \to \mathbf{J}_n^{\mathfrak{ab}}$.)

(c). Let $p_n : \mathbf{J}_n^{\mathfrak{ab}} \to \mathbf{P}_n^{\mathfrak{ab}}$, $n = 3, 4$, be induced by the composition $\mathbf{J}_n \to \mathbf{P}_n \to \mathbf{P}_n^{\mathfrak{ab}}$. Then

$$\operatorname{im} p_n = \{\sum x_{ij} A_{ij} \mid \sum x_{ij} = 0\}, \qquad \ker p_n = \langle \bar u^3, \bar t^3 \rangle.$$

Proof. (a). The result is obvious for $n = 2$ and it follows from Lemma 1.3 for $n \ge 5$.

For $n = 3$, the result follows from the following argument proposed by the referee. We have $\mathbf{B}_3' \cong \pi_1(\Gamma)$ where Γ is the bouquet $S^1 \vee S^1$. Since $|\mathbf{B}_3'/\mathbf{J}_3| = 3$ (see (1)), we have $\mathbf{J}_3^{\mathfrak{ab}} \cong H_1(\tilde\Gamma)$ where $\tilde\Gamma \to \Gamma$ is a connected 3-fold covering. Then the Euler characteristic of $\tilde\Gamma$ is $\chi(\tilde\Gamma) = 3\chi(\Gamma) = -3$ whence $\operatorname{rk} H_1(\tilde\Gamma) = 4$.

The group $\mathbf{J}_4^{\mathfrak{ab}}$ can be easily computed by the Reidemeister–Schreier method either as $\ker \mu'$ using Gorin and Lin's [11] presentation for $\mathbf{B}_4'$, or as $\ker(e|_{\mathbf{P}_4})$ using Artin's presentation [1] of $\mathbf{P}_4$. Here is the GAP code for the first method:

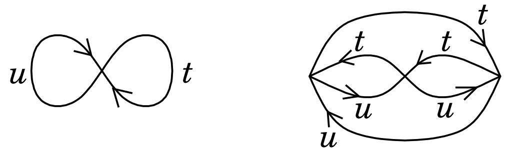

FIGURE 1. The graphs Γ and $\tilde\Gamma$ in the proof of Proposition 1.4

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f:=FreeGroup(4); u:=f.1; v:=f.2; w:=f.3; c:=f.4;
g:=f/[u*c/u/w, u*w/u/w*c/w/w, v*c/v/w*c, v*w/v/w*c*c/w*c/w*c/w*c];
u:=g.1; v:=g.2; w:=g.3; c:=g.4;     # group B'(4) according to [11]
s:=SymmetricGroup(4); t1:=(1,2); t2:=(2,3); t3:=(3,4);
U:=t2*t1; V:=t1*t2; W:=t2*t3*t1*t2; C:=t3*t1; # U=mu(u),V=mu(v),...
mu:=GroupHomomorphismByImages(g,s,[u,v,w,c],[U,V,W,C]);
AbelianInvariants(Kernel(mu));     # should be [0,0,0,0,0,0,0]
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(b) for $n=3$. In Figure 1 we show the graphs Γ and $\tilde\Gamma$ discussed above. We see that the loops in $\tilde\Gamma$ represented by the elements of $\mathcal{E}_3$ form a base of $H_1(\tilde\Gamma)$.

(c) for $n=3$. The claim about $\operatorname{im} p_3$ is evident and a computation of the linking numbers shows that $p_3(\bar u^3)=p_3(\bar t^3)=0$.

(b,c) for $n=4$. The claim about $\operatorname{im} p_4$ is evident and a computation of the linking numbers shows that $p_4(\mathcal{E}_4\setminus\{\bar t^3,\bar u^3\})$ is a base of $\operatorname{im} p_4$. One can check that the homomorphism $\mathbf{B}_4'\to\mathbf{B}_4'/\mathbf{K}_4\cong\mathbf{B}_3'$ maps $\mathbf{J}_4$ to $\mathbf{J}_3$. Hence it induces a homomorphism $\mathbf{J}_4^{\mathfrak{ab}}\to\mathbf{J}_3^{\mathfrak{ab}}$ which takes $\bar u^3$ and $\bar t^3$ of $\mathbf{J}_4^{\mathfrak{ab}}$ to $\bar u^3$ and $\bar t^3$ of $\mathbf{J}_3^{\mathfrak{ab}}$. Hence $\operatorname{rk}(\ker p_4)\ge\operatorname{rk}\langle\bar u^3,\bar t^3\rangle=2$. Since $\operatorname{rk}\mathbf{J}_4^{\mathfrak{ab}}=7$ and $\operatorname{rk}(\operatorname{im} p_4)=5$, we conclude that $\ker p_4=\langle\bar u^3,\bar t^3\rangle$. $\square$

Remark 1.5. Note that the braid closures of both u^3 and t^3 are Borromean links. So, maybe, it could be interesting to study how the considered base of $\mathbf{J}_3^{\mathfrak{ab}}$ is related to Milnor's μ-invariant.

1.3. Mixed braid groups and the cabling map. Let $n\ge1$ and $\vec m=(m_1,\dots,m_k)$, $m_1+\dots+m_k=n$, $m_i\in\mathbb{Z}$, $m_i>0$.

The *mixed braid group* $\mathbf{B}_{\vec m}$ (see [19], [20], [10]) is defined as $\mu^{-1}(S_{\vec m})$ where $S_{\vec m}$ is the stabilizer of the following vector under the natural action of $\mathbf{S}_n$ on $\mathbb{Z}^n$:

$$(\underbrace{1,\dots,1}_{m_1},\underbrace{2,\dots,2}_{m_2},\dots,\underbrace{k,\dots,k}_{m_k}).$$

We emphasize two particular cases: $\mathbf{B}_{1,\dots,1}$ is the pure braid group and $\mathbf{B}_{n-1,1}$ is the Artin group corresponding to the Coxeter group of type B_{n-1}.

We define the *cabling map* $\psi=\psi_{\vec m}:\mathbf{B}_k\times(\mathbf{B}_{m_1}\times\dots\times\mathbf{B}_{m_k})\to\mathbf{B}_n$ by sending $(X;X_1,\dots,X_k)$ to the braid obtained by replacing each strand of X by a geometric braid representing X_i embedded into a small tubular neighbourhood of this strand.

Note that $\psi_{\vec m}$ is not a homomorphism but its restriction to $\mathbf{P}_k\times\prod_i\mathbf{B}_{m_i}$ is. We have $\psi(\mathbf{P}_k\times\prod_i\mathbf{P}_{m_i})\subset\mathbf{P}_n$ and $\psi(\mathbf{P}_k\times\prod_i\mathbf{B}_{m_i})\subset\mathbf{B}_{\vec m}$.

2. $\mathbf{J}_n^{\mathfrak{ab}}$ AS AN $\mathbf{A}_n$-MODULE AND ITS AUTOMORPHISMS

Let $n\ge5$. As we mentioned already, by [15; Theorem D], $\mathbf{J}_n$ is a characteristic subgroup of $\mathbf{B}_n'$, i.e., $\mathbf{J}_n$ is invariant under any automorphism of $\mathbf{B}_n'$ (in fact a stronger statement is proven in [15]).

Lemma 2.1. *Let $\varphi \in \operatorname{Aut}(\mathbf{B}_n')$ be such that $\mu'\varphi = \mu'$. Let φ_* be the automorphism of $\mathbf{J}_n^{\mathfrak{ab}}$ induced by $\varphi|_{\mathbf{J}_n}$. Then $\varphi_* = \pm\operatorname{id}$.*

Proof. The exact sequence $1 \to \mathbf{J}_n \to \mathbf{B}_n' \to \mathbf{A}_n \to 1$ (see (1)) defines an action of $\mathbf{A}_n$ on $\mathbf{J}_n^{\mathfrak{ab}}$ by conjugation. The condition $\mu'\varphi = \mu'$ implies that φ_* is $\mathbf{A}_n$-equivariant. Let V be a complex vector space with base $e_1, \dots, e_n$ endowed with the natural action of $\mathbf{S}_n$ induced by the action on the base. We identify $\mathbf{P}_n^{\mathfrak{ab}}$ with its image in the symmetric square $\operatorname{Sym}^2 V$ by the homomorphism $A_{ij} \mapsto e_ie_j$. Then, by Lemma 1.3, we may identify $\mathbf{J}_n^{\mathfrak{ab}}$ with $\{\sum c_{ij}e_ie_j \mid \sum c_{ij} = 0\}$. These identifications are compatible with the action of $\mathbf{A}_n$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we denote the corresponding irreducible representation of $\mathbf{S}_n$ over $\mathbb{C}$ (the $\mathbb{C}\mathbf{S}_n$-module) by V_λ, see, e.g., [7; §4]. For an element v of a $\mathbb{C}\mathbf{S}_n$-module, let $\langle v\rangle_{\mathbb{C}\mathbf{S}_n}$ be the $\mathbb{C}\mathbf{S}_n$-submodule generated by v. We set $e_0 = e_1 + \dots + e_n$, $U = \langle e_0\rangle_{\mathbb{C}\mathbf{S}_n} = \mathbb{C}e_0$, and $U^\perp = \langle e_1 - e_2\rangle_{\mathbb{C}\mathbf{S}_n}$. Consider the following $\mathbb{C}\mathbf{S}_n$-submodules of $\operatorname{Sym}^2 V$:

$$W_0 = \langle e_1^2\rangle_{\mathbb{C}\mathbf{S}_n}, \quad W_1 = \langle w\rangle_{\mathbb{C}\mathbf{S}_n} = \mathbb{C}w \quad \text{where } w = \sum_{i<j} e_ie_j,$$
$$W_2 = \langle (e_1 - e_2)(e_3 + \dots + e_n)\rangle_{\mathbb{C}\mathbf{S}_n}, \quad W_3 = \langle (e_1 - e_2)(e_3 - e_4)\rangle_{\mathbb{C}\mathbf{S}_n}.$$

We have $\operatorname{Sym}^2 V = \operatorname{Sym}^2(U \oplus U^\perp) = \operatorname{Sym}^2 U \oplus \operatorname{Sym}^2 U^\perp \oplus (U \otimes U^\perp)$ and $U^\perp \cong V_{n-1,1}$ (that is V_λ for $\lambda = (n-1,1)$). It is known (see [17; Lemma 2.1] or [7; Exercise 4.19]) that $\operatorname{Sym}^2 V_{n-1,1} \cong U \oplus V_{n-1,1} \oplus V_{n-2,2} \cong V \oplus V_{n-2,2}$. Thus

$$\operatorname{Sym}^2 V \cong V \oplus V \oplus V_{n-2,2}. \tag{3}$$

Let $W = \mathbf{J}_n^{\mathfrak{ab}} \otimes \mathbb{C}$. It is clear that $\operatorname{Sym}^2 V = W_0 \oplus W_1 \oplus W$. Since $W_0 \cong V$ and $W_1 \cong U$, we obtain $W \cong U^\perp \oplus V_{n-2,2}$ by cancelling out $U \oplus V$ in (3). Note that $(e_1 - e_2)(e_3 + \dots + e_n) = (e_1 - e_2)(e_0 - (e_1 + e_2)) = (e_1 - e_2)e_0 - (e_1^2 - e_2^2)$, hence the mapping $e_i - e_j \mapsto (e_i - e_j)e_0 - (e_i^2 - e_j^2)$ induces an isomorphism of $\mathbb{C}\mathbf{S}_n$-modules $U^\perp \cong W_2$. The identity

$$(n-2)(e_1 - e_2)e_3 = (e_1 - e_2)(e_3 + \dots + e_n) + \sum_{i\ge 4}(e_1 - e_2)(e_3 - e_i) \tag{4}$$

shows that $W_2 + W_3 = \langle (e_1 - e_2)e_3\rangle_{\mathbb{C}\mathbf{S}_n} = W$. One easily checks that W_2 and W_3 are orthogonal to each other with respect to the scalar product on $W + W_1$ for which $\{e_ie_j\}_{i,j}$ is an orthonormal basis. Therefore $W = W_2 \oplus W_3$ is the decomposition of W into irreducible factors.

We have $W_2 \cong V_{n-1,1}$ and $W_3 \cong V_{n-2,2}$. Since the corresponding Young diagrams are not symmetric, W_2 and W_3 are irreducible as $\mathbb{C}\mathbf{A}_n$-modules (see [7; §5.1]). Since $\dim W_2 \ne \dim W_3$ and φ_* is $\mathbf{A}_n$-equivariant, Schur's lemma implies that $\varphi_*|_{W_k}$, $k = 2, 3$, is multiplication by a constant c_k. Moreover, since φ_* is an automorphism of $\mathbf{J}_n^{\mathfrak{ab}}$ (a discrete subgroup), we have $c_k = \pm 1$. If $c_3 = -c_2 = \pm 1$, then (4) contradicts the fact that $\varphi_*((e_1 - e_2)e_3) \in \mathbf{J}_n^{\mathfrak{ab}}$. □

Let $\nu \in \operatorname{Aut}(\mathbf{S}_6)$ be defined by $(12) \mapsto (12)(34)(56)$, $(123456) \mapsto (123)(45)$. It is well known that ν represents the only nontrivial element of $\operatorname{Out}(\mathbf{S}_6)$.

Lemma 2.2. *Let* $\varphi \in \mathrm{Aut}(\mathbf{B}_6')$*. Then* $\mu'\varphi \neq \nu\mu'$*.*

Proof. Given a commutative ring k and a $k\mathbf{A}_6$-module V corresponding to a representation $\rho : \mathbf{A}_6 \to \mathrm{GL}(V, k)$, we denote the $k\mathbf{A}_6$-module corresponding to the representation $\rho\nu$ by $\nu^*(V)$. It is clear that ν^* is a covariant functor which preserves direct sums (hence irreducibility), tensor products, symmetric powers etc.

Suppose that $\mu'\varphi = \nu\mu'$. As in the proof of Lemma 2.1, we endow $\mathbf{J}_6^{\mathfrak{ab}}$ with the action of $\mathbf{A}_6$. The condition $\mu'\varphi = \nu\mu'$ implies that φ induces an isomorphism of $\mathbf{A}_6$-modules $\mathbf{J}_6^{\mathfrak{ab}} \cong \nu^*(\mathbf{J}_6^{\mathfrak{ab}})$. Let us show that these modules are not isomorphic.

We have $\mathbf{J}_6^{\mathfrak{ab}} \otimes \mathbb{C} \cong V_{5,1} \oplus V_{4,2}$ (see the proof of Lemma 2.1). Hence $\nu^*(\mathbf{J}_6^{\mathfrak{ab}}) \otimes \mathbb{C} \cong \nu^*(V_{5,1}) \oplus \nu^*(V_{4,2})$. We have $\dim V_{5,1} = 5 \neq 9 = \dim V_{4,2}$, thus, to complete the proof, it is enough to show that $V_{5,1} \not\cong \nu^*(V_{5,1})$ (note that $V_{4,2} \cong \nu^*(V_{4,2})$). Indeed, ν exchanges the conjugacy classes of the permutations $a = (123)$ and $b = (123)(456)$, hence we have $\chi(a) = 2 \neq -1 = \chi(b) = \chi\nu(a)$ where χ and $\chi\nu$ are the characters of $\mathbf{A}_6$ corresponding to $V_{5,1}$ and to $\nu^*(V_{5,1})$ respectively. $\square$

3. Centralizers of pure braids

Centralizers of braids are computed by González-Meneses and Wiest [10]. For pure braids the answer is much simpler and it can be easily obtained as a specialization of the results of [10].

3.1. Nielsen-Thurston trichotomy. The following definitions and facts we reproduce from [10; Section 2] where they are taken from different sources, mostly from the book [12] which can be also used as a general introduction to the subject.

Let $\mathbb{D}$ be a disk in $\mathbb{C}$ that contains $X_n = \{1, \dots, n\}$. The elements of X_n will be called *punctures*. It is well known that $\mathbf{B}_n$ can be identified with the mapping class group $\mathcal{D}/\mathcal{D}_0$ where $\mathcal{D}$ is the group of diffeomorphisms $\beta : \mathbb{D} \to \mathbb{D}$ such that $\beta|_{\partial\mathbb{D}} = \mathrm{id}_{\partial\mathbb{D}}$ and $\beta(X_n) = X_n$, and $\mathcal{D}_0$ is the connected component of the identity. Sometimes, by abuse of notation, we shall not distinguish between braids and elements of $\mathcal{D}$. For $A, B \subset \mathbb{D}$, we write $A \sim B$ if $\beta_0(A) = B$ for some $\beta_0 \in \mathcal{D}_0$.

An embedded circle in $\mathbb{D} \setminus X_n$ is called an *essential curve* if it encircles more than one but less than n points of X_n. A *multicurve* in $\mathbb{D} \setminus X_n$ is a disjoint union of embedded circles. It is called *essential* if all its components are essential.

Let $\beta \in \mathcal{D}$. We say that a multicurve C in $\mathbb{D} \setminus X_n$ is *stabilized* or *preserved* by β if $\beta(C) \sim C$ (the components of C may be permuted by β). The braid represented by β is called *reducible* if β stabilizes some essential multicurve.

A braid β is called *periodic* if some power of β belongs to $Z(\mathbf{B}_n)$. If a braid is neither periodic nor reducible, then it is called *pseudo-Anosov*; see [12].

3.2. Canonical reduction systems. Tubular and interior braids. An essential curve C is called a *reduction curve* for a braid β if it is stabilized by some power of β and any other curve stabilized by some power of β is isotopic in $\mathbb{D} \setminus X_n$ to a curve disjoint from C. An essential multicurve is called a *canonical reduction system* (CRS) for β if its components represent all isotopy classes of reduction curves for β (each class being represented once). It is known that there exists a canonical reduction system for any braid and that it is unique up to isotopy, see [2], [12; §7], [10; §2]. If a braid is periodic or pseudo-Anosov, the CRS is empty. The following properties of CRS are immediate consequences of their existence and uniqueness.

Proposition 3.1. *Let* C *be the CRS for* $\beta \in \mathcal{D}$*. Then* C *is the CRS for* β^{-1}*.* $\square$

Proposition 3.2. *Let $\beta, \gamma \subset \mathcal{D}$ and let C be the CRS for β. Then $\gamma^{-1}(C)$ is the CRS for β^γ.* □

Proposition 3.3. *Let $\beta, \gamma \in \mathcal{D}$ represent commuting braids. Then:*

(a). γ preserves the CRS of β.

(b). If γ is pure, then it preserves each reduction curve of β.

Proof. (a). Follows from Proposition 3.2.
(b). Follows from (a). □

We say that a braid is in *almost regular form* if its CRS is a union of round circles ('almost' because the definition of regular form in [10] includes some more conditions which we do not need here). By Proposition 3.2 any braid is conjugate to a braid in almost regular form.

Let β be an element of $\mathcal{D}$ which represents a reducible braid in almost regular form and let C be a CRS for β. Without loss of generality we may assume that $\beta(C) = C$ and C is a union of round circles. Let $R = R' \cup R''$ where R' is the union of the outermost components of C and R'' is the union of small circles around the points of X_n not encircled by curves from R'. Let $C_1, \dots, C_k$ be the connected components of R numbered from left to right.

Recall that the geometric braid (a union of strings in the cylinder $[0,1] \times \mathbb{D}$) is obtained from β as follows. Let $\{\beta_t : \mathbb{D} \to \mathbb{D}\}_{t\in[0,1]}$ be an isotopy such that $\beta_0 = \beta$, $\beta_1 = \mathrm{id}_\mathbb{D}$, and $\beta_t|_{\partial\mathbb{D}} = \mathrm{id}_{\partial\mathbb{D}}$ for any t. Then the i-th string of the geometric braid is the graph of the mapping $t \mapsto \beta_t(i)$ and the whole geometric braid is $\bigcup_t \big(\{t\} \times \beta_t(X_n)\big)$. Similarly, starting from the circles C_i, we define the embedded cylinders (tubes) $\bigcup_t \big(\{t\} \times \beta_t(C_i)\big)$, $i = 1, \dots, k$.

Let m_i be the number of punctures encircled by C_i. Following [10; §5.1], we define the *interior braid* $\beta_{[i]} \in \mathbf{B}_{m_i}$, $i = 1, \dots, k$, as the element of $\mathbf{B}_{m_i}$ corresponding to the union of strings contained in the i-th tube, and we define the *tubular braid* $\hat\beta$ of β as the braid obtained by shrinking each tube to a single string. Let $\vec m = (m_1, \dots, m_k)$ and let $\psi_{\vec m}$ be the cabling map (see §1.3). Then we have $\beta = \psi_{\vec m}(\hat\beta; \beta_{[1]}, \dots, \beta_{[k]})$.

Recall that C is a CRS for β. Let a be an open connected subset of $\mathbb{D}$ such that $\partial a \subset C \cup \partial\mathbb{D}$. With each such a we associate the braid which is the union of the strings of β starting at a and the strings obtained by shrinking the tubes corresponding to the interior components of ∂a. We denote this braid by $\beta_{[a]}$. For example, if a is the exterior component of $\mathbb{D} \setminus C$, then $\beta_{[a]} = \hat\beta$.

3.3. Periodic and reducible pure braids. The structure of the centralizers of periodic and reducible braids becomes extremely simple if we restrict our attention to pure braids only. The following fact immediately follows from a result due to Eilenberg [6] and Kerékjártó [13] (see [10; Lemma 3.1]).

Proposition 3.4. *A pure braid is periodic if and only if it is a power of Δ^2.* □

The following fact can be considered as a specialization of the results of [10].

Proposition 3.5. *Let β be a pure n-braid.*

(a). If β is periodic, then $Z(\beta; \mathbf{P}_n) = \mathbf{P}_n$.

(b). If β is pseudo-Anosov, then $Z(\beta; \mathbf{P}_n)$ is the free abelian group generated by Δ^2 and some pseudo-Anosov braid which may or may not coincide with β.

(c). If β is reducible non-periodic and in almost regular form, then $\psi_{\vec m}$ maps $Z(\hat\beta;\mathbf{P}_k)\times Z(\beta_{[1]};\mathbf{P}_{m_1})\times\dots\times Z(\beta_{[k]};\mathbf{P}_{m_k})$ isomorphically onto $Z(\beta;\mathbf{P}_n)$ (see §3.2).

Proof. (a). Follows from Proposition 3.4.

(b). Follows from [10; Proposition 4.1].

(c). (See also the proof of [10; Proposition 5.17]). By Proposition 3.3 we have $Z(\beta;\mathbf{P}_n)\subset\psi_{\vec m}(\mathbf{P}_k\times\prod\mathbf{P}_{m_i})$. The injectivity of the considered mapping and the fact that $\psi_{\vec m}^{-1}(Z(\beta;\mathbf{P}_n))$ is as stated, are immediate consequences from the following observation: if two geometric braids are isotopic, then the braids obtained from them by removal of some strings are isotopic as well. □

Lemma 3.6. *Let $\vec m=(m_1,\dots,m_k)$, $m_1+\dots+m_k=n$, and $p\in\mathbb{Z}$. Then $\psi_{\vec m}(\Delta_k^p;\Delta_{m_1}^p,\dots,\Delta_{m_k}^p)=\Delta_n^p$.*

Proof. The result immediately follows from the geometric characterization of Δ as a braid all whose strings lie on a half-twisted band. Note that the sub-bands of the half-twisted band arising from consecutive strings also consist of half-twisted bands. □

If X is a periodic pure braid, then $X=\Delta^{2d}$, $d\in\mathbb{Z}$, by Proposition 3.4. In this case we set $d=\deg X$, the *degree* of X. It is clear that $\mathrm{lk}_{ij}(X)=d$ for any $i<j$.

Lemma 3.7. *Let C be the CRS for a reducible pure braid represented by $\beta\in\mathcal{D}$. Let a and b be two neighboring components of $\mathbb{D}\setminus C$ and let $X=\beta_{[a]}$ and $Y=\beta_{[b]}$ be the braids associated to a and b (see the end of §3.2). Suppose that each of X and Y is periodic. Then* $\deg X\neq\deg Y$.

Proof. Suppose that $\deg X=\deg Y=p$, i. e., $X=\Delta_k^{2p}$ and $Y=\Delta_m^{2p}$ for some $k,m\ge 2$. Let C_i be the component of C that separates a and b. We may assume that a is exterior to C_i. Let c be the closure of $a\cup b$. Then we have

$$\beta_{[c]}=\psi_{1,\dots,1,m,1,\dots,1}(\Delta_k^{2p};1,\dots,1,\Delta_m^{2p},1,\dots,1)=\Delta_{k+m-1}^{2p}$$

by Lemma 3.6. Hence $\beta_{[c]}$ preserves any closed curve, in particular a curve which separates some two strings of $\beta_{[b]}$ and encircles a string of $\beta_{[c]}$ not belonging to $\beta_{[b]}$. Such a curve is not isotopic to any curve disjoint from C_i. This fact contradicts the condition that C_i is a reduction curve. □

Lemma 3.8. *$Z(\sigma_1^2\sigma_3^{-2};\mathbf{J}_n)\cong\mathbf{P}_{n-2}\times\mathbb{Z}$ for $n\ge 4$.*

Proof. The CRS for $\sigma_1^2\sigma_3^{-2}$ consists of two round circles: one of them encircles the punctures 1 and 2, and the other one encircles the punctures 3 and 4. Then Proposition 3.4(c) implies that $\psi=\psi_{\vec m}:\mathbf{P}_{n-2}\times(\mathbf{P}_2\times\mathbf{P}_2)\to\mathbf{P}_n$, $\vec m=(2,2,1_{n-4})$, is injective and $\operatorname{im}\psi=Z(\sigma_1^2\sigma_3^{-2};\mathbf{P}_n)$. One easily checks that the mapping $\mathbf{P}_{n-2}\times\mathbf{P}_2\to Z(\sigma_1^2\sigma_3^{-2};\mathbf{J}_n)$, $(X,\sigma_1^k)\mapsto\psi(X;\sigma_1^k,\sigma_1^{-m})$, $m=e(\psi(X;1,1))+k$ is an isomorphism. Indeed, any element Y of $Z(\sigma_1^2\sigma_3^{-2};\mathbf{P}_n)$ is of the form $Y=\psi(X;\sigma_1^k,\sigma_1^{-m})$ and the condition $e(Y)=0$ becomes $e(\psi(X;1,1))+k-m=0$. □

Lemma 3.9. *Let β be a reducible n-braid in almost regular form. Suppose that $\hat\beta\in\mathbf{P}_k$ and that the $\beta_{[i]}$'s* (see §3.2) *are pairwise non-conjugate. Then $\psi_{\vec m}$ maps $Z(\hat\beta;\mathbf{P}_k)\times Z(\beta_{[1]};\mathbf{B}_{m_1})\times\dots\times Z(\beta_{[k]};\mathbf{B}_{m_k})$ isomorphically onto $Z(\beta;\mathbf{B}_n)$*

Proof. See the proof of Proposition 3.5(c). □

4. Proof of Theorem 1 for $n \geq 5$

4.1. Invariance of the conjugacy class of $\sigma_1\sigma_3^{-1}$. Suppose that $n \geq 5$. Let $\varphi \in \mathrm{Aut}(\mathbf{B}'_n)$ be such that $\mu'\varphi = \mu'$ and $\varphi_* = \mathrm{id}$ where φ_* is as in Lemma 2.1. Then we have

$$\mathrm{lk}_{i,j}(X) = \mathrm{lk}_{i,j}(\varphi(X)), \qquad X \in \mathbf{J}_n, \quad 1 \leq i < j \leq n. \tag{5}$$

Let $\tau = \psi_{2,n-2}(1; \sigma_1^{(n-2)(n-3)}, \Delta^{-2})$. We have $\tau \in \mathbf{J}_n$.

Lemma 4.1. *Let X be $\sigma_1^2\sigma_3^{-2}$ or τ. Let $\alpha \in \mathcal{D}$ represent $\varphi(X)$. Let C be a simple closed curve preserved by α. Suppose that C encircles at least two punctures. Then the punctures 1 and 2 are in the same component of $\mathbb{D} \setminus C$.*

Proof. Suppose that 1 and 2 are separated by C. Without loss of generality we may assume that 1 is outside C and 2 is inside C. Let p be another puncture inside C. Then we have $\mathrm{lk}_{1,p}(\alpha) = \mathrm{lk}_{1,2}(\alpha)$ which contradicts (5) because $\mathrm{lk}_{1,2}(X) \neq 0$ and $\mathrm{lk}_{1,p}(X) = 0$ for any $p \neq 2$. □

Lemma 4.2. *Let $\alpha \in \mathcal{D}$ represent $\varphi(\sigma_1^2\sigma_3^{-2})$. Then the CRS for α is invariant under some element of $\mathcal{D}$ which exchanges $\{1,2\}$ and $\{3,4\}$.*

Proof. Follows from Propositions 3.1 and 3.2 because α is conjugate to α^{-1} and the conjugating element of $\mathcal{D}$ exchanges $\{1,2\}$ and $\{3,4\}$. □

Lemma 4.3. *Let $\alpha \in \mathcal{D}$ represent $\varphi(\tau)$. Let C be a component of the CRS for α. Then C cannot separate i and j for all $3 \leq i < j \leq n$.*

Proof. Let $\beta \in \mathcal{D}$ represent $\varphi(\sigma_{ij}^2\sigma_1^{-2})$. Since α and β commute, β preserves C by Proposition 3.3(b). Hence C cannot separate i and j by Lemma 4.1 applied to β (note that β is conjugate to $\sigma_1^2\sigma_3^{-2}$; see the beginning of §4.2). □

Lemma 4.4. *Let $\alpha \in \mathcal{D}$ represent $\varphi(\sigma_1^2\sigma_3^{-2})$. Suppose that $n \geq 6$. Let C be a component of the CRS for α. Then:*

(a). C cannot separate 1 and 2. It cannot separate 3 and 4.

(b). C cannot separate i and j for $5 \leq i < j \leq n$.

(c). C cannot separate $\{1,2,3,4\}$ from $\{5,\dots,n\}$.

(d). C cannot encircle $5,\dots,n$.

Proof. (a). Follows from Lemma 4.1 and Lemma 4.2.

(b). Let $\beta \in \mathcal{D}$ represent $\varphi(\sigma_{ij}^2\sigma_1^{-2})$. Since α and β commute, β preserves C by Proposition 3.3(b). Hence C cannot separate i and j by Lemma 4.1 applied to β (see the proof of Lemma 4.3).

(c). Suppose that C separates $1,2,3,4$ from $5,6,\dots,n$. Let $\beta \in \mathcal{D}$ represent $\varphi(\sigma_1^2\sigma_5^{-2})$. Then β is conjugate to α. Let $\gamma \in \mathcal{D}$ be a conjugating element. Then $\gamma(C)$ is a component of the CRS for β and it separates the punctures $1,2,5,6$ from all the other punctures. Since α and β commute, β preserves C. This is impossible because the geometric intersection number of C and $\gamma(C)$ is nonzero.

(d). Combine (a), (c), and Lemma 4.2. □

Lemma 4.5. *Let $\alpha \in \mathcal{D}$ represent $\varphi(\sigma_1^2\sigma_3^{-2})$. Suppose that α is reducible non-periodic. Then the CRS for α has exactly two components: one of them encircles* 1 *and* 2, *and the other one encircles* 3 *and* 4.

Proof. If $n \ge 6$, the result follows from Lemma 4.2 and Lemma 4.4. Suppose that $n = 5$ and the CRS is not as stated. By combining Lemma 4.2 with Lemma 3.7, we conclude that the CRS consists of a single circle which encircles 1,2,3,4. The interior braid cannot be periodic by (5), hence it is pseudo-Anosov. Therefore, $Z(\alpha;\mathbf{P}_5) \cong \mathbb{Z}^2$ by Proposition 3.5(b) whence $Z(\alpha;\mathbf{J}_5) = \mathbb{Z}$. This contradicts Lemma 3.8. □

Lemma 4.6. *$\varphi(\sigma_1\sigma_3^{-1})$ is conjugate in $\mathbf{B}_n$ to $\sigma_1\sigma_3^{-1}$.*

Proof. Let $\alpha \in \mathcal{D}$ represent $\varphi(\sigma_1^2\sigma_3^{-2})$. If α is pseudo-Anosov, then $Z(\alpha;\mathbf{P}_n) \cong \mathbb{Z}^2$ by Proposition 3.5(b), hence $Z(\alpha;\mathbf{J}_n)$ is abelian which contradicts Lemma 3.8. If α is periodic, then it is a power of Δ^2 by Proposition 3.4. This contradicts (5), hence α is reducible non-periodic and its CRS is as stated in Lemma 4.5.

Suppose that $\hat\alpha$ is pseudo-Anosov. Then $Z(\hat\alpha;\mathbf{P}_n) \cong \mathbb{Z}^2$ by Proposition 3.5(b) whence $Z(\alpha;\mathbf{P}_n) \cong \mathbb{Z}^4$ by Proposition 3.5(c) and therefore $Z(\alpha;\mathbf{J}_n)$ is abelian which contradicts Lemma 3.8. Thus, $\hat\alpha$ is periodic. By Proposition 3.4 this means that $\hat\alpha$ is a power of Δ^2. This fact combined with (5) implies $\hat\alpha = 1$. It follows that $\varphi(\sigma_1^2\sigma_3^{-2})$ is conjugate to $\sigma_1^{2k}\sigma_3^{-2k}$ for some k, and we have $k = 1$ by (5). The uniqueness of roots up to conjugation [9] implies that $\varphi(\sigma_1\sigma_3^{-1})$ is conjugate to $\sigma_1\sigma_3^{-1}$. □

Lemma 4.7. *$\varphi(\tau)$ is conjugate in $\mathbf{P}_n$ to τ.*

Proof. Let $\alpha \in \mathcal{D}$ represent $\varphi(\tau)$. By Proposition 3.3, it cannot be pseudo-Anosov because it commutes with $\varphi(\sigma_1\sigma_3^{-1})$ which is reducible non-periodic by Lemma 4.6. If α were periodic, then it would be a power of Δ^2 by Proposition 3.4. This contradicts (5), hence α is reducible.

Let C be the CRS for α. By Lemmas 4.1 and 4.3, one of the following three cases occurs.

Case 1. C is connected, the punctures 1 and 2 are inside C, all the other punctures are outside C. Then the tubular braid $\hat\alpha$ cannot be pseudo-Anosov because α commutes with $\varphi(\sigma_1\sigma_3^{-1})$, hence it preserves a circle which separates 3 and 4 from $5,\dots,n$. Hence $\hat\alpha$ is periodic which contradicts (5) combined with Proposition 3.4. Thus this case is impossible.

Case 2. C is connected, the punctures 1 and 2 are outside C, all the other punctures are inside C. This case is also impossible and the proof is almost the same as in Case 1. To show that $\hat\alpha$ cannot be pseudo-Anosov, we note that α preserves a curve which encircles only 1 and 2.

Case 3. C has two components: c_1 and c_2 which encircle $\{1,2\}$ and $\{3,\dots,n\}$ respectively. The interior braid $\alpha_{[2]}$ cannot be pseudo-Anosov by the same reasons as in Case 1, because α preserves a circle separating 3 and 4 from $5,\dots,n$. Hence $\alpha_{[2]}$ is periodic. Using (5), we conclude that α is a conjugate of τ. Since the elements of $Z(\tau;\mathbf{B}_n)$ realize any permutation of $\{1,2\}$ and $\{3,\dots,n\}$, the conjugating element can be chosen in $\mathbf{P}_n$. □

Lemma 4.8. *There exists $\gamma \in \mathbf{P}_n$ such that*

$$\varphi(\sigma_1\sigma_i^{-1}) = (\sigma_1\sigma_i^{-1})^\gamma \quad \text{for } i = 3,\dots,n. \tag{6}$$

Proof. Due to Lemma 4.7, without loss of generality we may assume that $\varphi(\tau)=\tau$ and $\tau(C)=C$ where C is the CRS for τ consisting of two round circles c_1 and c_2 which encircle $\{1,2\}$ and $\{3,\dots,n\}$ respectively.

By Lemma 3.9, $\psi_{2,n-2}$ restricts to an isomorphism $\psi:\mathbf{P}_2\times\mathbf{B}_2\times\mathbf{B}_{n-2}\to Z(\tau):=Z(\tau;\mathbf{B}_n)$. Let $\pi_1:Z(\tau)\to\mathbf{P}_2$ and $\pi_3:Z(\tau)\to\mathbf{B}_{n-2}$ be defined as $\pi_i=\mathrm{pr}_i\circ\psi^{-1}$.

Let $H=\pi_1^{-1}(1)\cap\mathbf{B}'_n$; note that the elements of $\pi_1^{-1}(1)$ correspond to geometric braids whose first two strings are inside the cylinder $[0,1]\times c_1$ and the other strings are inside the cylinder $[0,1]\times c_2$. Then $\pi_3|_H:H\to\mathbf{B}_{n-2}$ is an isomorphism and its inverse is given by $Y\mapsto\psi_{2,n-2}(1,\sigma_1^{-e(X)},Y)$, that is $\sigma_i\mapsto\sigma_1^{-1}\sigma_{i+2}$, $i=1,\dots,n-3$.

Let us show that $\varphi(H)=H$. Indeed, let $X\in H$. Since $X\in Z(\tau;\mathbf{B}'_n)$ and $\varphi(\tau)=\tau$, we have $\varphi(X)\in Z(\tau;\mathbf{B}'_n)$. The fact that $\pi_1(X)=1$ follows from (5) applied to a power of X belonging to $\mathbf{J}_n$. Hence $\varphi(H)\subset H$. By the same arguments $\varphi^{-1}(H)\subset H$.

Thus $\varphi|_H$ is an automorphism of H and we have $H\cong\mathbf{B}_{n-2}$. Hence, by Dyer and Grossman's result [5] cited after the statement of Theorem 1, there exists $\gamma\in H$ such that $\tilde\gamma\varphi|_H$ is either id_H or $\Lambda|_H$. The latter case is impossible by (5). Thus there exists $\gamma\in\mathbf{B}_n$ such that (6) holds.

It remains to show that γ can be chosen in $\mathbf{P}_n$. By replacing γ with $\sigma_1\gamma$ if necessary, we may assume that 1 and 2 are fixed by γ. By combining (2), (5), and (6), we conclude that $\gamma(\{i,j\})=\{i,j\}$ for any $i,j\in\{3,\dots,n\}$ and the result follows. $\square$

4.2. Conjugates of σ_1 and simple curves which connect punctures. We fix $n\ge 2$ and we consider $\mathbb{D}$ and the set of punctures $X_n=\{1,\dots,n\}\subset\mathbb{D}$ as above. Let $\mathcal{I}$ be the set of all smooth simple curves (embedded segments) $I\subset\mathbb{D}$ such that $\partial I\subset X_n$ and $I^\circ\subset\mathbb{D}\setminus X_n$. Here we denote $I^\circ=I\setminus\partial I$ and $\partial I=\{a,b\}$ where a and b are the ends of I. Recall that we write $I\sim I_1$ if $I_1=\alpha(I)$ for some $\alpha\in\mathcal{D}_0$ (see §3.1), i. e., if I and I_1 belong to the same connected component of $\mathcal{I}$.

Let $I\in\mathcal{I}$ and let $\beta\in\mathcal{D}$ be such that $\beta(I)$ is the straight line segment $[1,2]$. Then we define the braid σ_I as σ_1^β. It is easy to see that σ_I depends only on the connected component of $\mathcal{I}$ that contains I. The CRS for σ_I is a single closed curve which encloses I and separates it from $X_n\setminus\partial I$. By definition, all conjugates of σ_1 are obtained in this way. In particular, we have $\sigma_i=\sigma_{[i,i+1]}$ and $\sigma_{ij}=\sigma_I$ for an embedded segment I which connects i to j passing through the upper half-plane.

Lemma 4.9. *For any $\beta\in\mathcal{D}$, $I\in\mathcal{I}$, we have $\sigma_{\beta(I)}^\beta=\sigma_I$.* $\square$

With this notation, a corollary of Lemma 4.8 can be formulated as follows.

Lemma 4.10. *Let $n\ge 5$ and let $\varphi\in\mathrm{Aut}(B'_n)$ be as in §4.1.*

(a). Let $I,J\in\mathcal{I}$ be such that $\mathrm{Card}(I\cap J)=\mathrm{Card}(\partial I\cap\partial J)=1$ *(i.e., $I\cap J$ is a common endpoint of I and J). Then there exist $I_1,J_1\in\mathcal{I}$ such that $I_1\cup J_1$ is homeomorphic to $I\cup J$ and*

$$\varphi(\sigma_I\sigma_J^{-1})=\sigma_{I_1}\sigma_{J_1}^{-1},\qquad\varphi(\sigma_I^{-1}\sigma_J)=\sigma_{I_1}^{-1}\sigma_{J_1}.\tag{7}$$

(b). Let $I,J\in\mathcal{I}$, $I\cap J=\varnothing$. Then the conclusion is the same as in Part (a).

(c). Let I and J be as in Part (a) and let $K\in\mathcal{I}$ be such that $K\cap(I\cup J)=\varnothing$. Then there exist $I_1,J_1,K_1\in\mathcal{I}$ such that $I_1\cup J_1\cup K_1$ is homeomorphic to $I\cup J\cup K$,

and (7) holds as well as

$$\varphi(\sigma_K\sigma_I^{-1}) = \sigma_{K_1}\sigma_{I_1}^{-1}, \qquad \varphi(\sigma_K\sigma_J^{-1}) = \sigma_{K_1}\sigma_{J_1}^{-1}. \tag{8}$$

Proof. (c). Let γ be as in Lemma 4.8 and let $\beta \in \mathcal{D}$ be such that $\beta(K) = [1,2]$, $\beta(I) = [3,4]$, and $\beta(J) = [4,5]$. We set $K_1 = \alpha^{-1}(K)$, $I_1 = \alpha^{-1}(I)$, $J_1 = \alpha^{-1}(J)$ where $\alpha = \beta^{-1}\gamma\varphi(\beta)$. Then we have

$$\begin{aligned} \varphi(\sigma_K\sigma_I^{-1}) &= \varphi((\sigma_1\sigma_3^{-1})^{\beta}) && \text{by definition of } \sigma_I \text{ and } \sigma_K \\ &= (\sigma_1\sigma_3^{-1})^{\gamma\varphi(\beta)} && \text{by Lemma 4.8} \\ &= \sigma_{K_1}\sigma_{I_1}^{-1} && \text{by Lemma 4.9} \end{aligned}$$

and, similarly, $\varphi(\sigma_K\sigma_J^{-1}) = \sigma_{K_1}\sigma_{J_1}^{-1}$. Since σ_K commutes with σ_I and σ_J, we have $\sigma_I^{\varepsilon}\sigma_J^{-\varepsilon} = (\sigma_K\sigma_I^{-1})^{-\varepsilon}(\sigma_K\sigma_J^{-1})^{\varepsilon}$, $\varepsilon = \pm 1$, thus (8) implies (7).

(a). Since $\operatorname{Card}(\partial I \cup \partial J) = 3$ and $n \ge 5$, we can choose $K \in \mathcal{I}$ disjoint from $I \cup J$ (which is an embedded segment, hence its complement is connected) and the result follows from (c).

(b). The same proof as for Part (c) but with $\beta(I) = [1,2]$ and $\beta(J) = [3,4]$. □

Lemma 4.11. *Let $I, J, I_1, J_1 \in \mathcal{I}$ be such that $I \cap J = I_1 \cap J_1 = \varnothing$. Suppose that $\sigma_I\sigma_J^{-1} = \sigma_{I_1}\sigma_{J_1}^{-1}$. Then $I \sim I_1$ and $J \sim J_1$.*

Proof. It is enough to observe that the CRS for $\sigma_I\sigma_J^{-1}$ is $\partial U_I \cup \partial U_J$ where U_I and U_J are ε-neighbourhoods of I and J for $0 < \varepsilon \ll 1$ (this fact follows, for example, from Lemma 4.5 and Proposition 3.2). □

Note that when $[\sigma_I, \sigma_J] \ne 1$, the statement of Lemma 4.11 is wrong. Indeed, in this case by Lemma 4.9 we have $\sigma_I\sigma_J^{-1} = \sigma_{\gamma(I)}\sigma_{\gamma(J)}^{-1}$ for $\gamma = \sigma_I\sigma_J^{-1}$ whereas $\sigma_I \ne \sigma_{\gamma(I)}$ and $\sigma_J \ne \sigma_{\gamma(J)}$.

Given $I, J \in \mathcal{I}$, the *geometric intersection number* $I \cdot J$ of I and J is defined as the minimum of the number of intersection points of I_1° and J_1° over all pairs $(I_1, J_1) \in \mathcal{I}^2$ such that $I \sim I_1$, $J \sim J_1$, and I_1 is transverse to J_1. In this case we say that I_1 and J_1 *realize* $I \cdot J$.

FIGURE 2. Digon removal (p is a puncture)

If $I, J \in \mathcal{I}$ are transverse to each other, we say that a closed embedded disk D is a *digon between I and J* if D is the closure of a component of $\mathbb{D} \setminus (I \cup J)$, and ∂D is a union of an arc of I and an arc of J. The common ends of these arcs are called the *corners* of D. We say that (I', J') is obtained from (I, J) by a *digon removal* if it is obtained by one of the modifications in Figure 2 performed in a neighbourhood of a digon between I and J one of whose corners is not in X_n. The inverse operation is called a *digon insertion*.

The following two lemmas have a lot of analogs in the literature but it is easier to write (and to read) a proof than to search for an appropriate reference.

Lemma 4.12. *Let $I, J \in \mathcal{I}$ be transverse to each other. Then a pair of segments realizing $I \cdot J$ can be obtained from (I, J) by successive digon removals.*

Proof. Isotopies of I and of J which transform (I, J) to a pair of segments realizing $I \cdot J$ can be perturbed into a sequence of digon removals and digon insertions. So, it is enough to prove the following "diamond lemma": if (I_1, J_1) and (I_2, J_2) are obtained from (I, J) by two different digon removals, then either the pair $(I_1 \cup J_1, I_1)$ is isotopic to $(I_2 \cup J_2, I_2)$, or (I_1, J_1) and (I_2, J_2) admit digon removals with the same result. We leave it to the reader to check this statement (see Figure 3). □

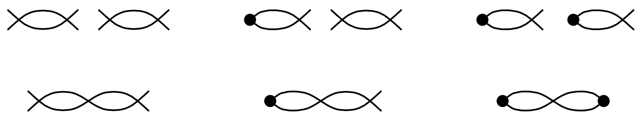


FIGURE 3. Cases to consider in the diamond lemma

Lemma 4.13. *Let $I_1, \dots, I_m \in \mathcal{I}$. Then there exist $I'_1, \dots, I'_m \in \mathcal{I}$ such that $I_i \sim I'_i$ for any $i = 1, \dots, m$, and (I'_i, I'_j) realizes $I_i \cdot I_j$ for any distinct $i, j = 1, \dots, m$.*

Proof. Induction on the total number of intersection points. If (I_i, I_j) does not realize $I_i \cdot I_j$, then by Lemma 4.12 there is a digon D between I_i and I_j. We can remove D so that the union of all segments is modified only near the corners of D. Then the total number of intersection points strictly decreases. □

4.3. End of the proof. Now we are ready to complete the proof of Theorem 1 for $n \geq 5$. First note that the injectivity of the restriction homomorphism $\mathrm{Aut}(B_n) \to \mathrm{Aut}(B'_n)$ is almost evident for any $n \geq 3$. Indeed, Let φ be an automorphism of $\mathbf{B}_n$ such that $\varphi|_{\mathbf{B}'_n} = \mathrm{id}$. By [5], we have $\varphi = \Lambda^k \tilde{\beta}$ with $\beta \in \mathbf{B}_n$ and $k = 0$ or 1 (see the introduction). Hence, for any $X \in \mathbf{B}'_n$, we have $\Lambda^k \tilde{\beta}(X) = X$, i. e., $\tilde{\beta}(X) = \Lambda^k(X)$. In particular, for $X_i = \sigma_i^{n(n-1)} \Delta^{-2}$, $1 \leq i < n$, we have $\tilde{\beta}(X_i) = X_i$ because $\Lambda(X_i) = X_i$. Hence the CRS of each X_i (which is a round circle containing the punctures i and $i+1$) is preserved by β; see Proposition 3.2(b). Hence β commutes with all σ_i for $i = 1, \dots, n-1$ whence $\beta \in Z(\mathbf{B}_n)$, i. e., $\tilde{\beta} = \mathrm{id}$. Thus $\varphi = \Lambda^k$. Since $\Lambda|_{\mathbf{B}'_n} \neq \mathrm{id}$, we conclude that $k = 0$, i. e., $\varphi = \mathrm{id}$.

Now let us prove that the restriction homomorphism $\mathrm{Aut}(B_n) \to \mathrm{Aut}(B'_n)$ is surjective for $n \geq 5$. So, let $n \geq 5$ and let φ be an automorphism of $\mathbf{B}'_n$. By [15; Theorem C], we may assume that either $\mu'\varphi = \mu'$, or $n = 6$ and $\mu'\varphi = \nu\mu'$ where ν is as in §2. However, $\mu'\varphi \neq \nu\mu'$ by Lemma 2.2. So, we assume that $\mu'\varphi = \mu'$. Then Lemma 2.1 implies that the automorphism φ_* of $\mathbf{J}_n^{\mathfrak{ab}}$ induced by φ is $\pm\,\mathrm{id}$. By composing φ with Λ if necessary, we may assume that $\varphi_* = \mathrm{id}$ (recall that Λ is the automorphism of $\mathbf{B}_n$ which takes each σ_i to σ_i^{-1}). By Lemma 4.8 we may also assume that

$$\varphi(\sigma_1\sigma_i^{-1}) = \sigma_1\sigma_i^{-1} \quad \text{for all } i = 3, \dots, n-1 \tag{9}$$

(otherwise we compose φ with $\tilde{\gamma}$ for the element γ given by Lemma 4.8). Hence

$$\varphi(\sigma_i\sigma_j^{-1}) = \sigma_i\sigma_j^{-1} \text{ and } \varphi(\sigma_i^{-1}\sigma_j) = \sigma_i^{-1}\sigma_j \quad \text{for all } i, j \in \{3, \dots, n-1\}. \tag{10}$$

Indeed, $\sigma_i\sigma_j^{-1} = (\sigma_1\sigma_i^{-1})^{-1}(\sigma_1\sigma_j^{-1})$ and $\sigma_i^{-1}\sigma_j = (\sigma_1\sigma_i^{-1})(\sigma_1\sigma_j^{-1})^{-1}$.

Let $I_1, J_1, K_1 \in \mathcal{I}$ be as in Lemma 4.10(c) where we set $I = [1,2]$, $J = [2,3]$, and $K = [4,5]$. By combining (8) with (9) for $i = 4$, we obtain $\sigma_1\sigma_4^{-1} = \sigma_{I_1}\sigma_{K_1}^{-1}$. Hence $I_1 \sim [1,2]$ and $K_1 \sim [4,5]$ by Lemma 4.11. Thus, if we set $L = J_1$, then (7) reads as

$$\varphi(\sigma_1\sigma_2^{-1}) = \sigma_1\sigma_L^{-1} \quad \text{and} \quad \varphi(\sigma_1^{-1}\sigma_2) = \sigma_1^{-1}\sigma_L. \tag{11}$$

By (5) for $\sigma_2^2\sigma_4^{-2}$ and by the claim $I_1 \cup J_1 \cong I \cup J$ of Lemma 4.10(c) we also have

$$\partial L = \{2,3\} \quad \text{and} \quad L \cdot [1,2] = 0. \tag{12}$$

By combining (9) with (11), we obtain

$$\varphi(\sigma_i\sigma_2^{-1}) = \sigma_i\sigma_L^{-1} \quad \text{and} \quad \varphi(\sigma_i^{-1}\sigma_2) = \sigma_i^{-1}\sigma_L \quad \text{for all } i = 3,\dots,n-1. \tag{13}$$

This fact combined with Lemma 4.10(b) and Lemma 4.11 implies

$$L \cdot [i, i+1] = 0 \qquad \text{for all } i = 4,\dots,n-1. \tag{14}$$

Indeed, for any $i = 4,\dots,n-1$, by Lemma 4.10(b) we have $\varphi(\sigma_2\sigma_i^{-1}) = \sigma_{I_1}\sigma_{J_1}^{-1}$ for some disjoint $I_1, J_1 \in \mathcal{I}$. On the other hand, $\varphi(\sigma_2\sigma_i^{-1}) = \sigma_L\sigma_i^{-1}$ by (13). Hence $I_1 \sim L$ and $J_1 \sim [i, i+1]$ by Lemma 4.11 whence (14) because $I_1 \cap J_1 = \varnothing$.

In the proof of the next lemma for any $n \ge 5$, we use Lemma 6.1 whose proof is based on Garside theory. However, for $n \ge 6$ we also give another proof which uses the material of this section only.

Lemma 4.14. $L \cdot [3,4] = 0$.

Proof. By Lemma 4.10(a) applied to $I = [2,3]$ and $J = [3,4]$, there exist I_1, J_1 such that $I_1 \cup J_1 \cong [2,4]$ (thus $I_1 \cdot J_1 = 0$) and $\varphi(\sigma_2\sigma_3^{-1}) = \sigma_{I_1}\sigma_{J_1}^{-1}$. By combining this fact with (13) for $i = 3$, we obtain $\sigma_L\sigma_3^{-1} = \sigma_{I_1}\sigma_{J_1}^{-1}$. Hence, by Lemma 6.1, there exists $\gamma \in \mathcal{D}$ such that $\sigma_L^\gamma = \sigma_{I_1}$ and $\sigma_3^\gamma = \sigma_{J_1}$ whence $\gamma(I_1) \sim L$ and $\gamma(J_1) \sim [2,3]$ by Lemma 4.9. Thus $L \cdot [3,4] = I_1 \cdot J_1 = 0$. □

Proof of Lemma 4.14 for $n \ge 6$ not using Garside theory. Let $n \ge 6$. We apply the same arguments that we used to obtain (11)–(13) but we set here $I = [3,4]$, $J = [2,3]$, $K = [5,6]$. So, let I_1, J_1, K_1 be as in Lemma 4.10(c) for the given choice of I, J, K. By combining (8) with (10) and (13), we obtain $\sigma_{I_1}\sigma_{K_1}^{-1} = \sigma_3\sigma_5^{-1}$ and $\sigma_{J_1}\sigma_{K_1}^{-1} = \sigma_L\sigma_5^{-1}$. Then Lemma 4.11 yields $I_1 = [3,4]$, $J_1 = L$, and $K_1 = [5,6]$. By Lemma 4.10(c), $I_1 \cup J_1$ is homeomorphic to $I \cup J$, hence $L \cdot [3,4] = I \cdot J = 0$. □

Further, Lemma 4.14 combined with (12) and (14), yields $L \cdot [i, i+1] = 0$ for any $i \in \{1\} \cup \{3,\dots,n-1\}$. By Lemma 4.13 this implies that $L \sim L_1$ where $L_1 \in \mathcal{I}$ is such that $[1,2] \cup L_1 \cup [3,n]$ is homeomorphic to a segment. Hence, up to composing φ with $\tilde\beta$ where $\beta \in \mathcal{D}$, $\beta([1,n]) = [1,2] \cup L_1 \cup [3,n]$, we may assume that $\sigma_L = \sigma_2$ in (9)–(11) and (13). This means that $\varphi(\sigma_i^\varepsilon\sigma_j^{-\varepsilon}) = \sigma_i^\varepsilon\sigma_j^{-\varepsilon}$ for any $\varepsilon = \pm 1$ and any $i, j \in \{1,\dots,n-1\}$. To complete the proof of Theorem 1 for $n \ge 5$, it remains to note that the elements $\sigma_i^\varepsilon\sigma_j^{-\varepsilon}$, $\varepsilon = \pm 1$, generate $\mathbf{B}'_n$. Indeed, it is shown in [11] (see also [15; §1.8]) that $\mathbf{B}'_n$ is generated by $u = \sigma_2\sigma_1^{-1}$, $v = \sigma_1\sigma_2\sigma_1^{-2} = (\sigma_2^{-1}\sigma_1)(\sigma_2\sigma_1^{-1})$, $w = (\sigma_2\sigma_1^{-1})(\sigma_3\sigma_2^{-1})$, and $c_i = \sigma_i\sigma_1^{-1}$, $i = 3,\dots,n-1$.

Remark 4.15. Our proof of Theorem 1 for $n \ge 5$ essentially uses Lemma 4.8 which is based on Dyer-Grossman's result [5] about $\mathrm{Aut}(B_n)$. If $n \ge 6$, Lemma 4.8 can be replaced by Lemma 6.2 (see below).

5. The case $n = 4$

Recall that $\mathbf{B}'_3$ is freely generated by $u = \sigma_2\sigma_1^{-1}$ and $t = \sigma_1^{-1}\sigma_2$ (see the Introduction). The group $\mathbf{B}'_4$ was computed in [11], namely $\mathbf{B}'_4 = \mathbf{K}_4 \rtimes \mathbf{B}'_3$ where $\mathbf{K}_4$ is the kernel of the homomorphism $\mathbf{B}_4 \to \mathbf{B}_3$, $\sigma_1, \sigma_3 \mapsto \sigma_1$, $\sigma_2 \mapsto \sigma_2$. The group $\mathbf{K}_4$ is freely generated by $c = \sigma_3\sigma_1^{-1}$ and $w = \sigma_2 c\,\sigma_2^{-1}$. The action of $\mathbf{B}'_3$ on $\mathbf{K}_4$ by conjugation is given by

$$ucu^{-1} = w, \qquad uwu^{-1} = w^2c^{-1}w, \qquad tct^{-1} = cw, \qquad twt^{-1} = cw^2. \tag{15}$$

Besides the elements c, w, u, t of $\mathbf{B}'_4$, we consider also

$$d = \psi_{2,2}(\sigma_1^{-1}; \sigma_1^2, \sigma_1^2) = \sigma_1^3\sigma_3^3\Delta^{-1}.$$

Lemma 5.1.
(a). $Z(d^2; \mathbf{B}'_4)$ is a semidirect product of infinite cyclic groups $\langle c\rangle \rtimes \langle d\rangle$ where d acts on $\langle c\rangle$ by $dcd^{-1} = c^{-1}$.
(b). $\langle c\rangle$ is a characteristic subgroup of $Z(d^2; \mathbf{B}'_4)$.

Proof. Let $G = Z(d^2; \mathbf{B}'_4)$.

(a). We have $G = Z(d^2; \mathbf{B}_4) \cap \ker e$ and, by [10; §5], $Z(d^2; \mathbf{B}_4)$ is the semidirect product $\langle \sigma_1, \sigma_3\rangle \rtimes \langle d\rangle$ where d acts on $\langle \sigma_1, \sigma_3\rangle$ by $\sigma_1^d = \sigma_3$, $\sigma_3^d = \sigma_1$.

(b). Let x be the image of c by an automorphism of G. Then (a) implies that x generates a normal subgroup of G and x is not a power of another element of G. It follows that $x \in \{c, c^{-1}\}$. □

Lemma 5.2. *All the conjugacy classes of $\mathbf{B}_4$ which are contained in $\mathbf{B}'_4$ are presented in Table 1. The corresponding centralizers are isomorphic to the groups indicated in this table.*

Proof. First, note that $\mathbf{B}'_n$ is normal in $\mathbf{B}_n$, hence for $X \in \mathbf{B}'_n$, the centralizer $Z(X; \mathbf{B}'_n)$ depends only on the conjugacy class of X in $\mathbf{B}_n$ (though this class may split into several classes in $\mathbf{B}'_n$).

The centralizers in $\mathbf{B}_4$ can be computed by a straightforward application of [10; Propositions 4.1] and Proposition 3.3. In the computation of $Z(\Delta^{2k+1}\sigma_2^{-12k-6})$ (which is, by the way, generated by Δ and σ_2), we use the fact that $Z(\Delta_3^{2k+1}; \mathbf{B}_3) = \langle\Delta\rangle$. This fact can be derived either from [10; Proposition 3.5] or from the uniqueness of Garside normal form in $\mathbf{B}_3$.

In all the cases except, maybe, the following two ones, the computation of $Z(X; \mathbf{B}'_4)$ is evident.

1). $X = Y^k$, $Y = \psi_{3,1}(\sigma_1^{-2}; \Delta_3^2, 1)$, $k \neq 0$. The group $Z(X; \mathbf{B}_4)$ is generated by $\psi_{3,1}(\sigma_1^2; 1, 1)$, σ_1, and σ_2. Since $\psi_{3,1}(\sigma_1^2; 1, 1) = Y\Delta_3^2$ and $\Delta_3 \in \mathbf{B}_3$, we can choose Y, σ_1, σ_2 for a generating set, and the result follows because $e(Y) = 0$.

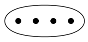
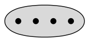
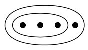
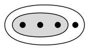
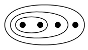
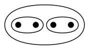
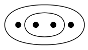
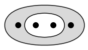
2). $X = \Delta^{2k}\sigma_1^{-6k}$, $k \neq 0$. The group $Z(X; \mathbf{B}_4)$ is the isomorphic image of $\mathbf{B}_{2,1} \times \mathbb{Z}$ under the mapping $f : (X, m) \mapsto \psi_{2,1}(X; \sigma_1^m, 1)$. Hence $Z(X; \mathbf{B}'_4)$ is the isomorphic image of $\mathbf{B}_{2,1}$ under the mapping $X \mapsto f(X, -e(f(X, 0)))$. □

Let $\varphi \in \mathrm{Aut}(\mathbf{B}'_4)$.

Lemma 5.3. *$\varphi(d)$ is conjugate in $\mathbf{B}_4$ to $d^{\pm 1}$.*

Proof. Let $x = \varphi(d^2)$. Since $Z(x; \mathbf{B}'_4) \cong Z(d^2; \mathbf{B}'_4)$, we see in Table 1 that $\varphi(\langle d^2\rangle) \subset \langle d^2\rangle$. By the same reasons we have $\varphi^{-1}(\langle d^2\rangle) \subset \langle d^2\rangle$, thus $\varphi(d^2) = d^{\pm 2}$ and the result follows from the uniqueness of roots up to conjugation [9]. □

Table 1. Centralizers of elements of $\mathbf{B}_4'$;
white/grey region $\Rightarrow$ the associated braid is periodic/pseudo-Anosov;
in $Z(d^{2k};\mathbf{B}_4)$ we mean $f:\mathbb{Z}\to\mathrm{Aut}(\mathbb{Z}\times\mathbb{Z})$, $f(1)(x,y)=(y,x)$.

CRS	X	$Z(X;\mathbf{B}_4)$	$Z(X;\mathbf{B}_4')$	$Z(X;\mathbf{B}_4')^{\mathfrak{ab}}$
	1	$\mathbf{B}_4$	$\mathbf{B}_4'$	
		$\mathbb{Z}^2$	$\mathbb{Z}$	
	$\psi_{3,1}(\sigma_1^{-2k};\Delta_3^{2k},1)$, $k\neq 0$	$\mathbf{B}_3\times\mathbb{Z}$	$\mathbf{B}_3'\times\mathbb{Z}$	$\mathbb{Z}^3$
		$\mathbb{Z}^3$	$\mathbb{Z}^2$	
		$\mathbb{Z}^3$	$\mathbb{Z}^2$	
	d^{2k}, $k\neq 0$ $d^{2k}c^l$, $l\notin\{0,\pm 6k\}$ d^{2k+1}	$\mathbb{Z}^2\rtimes_f\mathbb{Z}$ $\mathbb{Z}^3$ $\mathbb{Z}^2$	$\mathbb{Z}\rtimes\mathbb{Z}$ $\mathbb{Z}^2$ $\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}_2$
	$\Delta^{2k}\sigma_1^{-12k}$, $k\neq 0$ $\Delta^{2k+1}\sigma_2^{-12k-6}$	$\mathbf{B}_{2,1}\times\mathbb{Z}$ $\mathbb{Z}^2$	$\mathbf{B}_{2,1}$ $\mathbb{Z}$	$\mathbb{Z}^2$
		$\mathbb{Z}^3$	$\mathbb{Z}^2$	

Lemma 5.4. *If $\varphi(d)=d$, then $\varphi(c)=c^{\pm 1}$.*

Proof. If $\varphi(d)=d$, then $\varphi(Z(d^2;\mathbf{B}_4'))=Z(d^2;\mathbf{B}_4')$, and we apply Lemma 5.1. $\square$

Lemma 5.5. $\mathbf{K}_4$ *is a characteristic subgroup in* $\mathbf{B}_4'$.

Proof. Lemma 5.3 combined with Lemma 5.4 imply that $\varphi(c)$ is conjugate to c in $\mathbf{B}_4$. Since $\mathbf{K}_4$ is the normal closure of c in $\mathbf{B}_4$, it follows that $\varphi(c)\in\mathbf{K}_4$. The same arguments can be applied to any other automorphism of $\mathbf{B}_4'$, in particular, to $\varphi\tilde\sigma_2$ whence $\varphi\tilde\sigma_2(c)\in\mathbf{K}_4$. It remains to recall that $\varphi\tilde\sigma_2(c)=\varphi(w)$ and $\mathbf{K}_4=\langle c,w\rangle$. $\square$

Let

$$S_1=\begin{pmatrix}1&-1\\0&1\end{pmatrix},\ S_2=\begin{pmatrix}1&0\\1&1\end{pmatrix},\ T=S_1^{-1}S_2=\begin{pmatrix}2&1\\1&1\end{pmatrix},\ U=S_2S_1^{-1}=\begin{pmatrix}1&1\\1&2\end{pmatrix}.$$

Lemma 5.6. *T and U generate a free subgroup of* $\mathrm{SL}(2;\mathbb{Z})$.

Proof. It is well known that the correspondence $\sigma_1\mapsto S_1$, $\sigma_2\mapsto S_2$ defines an isomorphism $\mathbf{B}_3/\langle\Delta^2\rangle\to\mathrm{PSL}(2,\mathbb{Z})$, see, e.g., [18; §3.5] (this mapping is also a specialization of the reduced Burau representation). Since $u\mapsto U$ and $t\mapsto T$, the image of $\mathbf{B}_3'=\langle u,t\rangle$ is $\langle U,T\rangle$. Hence $\langle U,T\rangle$ is free. $\square$

Lemma 5.7. *If $\varphi|_{\mathbf{K}_4}=\mathrm{id}$, then $\varphi=\mathrm{id}$.*

Proof. Let $\varphi|_{\mathbf{K}_4}=\mathrm{id}$. Since $\mathbf{B}_4'=\mathbf{K}_4\rtimes\mathbf{B}_3'$, we may write $\varphi(u)=u_1a$ and $\varphi(t)=t_1b$ with $u_1,t_1\in\mathbf{B}_3'$ and $a,b\in\mathbf{K}_4$. For $x\in\mathbf{K}_4$, we have $\varphi\tilde u(x)=\varphi(uxu^{-1})=u_1axa^{-1}u_1^{-1}=\tilde u_1\tilde a(x)$. Since $\tilde u(x)\in\mathbf{K}_4$ and $\varphi|_{\mathbf{K}_4}=\mathrm{id}$, we conclude that $\tilde u(x)=\varphi\tilde u(x)=\tilde u_1\tilde a(x)$. Similarly, $\tilde t(x)=\tilde t_1\tilde b(x)$. Thus,

$$\tilde u\,|_{\mathbf{K}_4}=\tilde u_1\tilde b\,|_{\mathbf{K}_4}\qquad\text{and}\qquad\tilde t\,|_{\mathbf{K}_4}=\tilde t_1\tilde a\,|_{\mathbf{K}_4}\tag{16}$$

Consider the homomorphism $\pi : \mathbf{B}_4' \to \mathrm{Aut}(\mathbf{K}_4^{\mathfrak{ab}}) = \mathrm{GL}(2,\mathbb{Z})$, $x \mapsto (\tilde{x})_*$; here we identify $\mathrm{Aut}(\mathbf{K}_4^{\mathfrak{ab}})$ with $\mathrm{GL}(2,\mathbb{Z})$ by choosing the images of c and w as a base of $\mathbf{K}_4^{\mathfrak{ab}}$. It is clear that $\pi(a) = \pi(b) = 1$ and it follows from (15) that $\pi(tu^{-1}) = T$ and $\pi(t) = U$. Thus, by Lemma 5.6, the restriction of π to $\mathbf{B}_3'$ is injective. It follows from (16) that $\pi(u_1) = \pi(u)$ and $\pi(t_1) = \pi(t)$. Hence $u_1 = u$ and $t_1 = t$ by the injectivity of π. Then it follows from (16) that $\tilde{a}\,|_{\mathbf{K}_4} = \tilde{b}\,|_{\mathbf{K}_4} = \mathrm{id}$. Since $\mathbf{K}_4$ is free, its center is trivial, and we obtain $a = b = 1$. Thus $\varphi = \mathrm{id}$. □

Proof of Theorem 1 for $n = 4$. The injectivity of the restriction homomorphism $\mathrm{Aut}(B_4) \to \mathrm{Aut}(B_4')$ is already proven in the beginning of §4.3, so let us prove the surjectivity. Let $\varphi \in \mathrm{Aut}(\mathbf{B}_4')$. By Lemma 5.3, we may assume that $\varphi(d) = d^{\pm 1}$. Then, by Lemma 5.4, we may assume that $\varphi(c) = c^{\pm 1}$. Since $c^{\Delta} = c^{-1}$, we may further assume that $\varphi(c) = c$. By Lemma 5.5, $\varphi(c)$ and $\varphi(w)$ is a free base of $\mathbf{K}_4$. Since $\varphi(c) = c$, it follows that $\varphi(w) = c^p w^{\pm 1} c^q$, $p, q \in \mathbb{Z}$, see [18; §3.5, Problem 3]. We have $\tilde{\sigma}_1(c) = c$ and $\tilde{\sigma}_1(w) = 123\bar{1}\bar{2}\bar{1} = 123\bar{2}\bar{1}\bar{2} = 1\bar{3}231\bar{2} = c^{-1}w$. (here $1, \bar{1}, 2, \dots$ stand for $\sigma_1, \sigma_1^{-1}, \sigma_2, \dots$). Thus, by composing φ with a power of $\tilde{c}$ and a power of $\tilde{\sigma}_1$ if necessary, we may assume that $\varphi(w) = w^{\pm 1}$. For $\Phi = \Lambda\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\Delta}$, we have $\Phi(c) = c$ and $\Phi(w) = \bar{1}\bar{3}\bar{2}3\bar{1}213 = \bar{1}2\bar{3}\bar{2}21\bar{2}3 = \bar{1}21\bar{3}2\bar{3} = 21\bar{2}2\bar{3}\bar{2} = w^{-1}$ hence, by composing φ with Φ if necessary, we may assume that $\varphi(c) = c$ and $\varphi(w) = w$, thus $\varphi|_{\mathbf{K}_4} = \mathrm{id}$ and the result follows from Lemma 5.7. □

6. Appendix. Garside-theoretic lemmas

Here, using Garside theory, we prove two statements one of which (Lemma 6.1) is used only in the proof of Theorem 1 for $n = 5$, see the proofs of Lemma 4.14, and the other one (Lemma 6.2) can be used in the proof of Theorem 1 for $n \geq 6$ instead of Dyer-Grossman theorem, see Remark 4.15.

Let $n \geq 3$ and let $\mathcal{I}$ and $\sigma_I \in \mathbf{B}_n$ for $I \in \mathcal{I}$ be as in §4.2.

Lemma 6.1. *Let $k, l \in \mathbb{Z} \setminus \{0\}$ and $I, J \in \mathcal{I}$. Suppose that $\sigma_I^k \sigma_J^l$ is conjugate to $\sigma_1^k \sigma_2^l$. Then there exists $u \in \mathbf{B}_n$ such that $\sigma_I^u = \sigma_1$ and $\sigma_J^u = \sigma_2$, in particular, $I \cdot J = 0$.*

Proof. It follows from Corollary 6.4 that there exists $u \in \mathbf{B}_n$ and $p, q, r, s \in \mathbb{Z} \cap [1, n]$ such that $\sigma_I^u = \sigma_{pq}$ and $\sigma_J^u = \sigma_{rs}$, This means that $\sigma_I^u = \sigma_{I_1}$ and $\sigma_J^u = \sigma_{J_1}$ where $I_1, J_1 \in \mathcal{I}$ satisfy one of the following conditions:

(a) $I_1 \cap J_1$ is a common endpoint of I_1 and J_1;
(b) $\mathrm{Card}(\partial I_1 \cup \partial J_1) = 4$.

It is enough to exclude Case (b). Indeed, in this case $\beta = \sigma_1^k \sigma_2^l$ cannot be conjugate to $\beta_1 = \sigma_{I_1}^k \sigma_{J_1}^l$, because $\mathrm{lk}_{i,j}(\beta^2) = 0$ for $i \notin \{1, 2, 3\}$ and any j whereas $\mathrm{lk}_{p,q}(\beta_1^2) = k$ and $\mathrm{lk}_{r,s}(\beta_1^2) = l$ for pairwise distinct p, q, r, s. □

Lemma 6.2. *Let X and Y be two distinct conjugates of σ_1 in $\mathbf{B}_n$, $n \geq 3$. If $XYX = YXY$, then there exists $u \in \mathbf{B}_n$ such that $X^u = \sigma_1$ and $Y^u = \sigma_2$.*

Proof. Follows from Lemma 6.5. □

When speaking of Garside structures on groups, we use the terminology and notation from [21]. Let $(G, \mathcal{P}, \delta)$ be a symmetric homogeneous square-free Garside structure with set of atoms $\mathcal{A}$ (for example, Birman-Ko-Lee's Garside structure [3] on the braid group, i. e., $G = \mathbf{B}_n$, $\mathcal{A} = \{\sigma_{ij}\}_{1 \leq i < j \leq n}$, $\mathcal{P} = \{x_1 \dots x_m \mid x_i \in \mathcal{A}, m \geq 0\}$, $\delta = \sigma_{n-1}\sigma_{n-2}\dots\sigma_2\sigma_1$).

For $a, b \in G$ we set $b^G = \{b^a \mid a \in G\}$, and we write $a \sim b$ if $a \in b^G$ and $a \preccurlyeq b$ if $a^{-1}b \in \mathcal{P}$. We define the set of *simple elements* of G as $[1, \delta] = \{s \in G \mid 1 \preccurlyeq s \preccurlyeq \delta\}$. For $X \in G$, the *canonical length* of X (denoted by $\ell(X)$) is the minimal r such that $X = \delta^p A_1 \dots A_r$ for some $p \in \mathbb{Z}$, $A_1, \dots, A_r \in [1, \delta] \setminus \{1\}$. The *summit length* of X is defined as $\ell_s(X) = \min\{\ell(Y) \mid Y \in X^G\}$. We denote the *cyclic sliding* of X and the *set of sliding circuits* of X by $\mathfrak{s}(X)$ and $\mathrm{SC}(X)$ respectively (these notions were introduced in [8], see also [21; Definition 1.12]).

The following result was, in a sense, proven in [21] without stating it explicitly.

Theorem 6.3. *Let $k, l \in \mathbb{Z} \setminus \{0\}$, $x, y \in \mathcal{A}$, and let $Z = XY$ where $X \sim x^k$ and $Y \sim y^l$. Then there exists $u \in G$ such that one of the following possibilities holds:*

(i) *$X^u = x_1^k$ and $Y^u = y_1^l$ with $x_1 \in x^G \cap \mathcal{A}$ and $y_1 \in y^G \cap \mathcal{A}$, or*

(ii) *$\ell(Z^u) = \ell(X^u) + \ell(Y^u)$ and $Z^u \in \mathrm{SC}(Z)$.*

Proof. If the statement is true for (k, l), then it is true for $(-k, -l)$, therefore we may assume that $l > 0$. Then the proof of [21; Corollary 3.5] may be repeated almost word by word in our setting if we define $\mathcal{Q}_m$ as $\{Z^u \mid u \in \mathcal{U}_m\}$ where $\mathcal{U}_m = \{u \mid \ell(X^u) \le 2m + |k|, \ell(Y^u) = l\}$. Namely, let m be minimal under the assumption that $\mathcal{Q}_m \ne \varnothing$. If $m = 0$, then (i) occurs. If $m > 0$, then, similarly to [21; Lemma 3.3] we show that if $u \in \mathcal{U}_m$, then $\ell(Z^u) = \ell(X^u) + \ell(Y^u)$, and similarly to [21; Lemma 3.4] we show that $\mathfrak{s}(\mathcal{Q}_m) \subset \mathcal{Q}_m$. whence $\mathcal{Q}_m \cap \mathrm{SC}(Z) \ne \varnothing$ which implies (ii). □

Corollary 6.4. *With the hypothesis of Theorem 6.3, assume that Z is conjugate to $x^k y^l$. Then there exists $u \in G$ such that (i) holds.*

Proof. Suppose that (ii) occurs. Since $\ell(Z) = \ell_s(Z) \le \ell(x^k y^l) \le |k| + |l|$, we have $\ell(X^u) + \ell(Y^u) \le |k| + |l|$. By combining this fact with $\ell(X^u) \ge |k|$ and $\ell(Y^u) \ge |l|$, we obtain $\ell(X^u) = |k|$ and $\ell(Y^u) = |l|$, and the result follows from [21; Theorem 1a]. □

Lemma 6.5. *Let $X \sim x$ and $Y \sim y$ where $x, y \in \mathcal{A}$. If $XYX = YXY$, then there exists $u \in G$ such that $X^u, Y^u \in \mathcal{A}$.*

Proof. Without loss of generality we may assume that $Y = y \in \mathcal{A}$. By [21; Theorem 1a], the left normal form of X is $\delta^{-p} \cdot A_p \cdot \ldots \cdot A_1 \cdot x \cdot B_1 \cdot \ldots \cdot B_p$ where $A_i, B_i \in [1, \delta] \setminus \{1\}$, $A_i \delta^{i-1} B_i = \delta^i$ for $i = 1, \dots, p$. By symmetry, the right normal form of X is $C_p \cdot \ldots \cdot C_1 \cdot x \cdot D_1 \cdot \ldots \cdot D_p \cdot \delta^{-p}$ again with $C_i, D_i \in [1, \delta] \setminus \{1\}$, $C_i \delta^{i-1} D_i = \delta^i$ for $i = 1, \dots, p$. We have $\sup yXy \le 2 \sup y + \sup X = 3 + p$. Thus, if $p > 0$, then $\sup X + \sup y + \sup X = 3 + 2p > 3 + p = \sup xYx = \sup YxY$. Then, by [22; Lemma 2.1b], either $B_p y$ or yC_p is a simple element. Without loss of generality we may assume that $B_p y \in [1, \delta]$.

Since the Garside structure is symmetric and $B_p y$ is simple, there exists an atom y_1 such that $B_p y = y_1 B_p$. Thus, for $v = B_p^{-1}$, we have $y^v \in \mathcal{A}$ and the left normal form of X^v is $\delta^{p-1} \cdot A_{p-1} \cdot \ldots \cdot A_1 \cdot x \cdot B_1 \cdot \ldots \cdot B_{p-1}$. Therefore, the induction on p yields $X^u = x$ and $Y^u = z \in \mathcal{A}$ for $u = (B_1 \dots B_p)^{-1}$. □

Remark 6.6. (Compare with [4, 14]). Lemma 6.5 admits the following generalization which can be proven using the results and methods of [21, 22]. *Let $X \sim x$ and $Y \sim y$ for $x, y \in \mathcal{A}$. Then either X and Y generate a free subgroup of G, or there exists $u \in G$ such that $X^u, Y^u \in \mathcal{A}$. In the latter case, the subgroup* *generated by* X, Y *is either free or isomorphic to Artin group of type* $I_2(p)$, $p \geq 2$. In particular, for $G = \mathbf{B}_n$, *if* X *and* Y *are two conjugates of* σ_1, *then either* X *and* Y *generate a free subgroup of* $\mathbf{B}_n$, *or there exists* $u \in \mathbf{B}_n$ *such that* $X^u = \sigma_1$ *and* $Y^u = \sigma_i$ *for some* i. Maybe, I will write a proof of this fact in a future paper.

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