THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF A PLANE ALGEBRAIC CURVE

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Abstract. $\pi_1(\mathbf{C}^2 - K)$ is computed where K is an algebraic curve having only simple double points and satisfying certain restrictions at infinity. These restrictions are satisfied, for example, for a generic curve paramertized by polynomials of given degrees, and also for a generic curve with a given Newton polygon. As corollary, a new proof of the Fulton-Deligne theorem which states that $\pi_1(\mathbf{C}P^2 - K)$ is abelian is obtained, if K has only simple double points in $\mathbf{C}P^2$.

Introduction

The interest in the fundamental group of the complement of an algebraic curve in $\mathbf{C}P^2$ has appeared beginning with Zariski's work in the twenties and the thirties (cf. [1]). In particular, Zariski proved that the problem of computing the fundamental group of the complement of an algebraic hypersurface in $\mathbf{C}P^n$ can be reduced to the problem of computing the fundamental group of the complement of an appropiate plane curve. He also computed the fundamental group of some series of plane curves.

Recently Fulton [2] and Deligne [3] have proved the following Zariski's conjecture: if all the singularities of a curve C in $\mathbf{C}P^2$ (possibly reducible) are nodes (i.e. points at which two analytically irreducible smooth branches intersect transversally), then the fundamental group $\pi_1(\mathbf{C}P^2 - C)$ is abelian. A survey of other examples in which it is possible to compute the fundamental group of the complement of a curve can be found in [4].

This paper is devoted to the study of the group $\pi_1(\mathbf{C}^2 - K)$ where K is an algebraic curve satisfying the negativity condition at the infinity (see §1). Our main result is the following.

Theorem A. *Let an algebraic curve K in $\mathbf{C}^2$ satisfy the negativity condition at the infinity (see §1), and assume that all its singularities (lying in $\mathbf{C}^2$) are nodes. Then to each irreducible component of K ane can associate an element of $\pi_1(\mathbf{C}^2 - K)$ in such a way that these elements generate the whole group, and if two irreducible components intersect each other then the corresponding elements commute.*

In §1, the negativity condition at the infinity is formulated, and examples of curves satisfying this condition are considered. In particular, if C is a curve in $\mathbf{C}P^2$ then the curve $C - (L \cap C)$ in $\mathbf{C}P^2 - L$ satisfies the negativity condition at the infinity for almost all lines L. Therefore, Theorem A yields a new proof of the theorem of Fulton and Deligne. (The inclusion $\mathbf{C}P^2 - C - L \to \mathbf{C}P^2 - L$ induces an epimorphism of the fundamental groups.)

§§2–5 are devoted to the proof of Theorem A. In passing, we give a presentation of $\pi_1(\mathbf{C}^2 - K)$ by generators and relations. This presentation is analogous to Wirtinger's presentation of the knot group. It allows one to find numerically the group of the complement of a curve which is explicitly defined and whose degree is not very large.

A sketch of the algorithm is given in §6. In the same section, the result of a computation is given. The computation has been executed on the PDP-11/70 computer in the Laboratory of Computer Methods of the Faculty of Mathematics and Mechanics in Moscow State University. I thank thank the collaborators of this laboratory for their help.

The results of the computation allows us to conjecture that Theorem A holds without the negativity condition. This conjecture agrees with the results of Libgober [4] on Alexander polynomial of plane algebraic curves.

Actually, Theorem A is valid for sets obtained from complex algebraic curves by means of small real-differentiable perturbations, if the resulting set is algebraic in some neighbourhoods of the infinity and of singular points. If Theorem A without the negativity condition were proved for such sets, this should yield (using the easy Lemma 4.2 from [7]) a proof of the well-known Jacobian Conjecture (see [6]).

Added in 2003: This reduction of the Jacobian Conjecture is wrong.

§1. Negativity condition

Let K be an algebraic curve in $\mathbf{C}^2$. We introduce an affine coordinate system (z,w) in $\mathbf{C}^2$. Then K defines the multivalued algebraic function $F(z) = \{w \mid (z,w) \in K\}$ which can be expanded into a Puiseux series at the infinity, i.e. there exists a positive integer d and a neighbourhood U of the infinite point of the z-axis such that the restriction of F to $U - [0,+\infty)$ consists of n distinct single-valued analytic branches $f_1,\dots,f_n$ of the form $f_j(z) = g_j(\tau(z))$, where every g_j is a function, meromorphic and single-valued at the point $\tau = 0$, and τ is a single-valued (in $U - [0,+\infty)$) branch of the function $z^{-1/d}$. If d is the minimal possible integer such that the above presentation is possible, then the set $\{g_1,\dots,g_n\}$ is uniquely determined by K and the coordinate system (z,w).

We say that K *satisfies the negativity condition at the infinity with respect to the coordinate system* (z,w) if $k \neq l$ implies that the function $g_k(\tau) - g_l(\tau)$ does not vanish at $\tau = 0$ (i.e. it has either a pole or a finite limit there). It is easy to see that if a curve satisfies the negativity condition at the infinity with respect to one coordinate system, then it satisfies the negativity condition at the infinity with respect to almost all coordinate systems. If such a coordinate system exists, we say that the curve *satisfies the negativity condition at the infinity* (in this section we shall write NC).

Using the terminology from [5], the NC can be reformulated as follows: a curve satisfies the NC if the braid corresponding to a sufficiently large circe is positive.

Example 1. (see the Introduction). If C is a curve in $\mathbf{C}P^2$, then for almost all lines L the curve $C - L$ in $\mathbf{C} = \mathbf{C}P^2 - L$ satisfies the NC.

Example 2. Consider the family of all curves parametrized by two polynomials of given degrees in one variable. A generic curve of this family satisfies the NC.

Example 3. A generic curve with a given Newton polygon satisfies the NC.

Example 4. A curve satisfies the NC if each of its singularities at the infinity (we assume that $\mathbf{C}^2$ is embedded into $\mathbf{C}P^2$) is either a cusp defined by $u^m = v^n$ tangent to the infinite line, or a transversal intersection of nonsingular branches one of which may be tangent to the infinite line.

Example 5. A curve satisfies the NC if each of its singularities at the infinity is analytically irreducible, and near this point the curve can be expanded as follows: $x = t^{-n}$ and $y = \sum_{-m}^{+\infty} a_i t^i$ (x and y are coordinates in $\mathbf{C}^2$), where all the characteristic Puiseux exponents are negative, i.e. $\gcd\{i > 0 \mid a_i \neq 0\} = 1$.

§2. Nondegenerate coordinate systems

Let K be an algebraic curve in $\mathbf{C}^2$ satisfying the hypothesis of Theorem A. Every affine coordinate system (z, w) defines the multi-valued function $F(z) = \{w \mid (z, w) \in K\}$. Let n be the number of values of F at a generic point of the z-axis. Denote by B the set of points of the z-axis at which F has less than n values. We say that a coordinate system (z, w) is *nondegenerate with respect to* K if the following conditions (N1)–(N8) hold.

I. *Nondegeneracy of the w-axis.*

(N1) The w-axis is not parallel to any line tangent to K at a singular point or at a point of inflextion.

(N2) The w-axis is parallel neither to a line passing through two singular points, nor to a line passing through a singular point and tangent to K, nor to a bitangent.

(N3) The infinite point of the w-axis does not belong to the closure of K in $\mathbf{C}P^2$.

(N4) K satisfies the negativity condition at the infinity with respect to the coordinate system (z, w).

II. *Nondegeneracy of the axis* $\operatorname{Re} w$.

(N5) If $z \in B$ and $\{w_1, \dots, w_k\} = F(z)$, then $\operatorname{Re} w_i \neq \operatorname{Re} w_j$ for $w_i \neq w_j$.

(N6) If U is a neighbourhood of the infinity on the z-axis and f_1 and f_2 are two branches of F, single-valued in $U - [0, +\infty)$, such that

$$\lim_{z \to \infty} (f_1(z) - f_2(z)) = A \neq \infty,$$

then $\operatorname{Re} A \neq 0$.

(N7) If U is an open simply connected subset of $\mathbf{C} - B$ and f_1 and f_2 are two distinct single-valued branches of F in U, then the set $\{z \in U \mid \operatorname{Re} f_1(z) = \operatorname{Re} f_2(z)\}$ is a smooth real algebraic subvariety of U (let us denote it by $B_U(f_1, f_2)$).

(N8) If U is an open simply connected subset of $\mathbf{C} - B$ and f_1, f_2, f_3, f_4 are single-valued branches of F in U such that $f_1 \neq f_2$, $f_3 \neq f_4$, and $\{f_1, f_2\} \neq \{f_3, f_4\}$ then the curves $B_U(f_1, f_2)$ and $B_U(f_3, f_4)$ have only transversal intersections ($\{\,,\,\}$ denotes a set consisting of two elements).

It is clear that we can always choose a coordinate system satisfying (N1)–(N4).

The condition (N3) implies that B is the projection onto the z-axis of the finite set $\tilde{B}$ consisting of the singular points of K at which the tangent to $\tilde{K}$ is parallel to the w-axis (branch points of F). (N2) implies that distinct points of $\tilde{B}$ are projected to distinct points of B. Therefore, we have

Proposition 2.1. *If the coordinate system is nondegenerate with respect to K, then the function F has exactly $n-1$ values at every point of B.*

The rest of this section is devoted to a proof of the following statement.

Lemma 2.2. *The curve K possesses a nondegenerate coordinate system. Moreover, if a coordinate system (z,w) satisfies (N1)-(N4) then for almost all $\theta\in[0,\pi]$ (except for a finite subset) the coordinate system $(z,\exp(i\theta)w)$ is nondegenerate.*

Lemma 2.2 is an immediate consequence of the following three lemmas.

Lemma 2.3. *Let f be an algebraic function, single-valued in a domain $U\subset\mathbf{C}$ and nonvanishing in U. Then there exists a finite set Θ, such that for $\theta\in[0,\pi]-\Theta$ the set $L_\theta=\{z\mid\operatorname{Re}(\exp(i\theta)f(z))=0\}$ is a smooth real algebraic subvariety of U.*

Proof. If the subvariety L_θ is not smooth at a point z, then $f'(z)=0$. Let $\Theta=\{\,\pi/2-\operatorname{Arg}f(z)\mid f'(z)=0\,\}$. It is clear that the set Θ is finite and L_θ is a smooth curve for $\theta\notin\Theta$. Q.E.D.

Lemma 2.4. *Let f_1 and f_2 be algebraic functions, single-valued in a domain $U\subset\mathbf{C}$ and nonvanishing in U. Let $f_1/f_2\neq$ const. Then there exists a finite set Θ such that for $\theta\in[0,\pi]-\Theta$ the real curves*

$$\{\operatorname{Re}(\exp(i\theta)f_1(z))=0\}\qquad\textit{and}\qquad\{\operatorname{Re}(\exp(i\theta)f_2(z))=0\}$$

intersect transversally.

Proof. Denote the curve $\{\operatorname{Re}(\exp(i\theta)f_j(z))=0\}$ by L_j^θ. If one of the functions is identically equal to a constant c, then for $\theta\not\equiv\pi/2-\operatorname{Arg}c \mod \pi$ the corresponding set L_j^θ is empty. Therefore we shall assume that neither function is constant.

If L_1^θ and L_2^θ intersect at a point z, then the complex numbers $f_1(z)$ and $f_2(z)$ are linearly dependant over $\mathbf{R}$. If these curves intersect at z nontransversally, then the complex numbers $f_1'(z)$ and $f_2'(z)$ are linearly dependant over $\mathbf{R}$ (because $\operatorname{grad}\operatorname{Re}f(z)=f'(z)$). Denote by S_0 (respectively, S_1) the set of those z for which $f_1(z)$ and $f_2(z)$ (respectively, $f_1'(z)$ and $f_2'(z)$) are linearly dependant over $\mathbf{R}$. Let $S=S_0\cap S_1$.

Let $\Theta=\{\,\pi/2-\operatorname{Arg}f_2(z)\mid z\in S\}$. It is clear that if $\theta\notin\Theta$, then $L_2^\theta\cap S=\varnothing$; hence the curves L_1^θ and L_2^θ intersect transversally. Therefore, it is sufficient to prove that the set Θ is finite. Being a real algebraic variety, S is the union of a finite number of points and smooth arcs. We shall prove that if a smooth arc defined by $z=\varphi(t)$ lies in $S-\{f_2'(z)=0\}$, then the function $\operatorname{Arg}f_2(\varphi(t))$ is constant (this implies the finiteness of Θ, and hence, completes the proof). Indeed, along φ the following identities hold:

$$f_1(\varphi(t))=\alpha(t)f_2(\varphi(t)),\qquad f_1'(\varphi(t))=\beta(t)f_2'(\varphi(t)),$$

where α and β are smooth real functions (since $f_1/f_2\neq$ const, we have $\alpha'(t)\neq0$). Derivating the first identity with respect to t and substracting the result from the second one multiplied by $\varphi'(t)$, we obtain

$$(\beta(t)-\alpha(t))\frac{d}{dt}f_2(\varphi(t))=\alpha'(t)f_2(\varphi(t)).$$

Hence, since $\alpha'(t) \neq 0$ and $f_2'(\varphi(t)) \neq 0$, we have

$$\frac{d}{dt}\operatorname{Arg} f_2(\varphi(t)) = \frac{d}{dt}\operatorname{Im}\ln f_2(\varphi(t)) = \operatorname{Im}\left(\frac{df_2(\varphi(t))/dt}{f_2(\varphi(t))}\right)$$
$$= \operatorname{Im}\alpha'(t)/(\alpha(t)-\beta(t)) = 0.$$

The lemma is proved.

Lemma 2.5. *Let f_1, f_2, f_3, f_4 be single-valued branches of F in a simpli connected domain U. Let $f_1 \neq f_2$, $f_3 \neq f_4$, and $\{f_1, f_2\} \neq \{f_3, f_4\}$, and $(f_1 - f_2)/(f_3 - f_4) =$ const. Then $f_1 - f_2 =$ const.*

Proof. Let R be the Riemann surface of the function $h = f_1 - f_2$. It is sufficient to prove that this function is nonvanishing in R. Indeed, by virtue of the negativity condition h does not vanish at the points lying over the infinity, and, according to Proposition 2.1, h does not vanish at other points. Indeed, if some branch of h vanished at some point, so would the corresponding branch of $f_2 - f_4$. Therefore, the number of values of F at that point would be less than $n-1$. The lemma is proved.

§3. Generators and relations of $\pi_1(\mathbf{C}^2 - K)$

As above, let a curve K satisfy the hypothesis of Theorem A, and let (z, w) be a coordinate system that is nondegenerate with respect to K. Denote by B_+ the set $B \cup \{z \mid \exists w_1, w_2 \in F(z)$ such that $w_1 \neq w_2$ and $\operatorname{Re} w_1 = \operatorname{Re} w_2\}$. It is a one-dimensional real semialgebraic set. Let B_0 be the nonsingular part of $B_+ - B$.

Define the integer-valued function N on B_0 as follows. Let $z \in B_0$. According to (N8) (see §2), in some neighbourhood U of z there exist uniquely defined (up to indexing) branches f_1 and f_2 of F, single-valued in U, such that $B_+ \cap U = \{\operatorname{Re} f_1 = \operatorname{Re} f_2\}$, and the real parts of the remaining values of F at z are pairwise distinct and differ from $\operatorname{Re} f_1(z)$. Let $N(z) = 1 + \#\{w \in F(z) \mid \operatorname{Re} w < \operatorname{Re} f_1(z)\}$. In other words, if the values of F at z are indexed in such a way that their real parts increase, then $N(z)$ and $N(z)+1$ are the indices of those values whose real parts coincide. The function N is continuous, hence, locally constant on B_0. Introduce on B_0 the orientation assuming that the tangent vector to B_0 at z is positive if and only if $|f_1 - f_2|$ decreases along it (such a vector exists since z is a nonsingular point of B_+).

The set B_+, together with the orientation and the function N, defines the pair $(\mathbf{C}^2, K)$ uniquely up to a homeomorphism. The above construction is due to Rudolph [5]. In that paper there are also examples of B_+ for some curves. Another example is considered in §6 below.

Now we shall describe the behavior of the set B_+ near points of B (cf. [5]). If z is a branch point of F (according to (N1) the ramification index should be equal to 2), then in appropriate local coordinates near z the set B_+ is a half-line which starts at z and which is oriented towards z (note that according top (N5) no other curve from B_+ passes through z). If z is the projection of a singular point of K (which must be a node), then near z the set B_+ is a smooth real curve also oriented towards z (at z the orientation of B_+ changes).

Now let us describe the group $\pi_1(\mathbf{C}^2 - K)$ in terms of the set B_+. As a base point we shall consider the infinite point of teh positive part of the axis $\operatorname{Im} w$.

Consider one of the parts of which the set $\mathbf{C} - B_+$ consists, and denote it by U. For such a domain define $\alpha_1(U), \dots, \alpha_n(U) \in \pi_1(\mathbf{C}^2 - K)$ as follows. Choose any point $z_0 \in U$. Let $w_1, \dots, w_n$ be the values of F at z_0, indexed in such a way that their real parts increase. Denote by $\alpha_j(U)$ the path which goes to (z_0, w_j) from the infinity along the ray $\{z = z_0, \operatorname{Re} w = \operatorname{Re} w_j, \operatorname{Im} w = \operatorname{Im} w_j\}$, goes around (z_0, w_j) counterclockwise along a small circle in the complex line $z = z_0$, and goes back to the infinity along the same ray. Clearly, the homotopy class of $\alpha_j(U)$ depends only on the domain U and not on the choice of the point z_0 on it. It is easy to show (see for instance [1]) that for any U the elements $\alpha_1(U), \dots, \alpha_n(U)$ generate $\pi_1(\mathbf{C}^2 - K)$.

Let U be a domain on a plane, and G a piece of its border. Suppose that the plane and the curve G are priented. We say that U is to the right of G if the base (e_1, e_2) is positively oriented, where e_1 is a vector beginning at some point of G and directed into U, and e_2 is a vector tangent to G at the same point and directed positively with respect to the orientation of G.

Lemma 3.1. *Let G be a connected component of B_0 and let U and V be the connected components of $\mathbf{C} - B_+$ having G as a common piece of boundary, so that U is to the left of G and V is to the right (possibly, $U = V$). Let $i = N(G)$. Then*

$$\alpha_j(U) = \alpha_j(V) \qquad \text{for } j \neq i, i+1, \tag{1}$$

$$\alpha_{i+1}(U) = \alpha_i(V) \tag{2}$$

(let us denote this element by β), and

$$\alpha_i(U) = \beta \alpha_{i+1}(V) \beta^{-1}. \tag{3}$$

The proof is omitted because it coincides with the proof of Wirtinger's relations from knot theory.

Remark 3.2. Denote the commutator $[\alpha_i(V), \alpha_{i+1}(V)]$ by $k(G)$. Then (3) can be rewritten as follows:

$$\alpha_i(U) = k(G)\, \alpha_{i+1}(V). \tag{3$'$}$$

Remark 3.3. One can show that the generators $\alpha_j(U)$, where $j = 1, \dots, n$ and U ranges over all components of $\mathbf{C} - B_+$, and relation (1)–(3) written for each connected component of B_0, completely define the group $\pi_1(\mathbf{C}^2 - K)$, i.e. any other relation on the elements $\alpha_j(U)$ is a consequence of those relations.

Remark 3.4. Actually, Lemma 3.1 can be applied to any curve for which the Lemma 2.5 holds (if Lemma 2.5 does not hold for some curve then Lemma 3.1 just does not make sense, because in this case the function N is not defined). As an example of a curve for which Lemma 2.5 does not hold, one can consider the complexification of two concentric circles.

Added in 2003: As it is shown in the paper [S.Yu. Orevkov, *The commutant of the fundamental group of the complement of a plane algebraic curve,* Russ. Math. Surveys, **??**], the curve in Remark 3.4 is the only case when the statement of Lemma 2.5 is not true; see also the appedix to the paper [S.Yu. Orevkov, *Rudolph diagrams and analytic realization of Vitushkin's covering*, Math. Notes, **60**(1996), 153-164].

§4. Proof of Theorem A under a certain additional condition

In this section we shall assume that a curve K satisfies the hypothesis of Theorem A, a coordinate system (z, w) is nondegenerate with respect to K (see §2), and the following condition holds.

(N9) The set of real parts of values of F at any point consists of at least $n-2$ pairwise distinct real numbers.

Lemma 4.1. *If a curve K satisfies* (N9) *then Theorem A is valid.*

This section is devoted to the proof of this lemma.

Condition (N9) ipmplies that at most three real curves from B_+ intersect in a point. Moreover, if at some point the curve $\{\operatorname{Re} f_1 = \operatorname{Re} f_2\}$ meets the curve $\{\operatorname{Re} f_3 = \operatorname{Re} f_4\}$, where f_1, f_2, f_3, f_4 are pairwise distinct branches of F, then no other curve passes through this point. If the curve $\{\operatorname{Re} f_1 = \operatorname{Re} f_2\}$ meets the curve $\{\operatorname{Re} f_2 = \operatorname{Re} f_3\}$, then evidently the curve $\{\operatorname{Re} f_1 = \operatorname{Re} f_3\}$ passes through the intersection point, but (N9) implies that no other curve does. Such points will be called *double* and *triple* points of B_+ respectively.

The last paragraph and the condition (N7) give the following result.

Proposition 4.2. *Every singular point of B_+ either belongs to B, or is a double or a triple point of B_+.*

Lemma 4.3. *Let $z \in B$, and let U be a connected component of $\mathbf{C} - B_+$ whose closure contains z. Let i and $i+1$ be the indices of the values of F coinciding at z (the values are assumed to be indexed so that their real parts increase). Then the elements $\alpha_i(U)$ and $\alpha_{i+1}(U)$ commute.*

Proof. Let $w = w_i = w_{i+1}$ be the singular value of F at z, i.e. (z, w) is either a singular point of K or a point of K at which the tangent to K is vertical. Choose a sufficiently small neighbourhood Z of z, and let I be an interval of the real axis containing $\operatorname{Re} w$ and not containing the real parts of the other values of F at z. Let $W = \{(z, w) \mid z \in Z, \operatorname{Re} w \in I\}$. Then the elements $\alpha_i(U)$ and $\alpha_{i+1}(U)$ are represented by loops lying in W. On the other hand, if Z and I are small enough, then $\pi_1(W - K)$ is abelian, because $W - K$ is homeomorphic either to $\mathbf{C} \times (\mathbf{C} - 0)$ (if (z, w) is a smooth point of K), or to $(\mathbf{C} - 0) \times (\mathbf{C} - 0)$ (if (z, w) is a singular point of K).

The lemma is proved.

Lemma 4.4. *Let G be a connected component of the set B_0, and let U and V be the connected components of $\mathbf{C} - B_+$ for which G is a common part of boundary. Denote $N(G)$ (cf. §3) by i. Then*

$$\alpha_i(U) = \alpha_{i+1}(V), \qquad \alpha_{i+1}(U) = \alpha_i(V). \tag{4}$$

Proof. According to Lemma 3.1 the statement (4) is equivalent to $k(G) = 1$ (see Remark 3.2).

For every connected component G of B_0 set $L(G) = \inf_G |f|$, where f is the difference of the branches of F whose real parts coincide on G. Assume that the components of B_0 are indexed so that $L(G)$ increases and use the induction.

First let us prove the lemma for $L(G) = 0$. In this case the negativity condition (see §1) implies that G has an end at some point $z \in B$. Hence, by Lemma 4.3, $k(G) = 1$, i.e. for $L(G) = 0$ the lemma is proved.

Now let us assume that the lemma is proved for all G' with $L(G') < L(G)$, and let us prove it for G. According to (N6) (see §2), moving along G in the positive direction ($|f|$ decreasing), we cannot reach the infinity. Hence, we reach $\inf_G |f|$ at some point $z_0 \in B_+ - B$. By Proposition 4.2, z_0 is either a double point or a triple point of B_+. Assume that z_0 is a double point. Denote the continuation of G through z_0 by G', and the components of $\mathbf{C} - B_+$ for which G is the common part of the boundary by U' and V' (U and U' being on one side of $G \cup G'$, and V and V' on the other). From Lemma 3.1 and the definition of a double point it follows that $\alpha_i(U) = \alpha_i(U')$ and $\alpha_{i+1}(U) = \alpha_{i+1}(U')$; hence $k(G) = k(G')$. But clearly, $L(G') < L(G)$, hence by the induction hypothesis $k(G') = 1$. In the case of a double point the induction step is proved.

Now let z_0 be a triple point of B_+. Denote by f_1, f_2, f_3 the branches of F with $\operatorname{Re} f_1(z_0) = \operatorname{Re} f_2(z_0) = \operatorname{Re} f_3(z_0)$, assuming them to be indexed so that

$$\operatorname{Im} f_1(z_0) < \operatorname{Im} f_2(z_0) < \operatorname{Im} f_3(z_0). \tag{5}$$

Let G_{12}, G_{23}, G_{13} be the connected components of B_0 ending at z_0 (i.e. containing z_0 as an end point and oriented towards it), such that $G_{ij} = \{\operatorname{Re} f_i = \operatorname{Re} f_j\}$. Let G'_{ij} be the continuation of G_{ij} through z_0. Denote the vectors $-\operatorname{grad} |f_i - f_j|$ by v_{ij} (the vector v_{ij} is tangent to G'_{ij}). We assume that the base (v_{12}, v_{23}) is positively oriented (the case when it is negatively oriented is absolutely analogous). It is clear that $v_{13} = v_{12} + v_{23}$; hence, moving around z_0 counterclockwise, we shall meet these curves in the following order:

$$G_{12}, G_{13}, G_{23}, G'_{12}, G'_{13}, G'_{23}. \tag{6}$$

Consider the mapping $h : \mathbf{C} \to \mathbf{C}$ defined by the function $f_3 - f_1$. It takes z_0 into a point $h(z_0)$ lying on the imaginary axis higher than zero, and v_{13} into a vector directed from $h(z_0)$ to zero. The mapping h maps the positively oriented base (v_{12}, v_{13}) into a positively oriented base. Therefore, $\operatorname{Re} h(z) < 0$ for $z \in G'_{12}$; hence, for $z \in G'_{12}$ the real parts of the numbers $f_i(z)$ are arranged in the following way: $\operatorname{Re} f_3 < \operatorname{Re} f_1 = \operatorname{Re} f_2$.

Analogously one can find the arrangements of the real parts of these functions on the other arcs G_{ij} and G'_{ij}. These arguments prove that if $N(G_{12}) = j$ (see the beginning of §3 for the definition of N), then

$$N(G'_{13}) = N(G_{23}) = N(G_{12}) = j, \qquad N(G_{13}) = N(G'_{23}) = N(G'_{12}) = j + 1. \tag{7}$$

Denote by U the domain (connected component of $\mathbf{C} - B_+$) lying between G'_{12} and G_{23}, and let

$$\alpha = \alpha_j(U), \qquad \beta = \alpha_{j+1}(U), \qquad \gamma = \alpha_{j+2}(U).$$

Lemma 3.1 and formulas (6) and (7) imply

$$k(G_{12}) = \alpha[\beta, \gamma]\alpha^{-1}, \qquad k(G'_{12}) = [\beta, \gamma], \tag{8}$$

$$k(G_{13}) = [\alpha, \gamma], \qquad k(G'_{13}) = [\alpha, \beta\gamma\beta^{-1}], \tag{9}$$

$$k(G_{23}) = [\alpha, \beta], \qquad k(G'_{23}) = [\alpha, \beta]. \tag{10}$$

Recall that we are proving the equality

$$k(G) = 1 \tag{11}$$

for a curve G ending at z_0, under the assumption that $k(G') = 1$ if $L(G') < L(G)$. There are three possible cases: $G = G_{12}$, $G = G_{23}$, or $G = G_{13}$.

In the first two cases (11) follows from (8) and (10) by the induction assumption and the evident inequality

$$L(G'_{ij} < L(G_{ij}) \qquad (ij = 12, 23, 13). \tag{12}$$

In the case when $G = G_{13}$, note that

$$L(G_{12}) = |\operatorname{Im}(f_2(z_0) - f_1(z_0))|, \qquad L(G_{13}) = |\operatorname{Im}(f_3(z_0) - f_1(z_0))|.$$

Therefore (5) implies $L(G'_{12}) < L(G_{12}) < L(G_{13})$; hence by the induction assumption we have $k(G'_{12}) = 1$, and, according to (8), $\gamma = \beta\gamma\beta^{-1}$. The induction assumption and the inequality (12) for $ij = 13$ implies $k(G'_{13}) = 1$. Therefore, by (9) we have

$$[\alpha, \beta\gamma\beta^{-1}] = 1.$$

Hence, $k(G) = k(G_{13}) = [\alpha, \gamma] = [\alpha, \beta\gamma\beta^{-1}] = 1$. The lemma is proved.

Denote the set $\{(z, w) \in K \mid z \in B_+\}$ by $\tilde{B}_+$. This is a real semialgebraic subset of K. Let us associate to each connected component U of $K - \tilde{B}_+$ the element $\alpha(U) \in \pi_1(\mathbf{C}^2 - K)$ which corresponds to the following path. It comes along the ray $\{(z, w) \mid z = z_0, \operatorname{Re} w = \operatorname{Re} w_0, \operatorname{Im} w > \operatorname{Im} w_0\}$ to some point $(z_0, w_0) \in U$, goes around this point in the complex line $z = z_0$ counterclockwise, and then goes back to the infinity. In the notation of §3 we have $\alpha(U) = \alpha_j(pU)$, where p is the projection onto the z-axis, and $i = \#\{\, w \in F(z_0) \mid \operatorname{Re} w \le \operatorname{Re} w_0 \,\}$ for some point $(z_0, w_0) \in U$.

Lemma 4.5. *Let K_1 be an irreducible component of K, and let U and V be connected components of $K_1 - \tilde{B}_+$. Then $\alpha(U) = \alpha(V)$. (Denote this element by $\alpha(K_1)$).)*

Proof. If U and V heve a common boundary part, then, by Lemma 4.3 we have $\alpha(U) = \alpha(V)$. Otherwise let us join some point of U with some point of V by a path lying in $K_1 - \tilde{B}$ (which is a smooth connected manifold). Then along this path there is a chain $U_1, \dots, U_k$ of components of $K_1 - \tilde{B}_+$ such that $U_1 = U$, $U_k = V$, and U_i and U_{i+1} share a common boundary part. The lemma is proved.

Proof of Lemma 4.1. By Lemma 4.5, to each irreducible component K_1 of K corresponds a unique element $\alpha(K_1)$. If the components intersect then by Lemma 4.3 these elements commute.

§5. Completion of the proof of Theorem A

In this section we use the same notation as in §§2 and 3. Condition (N9) is not assumed to hold.

Let $z_1, \dots, z_s$ be the singular points of B_+ not contained in B. Choose closed disks $U_1, \dots, U_s$ centered at these points, in such a way that all their pairwise intersections are empty, and none of them contains a point of B. Choose also in every disk U_i a smaller concentrical disk V_i. Denote $\bigcup U_i$ and $\bigcup V_i$ by U and V.

Let H be the space of all functions holomorphic in a neighbourhood of U, and set $f = (f_1, \dots, f_n) \in H^n$ where $f_1, \dots, f_n$ are analytic branches of F, single-valued in U. Fix a smooth real function φ which is equal to 1 on V and 0 on $\mathbf{C} - U$. For $h = (h_1, \dots, h_s) \in H^n$, denote by $K(h)$ a "perturbation of K", i.e. the set $\Gamma \cup (K \cap ((\mathbf{C} - U) \times \mathbf{C}))$ where Γ is the union of the graphs of the functions $(f_i + h_i\varphi)$. Let us denote by $F(h)$ and $B_+(h)$ the objects analogous to F and B_+ (see §§2 and 3), but constructed for $K(h)$.

Let H_1 be the set of $h \in H^n$ such that $F(h)$ satisfies condition (N9) from §4, and H_2 the set of all $h \in H^n$ satisfying the following conditions:

(a) There exists a neighbourhood O of B_+ such that the mapping

$$(f_i - f_j) + (h_i - h_j)\varphi : U \cap O \to \mathbf{C}$$

is orientation preserving.

(b) A one-parameter family of inclusions of $K(th)$ into $\mathbf{C}^2$, $t \in [0, 1]$, realizes an isotopy between the inclusion of K into $\mathbf{C}^2$ and the inclusion of $K(h)$ into $\mathbf{C}^2$.

(c) A one-parameter family of inclusions of $B_+(th)$ into $\mathbf{C}$, $t \in [0, 1]$, realizes an isotopy between the inclusion of B_+ into $\mathbf{C}$ and the inclusion of $B_+(h)$ into $\mathbf{C}$.

It is easy to see that in the topology $\mathcal{C}^2$ the set H_1 is everywhere dense, and the set H_2 is open.

To complete the proof of Theorem A it suffices to note that if $h \in H_1 \cap H_2$ then, first, $\pi_1(\mathbf{C}^2 - K) = \pi_1(\mathbf{C}^2 - K(h))$, and, second, all the arguments from §§2–4 can be applied to the curve $K(h)$.

§6. Numerical computation of the fundamental group of a curve

Let K be an algebraic curve in $\mathbf{C}^2$ for which Lemma 2.5 holds. Then (see Remark 3.4) Lemma 3.1 gives us a presentation of $\pi_1(\mathbf{C}^2 - K)$ in terms of the set B_+.

If a curve K is defined by an equation with given concrete coefficients, then the set B_+ can be computed numerically. To this end one has to expand F at the infinity into a Puiseux series, to find the points of intersection of B_+ with a sufficiently large ("infinite") circle, and, then, for each of these points, to draw a curve (i.e. an irreducible component B_+) passing through this point. The curve should be drawn either up to the singularity of F or up to the return to the "infinite" circle. At each step along the curve it is essential not to mix the single-valued branches of F.

If at a particular step, z_0 is replaced by $z = z_0 + h$ and the values $f_i(z_0)$ ($i = 1, \dots, n$) are known, then the values at z of the analytic continuation of the f_i along the segment $[z_0, z]$ can be computed in the following way. As initial approximation for $f_i(z)$ take $f_i(z_0) + f_i'(z_0)h$ (the derivative can be found by the implicit function theorem). Then this approximation is refined by Newton's method ($f_i(z)$ is a root of a polynomial with coefficients depending on z).

I have done the computations for the curve defined parametrically by

$$z = t^4 + 2t^2 + 2t + 1, \qquad w = 2t^6 + 6t^4 + 6t^3 + 6t^2 + 6t + 5.$$

These are polynomials of the least degrees for which is possible that the parametrized curve does not satisfy the negativity condition at the infinity. On the other hand, among the curves parametrized by polynomials of the fourth and the sixth degrees, not satisfying the negativity condition, this curve is "the worst", i.e. its B_+ has the maximal possible number of curves oriented towards the infinity.

In the variables (z, w) this curve is defined by

$$(4z^3 + 12z + 3 - w^2)^2 - 8w - 48 + 52 = 0;$$

it has three nodes. The z-coordinates of the nodes, and, respectively, the values of z for which $w(z)$ is ramified, are the roots of

$$z^3 - 4z^2 - 4 = 0 \qquad \text{and} \qquad 16z^3 - 16z^2 + 72z - 37 = 0.$$

The expansion of $F(z)$ into a Puiseux series at the infinity is

$$z = t^{-4}, \qquad w = 2t^{-6} + 3t^2 - t^3 + \dots .$$

This implies that in a neighbourhood of the infinity, B_+ consists of three real curves oriented outwards and of twelve curves (*grouped in three pencils of four curves*) oriented inwards. The topology of B_+, constructed numerically, is shown in Figure 1. The integers written near the curves are the values of the function N.

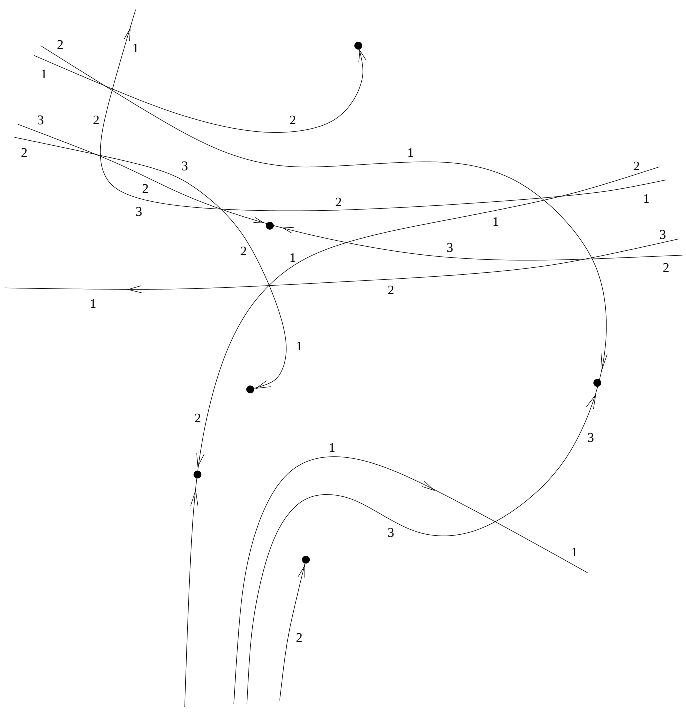

FIGURE 1

For the numerical experiment I used a PDP-11/70 computer; the computations were performed with single precision; on each curve were built about 1000–2000 points; as the "infinite" circle I chosed the circle of radius 8 centered at the origine. F was expanded at the infinity into a Puiseux series up to the 40th term.

The fundamental group of the complement of teh described curve turned out to be abelian. To see this, it is enough to look at Figure 1 and to apply Lemma 3.1.

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