PSEUDOHOLOMORPHIC, ALGEBRAICALLY UNREALIZABLE CURVES

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To Vladimir Igorevich Arnold with admiration

Abstract. We show that there exists a real non-singular pseudoholomorphic sextic curve in the affine plane which is not isotopic to any real algebraic sextic curve. This result completes the isotopy classification of real algebraic affine M-curves of degree 6. Comparing with the isotopy classification of real affine pseudoholomorphic sextic M-curves, obtained earlier by the first author, one finds three pseudoholomorphic isotopy types which are algebraically unrealizable. In a similar way we find a real pseudoholomorphic, algebraically unrealizable $(M-1)$-curve of degree 8 on a quadratic cone with a special position with respect to a generating line. The proofs are based on the Hilbert-Rohn-Gudkov approach developed by the second author and the cubic resolvent method developed by the first author.

Introduction

This paper can be considered as a continuation of [15]. The study of plane real pseudoholomorphic curves has been initiated by the first author [11, 3, 12]. It gives a new insight on the classical Hilbert 16th problem whose first part is the question of classification of the oval arrangements of real plane non-singular algebraic curves. This classification is completed only for small degrees. The global rigidity of the symplectic structure, which especially becomes apparent in Gromov's theory of pseudoholomorphic curves [5], makes the isotopy classification[1] of real pseudoholomorphic and algebraic curves rather similar. On the other hand, the difference between pseudoholomorphic and algebraic curves and methods which allow one to distinguish these classes are of a natural interest. This can be viewed as an analogue of the question of Viro [25], who introduced *flexible curves* and asked if there is a real flexible non-singular curve in $\mathbf{R}P^2$ which is not isotopic to a homologous real algebraic curve, the question which still is open.

In the present paper, we provide two examples of real pseudoholomorphic curves which are not isotopic to any real algebraic curve in the corresponding class. One example is an affine M-curve of degree 6, and this is a completion of the isotopy classification of real algebraic affine M-sextics. The other example is a real

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[1]Under isotopy as usual we understand a smooth isotopy in the real part of the ambient algebraic surface

$(M-1)$-curve of degree 8 on a quadratic cone. Previously known examples of real pseudoholomorphic, algebraically unrealizable curves have been found in [3,15,13]. We should like also to mention an infinite series of examples of real flexible curves on Hirzebruch surfaces found by Welschinger [26] which are are not realizable neither algebraically, nor pseudoholomorphically.

Similarly to [15], we prove the nonexistence of a real algebraic curve with a specific arrangement of its connected components in two steps. First, we show that a hypothetical curve of such a type must degenerate into a highly singular real algebraic curve (Section 2). A degeneration is obtained in the framework of the Hilbert-Rohn-Gudkov approach: we consider a one-parametric equisingular deformation of a given curve such that some of the curve geometric characteristics changes monotonically, and this necessarily leads to a degeneration. If the classical Hilbert-Rohn-Gudkov method deals with only one-dimensional families of nodal curves passing through fixed points [7,17,18,21,15], in the present paper we use higher-dimensional equisingular families of curves with arbitrary singularities A_k, $k \geq 1$. One cannot reduce the dimension of the latter families to 1 by fixing extra points without loss of nice geometric properties (like smoothness), thus, inside these families, we construct piece-wise algebraic one-dimensional paths which may consist of infinitely many pieces. The construction of such paths and the study of their limits as the main technical novelty in our development of the Hilbert-Rohn-Gudkov method. Notice also that a rather complicated tree of possible degenerations in the above deformations can be substantially reduced by applying prohibitions provided by the braid group techniques from [11]. The second step of our consideration (see Section 3) is the prohibition of the hypothetical singular algebraic curves obtained in the first stage. In contrast to [15], the main tool used here is rather simpler and is based on the cubic resolvent method suggested by the first author (this method was used already in [13]). Namely, a hypothetical algebraic curve is represented as a ramified four-fold covering of the line, i.e., as an algebraic family of real equations of degree 4, and the contradiction is extracted from the properties of the family of the cubic resolvents of these equations.

1. Statement of results

1.1. Real plane affine sextic M-curves. A real plane non-singular pseudoholomorphic or algebraic curve of degree m is called M-curve if its real part has the maximal possible for degree m number of connected components. This number is $g+1 = (m-1)(m-2)/2+1$ for a projective curve, and $g+m = (m-1)(m-2)/2+m$ for an affine curve. The projective closure of an affine M-curve is a projective M-curve which has a connected component crossing the infinite line at m points. The isotopy classification of real plane projective M-curves is known for degree ≤ 7 in the algebraic case [23,24], and for degree ≤ 8 in the pseudoholomorphic case [12]. The isotopy classification of real plane affine M-curves has been known for degree ≤ 5 in the algebraic case [16], and for pseudoholomorphic curves of degree ≤ 6 [11]. We complete the classification of affine algebraic M-sextics and, for the reader convenience, formulate both algebraic and pseudoholomorphic classification. The notation for isotopy types of affine M-sextics, which we represent as pairs of a projective sextic and a real line in $\mathbf{R}P^2$ (the line at infinity) is shown in Figure 1, where a, b, c denote the number of empty ovals[2] in the corresponding domains.

[2] An oval is called *empty* if it does not surround any other oval.

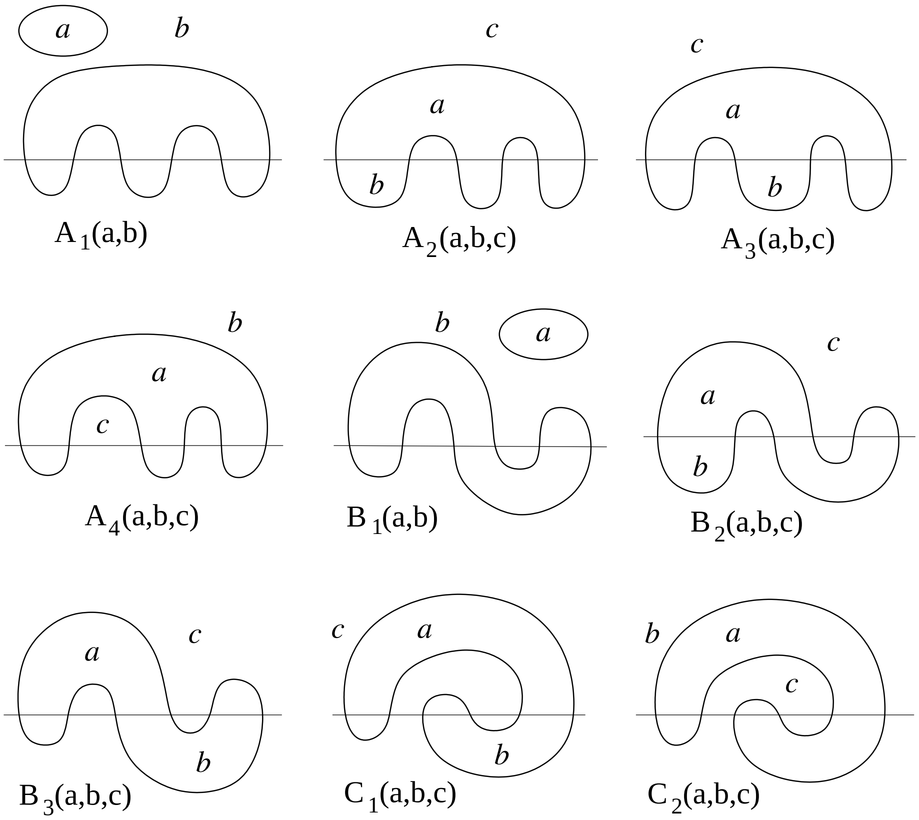

FIGURE 1

Theorem 1.1. *(a) Any real affine pseudoholomorphic M-sextic belongs to one of the following 35 isotopy types:*

$$\left\{\begin{array}{ll} A_1(a,b), & (a,b)=(1,8),(5,4), \\ A_2(a,b,c), & (a,b,c)=(1,8,1),(8,1,1),(0,5,5),(1,4,5), \\ & \qquad\qquad (4,1,5),(5,0,5),(0,1,9),(1,0,9), \\ A_3(a,b,c), & (a,b,c)=(4,5,1),(7,2,1),(2,3,5),(4,1,5),(0,1,9), \\ & \qquad\qquad (0,5,5), \\ A_4(a,b,c), & (a,b,c)=(1,8,1),(5,4,1), \\ B_1(a,b), & (a,b)=(1,8),(5,4), \\ B_2(a,b,c), & (a,b,c)=(1,8,1),(0,5,5),(5,0,5),(0,1,9),(1,0,9), \\ B_3(a,b,c), & (a,b,c)=(3,6,1),(1,4,5),(2,3,5), \\ C_1(a,b,c), & (a,b,c)=(0,9,1),(7,2,1),(0,5,5),(3,2,5),(0,1,9), \\ C_2(a,b,c), & (a,b,c)=(1,7,2),(5,3,2) \end{array}\right. \tag{1}$$

and

$$A_4(1,4,5), \quad B_2(1,4,5), \quad C_2(1,3,6)\ . \tag{2}$$

Each of the types (1), (2) is represented by a real affine pseudoholomorphic sextic.

(b) Each of the types (1) is represented by a real affine algebraic sextic. The types (2) are not realizable by real affine algebraic sextics.

The nonexistence of pseudoholomorphic sextics of the types different from (1) and (2) is proved in [11], a construction of algebraic sextics of types (1) can be found in [9, 10], a construction of pseudoholomorphic sextics of types (2) can be found in [3, 11]

The nonexistence of algebraic sextics of type $B_2(1,4,5)$ is proved in [3] by means of the use of pencils of algebraic cubics, the nonexistence of algebraic sextics of type $A_4(1,4,5)$ is proved in [15]. Thus, to complete the proof of Theorem 1.1, we have to prohibit only algebraic sextics of type $C_2(1,3,6)$, what is done in the present paper (combine Lemma 2.10 and Section 3.4).

1.2. Real curves on a quadratic cone. In [15], we studied curves of bidegree $(4,8)$ on a quadratic cone which have an oval arranged in a certain way with respect to a generating line of the cone. We have completely classified the M-curves (both, algebraic and pseudoholomorphic) and also pseudoholomorphic $(M-1)$-curves. Algebraic realizability of two pseudoholomorphically realizable $(M-1)$-curves remained open. Here we exclude one of them.

Blowing up the singular point of a cone, we obtain the Hirzebruch surface $\mathcal{F}_2$. We recall that the *Hirzebruch surface* (relatively minimal rational ruled surface) $\mathcal{F}_n$ $(n \geq 1)$ is the fiberwise compactification of the line bundle $\mathcal{O}(n)$ over $\mathbf{P}^1$. There is a fibration $\pi_n : \mathcal{F}_n \to \mathbf{P}^1$ with fibers isomorphic to $\mathbf{P}^1$. This fibration has one exceptional section which we denote by E. Its self-intersection is equal to $(-n)$. The surface $\mathcal{F}_n$ can be defined by gluing of four copies of $\mathbf{C}^2$ with coordinate systems $(x_0,y_0),\dots,(x_3,y_3)$ and the coordinate changes

$$x_1 = x^{-1},\ \ y_1 = y_0x_0^{-n} \qquad x_2 = x_0,\ \ y_2 = y_0^{-1} \qquad x_3 = x_2^{-1},\ \ y_3 = y_2x_2^n.$$

The projection π is defined by $(x_j,y_j) \to x_j$ for each $j = 1,\dots,4$. The exceptional section E is defined by $y_2 = 0$ and by $y_3 = 0$. The coordinate system (x_0,y_0) is called *standard*.

The group of classes of divisors (the Picard group) of $\mathcal{F}_n$ is generated by E and a fiber F. If a curve belongs to the linear system $|nE + kF|$, we say that it has *bidegree* (n,k).

So, we consider curves of bidegree $(4,8)$ on $\mathcal{F}_2$, which have 9 ovals located with respect to some fibre F_0 as shown in Figure 2(a) (type $A(3,1,4)$ in the notation of [15]). These curves can be represented as affine plane curves defined by a polynomial with Newton triangle spanned by the points $(0,0)$, $(8,0)$, $(0,4)$, and having branches located as shown in Figure 2(b). Such a pseudoholomorphic curve has been constructed in [15]. We state

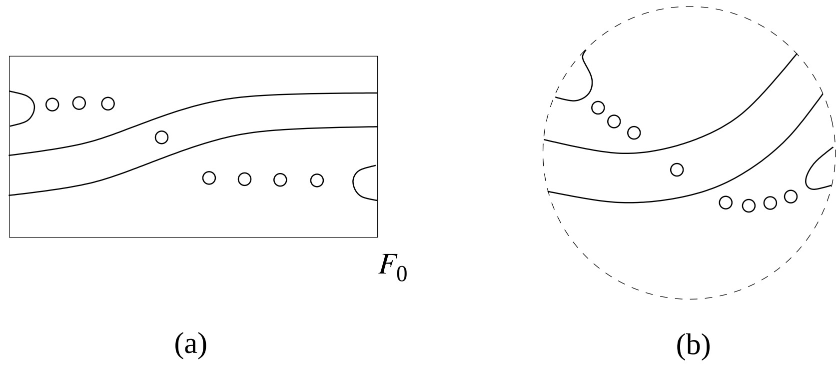

FIGURE 2

Theorem 1.2. *There is no real algebraic curve of bidegree (4,8) on $\mathcal{F}_2$ realizing the isotopy type $A(3,1,4)$.*

2. Degeneration of real algebraic curves: The Hilbert-Rohn-Gudkov approach

2.1. Equisingular families of curves with singularities of type A_k.

Let z be a singular point of type A_k, $k \ge 1$, on an algebraic curve C in a smooth algebraic surface Σ. In appropriate local coordinates x, y in a small neighborhood $U(z)$ of $z = (0,0)$ in Σ, the curve C is given by an equation $y^2 \pm x^{k+1} = 0$. We identify $\mathcal{O}_{\Sigma,z}$ with the ring $\mathbf{C}\{x,y\}$ of germs of holomorphic functions in x, y at the origin. Any curve $C' \in |C|$ defines a non-trivial element of $\mathcal{O}_{\Sigma,z}$ up to a non-zero constant factor, and without further confusion we shall denote one of these elements by the same symbol.

Introduce the ideals in the ring $\mathcal{O}_{\Sigma,z}$

$$I(C,z) = \langle y^2,\ yx^{[(k-1)/2]},\ x^k\rangle, \quad I_0(C,z) = \langle y^2,\ yx^{[(k+1)/2]},\ x^{k+1}\rangle\ ,$$

$$I_0'(C,z) = \langle y^2,\ yx^{[k/2]+1},\ x^{k+1}\rangle\ .$$

Clearly, $C \in I_0'(C,z) \subset I_0(C,z) \subset I(C,z)$, and elements $C' \in I_0'(C,z)$ close to C have at z singularity A_k. Observe also that $I_0(C,z) = I_0'(C,z)$ as k is odd. Denote by $V_0(C,z)$ and $V(C,z)$ germs at C of the subsets of $\mathcal{O}_{\Sigma,z}$

$$\{\varphi(x, y + ux^{[(k+1)/2]})\ :\ \varphi \in I_0'(C,z),\ u \in \mathbf{C}\}\ , \tag{3}$$

$$\{\varphi(x+t, y + ux^{[(k+1)/2]} + vx^{[(k-1)/2]})\ :\ \varphi \in I_0'(C,z),\ t,u,v \in \mathbf{C}\}\ , \tag{4}$$

respectively, where, for each $\varphi \in I_0'(C,z)$, t, u, v are obstructed by the condition that the elements of $V_0(C,z), V(C,z)$ remain holomorphic at z.

Proposition 2.1. *(a) If $\Lambda \subset \mathcal{O}_{\Sigma,z}$ is a finite-dimensional linear subspace, which contains C and intersects transversally with the ideal $I(C,z)$ (resp., $I_0(C,z)$) then $V(C,z) \cap \Lambda$ (resp. $V_0 \cap \Lambda$) is a germ at C of a smooth subvariety with the tangent space $T_C(V(C,z) \cap \Lambda) = I(C,z) \cap \Lambda$ (resp., $T_C(V_0(C,z) \cap \Lambda) = I_0(C,z) \cap \Lambda$).*

(b) Any element $C' \in V_0(C,z)$ has singularity A_k at z, and any element $C' \in V(C,z)$ has a singular point of type A_k in $U(z)$, which lies on the line $\gamma_z = \{y = 0\}$ as $k \ge 3$. In addition, the total intersection of C and C' in $U(z)$ (which reduces here to a common convergence neighborhood) is

$$(C' \cdot C)_{U(z)} \ge k + 2\left[\frac{k+1}{2}\right] + \begin{cases} 1, & C' \in V_0(C,z), \\ -1, & C' \in V(C,z), \end{cases} \tag{5}$$

where $U(z)$ reduces up to z if $C' \in V_0(C,z)$ and $\mu(C,z)$ is odd.

Remark 2.2. (1) The ideals $I(C,z)$, $I_0(C,z)$ define zero-dimensional schemes $Z(C,z)$, $Z_0(C,z) \subset \Sigma$ of degrees $\deg Z(C,z) = k-1+[(k+1)/2]$, $\deg Z_0 = k+1+[(k+1)/2]$.

(2) The line γ_z depends in general on the choice of local coordinates. We fix just one of these lines.

Proof of Proposition 2.1. Since we consider finite-dimensional subspaces in $\mathcal{O}_{\Sigma,z}$, we can restrict this ring up to the ring (denoted by the same symbol) of functions

$\varphi(x,y)$, which are holomorphic in a fixed neighborhood U of z in Σ and such that $\varphi(x+t,y+ux^{(k+1)/2}+vx^{(k-1)/2})$ are holomorphic in U as well, for all sufficiently close to zero $t,u,v\in\mathbf{C}$. Similarly we restrict $I_0(C,z)$, $I_0'(C,z)$, $I(C,z)$, $I'(C,z)$, $V_0(C,z)$, $V(C,z)$.

For k odd, it is easy to check that $V_0(C,z)=I_0(C,z)$, and statement (a) of Proposition becomes trivial. For k even, (3) reads as

$$V_0(C,z)=\{\varphi(x,y)\in\mathcal{O}_{\Sigma,z}\ :\ \varphi(x,y+ux^{k/2})\in I_0'(C,z)\}\ .$$

Any element φ of $V_0(C,z)$ uniquely determines the parameter u, which is just the coefficient of $yx^{k/2}$ in the power series expansion of φ. The fibers of the projection $\varphi\mapsto u$,

$$I_u(C,z)=\{\varphi(x,y)\in\mathcal{O}_{\Sigma,z}\ :\ \varphi(x,y-ux^{k/2})\in I_0'(C,z)\}\ ,$$

are germs of ideals in $\mathcal{O}_{\Sigma,z}$ of the same codimension as $I_0'(C,z)$. Consider the intersection $\Lambda\cap V_0(C,z)$. Since Λ intersects with $I_0(C,z)$ transversally, so does the linear space $\Lambda'=\Lambda+\mathrm{Span}\{yx^{k/2}\}$ with $I_0'(C,z)$, which means $\dim\Lambda'/I_0'(C,z)=\operatorname{codim}I_0'(C,z)=3k/2+1$. Thus, $\Lambda'\cap V_0(C,z)$ fibers over $(\mathbf{C},0)$ into the germs of equidimensional linear spaces $\Lambda'\cap I_u(C,z)$, which are disjoint in a neighborhood of C. If $\Lambda_0\subset\Lambda'$ is a linear subspace, containing $yx^{k/2}$, which is transverse to $I_0(C,z)$ and has dimension $\dim\Lambda_0=\operatorname{codim}I_0(C,z)$, then it meets each ideal $I_u(C,z)$ only at zero. Thus, $C+\Lambda_0$ meets each ideal $I_u(C,z)$ at only one point, and the intersection points form a smooth curve with the tangent line spanned by

$$\left.\frac{\partial C(x,y+ux^{k/2})}{\partial u}\right|_{u=0}=2yx^{k/2}\ .$$

Hence $\Lambda'\cap V_0(C,z)$ is a germ of a smooth variety of codimension $3k/2$ and the tangent space $\Lambda'\cap(I_0'(C,z)+\mathrm{Span}\{yx^{k/2}\})=\Lambda'\cap I_0(C,z)$. Finally, notice that Λ is transverse to $T_C(\Lambda'\cap V_0(C,z))=\Lambda'\cap I_0(C,z)$ in Λ', which completes the proof of statement (a) of Proposition in the case considered.

The proof of this statement for $V(C,z)$ is a word-for-word copy of the above reasoning with the following changes:

(i) If k is odd then

(1) $V(C,z)=\{\varphi(x,y)\in\mathcal{O}_{\Sigma,z}\ :\ \varphi(x+t,y+vx^{(k-1)/2})\in I_0'(C,z)\}$,
(2) the fibers of the projection $\varphi\in V(C,z)\mapsto(t,v)\in(\mathbf{C}^2,0)$ are the germs of ideals

$$I_{t,v}(C,z)=\{\varphi(x,y)\in\widehat{\mathcal{O}}_{\Sigma,z}\ :\ \varphi(x-t,y-vx^{(k-1)/2})\in I_0(C,z)\}\ ,$$

(3) $\Lambda'=\Lambda+\mathrm{Span}\left\{yx^{(k-1)/2},\ x^k\right\}$,
(4) the intersection points of $C+\Lambda_0$ with $I_{t,v}(C,z)$ form a smooth surface with the tangent plane spanned by

$$\left.\frac{\partial C(x+t,y+vx^{(k-1)/2})}{\partial t}\right|_{t=v=0}=(k+1)x^k\ ,$$

$$\left.\frac{\partial C(x+t,y+vx^{(k-1)/2})}{\partial v}\right|_{t=v=0}=2yx^{(k-1)/2}\ .$$

(ii) If k is even then

(1) $V(C,z)=\{\varphi(x,y)\in\mathcal{O}_{\Sigma,z}\ :\ \varphi(x+t,y+ux^{k/2}+vx^{k/2-1})\in I'_0(C,z)\}$,

(2) the fibers of the projection $\varphi\in V(C,z)\mapsto(t,u,v)\in(\mathbf{C}^3,0)$ are the germs of ideals

$$I_{t,u,v}(C,z)=\{\varphi(x,y)\in\widehat{\mathcal{O}}_{\Sigma,z}\ :\ \varphi(x-t,y-ux^{k/2}-vx^{k/2-1})\in I_0(C,z)\}\ ,$$

(3) $\Lambda'=\Lambda+\mathrm{Span}\left\{yx^{k/2-1},\ yx^{k/2},\ x^k\right\}$,

(4) the intersection points of $C+\Lambda_0$ with $I_{t,u,v}(C,z)$ form a smooth three-manifold with the tangent space spanned by

$$\left.\frac{\partial C(x+t,y+ux^{k/2}+vx^{k/2-1})}{\partial t}\right|_{t=u=v=0}=(k+1)x^k\ ,$$

$$\left.\frac{\partial C(x+t,y+ux^{k/2}+vx^{k/2-1})}{\partial u}\right|_{t=u=v=0}=2yx^{k/2}\ ,$$

$$\left.\frac{\partial C(x+t,y+ux^{k/2}+vx^{k/2-1})}{\partial v}\right|_{t=u=v=0}=2yx^{k/2-1}\ .$$

The statement (b) follows from that fact that, for any local branch $C^{(i)}$ of C at z, and $C'\in\Lambda\cap V(C,z)$,

$$(C'\cdot C^{(i)})_{U(z)}\ge\min\{(\varphi\cdot C^{(i)})_z\ :\ \varphi\in T_C(\Lambda'\cap V(C,z)\}$$

(see [6]), and elementary computations with the generators of $I(C,z)$. $\Box$

Remark 2.3. The ideals $I(C,z)$, $I_0(C,z)$ can be described via their generic elements. Namely, if k is odd, then generic elements of $I_0(C,z)$ have $(k+1)/2$ infinitely near points common with C, all of multiplicity 2, and generic elements of $I(C,z)$ have the same infinitely near points, $(k-1)/2$ of them of multiplicity 2 and the last one of multiplicity 1. If k is even then generic elements of $I_0(C,z)$ have $k/2+1$ infinitely near points common with C, among them $k/2$ points of multiplicity 2 and one point of multiplicity 1, respectively, generic elements of $I(C,z)$ have the same infinitely near points, among them $k/2-1$ points of multiplicity 2 and two points of multiplicity 1. Furthermore, $I_0(C,z)$ and $I(C,z)$ are characterized by the above properties.

For zero-dimensional schemes determined by infinitely near points $w_1,...,w_p$ and multiplicities $k_1,...,k_p$ at them (so called *cluster* schemes) we introduce the notation $Z(k_1w_1,...,k_pw_p)$. Namely, the local ideal defining $Z(k_1w_1,...,k_pw_p)$ is generated by power series convergent in a neighborhood of w_1 which determine germs of analytic curves, having multiplicity k_1 at w_1 and multiplicity k_i at the infinitely near point w_i (appearing after $(i-1)$ blow-ups), $i=2,...,p$.

Proposition 2.4. *(a) Let C be an irreducible algebraic curve in a linear system $|D|$ on a smooth algebraic surface Σ with singular points $z_1,..,z_r$ of types $A_{k_1},...,A_{k_r}$, respectively. Assume that a germ of $|D|$ at C is embedded in a natural way into $\mathcal{O}_{\Sigma,z_i}$ for any $i=1,...,r$. Fix distinct non-singular points $w_1,...,w_s\in C$. If, for some $p\le r$,*

$$\sum_{i=1}^r k_i+2p-r+s<-DK_\Sigma\ , \tag{6}$$

then the germ W at C of the set of curves in $|D|$, passing through $w_1,...,w_s$ and belonging to $V_0(C,z_i)$, $1\le i\le p$, and to $V(C,z_i)$, $p<i\le r$, is smooth of dimension

$$\dim W = \frac{D^2-DK_\Sigma}{2} - \sum_{i=1}^{r}\left(k_i-1+\left[\frac{k_i+1}{2}\right]\right) - 2p - s \ . \tag{7}$$

(b) Under condition (6), the configuration $S(C)$ consisting of the points $w_1,...,w_s$ and, for each $1\le i\le p$, the first $[(k_i+1)/2]$ infinitely near to z_i points of C, can be moved in general position when varying C in its equisingular stratum in $|C|$.

Proof. Let Z be a zero-dimensional subscheme of Σ concentrated at the points $z_1,...,z_r$, $w_1,...,w_s$, and defined as follows. At w_i, $1\le i\le s$, Z is a reduced point, i.e., defined by the maximal ideal in $\mathcal{O}_{\Sigma,w_i}$); at z_i, $1\le i\le p$, Z coincides with $Z_0(C,z_i)$; at z_i, $p<i\le r$, Z coincides with $Z(C,z_i)$ (see Remark 2.2(1) for the definition of $Z_0(C,z)$ and $Z(C,z)$). Denote by $\mathcal{J}_{Z/\Sigma}$ the ideal sheaf of the subscheme Z in the surface Σ. We claim that

$$H^1(\Sigma, \mathcal{J}_{Z/\Sigma}\otimes\mathcal{O}_\Sigma(D)) = 0 \ . \tag{8}$$

Assuming condition (8), which will be proven later, we shall derive the statements of Proposition 2.4.

The meaning of (8) is that $H^0(\Sigma,\mathcal{O}_\Sigma(D))$ intersects transversally with $I_0(C,z_i)$ in $\mathcal{O}_{\Sigma,z_i}$ for each $i=1,...,p$, and intersects transversally with $I(C,z_i)$ in $\mathcal{O}_{\Sigma,z_i}$ for each $p<i\le r$. In particular, by Proposition 2.1, all the intersections $|D|\cap V_0(C,z_i)$, $1\le i\le p$, $|D|\cap V(C,z_i)$, $p<i\le r$, are smooth of regular codimension $\deg Z_0(C,z_i)$, $1\le i\le p$, $\deg Z(C,z_i)$, $p<i\le r$, respectively. Moreover, (8) implies that all the above variety germs and all the linear systems of curves $C'\in|D|$ passing through each of the points $w_1,...,w_s$ intersect transversally in $|D|$. In addition, under (8), [4], Theorem 6.1(ii), implies the formula $\dim|D|=(D^2-DK_\Sigma)/2$, the smoothness of W, and formula (7).

To prove statement (b) of Proposition, introduce the germ $\widetilde{W}$ at C of the set of curves in $|D|$, belonging to $V(C,z_i)$ for each $p<i\le r$, and having a singular point of type A_{k_i} in $U(z_i)$ for each $i=1,...,p$. The (projective) Zariski tangent space to the germ of curves on $|D|$ having a singular point of type A_{k_i} in $U(z_i)$ is the linear system $|\mathcal{J}_{Z_i^{es}/\Sigma}\otimes\mathcal{O}_\Sigma(D)|$ (see, for instance [4]), where the zero-dimensional scheme Z_i^{es} is defined at z_i by the Jacobian ideal generated by the derivatives of a local equation of C. Then

$$\dim|\mathcal{J}_{Z^{es}/\Sigma}\otimes\mathcal{O}_\Sigma(D)| \ge \dim|D| - \deg Z^{es} = \dim|D| - k_i \ ;$$

hence

$$\dim\widetilde{W} \ge \dim|D| - \sum_{i=1}^{p}\operatorname{codim}V(C,z_i) - \sum_{p<i\le r}\operatorname{codim}|\mathcal{J}_{Z_i^{es}/\Sigma}\otimes\mathcal{O}_\Sigma(D)|$$

$$\ge \dim|D| - \sum_{i=1}^{p}k_i - \sum_{p<i\le r}\deg Z(C,z_i)$$

$$= \frac{D^2 - DK_\Sigma}{2} - \sum_{i=1}^{p} k_i - \sum_{p<i\le r} \left(k_i - 1 + \left[\frac{k_i+1}{2}\right]\right) . \tag{9}$$

A germ of the configuration space of $[(k_i+1)/2]$ infinitely near points to z_i is smooth and has dimension $[(k_i+1)/2]+1$. Consider the natural projection $\Pi : \widetilde{W} \to \mathcal{C}$, where $\mathcal{C}$ is the product of straight lines $L_1, ..., L_s$, respectively passing through $w_1, ..., w_s$ and transverse to C, and germs of the configuration spaces of $[(k_i+1)/2]$ infinitely near points to z_i, for each $i = 1, ..., p$ (here $\widetilde{W} \to L_l$ is defined as the unique intersection point of $C' \in \widetilde{W}$ and L_l in $U(z_i)$). Combining (9) with the formula $\dim \mathcal{C} = s + \sum_{i=1}^{p}[(k_i+1)/2] + p$, and comparing them with the dimension formula (7) for the central fibre W of Π, we derive that Π is a submersion, completing statement (b) of Proposition.

Now we prove (8). Let us blow up the following points:

- $w_1, ..., w_s$ each once,
- z_i we blow up $[(k_i+1)/2]$ times if $1 \le i \le p$, or $[(k_1-1)/2]$ times if $p < i \le r$.

The strict transform C^* of C passes through the points z_i, $1 \le i \le p$, with even k_i, has nodes at the points z_i, $p < i \le r$, with odd k_i, and has cusps at the points z_i, $p < i \le r$, with even k_i. Furthermore, C^* belongs to the linear system $|\pi^* D - \sum_j E'_j - 2\sum_l E''_l|$ on the blown-up surface Σ^*, where $\pi : \Sigma^* \to \Sigma$ is the total blowing up, E'_j are exceptional divisors coming from $w_1, ..., w_s$, and E''_l are the exceptional divisors coming from $z_1, ..., z_r$. In view of Remark 2.3, the linear system $|\mathcal{J}_{Z/\Sigma} \otimes \mathcal{O}_\Sigma(D)|$ on Σ transforms into the linear system $|\mathcal{J}_{Z^*/\Sigma^*} \otimes \mathcal{O}_{\Sigma^*}(C^*)|$ on Σ^*, where Z^* is the union of the scheme $\bigcup_{i=p+1}^{r} Z(C^*, z_i^*)$ with the scheme of simple points z_i, $1 \le i \le p$, having even k_i. Using the formula $K_{\Sigma^*} = \pi^* K_\Sigma + \sum E'_j + \sum E''_l$, one can easily compute that

$$-C^* K_{\Sigma^*} = -CK_\Sigma - 2\sum_{i=1}^{p}\left[\frac{k_i+1}{2}\right] - 2\sum_{p<i\le r} - s\left[\frac{k_i-1}{2}\right]$$

$$= -DK_\Sigma - \sum_{i=1}^{r} k_i - 2p + r - s + \#\{i = 1, ..., r \ : \ k_i \text{ is even}\}$$

$$> \#\{i = 1, ..., r \ : \ k_i \text{ is even}\}$$

the latter inequality being equivalent to (6). On the other hand, the relation obtained can be rewritten as

$$\#\{i = 1, ..., p \ : \ k_i \text{ is even}\} + \#(\text{cusps of } C^*) < -C^* K_{\Sigma^*} ,$$

which by [4], Theorem 6.1, implies

$$H^1(\Sigma^*, \mathcal{J}_{Z^*/\Sigma^*} \otimes \mathcal{O}_{\Sigma^*}(C^*)) = 0 . \tag{10}$$

Next, (10) yields that

$$\dim |\mathcal{J}_{Z/\Sigma} \otimes \mathcal{O}_\Sigma(D)| = \dim |\mathcal{J}_{Z^*/\Sigma^*} \otimes \mathcal{O}_{\Sigma^*}(C^*)|$$

$$= \frac{(C^*)^2 - C^* K_{\Sigma^*}}{2} - \#(\text{nodes of } C^*) - 2\#(\text{cusps of } C^*)$$

$$-\#\{i = 1, ..., p \ : \ k_i \text{ is even}\} .$$

At last, by a routine computation one can transform the right-hand side of (7) into $\dim |D| - \deg Z$, what finally gives (8). $\square$

Proposition 2.5. *In the notations of Proposition 2.4, assume that the curve C and the points $z_1,...,z_r$, $w_1,...,w_s$ are real. Suppose that C is rational (resp. elliptic) with one (resp., two) one-dimensional real component and such that*

$$\sum_{i=1}^{r} k_i + 2p - r + s = -DK_\Sigma - 2 \quad (resp., \quad = -DK_\Sigma - 1\ ). \tag{11}$$

Then:

(a) W is smooth, one-dimensional, and all (resp., all but one) of the intersection points of C with $C' \in W$ are concentrated at $w_1,...,w_s$ and in neighborhoods of $z_1,...,z_r$ as defined in Proposition 2.1(b).

(b) There is an one-dimensional connected component c of C such that the intersection points of C with $C' \in W\backslash\{C\}$ in a neighborhood of c are located in neighborhoods of singular points of c; moreover, the intersection numbers of C and C' in neighborhoods of singular points of c are equal to the corresponding right-hand sides of (5).

(c) Let c be as in parts (b). Assume that all its singular points have odd Milnor numbers, and c contains a chain of loops, bounding $l \geq 1$ discs, as shown in Figure 3(a). If all the singular points in this chain belong to $\{z_1,...,z_p\}$, then c changes as shown on Figure 3(b) if moving along the real part of W in a certain direction, and changes as shown in Figure 3(c) if mowing along W in the opposite direction. If all but one of the singular points of the chain considered belong to $\{z_1,...,z_p\}$, then c changes as shown in Figure 3(d,e) depending on the direction of motion along W. At last, if all but two neighboring of the singular points of the chain belong to $\{z_1,...,z_p\}$, then c changes as shown in Figure 3(f,g) depending on the direction of motion along W.

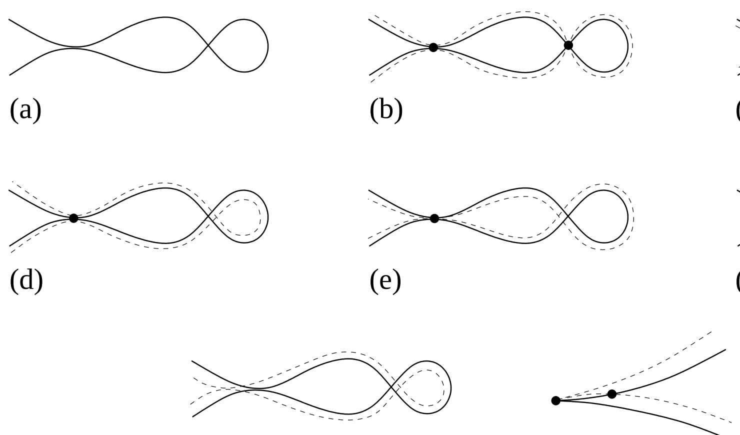

FIGURE 3

(d) Let c be as in parts (b). Assume that it has a singular point z_i, $1 \leq i \leq p$, with an even Milnor number. Then, in a neighborhood of z_i, $C' \in W\backslash\{C\}$ crosses c at z_i and at one more point $z \neq z_i$. The germ of c at z_i deforms as shown in Figure 3(h) when moving along W.

Proof. (a) Smoothness of W follows from Proposition 2.4, $\dim W = 1$ follows from (7), (11) and the formula for the geometric genus

$$g(C) = \frac{C^2 + CK_\Sigma}{2} + 1 - \sum_{i=1}^{r} \delta(C, z_i) = \frac{C^2 + CK_\Sigma}{2} + 1 - \sum_{i=1}^{r} \left[\frac{k_i+1}{2}\right] .$$

At last, by (5) the total intersection of C and $C' \in W \backslash \{C\}$ at $w_1, ..., w_s$ and neighborhood of $z_1, ..., z_r$ is

$$\geq s + \sum_{i=1}^{r} \left(k_i + 2\left[\frac{k_i+1}{2}\right]\right) + 2p - r = C^2 - g(C) .$$

(b) The second statement immediately follows from the preceding argument.

(c) Let the tangent line to W at C passes through C and $C_0 \in |C| \backslash \{C\}$. Clearly, for any singular point $z_i \in c$ of C, the intersection multiplicity of C, C_0 at z_i is given by the right-hand side of (5), and thus, C_0 meets c at only singular points of c. Introducing some coordinates in the (simply connected) part of Σ, containing the considered chain of loops of c, describe C and C_0 as zero loci of some functions. Then C_0 does not change sign along any component of the non-singular part of c. Another consequence of formulas (5) is that C_0 does not change sign when passing along C through a singular point z_i, $1 \leq i \leq p$, of c, and does when passing through a singular point $z_i \in c$, $i < p$. This completes the proof.

(d) Let z_i be of type A_{2m}, $m \geq 1$. Formula (3) shows that C and C' intersect at z_i with multiplicity $4m$ if $u \neq 0$, and with multiplicity $\geq 4m + 2$ if $u = 0$. In the situation considered, the total intersection multiplicity of C and C' in a neighborhood of z_i is $4m + 1$. Hence $u \neq 0$ and the statement follows. $\square$

In the situation of Proposition 2.5, we observe a strongly monotone change of some characteristics of the real part of C when moving along W in a certain direction. We want to extend it as much as possible.

Definition 2.6. Assume that $k_i \leq 2$, $i = p+1, ..., r$. Then, for any $C' \in W$, the germ of W at C' is the intersection of $V_0(C', z_i)$, $i = 1, ..., p$, $V(C', z_i')$ [3], $i = p+1, ..., r$, with the linear system of curves passing through $w_1, ..., w_s$. So, we can consider W as a germ of an one-dimensional algebraic subvariety M of $|C|$. The above mentioned monotonicity means that one necessarily comes to a degeneration when moving along M from C in a certain direction. By the extension of W we call the respectively oriented segment of M with endpoints at C and at the first occurring degeneration.

Assume that there is $k_i > 2$, $p < i \leq r$. Then the germ of W at $C' \in W \backslash \{C\}$ is no longer inside $V(C', z_i')$, and we construct an extension of W in a different way. First, fix an orientation of W. Consider the set $\widetilde{M}$ of real curves in $|C|$, isotopic and equisingular to C, passing through $w_1, ..., w_s$ and belonging to $V_0(C, z_i)$, $i = 1, ..., p$. For any $C' \in \widetilde{M}$, one can define a similar smooth one-dimensional germ $W' \subset |C|$, which we orient as W. Introduce a partial order in $\widetilde{M}$ by $C' \prec C''$, $C', C'' \in \widetilde{M}$, if there is a continuous path $[\alpha, \beta] \to \widetilde{M}_m$ connecting $C(\alpha) = C'$ and $C(\beta) = C''$, and a subdivision $\alpha = t_0 < t_1 < ... < t_n = \beta$ such that $[t_i, t_{i+1}]$ is embedded into the

[3] Here z_i' denotes the new position of a moving singular point z_i of C

respectively oriented half of the germ W_i defined for $C(t_i)$. In particular, the discs bounded by some loops of the real part of the current curve $C(t)$ grow or contract monotonically when moving along such paths. This yields that any well ordered set of curves in $\widetilde{M}$ has a maximal element belonging to the closure of $\widetilde{M}$, i.e., the existence of a well ordered line $C(t) \in \widetilde{M}$, $t \in [\alpha, \beta)$, whose closed segments are paths like described above and satisfying $\lim_{t\to\beta} C(t) = C(\beta) \notin \widetilde{M}$. Any such line we call an extension of W.

2.2. Auxiliary restrictions to curves in $\mathbf{C}P^2$ and $\mathcal{F}_2$.

Proposition 2.7. *Let $z_0 \in \mathcal{F}_2\backslash E$, $z_1, ..., z_5$ successive infinitely near to z_0 points in general position. Then there is neither curve of bidegree $(1,3)$ containing the scheme $Z(z_0, ..., z_5)$, nor curve of bidegree $(3,6)$ containing the scheme $Z(2z_0, ..., 2z_5)$*[4].

Proof. The first fact follows from $\dim|E + 3F| = 5$.

Now assume that $Z(2z_0, ..., 2z_5) \subset C \in |3E + 6F|$. Since $\dim|E + 3F| = 5$ there is a unique non-singular curve $C' \in |E+3F|$ through $z_0, ..., z_4$, which thereby does not pass through z_5 in view of general position. Then $(C \cdot C')_{z_0} \geq 2 \cdot 5 = 10 > (3E + 6F)(E + 3F) = 9$. Hence $C \supset C'$, and thereby $C = C' + C'' + E$, $C'' \in |E + 3F|$. Observe that, in appropriate local coordinates, the ideal of $Z(2z_0, ..., 2z_5)$ is $I = \{\sum_{i+6j\geq 12} a_{ij}x^iy^j\}$ and the germ of C' is given by $y + x^5$. Let C be given by $\sum b_{ij}x^iy^j$. Then the condition $C'C'' \supset Z(2z_0, ..., 2z_5)$, equivalent to $(y + x^5)(\sum b_{ij}x^iy^j) \in I$, reads as $b_{i0} = 0$, $i \leq 6$, and $b_{5,0} + b_{0,1} = 0$, implying $b_{01} = 0$, and hence $C \supset Z(2z_0, z_1, ..., z_5)$. On the other hand, $(C' \cdot C'')_{z_0} \geq 2 + 4 = 6 > (E + 3F)^2 = 4$. Hence $C'' = C'$ and cannot contain $Z(2z_0, z_1, ..., z_5)$. Contradiction. $\square$

Proposition 2.8. *There is no real algebraic projective plane sextic with a singular point of type A_3 and nine connected components located with respect to the tangent at the singular point as shown in Figure 4(b).*

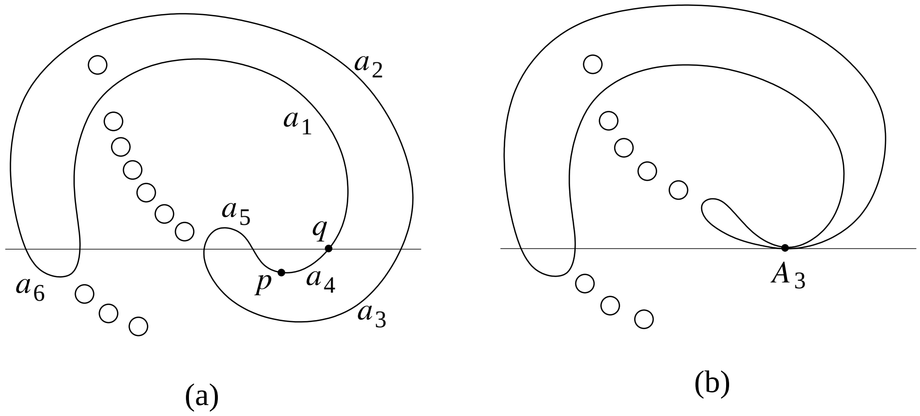

FIGURE 4

[4]The relation $Z \subset C$ for a curve C and a zero-dimensional scheme Z, concentrated at a point w, means that a power series, defining the germ $C, w)$ in local coordinates, belongs to the ideal of the scheme Z

For the proof see [15], Remark in Section 1.3.

Proposition 2.9. *Assume that there exists a non-singular real sextic curve with* 11 *ovals located with respect to a certain real straight line L as shown in Figure 4(1). Then, for any reduced degeneration of such a curve, its arcs a_1, a_2, a_3, a_4 in the complement to the straight line (see Figure 4(a)) remain non-singular.*

This follows from the results of [8].

2.3 Degeneration of plane sextics.

Lemma 2.10. *Assume that there exists a real non-singular plane curve C with* 11 *ovals located with respect to some straight line as shown in Figure 4(a). Then there exists one of the rational real sextic curves with singularities of type A_k located with respect to the fixed straight line L as shown in Figure 5.*

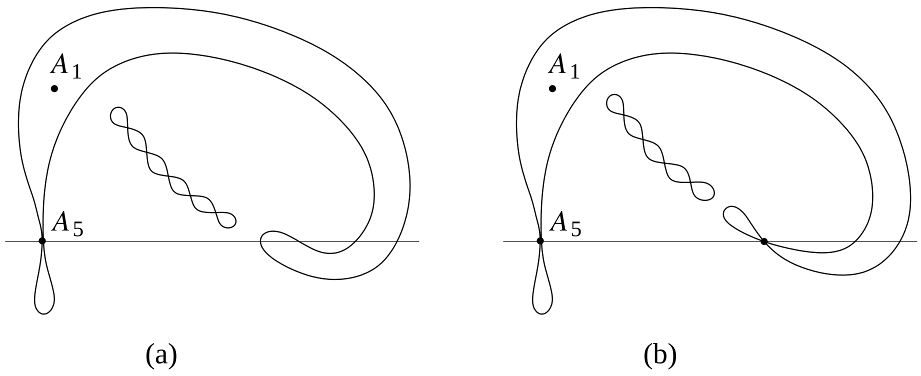

FIGURE 5

Proof. First, we degenerate C into an elliptic sextic with 9 nodes and with one of the shapes shown in Figure 6, and further on prove that they either do not exist, or necessarily degenerate to one of the sextics shown in Figure 5.

Denote by $\mathcal{M}$ the set of non-singular real plane sextics located with respect to the line L as shown in Figure 4(a) and passing through the point p. Let $\overline{\mathcal{M}}$ be its (metric) closure.

Step 1. To obtain one of the nodal degenerations shown in Figure 6, we deform the given curve C along a broken line in the space of real plane sextics, consisting of segments $C' + tC_3^2$, $t \geq 0$, where $C' \in \overline{\mathcal{M}}$ is an irreducible sextic with $m \leq 8$ nodes, the cubic C_3 passes through the nodes of C' and the point p (see Figure 4(a)), and it is assumed that the oriented domain bounded by the branch crossing the line L grows as $t \geq 0$ grows. Arguing exactly as in [15], section 3.8, we can obtain and irreducible sextic $C_0 \in \overline{\mathcal{M}}$ with 9 nodes. The only possible dispositions of nodes, by Proposition 2.9, are shown in Figure 6.

Step 2. We shall deform the curve C_0 along a continuous path in $\overline{\mathcal{M}}$, constructed as extensions in the sense of Definition 2.6. For the limit curve C^*, we have few possibilities: (i) C^* is non-reduced, (ii) C^* is reduced and reducible, (iii) C^* is irreducible elliptic, (iv) C^* is irreducible rational. The fact that a non-reduced C^* cannot occur in the degenerations considered, will be proved below in Step 6. Proposition 2.9 implies that C^* cannot be reducible and, moreover, that C^* must be an irreducible curve with singular points of type A_k.

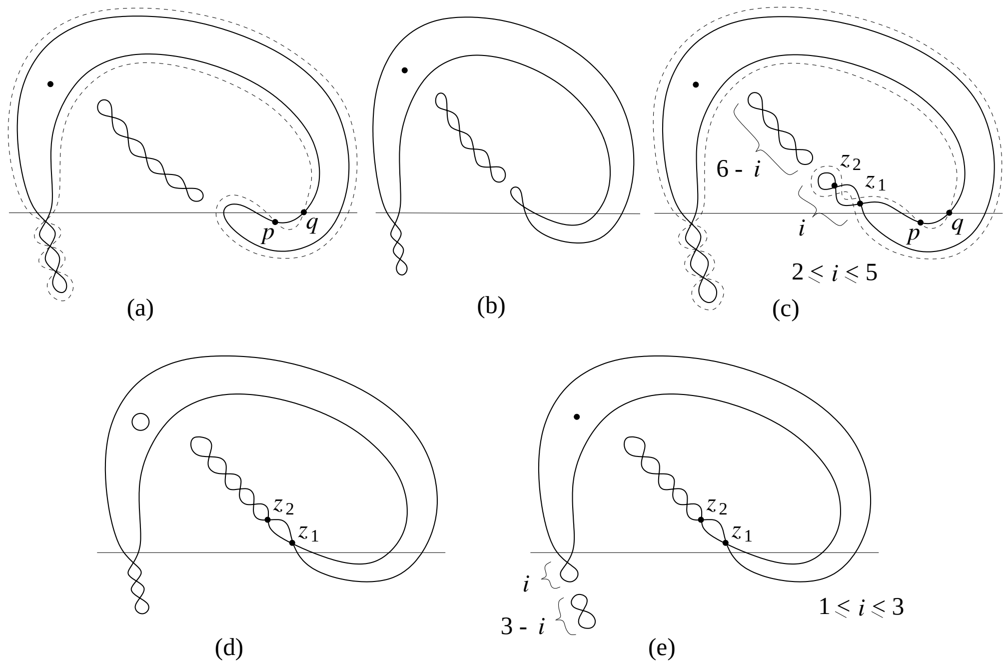

FIGURE 6

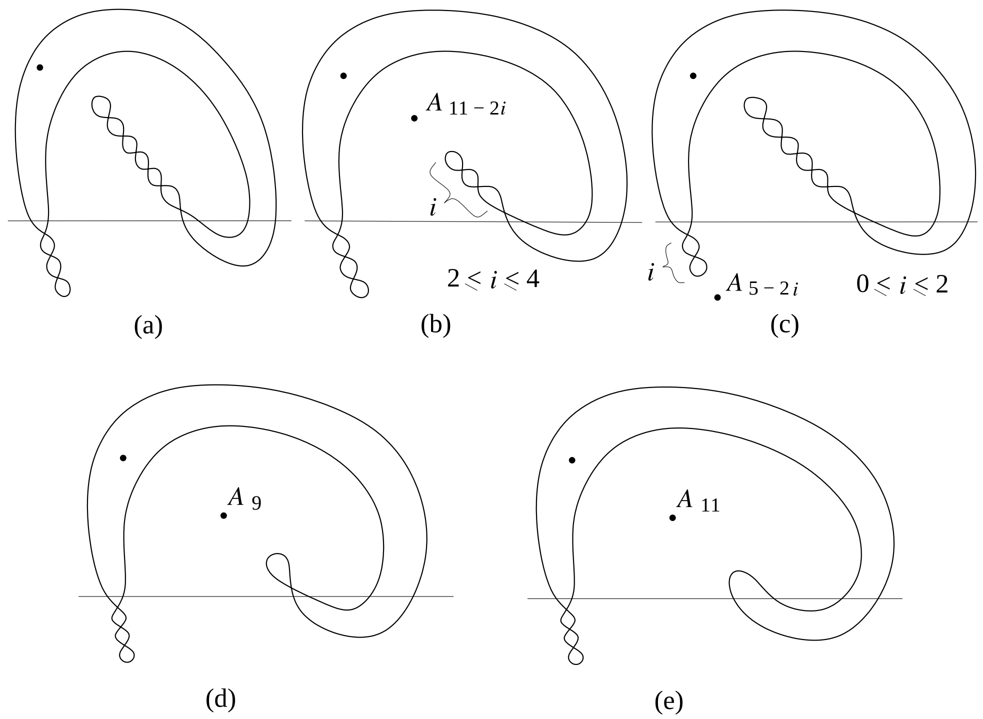

FIGURE 7

Step 3. Now we shall analyze the case of rational irreducible curve C^*. Since C^* has only singularities of type A_k and their number is at least three, one can easily check that the total Milnor number is less than $20 = 4(6-1)$; hence by [22], any deformations of singular points of C^* can independently be realized. Since C^* can be deformed into one of the elliptic nodal curves shown in Figure 6, one can deform C^* into one of the rational curves shown in Figure 7(see also Proposition 2.9 imposing necessary restrictions).

Assume that C_0 has a form depicted in Figure 7(a). Consider the germ W at C_0 consisting of sextics belonging to $V(C_0, z_i)$, $i = 1, 2, 3$ (see Figure 8(a)), $V_0(C_0, z)$, $z \in \mathrm{Sing}(C_0)\backslash\{z_1, z_2, z_3\}$, and passing through p, q. By Proposition 2.5, W is smooth one-dimensional and admits an extension realizing a deformation shown in Figure 8(a) by dashes. Such an extension cannot have a non-reduced limit endpoint, and the reduced one could only be a degeneration of collision of nodes z_1, z_2, impossible by Proposition 2.8, thereby prohibiting a sextic of the form in Figure 7(a).

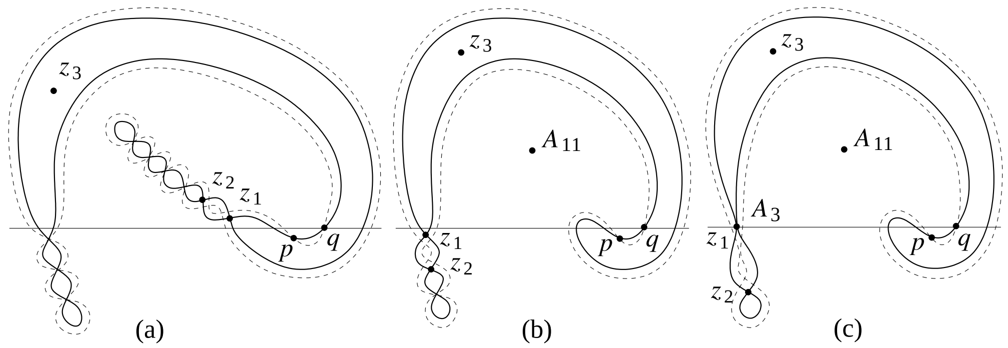

FIGURE 8

Similarly, one can prohibit sextics shown in Figures 7(b,c): we consider a germ W centered at a sextic of the form shown in Figures 7(b,c), and necessarily come to a curve prohibited by Proposition 2.8, or to collision of an isolated singular point with the one-dimensional real component, but the latter curve could be deformed into the prohibited above nodal curve in Figure 7(a).

Let C_0 have a form depicted in Figure 7(e). Consider the germ W_0 at C_0 of the family of sextics belonging to $V(C_0, z_i)$, $i = 1, 2, 3$ (see Figure 8(b)), $V_0(C_0, z)$, $z \in \mathrm{Sing}(C_0)\backslash\{z_1, z_2, z_3\}$, and passing through p, q. As above, by Proposition 2.5, W_0 is smooth one-dimensional and admits an extension realizing a deformation shown in Figure 8(b) by dashes. Such an extension cannot have a non-reduced limit endpoint, and the reduced one could only be a degeneration of collision of nodes z_1, z_2, resulting in a rational sextic C_1 shown in Figure 8(c) (as above, a collision of an isolated singular point with the one-dimensional real component is impossible, since the resulting curve could be deformed into a curve depicted in Figure 7(a) which is already prohibited). Then we consider the germ W_1 at C_1 of the family of sextics belonging to $V(C_0, z_i)$, $i = 1, 2, 3$ (see Figure 8(b)), $V_0(C_0, z)$, $z \in \mathrm{Sing}(C_0)\backslash\{z_1, z_2, z_3\}$, and passing through p, q. By Proposition 2.5, W_1 is smooth one-dimensional and admits an extension realizing a deformation shown in Figure 8(c) by dashes. Again such an extension cannot have a non-reduced limit endpoint, and the reduced one could only be a degeneration of collision of the node z_1 and the tacnode z_2, resulting in a sextic shown in Figure 5(a).

Similarly, the existence of a curve depicted in Figure 7(d) implies the existence of a sextic shown in Figure 5(b).

Step 4. Let C be an irreducible elliptic nodal curve of one of the types depicted in Figure 6(c,d,e). It has two one-dimensional real connected components, c_1 which crosses L and c_2 which does not. Consider the germ W at C consisting of sextics belonging to $V(C, z_i)$, $i = 1, 2$ (see Figure 6(c,d,e)), $V_0(C, z)$, $z \in \mathrm{Sing}(C)\backslash\{z_1, z_2)$, and passing through p, q, and one more point on c_2. By Proposition 2.5, W is smooth one-dimensional and admits an extension realizing a deformation of c_1 shown in Figure 6(c) by dashes. Since no collision of points z_1, z_2 can happen in view of Proposition 2.8, and no rational degeneration can occur by the results of Step 3, the only possibility is a degeneration of one of the nodes on c_2 into a cusp. Then we consider the germ W_1 at the corresponding curve C_1 consisting of sextics belonging to $V(C_1, z_i)$, $i = 1, 2$, $V_0(C_1, z)$, $z \in \mathrm{Sing}(C_1)\backslash\{z_1, z_2)$, and passing through p, q. By Proposition 2.5, W_1 is smooth one-dimensional and admits an extension realizing a deformation of c_1 as shown in Figure 6(c) by dashes. Again the only possible degeneration is turning one more node on c_2 into a cusp. Then we consider the germ W_2 at the obtained curve C_2, consisting of sextics belonging to $V(C_2, z_i)$, $i = 1, 2, 3$, where z_3 is one of the cusps on c_2, $V_0(C_2, z)$, $z \in \mathrm{Sing}(C_2)\backslash\{z_1, z_2, z_3)$, and passing through p, q, and one more nonsingular point on c_2. By Proposition 2.5, W_2 is smooth one-dimensional and admits an extension realizing a deformation of c_1 as shown in Figure 6(c) by dashes. Further possible degenerations are collisions of z_3 with remaining nodes of c_2, until we come to the curve with exactly two cusps on c_2 as only singularities, which then does not allow degeneration in contrary to the monotone deformation of c_1. This contradiction prohibits sextics of the form shown in Figure 6(c,d,e).

Step 5. Let C be an irreducible elliptic sextic of one of the forms shown in Figure 6(a,b). Define c_1 and c_2 as in Step 4. Consider the germ W at C consisting of sextics belonging to $V(C, z_i)$, $i = 1, 2$, where z_1 is the isolated node, z_2 is one of the nodes on c_2, $V_0(C, z)$, $z \in \mathrm{Sing}(C)\backslash\{z_1, z_2)$, and passing through p, q, and one more nonsingular point on c_2. By Proposition 2.5, W is smooth one-dimensional and admits an extension realizing a deformation of c_1 as shown in Figure 6(a) by dashes. The case of a rational degeneration was considered in Step 3, so we shall concentrate on possible elliptic degenerations, which can consist in only collision of singular points on c_2 or their turning into cusps. As in the previous step the maximal possible elliptic degeneration, which can be obtained along extensions of equisingular family germs we consider, is a curve C_1 with $\sigma = 1$ or 2 cusps as the only singularities on c_2. The we consider the germ W_1 at C_1 consisting of sextics belonging to $V(C_1, z)$, $z \in \mathrm{Sing}(C_1)\backslash c_1$, $V_0(C_1, z)$, $z \in \mathrm{Sing}(C_1) \cap c_1$, and passing through p, q and $\sigma - 1$ non-singular points on c_2. By Proposition 2.5, W_1 is smooth one-dimensional and admits an extension realizing a deformation of c_1 as shown in Figure 6(a) by dashes. The only possible reduced degeneration which may occur on the extension of W_1 is rational.

Step 6. To complete the proof we have to exclude non-reduced degenerations in the above one-parametric deformations. In all the cases we use the same argument, and to save the space we shall demonstrate it in one situation, which is the most involved. Namely, consider the one-dimensional deformation $\mathcal{M}$ of the sextic shown in Figure 8(c), described in the last paragraph of Step 3. Assume that the limit curve C^* of $\mathcal{M}$ is non-reduced, and derive a contradiction.

Let C be the initial curve of $\mathcal{M}$ (see Figure 9(a)). Here q_1, q_2 are the fixed points of $C' \in \mathcal{M}$. By Bézout's theorem, the singular point z_4 of type A_{11} lies in the domain $\Delta_1 = \Delta_1(C)$ bounded by the arc a_1 and the two tangent lines l_1, l_2 to C,

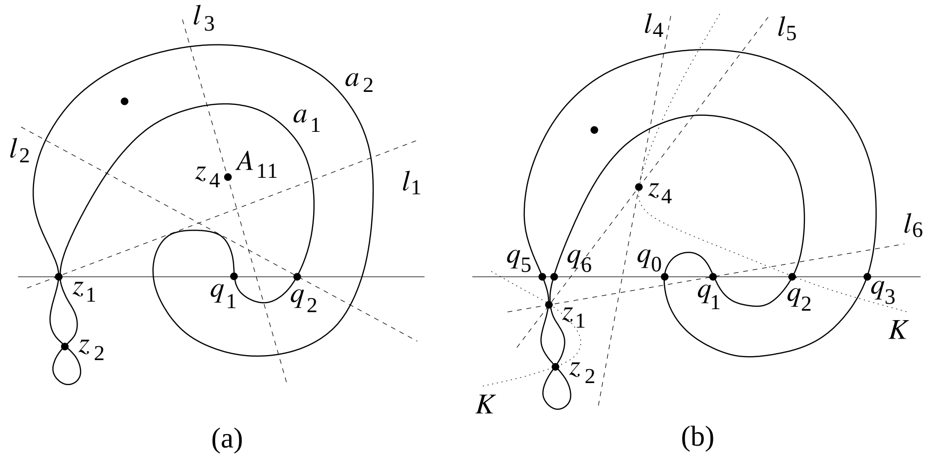

FIGURE 9

which pass through the singular point z_1 of type A_3 and the point q_2, respectively, as shown in Figure 9(a). Furthermore, since the domain $\Delta_1(C')$, similarly defined for a curve $C' \in \mathcal{M}\backslash\{C^*\}$, monotonically shrinks (see Figure 8(c)), the moving singular point $z_4(C')$ always belongs to $\Delta_1(C)$. Similarly, the moving singular point $z_2(C')$ remains inside the domain, bounded by the loop of the curve C with endpoints at $z_1 = z_1(C)$. In particular, $z_4^* = \lim_{C' \to C^*} z_4(C') \neq z_1^* = \lim_{C' \to C^*} z_1(C')$.

Now pick up a point q in the interval (q_1, q_2) in $L\backslash C$ and draw a line l_3 through q and $z_4(C')$. Since $z_4(C') \in \Delta_1(C)$, the line l_3 does not tend to L, and, on the other side, no intersection point of l_3 with C' approaches q (see Figure 8(b)). Hence $L \not\subset C^*$.

As one can see in Figure 8(b), the intersection points of $C' \in \mathcal{M}$ with L, different from q_1, q_2, do not approach q_1, q_2, and hence q_1, q_2 do not lie on a multiple component of C^*. Further on, no three intersection points of $C' \in \mathcal{M}$ with L can merge into one point, and hence the multiplicity of any component of C^* is at most 2. Altogether this means that either $C^* = C_2^2 \hat{C}_2$, where $C_2, \hat{C}_2$ are reduced conics with no component in common, or $C^* = C_1^2 \hat{C}_4$, where C_1 is a line, $\hat{C}_4 \not\supset C_1$ is a reduced quartic.

Given $C' \in \mathcal{M}$, denote by $z_{4,1}(C')$ the first infinitely near to $z_4(C')$ point of C'.

Let the straight line l_4 pass through $z_4(C'), z_{4,1}(C')$, i.e., is tangent to C' at $z_4(C')$, the singular point of type A_{11}. By Bézout's theorem, l_4 must cross C' as shown in Figure 9(b). Consider now the conic K passing through $q_2, z_4(C'), z_{4,1}(C')$, and $z_1(C'), z_2(C')$. In view of the disposition of l_4 and by Bézout's theorem, K must cross C' as shown in Figure 9(b). Put $K^* = \lim_{C' \to C^*} K$. Since K cannot approach the (growing) segment $[q_0, q_1]$ of L (see Figure 9(b)), $L \not\subset K^*$. Furthermore, the intersection points of K with L do not approach the intersection point of L with the straight line l_5 through $z_1(C'), z_4(C')$. Hence K^* does not contain the line $l_5^* = \lim_{C' \to C^*} l_5$ through z_1^*, z_4^*.

The preceding consideration implies that the conic K^* is non-singular at z_1^*, z_4^* and intersects with $l_5^* \supset \{z_1^*, z_4^*\}$ transversally. In particular, the line l_4^*, tangent to K^* at z_4^*, does not pass through z_1^*. By Bézout's theorem, the lines l_4 and l_5 cross the line L in the same interval $[q_0, q_4] \subset L\backslash\{q_1, q_2\}$ (see Figure 9(b)). Since $l_4^* \neq l_5^*$ and $l_4^* \cap l_5^* = \{z_4^*\}$, the intersection points $L \cap l_4^*$ and $L \cap l_5^*$ are distinct, and hence the interval $[q_0, q_4] \subset L$ does not collapse as $C' \to C^*$. This means that the multiple component of C^* might be only a straight line C_1 crossing L in the interval $[q_2, q_5] \subset L \backslash \{q_1, q_2\}$.

Since the point q_1 cannot lie on a multiple component of C^* as noticed above, the limit l_6^* of the line l_6, passing through q_1 and $z_1(C')$, differs from C_1. Note also that no arc of C' approaches l_6 in the growing domain bounded by the arcs a_1, a_2 and the line L (see Figures 8(b), 9(b)). Hence $l_6 \not\subset C^*$. On the other hand, the only z_1^* might be a multiple intersection point of l_6^* with C^*. Hence $z_1^* \in C_1$, and thus, $z_4^* \notin C_1$, that means the reduced quartic $\hat{C}_4$ must have at least A_{11} as a singularity at z_4^*, what is impossible. □

2.4. Degeneration of a curve of type $A(3,1,4)$**.** We shall prove the non-existence of curves of type $A(3,1,4)$ arguing by contradiction. In Lemma 3.3 [15], it was shown that the existence of such a curve yields the existence of the nodal rational curve on the surface $\mathcal{F}_2$ pictured in Figure 10(a).

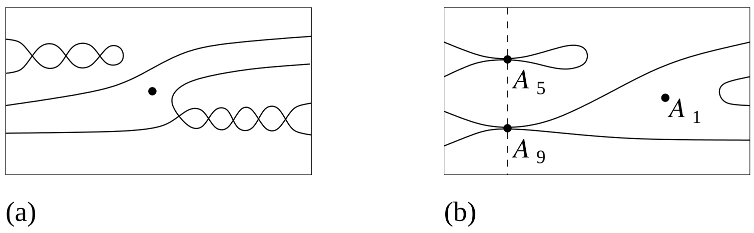

FIGURE 10

Lemma 2.11. *If there exists a real nodal algebraic curve C of bidegree $(4,8)$ on $\mathcal{F}_2$ as shown in Figure 10(a), then there exists a real rational curve with a disposition and singularity collection as in Figure 10(b) (a fibre passing through two singular points of such a curve is shown by dashes).*

Proof. We start with a real nodal curve C_0 of bidegree $(4,8)$ shown in Figure 10(a), and we shall describe how it degenerates into a curve shown in Figure 10(b).

Step 1. Denote by z_0 the isolated node of C_0 and by $z_1, ..., z_8$ the non-isolated nodes of C_0 as indicated in Figure 11(a). By Proposition 2.5(a), the germ

$$N_0 = V_0(C_0, z_0) \cap ... \cap V_0(C_0, z_6) \cap V(C_0, z_7) \cap V(C_0, z_8)$$

is smooth of dimension 1, and extends up to an one-dimensional variety $\widetilde{N}_0 \subset |4E + 8F|$. Furthermore, the movement along $\widetilde{N}_0$ in a certain direction implies a deformation shown in Figure 11(b) by dashes. The monotone change of some components of the complement of a curve in $\mathcal{F}_2$ yields the existence of a degeneration C_1, which is reduced by Proposition 2.12 below, and by Lemmas 3.5, 3.6, Corollary 2.9 and Proposition 2.11 [15], must have nodes at $z_0, ..., z_6$, a singular point z_7' of type A_3 and the real part as shown in Figure 11(c).

Step 2. Denote by N_1 the germ at C_1 of the set of real rational curves of bidegree $(4,8)$ having nodes at $z_0, ..., z_5$, a node at some point z_6', a singular point z_7' of type A_3, and the real part as shown in Figure 11(c). It is smooth, one-dimensional and admits an extension $\widetilde{N}_1$ along which the current curve is deformed as shown in Figure 11(c) by dashes. The monotone changes of the current curve in $\widetilde{N}_1$ implies the existence of a degeneration C_2 which is reduced by Proposition 2.12, and the

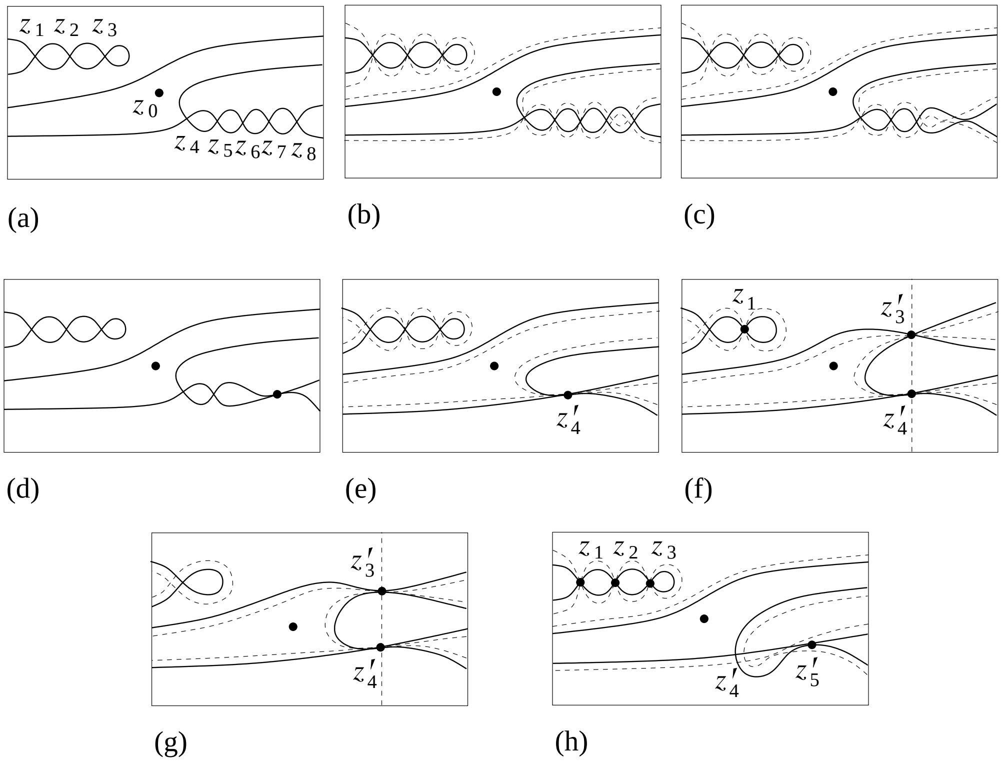

FIGURE 11

only possible singularities of C_2 are nodes at $z_0, ..., z_5$ and A_5 at some point z'_6, and the real part is shown in Figure 11(d).

In a similar manner we perform two more degenerations of C_2, first, gluing up the node z_5 and the singular point z'_6 of type A_3 into a singular point of type A_7 (denoted again by z_5 for the sake of notation) and resulting in a degeneration C_3 shown in Figure 11(h), and then moving C_3 along an appropriate one-dimensional family $\widetilde{N}_3$, which realizes a deformation shown by dashes in Figure 11(h) and leads to gluing up the (travelling) node z_4 and the (travelling) singular point z_5 of type A_7 into one singular point z'_4 of type A_9, and obtaining a curve C_4 with real part shown in Figure 11(e).

Step 3. Next consider the smooth one-dimensional germ

$$N_4 = V_0(C_4, z_1) \cap V_0(C_4, z_2) \cap V_0(C_4, z'_4) \cap V(C_4, z_0) \cap V(C_4, z_3)$$

and extend in up to an one-dimensional variety $\widetilde{N}_4 \subset |4E + 8F|$. Furthermore, the movement of C_4 along $\widetilde{N}_4$ in a certain direction induces a deformation shown in Figure 11(e) by dashes. The monotone change of some components of the complement of the current curve $C' \in \widetilde{N}_4$ in $\mathcal{F}_2$ yields the existence of a degeneration. However no degeneration is possible before the travelling node z'_3 appears on one fibre with z'_4 (see the resulting curve C_5 in Figure 11(f)). The reason is that C' cannot become non-reduced by Proposition 2.12, the projection of the travelling node z_3 on $E \backslash F_1$, where F_1 is the fibre through z_1, goes away from the projection of z_2, and the travelling isolated node z'_0 cannot join the real branch of C'.

Then consider the smooth one-dimensional germ

$$N_5 = V_0(C_5, z_1) \cap V_0(C_5, z'_3) \cap V_0(C_5, z'_4) \cap V(C_5, z'_0) \cap V(C_5, z_2)$$

and extend it up to an one-dimensional variety $\widetilde{N}_5 \subset |4E+8F|$. The above argument ensures that, moving C_5 along $\widetilde{N}_5$ in a certain direction (the corresponding deformation is shown in Figure 11(f) by dashes), one necessarily obtains a collision of the travelling node z_2' with the singular point z_3', i.e., a real rational curve C_6 of bidegree $(4,8)$ with a node at z_1, an isolated node at some point z_0', singularity A_3 at z_3' and singularity A_9 at z_4', where z_3', z_4' lie on the same fibre.

Finally, in the same manner, one considers the extension $\widetilde{N}_6 \subset |4E+8F|$ of the smooth one-dimensional germ

$$N_6 = V_0(C_6, z_3') \cap V_0(C_6, z_4') \cap V(C_6, z_1) \cap V(C_6, z_0') \ .$$

Then one moves C_5 along $\widetilde{N}_6$ so that the deformation of C_6 looks as shown in Figure 11(g) by dashes. The only possible (first) degeneration is a real rational curve C_7 shown in Figure 10(b), obtained as result of collision of the travelling node z_1' with the singular point z_3'. $\square$

Proposition 2.12. *The curves C_i, $i = 1, ..., 7$, appearing in the proof of Lemma 2.11 cannot be non-reduced.*

Proof. The curves C_i, $i = 1, ..., 7$, appear as the first degeneration in some deformations. We intend to show that these deformations do not lead to a non-reduced curves.

First of all, we notice that all our deformations consist of one-parametric arcs inside varieties of type $V_0(C, w_1) \cap ... \cap V_0(C, w_n) \cap V(C, w_1') \cap ... \cap V(C, w_m') \subset |4E+8F|$, where $C \in |4E+8F|$ is an irreducible curve with some singular points $w_1, ..., w_n$, $w_1', ..., w_m'$ of types $A_{2i_1+1}, ..., A_{2i_n+1}$, $A_{2j_1+1}, ..., A_{2j_m+1}$, respectively, with

$$(2i_1+2) + ... + (2i_n+2) + 2j_1 + ... + 2j_m \le 14 < -CK_{\mathcal{F}_2} \ .$$

Hence, by Proposition 2.4(b), the configuration

$$w_1, w_{1,1}, ..., w_{1,i_1}, \ ..., \ w_n, w_{n,1}, ..., w_{n,i_n}$$

of singular points $w_1, ..., w_n$ with the corresponding infinitely near points of C at them can be moved in general position keeping C in its equisingular stratum. In the sequel we always assume such a general position.

All the cases are treated in a similar way, and we consider in detail only the most difficult cases, C_4 and C_7.

Case C_4. Assume that C_4 contains a multiple component different from E. The two situations are possible (see the deformation along $\widetilde{N}_3$ leading to the degeneration C_4 in Figure 11(h)):

(1) the limit positions $\widetilde{z}_4, \widetilde{z}_5$ of the travelling points z_4' (node) and z_5' (singularity A_7) of a current curve $C' \in \widetilde{N}_3$, approaching C_4, are different,
(2) in the above notations $\widetilde{z}_4 = \widetilde{z}_5$.

We consider the latter situation.

On any fibre F of $\mathcal{F}_2$, crossing the loop with the endpoints at z_1, or at z_1, z_2, or at z_2, z_3, of a current curve $C' \in \widetilde{N}_3$ (see Figure 11(h)), at most two intersection points with C' can merge to a common limit point outside E. Hence a multiple component of C_4 different from E and a fibre could only be a non-singular curve of bidegree $(1,m)$, $2 \le m \le 4$.

Step 1. Assume that C_4 contains a double irreducible curve $D^{(1)}$ of bidegree $(1,m)$, $2\le m\le 4$. Then the intersection of the unique real branch of that component with fibres $F\in[F_0,F_1]$, where F_1 is tangent to the loop of C_3 with endpoints at z_1, lies in the domain Δ, bounded by F_0, F_1 and the corresponding arcs of C' (see Figure 12(a)). Since there is no smooth arc in Δ passing through z_1, z_2 or z_3 and transversal to the fibres $F\in[F_0,F_1]$, $D^{(1)}$ does not contain z_1,z_2,z_3. That means the domain Δ contracts to a (doubled) arc of $D^{(1)}$, and, on the other hand, contains the points z_1,z_2,z_3. Hence C_4 splits off double fibers through z_1, z_2 and z_3, and thus, cannot contain a double curve of bidegree $(1,m)$, $m\ge 2$. Contradiction.

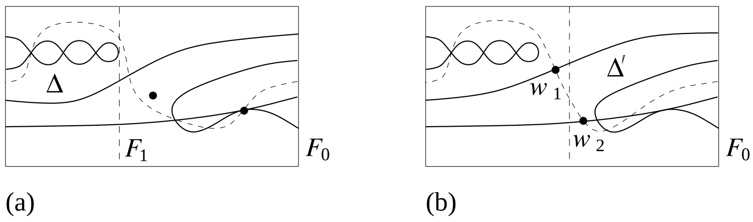

FIGURE 12

Step 2. Assume that multiple components of C_4 different from E are fibres of $\mathcal{F}_2$. We claim that in this case, the scheme $Z_0(C',z'_4)\cup Z(2z'_5,2z'_{5,1},z'_{5,2})\subset Z_0(C',z'_4)\cup Z_0(C',z'_5)$, where $z'_{5,1},z'_{5,2}$ are two infinitely near points of C' at z'_5 (recall that $Z_0(C',z'_5)$ imposes an extra condition of multiplicity 2 at $z'_{5,3}$), has a flat limit $Z(2\widetilde{z}_4,2\widetilde{z}_{4,1},2\widetilde{z}_{4,2},2\widetilde{z}_{4,3})$, where $\widetilde{z}_{4,i}$, are certain infinitely near points to $\widetilde{z}_4$. We derive such a result showing that there is a smooth curve through $z'_4,z'_5,z'_{5,1},z'_{5,2}$ which has a smooth limit transversal to the fibre through $\widetilde{z}_4$ as $C'\to C_4$.

Let $C'\in\widetilde{N}_3$. Consider the (unique) curve $C_{1,2}$ of bidegree $(1,2)$ passing through the singular points z_3,z'_4,z'_5 of C'. It is non-singular, and, since $C'C_{1,2}=8$, is located with respect to C' as shown in Figure 12(a) ($C_{1,2}$ is dashed). In particular, $(C_{1,2}\cdot C')_{z_3}=(C_{1,2}\cdot C')_{z'_4}=(C_{1,2}\cdot C')_{z'_5}=2$. Furthermore, the intersection of $C_{1,2}$) with the subsegment $[F_3,F_0]\subset[F_1,F_0]$, where F_3 is the fibre through z_3, is an arc in the domain Δ (see Figure 12(a)). Therefore, this arc cannot approach E, and hence the limit $\widetilde{C}_{1,2}$ of $C_{1,2}$ as $C'\to C_4$ is non-singular and transverse to the fibres of $\mathcal{F}_2$. This means that the points z'_4,z'_5 merge into a zero-dimensional scheme consisting of $\widetilde{z}_4$ and its first infinitely near point $\widetilde{z}_{4,1}$, which does not belong to the fibre through $\widetilde{z}_4$. Notice in addition that $\widetilde{C}_{1,2}$ does not pass through z_1,z_2, since the pair of non-singular intersection points of $C_{1,2}$ and C' do not approach each other as $C'\to C_4$. By the same reason $(\widetilde{C}_{1,2}\cdot C_4)_{\widetilde{z}_4}<6$, since, in addition, $\widetilde{C}_{1,2}\not\subset C_4$. The latter holds, because $\widetilde{C}_{1,2}$ crosses C_3 as shown in Figure 12(b), and on any fibre between the points $w_1,w_2\in\widetilde{C}_{1,2}\cap C_3$ (fibre is shown dotted in Figure 12(b)), the intersection points with the current curve C' do not approach $\widetilde{C}_{1,2}$. At last, $C_{1,2}$ separates the loop od C' with endpoints at z_1 from E (see Figure 12(a)), and hence C_4 cannot contain E as a multiple component.

The same argument results in similar statements when replacing z_3 by z_1 or z_2.

Step 3. Since $\dim|E+3F|=5$, there is a curve $C_{1,3}$ of bidegree $(1,3)$ passing through the points z_2,z_3,z'_4,z'_5 and $z'_{5,1}$ of a current curve $C'\in\widetilde{N}_3$. It cannot split off a fibre, since there is no curve of bidegree $(1,2)$ passing through four of the above five points. Also $C_{1,3}$ is unique, since $C^2_{1,3}=4$. Let $\widetilde{C}_{1,3}$ be the limit of $C_{1,3}$ as C' tends to C_4. It passes through z_2, z_3 and satisfies $(\widetilde{C}_{1,3} \cdot C_4)_{\widetilde{z}_5} \geq 6$. Then $\widetilde{C}_{1,3}$ is non-singular. Indeed, if $\widetilde{C}_{1,3}$ splits off a fibre then the remaining curve of bidegree $(1,2)$ has at least three common points with $\widetilde{C}_{1,2}$, and hence coincides with $\widetilde{C}_{1,2}$, but $(\widetilde{C}_{1,2} \cdot C_4)_{\widetilde{z}_4} < 6$ contradicting $(\widetilde{C}_{1,3} \cdot C_4)_{\widetilde{z}_4} \geq 6$. Note also that the real branch of $C_{1,3}$ crosses twice the boundary of the domain Δ', bounded by F_0 and arcs of C', and one of these intersection points does not approach $\widetilde{z}_4$, whereas the other point does not approach z_1, when C' tends to C_4. Hence $\widetilde{C}_{1,3}$ does not pass through z_1. The same reason yields that $(\widetilde{C}_{1,3} \cdot C_4)_{\widetilde{z}_4} < 8$, since, in addition $\widetilde{C}_{1,3} \not\subset C_4$. The latter can be shown in the same way as the relation $\widetilde{C}_{1,2} \not\subset C_4$ above. The properties of $\widetilde{C}_{1,3}$ altogether imply that the points $z'_4, z'_5, z'_{5,1}$ of C' merge into a zero-dimensional scheme consisting of $\widetilde{z}_4$ and its two infinitely near points $\widetilde{z}_{4,1}, \widetilde{z}_{4,2}$ outside the fibre through $\widetilde{z}_4$.

Again we observe that the same is valid when replacing the pair z_2, z_3 by z_1, z_3 or z_1, z_2 in the above construction.

Step 4. Since $\dim |E + 4F| = 7$ there is a curve $C_{1,4} \in |E + 4F|$ passing through the points $z_1, z_2, z_3, z'_4, z'_5, z'_{5,1}, z'_{5,2}$ of a current curve $C' \in \widetilde{N}_3$. Reasoning as in the study of $C_{1,2}, C_{1,3}$, one can show that $C_{1,4}$ is non-singular, unique and has a non-singular limit $\widetilde{C}_{1,4}$ as $C' \to C_4$. Notice that $\widetilde{C}_{1,4} \not\subset C_4$. Indeed, otherwise C_4 would contain $\widetilde{C}_{1,4}$ as multiple component, or would contain together with $\widetilde{C}_{1,4}$ either a double curve D of bidegree $(1,2)$ or a multiple fibre, and the only latter situation is allowed by the results of Step 1. If D_4 contains $\widetilde{C}_{1,4}$ and a double fibre through $\widetilde{z}_4$, then it contains a curve D of bidegree $(1,2)$ through z_1, z_2, z_3 and $\widetilde{z}_4$, but this is impossible, since then D would intersect C_3 at least at $9 > DC_3 = 8$ points: twice at each of z_1, z_2, z_3, once the boundary of Δ' at a point different from z_1, and twice the domain bounded by the arcs joining z'_4 and z'_5 (the domain containing $\widetilde{z}_4$). If C_4 contains $\widetilde{C}_{1,4}$ and a double fibre through z_1, z_2 or z_3, then it contains a curve D of bidegree $(1,2)$ passing through two points of z_1, z_2, z_3 and tangent to $\widetilde{C}_{1,4}$ at $\widetilde{z}_4$, but then D must coincide with $\widetilde{C}_{1,2}$ contradicting the fact that $\widetilde{C}_{1,2}$ passes through exactly one of the points z_1, z_2, z_3.

Thus, we obtain that the configuration of the points $z'_4, z'_5, z'_{5,1}, z'_{5,2}$ of a current curve $C' \in \widetilde{N}_3$ has a limit consisting of $\widetilde{z}_4$ and its infinitely near points $\widetilde{z}_{4,1}, \widetilde{z}_{4,2}, \widetilde{z}_{4,3}$ on $\widetilde{C}_{1,4}$. We, furthermore, conclude that

(1) C_4 cannot have four pairs of double fibres since there are ≥ 5 singular points with distinct projections to E,
(2) C_4 cannot have three pairs of double fibres, since otherwise the remaining curve of bidegree $(1,2)$ must have ≥ 2 singular points with distinct projections to E, what is impossible,
(3) C_4 cannot have two pairs of double fibres, since otherwise the remaining reduced curve of bidegree $(2,4)$ must have singular points with total δ-invariant ≥ 6 what is impossible,
(4) at last, C_4 cannot split into a double fibre, E and a reduced curve D of bidegree $(3,6)$, since otherwise D must have in $\mathcal{F}_2 \backslash E$ singular points with the total δ-invariant ≥ 7 what is impossible.

Case C_7. The curve C_7 is a degeneration of a family of curves $C' \in \widetilde{N}_6$ shown in Figure 11(g). Evolution of intersection points of C' with fibres close to that through z'_3, z'_4 ensures that C_7 is disjoint with E. On the other hand, evolution of intersection points of C' with fibres which cross the growing domain, bounded by a loop containing z_1', leaves the only non-reduced form of C_7 to be a double non-singular curve $C_{1,2}$ of bidegree $(1,2)$ and a reduced curve $C_{2,4}$ of bidegree $(2,4)$. If $C_{1,2}$ passes through z_3', then it does not go through z_4', where $C_{2,4}$ must have singularity with $\delta \geq 4$ what is impossible. If $C_{1,2}$ passes through z_4' then it cannot go through z_3' and does not contain all four common infinitely near points of curves $C' \in \widetilde{N}_6$ at z_4', which are in general position. Thus, $C_{2,4}$ passes through z_4' and is singular at z_3'; hence splits off a fibre through z_3', z_4' in contrary to $C_7 \cap E = \varnothing$. □

3. Application of the cubic resolvent

3.1. Cubic resolvent of a polynomial in one variable.

Definition 3.1. Let $F(y)$ be a polynomial of the form

$$F(y) = y^4 + a_2 y^2 + a_3 y + a_4. \tag{12}$$

Let $y_1, \dots, y_4$ be its roots. The *cubic resolvent* of F is the polynomial

$$R(z) = (z - z_1)(z - z_2)(z - z_3),$$

where

$$z_1 = (y_1 + y_2)(y_3 + y_4), \quad z_2 = (y_1 + y_3)(y_2 + y_4), \quad z_3 = (y_1 + y_4)(y_2 + y_3). \tag{13}$$

Since $y_1 + \dots + y_4 = 0$, one has

$$z_1 = -(y_1 + y_2)^2, \quad z_2 = -(y_1 + y_3)^2, \quad z_3 = -(y_1 + y_4)^2. \tag{14}$$

One can easily check that

$$R(z) = z^3 + b_1 z^2 + b_2 z + b_3, \quad \text{where } b_1 = -2a_2, \quad b_2 = a_2^2 - 4a_4, \quad b_3 = a_3^2. \tag{15}$$

Lemma 3.2. *$y_1, \dots, y_4$ are distinct if and only if z_1, z_2, z_3 are distinct.*

Proof. By (14), one has

$$z_1 - z_2 = -(y_1 + y_2)^2 + (y_1 + y_3)^2 = (y_3 - y_2)(2y_1 + y_2 + y_3) = (y_3 - y_2)(y_1 - y_4).$$

Similarly, $z_2 - z_3 = (y_4 - y_3)(y_1 - y_2)$ and $z_1 - z_3 = (y_4 - y_2)(y_1 - y_3)$. □

Lemma 3.3. *Suppose that the coefficients a_2, a_3, a_4 are real.*

(a). Suppose that all $y_1, \dots, y_4$ are real. Then:

1. *if $y_1 < y_2 < y_3 < y_4$ then $z_1 < z_2 < z_3 \leq 0$;*
2. *if $y_1 = y_2 < y_3 < y_4$ then $z_1 < z_2 = z_3 < 0$;*
3. *if $y_1 < y_2 = y_3 < y_4$ then $z_1 = z_2 < z_3 < 0$;*
4. *if $y_1 < y_2 < y_3 = y_4$ then $z_1 < z_2 = z_3 < 0$.*

(b). Suppose that y_1, y_2 are real but y_3, y_4 are not ($y_3 = \overline{y_4}$, $y_3 \neq y_4$). Then z_1 is real non-positive and $\operatorname{Re} z_2 = \operatorname{Re} z_3$. Moreover,

1. *if $y_1 \neq y_2$ then $z_2 = \overline{z_3}$, $z_2 \neq z_3$, i.e. z_2 and z_3 are non-real;*
2. *if $y_1 = y_2$ then $z_2 = z_3 > 0$.*

(c). Suppose that all $y_1, \dots, y_4$ are distinct and not real. Let $y_1 = \overline{y_2}$, $y_3 = \overline{y_4}$, $\operatorname{Im} y_1 > 0 > \operatorname{Im} y_2$, $\operatorname{Im} y_3 > 0 > \operatorname{Im} y_4$. Then $z_1 < 0 \leq z_3 < z_2$. Moreover, $z_3 = 0$ if and only if either $\operatorname{Im} y_1 = \operatorname{Im} y_3$ or $\operatorname{Re} y_1 = \operatorname{Re} y_3$.

Proof. This follows from (14) and the identities in the proof of Lemma 3.2. □

3.2. Cubic resolvent of a curve.

Definition 3.4. Let C be a curve on $\mathcal{F}_n$ of bidegree $(4,4n)$ which does not contain E as an irreducible component. Let us choose a standard system of coordinates (x,y). Then the equation of C is $F=0$ where F is defined by (12) with $a_j=a_j(x)$, $\deg_x a_j = nj$ (one can always kill the coefficient of y^3 by the standard trick). We define *the cubic resolvent of* C as the curve in $\mathcal{F}_{2n}$ of bidegree $(3,6n)$ given in some standard coordinates (x,z) by the equation $R(x,z)=0$ where $R(x,z)$ is defined by (15). The curve in $\mathcal{F}_{2n}$ which is defined in the same coordinates by the equation $z=0$ will be called *the core of* C.

It is easy to check that the definition of the resolvent of a curve on $\mathcal{F}_n$ does not depend on the choice of a standard coordinate system.

Lemma 3.5. *Let C be a curve of bidegree $(4,4n)$ on $\mathcal{F}_n$. Let L and R be the core and the the cubic resolvent of C respectively. Suppose that C has singularities of types A_{n-1} and A_{k-1} $(k>n)$ on the fiber $\pi_n^{-1}(x_0)$. Then*

(1) *R has a singularity of the type A_{k+n-1} at the point $p=L\cap\pi_{2n}^{-1}(x_0)$;*
(2) *$(R.L)_p = 2n$.*

Proof. Without loss of generality we may assume that $x_0=0$. The Puiseux expansion of the singular branches $y_1,\dots,y_4$ has the form:

$$y_{1,2}=f(x)\pm t^n g(x), \qquad y_{3,4}=-f(x)\pm t^k h(x) \tag{16}$$

where t is some branch of $\sqrt{x}$ and f, g, h are non-vanishing at 0 analytic functions.

Putting (16) into (13), we get

$$z_1=-f(x)^2, \qquad z_{2,3}=-x^n\left(g(x)^2+x^{k-n}h(x)^2\right)\mp 2t^{n+k}g(x)h(x)$$

and the result follows (recall that L is given by $z=0$). $\square$

Definition 3.6. Let $y=f(x)$ be a 4-valued real algebraic function. Suppose that the segments $I_1=[a_1,b_1]$ and $I_2=[a_2,b_2]$ of the real axis are such that

(1) $I_2\subset I_1$ and f has no pole on I_1;
(2) f has simple branching at the ends of the both segments and the values of $f(x)$ are distinct for any other $x\in I_1$;
(3) f has no real branch on $\operatorname{Int} I_2$ and two real branches on each component of $(\operatorname{Int} I_1)\setminus I_2$.

Let $f_j:[a_j,c_j]\to\mathbf{C}$, $j=1,2$, be the maximal single-valued analytic branch of f such that $\operatorname{Im} f_j>0$ (thus, (c_1,c_2) is some permutation of (b_1,b_2)). Let $V=\mathbf{R}\times\mathbf{C}=\{(x,y)\mid \operatorname{Im} x=0\}$ and let $S_j\subset V$ be the union of the graph of f_j with the segments $[(a_j,0),(a_j,f_j(a_j))]$ and $[(c_j,0),(c_j,f_j(c_j))]$, $j=1,2$. Under the above assumptions, we define the *self-linking number of f on I_1* as the linking number of the circles $S_1\cup r(S_1)$ and $S_2\cup r(S_2)$ where r is the rotation of V around the axis $y=0$ by 180°. The self-linking number is even when $(c_1,c_2)=(b_1,b_2)$ and it is odd when $(c_1,c_2)=(b_2,b_1)$.

Lemma 3.7. *Let C be a real algebraic curve on $\mathcal{F}_n$ of bidegree $(4,4n)$ which does not contain E as an irreducible components. Let $y = f(x)$ be the 4-valued algebraic function whose graph is C. Suppose that the conditions (1)–(3) of Definition 3.6 are satisfied for some intervals $I_2 \subset I_1$ and let k be the self-linking number of f on I_1. Then*

$$2|k| - 1 \leq 3n \tag{17}$$

Proof. We may suppose that C is defined by (12) where a_j is a polynomial in x of degree $\leq nj$. Let $R(x,z)$ be the cubic resolvent of $F(x,y)$. Let f_1, f_2 be as in Definition 3.6. By Lemma 3.3(c), there are $\geq 2|k| - 1$ values of x such that $R(x,0) = 0$. Indeed, at least k values where $\operatorname{Im} f_1 = \operatorname{Im} f_2$ and at least $|k| - 1$ values where $\operatorname{Re} f_1 = \operatorname{Re} f_2$. By the same reason, $R(x,0)$ cannot be identically zero unless $k = 0$. Indeed, if $\operatorname{Im} f_1 = \operatorname{Im} f_2$ or $\operatorname{Re} f_1 = \operatorname{Re} f_2$ identically, then the graphs of f_1 and f_2 are not linked. It remains to note, that $R(x,0) = a_3(x)^2$, hence $R(x,0)$ cannot have more than $\deg a_3(x) = 3n$ distinct roots. □

3.3. Prohibition of the curve in Figure 10(b).

Lemma 3.8. *Let C be a curve of bidegree $(4,8)$ on $\mathcal{F}_2$. Suppose that C has singularities A_1, A_5, A_9 and C is arranged on $\mathcal{F}_2$ as in Figure 10(b), in particular, A_5, A_9 lie on the same fiber. Let L and R be the core and the the cubic resolvent of C respectively. Then*

(a). *R has a singularity of the type A_{15} at some point p, and $(R.L)_p = 12$.*
(b). *R is arranged in $\mathcal{F}_4$ as in Figure 13,*

Proof. (a). Follows from Lemma 3.5. (b). Follows from Lemma 3.3.

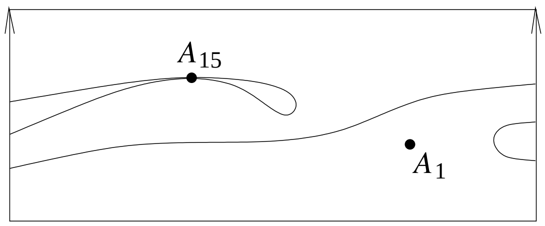

FIGURE 13

Lemma 3.9. *Let R and L be irreducible curves of bidegrees $(3,12)$ and $(1,4)$, respectively, on $\mathcal{F}_4$. Suppose that R has a singularity of the type A_n with $n \geq 15$ at a point p and $(R \cdot L)_p = 12$. Then there is a standard coordinate system (x,y) on $\mathcal{F}_4$ with the origine at p such that L is the horizontal axis $y = 0$ and R is symmetric with respect to the vertical axis $x = 0$.*

Proof. Let us choose a standard coordinate system (x,y) on $\mathcal{F}_4$ so that $p = (0,0)$ and L is defined by $y = 0$. Then R is defined by a polynomial $f(x,y) = \sum_{(k,l)\in\Delta} a_{k,l}x^k y^l$ with $\Delta = [(12,0),(0,4),(0,3)]$. We may assume that $a_{0,2} = 1$. After the change of coordinates

$$(x,y) \to \left(\frac{x}{1-ax}, \frac{y}{(1-ax)^4}\right), \qquad \text{where } a = \frac{a_{1,2}}{4}$$

we get $a_{1,2} = 0$. Note, that this change of coordinates is the composition of $(x, y) \to (x^{-1}, x^{-4}y)$, $(x, y) \to (x - a, y)$, and $(x, y) \to (x^{-1}, x^{-4}y)$. Hence, the new coordinate system is also standard and L is still defined by $y = 0$.

The fact that R has a singularity of the type A_n with $n \geq 15$ means that there exists a germ of analytic function $\gamma(x) = c_6x^6 + c_7x^7 + \dots$ such that the Newton diagram of $f(x, z - \gamma(x)) = \sum b_{k,l}x^k z^l$ is contained in $[(0, \infty), (0, 2), (16, 0), (\infty, 0)]$, i.e. $b_{k,0} = 0$ for $k = 0, \dots, 15$ and $b_{k,1} = 0$ for $k = 0, \dots, 7$. Rescaling x, y, and f, one can achieve that $c_6 = 1$.

Expanding $f(x, z - \gamma(x)) = f(x, z - x^6 - c_7x^7 - \dots)$, we can express each $b_{k,l}$ as a polynomial in $a_{k,l}$'s and c_j's. In particular,

$$b_{6,1} = a_{6,1} - 2, \qquad \text{hence} \qquad a_{6,1} = 2.$$

Substituting this into the expression for $b_{12,0}$, we get

$$b_{12,0} = a_{12,0} - 1, \qquad \text{hence} \qquad a_{12,0} = 1.$$

Continuing this process, we get successively:

$$\begin{aligned} b_{13,0} &= -a_{7,1}, && \text{hence} \quad a_{7,1} = 0; \\ b_{7,1} &= -2c_7, && \text{hence} \quad c_7 = 0; \\ b_{14,0} &= a_{2,2} - a_{8,1}, && \text{hence} \quad a_{2,2} = a_{8,1}; \\ b_{15,0} &= a_{3,2}, && \text{hence} \quad a_{3,2} = 0. \end{aligned}$$

Thus, $a_{7,1} = a_{3,2} = 0$, hence, $f(x, y)$ is a polynomial in x^2 and y. □

Corollary 3.10. *The arrangement in Figure 10(b) is algebraically unrealizable*

Proof. Combine Lemmas 3.8 and 3.9 with the fact that Figure 13 is asymmetric.

3.4. Prohibition of the affine sextic. Throughout this subsection we shall suppose that C_6 is a real pseudoholomorphic sextic curve on $\mathbf{R}P^2$ which has a singular point p of the type A_5 and which is arranged with respect to a real line L as in Figure 14(a) up to isotopy (note, that Figures 14(a) and 14(b) represent the same arrangement of $C_6 \cup L$). The goal of this subsection is to prove that such an arrangement is algebraically unrealizable. Let T be the tangent line to C_6 at p.

Lemma 3.11. *The interior epmty oval of C_6 is separated from the exterior ones by $L \cup T$.*

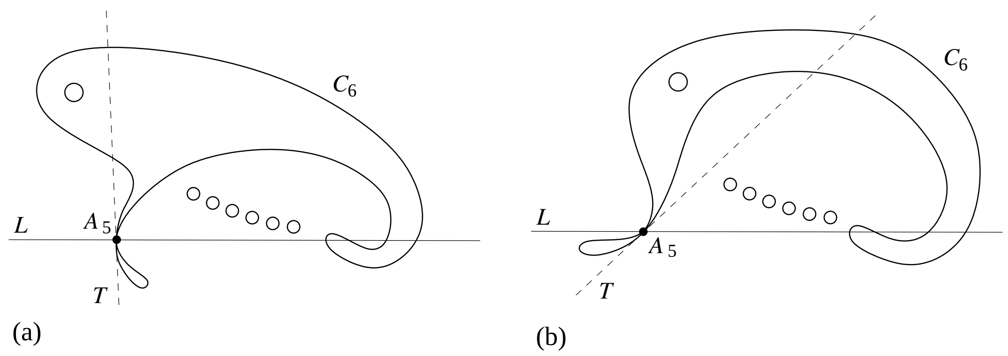

FIGURE 14

Proof. Let us sooth out the singularity A_5 so that three ovals appear. Then we obtain a projective M-sextic of the isotopy type $\langle 9 \sqcup 1\langle 1\rangle\rangle$ (Harnack M-sextic). At least two its exterior empty ovals (we denote them by V_1 and V_2) are connected by a vanishing cycle to the non-empty oval: the rightmost oval in Figures 14(a,b) and one of the ovals coming from A_5. Here a *vanishing cycle* between two ovals of a real curve A is a disk $D \subset \mathbf{C}P^2$ such that $D \cap \mathbf{C}A = \partial D$ and $D \cap \mathbf{R}P^2$ is an arc between the ovals.

If the interior empty oval were on the other side of T then the ovals V_1 and V_2 would be consecutive in the pencil of lines through the interior empty oval. This is impossible by [8]. □

Corollary 3.12. *The common arrangement of C_6, L, and T on $\mathbf{R}P^2$ up to isotopy is either as in Figure 14(a) or as in Figure 14(b).* □

Let us blow up twice the singular point of C_6 and then blow down the proper transform of T.[5] Let us denote the exceptional curves of the blowups by E_1 and E_2 (E_2 is the transform of a point of E_1).

We obtain a curve C_4 on $\mathcal{F}_2$ of bidegree $(4, 8)$ which has two nodes on the same fiber F_2 (which is the transform of E_2). Let us denote the proper transforms on $\mathcal{F}_2$ of the curves L and E_1 by F_2 and E. These are a fiber and the exceptional section respectively. The arrangements of $C_6 \cup L \cup T$ in Figures 14(a) and 14(b) correspond to the arrangements of $C_4 \cup F_1 \cup F_2 \cup E$ in Figures 15(a) and 15(b) respectively.

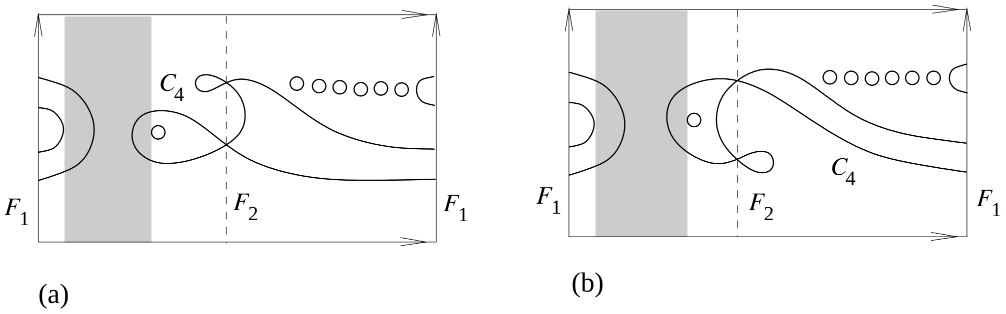

FIGURE 15

In the proof of the following lemma we use the approach based on braids (see [11]). We shall denote by $\sigma_1, \dots, \sigma_{m-1}$ the standard generators of B_m, the group of braids with m strings. For a braid $b = \prod_i \sigma_{j_i}^{k_i}$ we denote $e(b) = \sum_i k_i$ (the exponent sum). A braid is called *quasipositive* if it can be presented as $\prod a_i^{-1} \sigma_{j_i} a_i$. To any arrangement of a curve of bidegree (m, nm) on $\mathcal{F}_n$ with respect to the pencil of vertical lines we asociate a certain braid with m strings (this braid may depend on some unknown parameters; see detais in [11]). The arrangement is realizable by a pseudoholomorphic curve if and only if the corresponding braid is quasipositive.

Lemma 3.13. *(a) The arrangement in Figure 15(a) is realizable by a real pseudo-holomorphic curves. For any such a realization, the self-linking number of C_4 on the shadowed segment* (*see Definition 3.6*) *is equal to* -4.

[5] To do this in pseudoholomorphic case, we need to assume that the almost complex structure is standard (integrable) in some neighbourhood of T. Note, that pseudoholomorphic realizability of an isotopy type of $C_6 \cup L \cup T$ does not depend on this assumption.

(b) The arrangement in Figure 15(b) is not realizable by a real pseudoholomorphic curves.

Proof. (a). The braid associated to this arrangement (see [11]) is

$$b = \sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{1-k}\sigma_3^{1+k}\sigma_2^{-1}\tau_{2,3}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\tau_{2,3}\sigma_3^{-6}\Delta^2$$

where $\tau_{2,3} = \sigma_3^{-1}\sigma_2$, $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, and k is the self-linking number of C_4 on the shadowed segment. We have $e(b) = 0$. Thus, b is quasipositive if and only if b is trivial. The linking number of the first two strings is $k+4$. Hence, b is non-trivial for $k \neq -4$. One easily checks that b is trivial for $k = -4$.

(b). The braid is

$$b = \sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{1-k}\sigma_3^{1+k}\sigma_2^{-1}\sigma_1^{-2}\sigma_3^{-1}\tau_{1,3}\sigma_3^{-6}\Delta^2$$

where Δ and k are as above and $\tau_{1,3} = \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2$. We have $e(b) = 0$, but b is not trivial for any k. Indeed, let $\varphi : B_n \to S_n$ be the standard homomorphism to the symmetric group: $\sigma_i \mapsto (i, i+1)$. Then $\varphi(b) = (124)$ for k odd and $\varphi(b) = (234)$ for k even. □

Corollary 3.14. *The arrangement in Figure 15(a) is algebraically unrealizable. Hence, the arrangement in Figure 14(a) is also algebraically unrealizable.*

Proof. Combine Lemmas 3.7 and 3.13 (the inequality (17) does not hold for $|k| = 4$ and $n = 2$). □

Remark 3.15. The arrangements in Figures 14(a) and 15(a) are pseudoholomorphically realizable.

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