$\mathbb{C}$-BOUNDARY LINKS UP TO SIX CROSSINGS

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Abstract. An oriented link is called $\mathbb{C}$-*boundary* if it is realizable as $(\partial B, A \cap \partial B)$ where A is an algebraic curve in $\mathbb{C}^2$ and B is an embedded 4-ball. This notion was introduced by Michel Boileau and Lee Rudolph in 1995. In a recent joint paper with N. G. Kruzhilin we gave a complete classification of $\mathbb{C}$-boundaries with at most 5 crossings. In the present paper a more regular method of construction of $\mathbb{C}$-boundaries is proposed and the classification is extended up to 6 crossings.

1. Introduction

Boileau and Rudolph [2] defined $\mathbb{C}$-*boundary* as an oriented link in the 3-sphere $\mathbb{S}^3$ which can be realized as $(\partial B, A \cap \partial B)$ where A is an algebraic curve in $\mathbb{C}^2$ and B is an embedded 4-ball (with the boundary orientation of $A \cap \partial B$ induced from $A \cap B$). It is observed in [2] that Kronheimer-Mrowka Theorem [6] (former Thom Conjecture) implies that some knots are not $\mathbb{C}$-boundaries, for example, the figure-eight knot 4_1.

If B is pseudo-convex (for example, a standard 4-ball), then $(\partial B, A \cap \partial B)$ is a quasipositive link by [1]. In [7] we observed that there exist non-quasipositive $\mathbb{C}$-boundaries, the simplest ones being $L \,\#\, {-L^*}$ where L is a quasipositive link. Here and below, L^* denotes the mirror image of L, and $-L$ denotes L with the opposite orientation. In [7] we also corrected some errors in [2] and observed that it is often more efficient to apply the so-called Immersed Thom Conjecture in order to obtain restrictions on the $\mathbb{C}$-boundaries (see details in [7, §3] and in §3 here).

It is natural to distinguish the case when L is realizable as $A \cap \partial B$ as above, and moreover $A \setminus B$ is connected. Such links are called in [7] *strong* $\mathbb{C}$-*boundaries*. Examples of non-strong $\mathbb{C}$-boundaries are given in [7]. The simplest one is the split sum $2_1 \sqcup 2_1^*$ (we denote the positive two-component Hopf link by 2_1).

We gave in [7] a complete list of $\mathbb{C}$-boundaries and strong $\mathbb{C}$-boundaries with at most 5 crossings (including split and composite links). In the present paper a complete list of $\mathbb{C}$-boundaries with 6 crossings is given (see §3). For two of them it remains unknown if they are strong $\mathbb{C}$-boundaries. Very elementary constructions were used in [7]. They are not enough for links with 6 crossings, and in §2 we present a more systematic way to construct (strong) $\mathbb{C}$-boundaries. In contrary, the fact that some links are not (strong) $\mathbb{C}$-boundaries, is proven here by the same methods as in [7]. In §4.2 we show that 10 (among 15) slice knots with ≤ 9 crossings are $\mathbb{C}$-boundaries. Some open questions are posed in §4.3.

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2. A construction of $\mathbb{C}$-boundaries

2.1. Notation and terminology. We denote the segment $[0,1]$ by I and we set $I^2 = I \times I$. Let L be an oriented link. A *band attached to* L is an embedding $b : I^2 \to \mathbb{S}^3$ such that $b|_{I\times\partial I} \to L$ is an orientation preserving embedding where the orientation of $I \times \partial I$ is inherited from the boundary orientation of $\partial(I^2)$ (see Figure 1). Sometimes, abusing the language, we use this term for $b(I^2)$ rather than for b. The result of the *band-move* (or *band surgery*) on L along b is

$$L_b = \big(L \setminus b(I \times \partial I)\big) \cup b(\partial I \times I)$$

with the orientation coherent with L (see Figure 1). We say that two bands b, b' are disjoint if $b(I^2) \cap b'(I^2) = \varnothing$. If $\mathbf{b}$ is a collection of pairwise disjoint bands attached to L, let $L_{\mathbf{b}}$ denote the result of simultaneous band-moves on L along all these bands.

A band b attached to a link L is called *trivial* if L_b is a split sum $L' \sqcup O$ where O is an unknot composed of an arc of ∂b and an arc of L (see the right panel in Figure 1). In this case, $O \cap L$ is called the *vanishing arc* of $(L, \mathbf{b})$. A collection $\mathbf{b} = \{b_1, \dots, b_n\}$ of pairwise disjoint bands attached to L is *trivial* if all b_i are trivial for L and their vanishing arcs are pairwise disjoint. When such $\mathbf{b}$ is a subset of a larger collection $\mathbf{b}'$ of pairwise disjoint bands, we say that $\mathbf{b}$ is a *trivial subset of* $\mathbf{b}'$, if it is a trivial collection of bands and all its vanishing arcs are disjoint from the bands belonging to $\mathbf{b}' \setminus \mathbf{b}$.

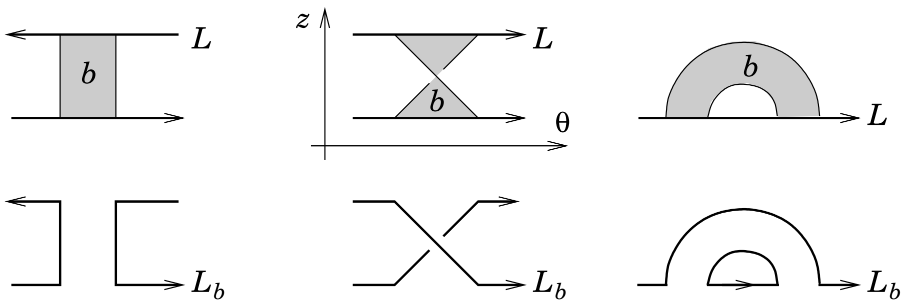

Figure 1. A band b attached to a link L. The link L_b.
In the middle: positive θ-monotone band. On the right: trivial band.

Now we identify $\mathbb{S}^3$ with a one-point compactification of $\mathbb{R}^3$ with standard coordinates (x, y, z). Let us fix cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$, that is $x = r\cos\theta$, $y = r\sin\theta$. We say that a link L is *θ-monotone* if $r|_L > 0$ and $d\theta$ is positive on L. We also call such links *geometric closed braids*.

Let L be a θ-monotone link. A band b attached to L is called *positive θ-monotone* if L_b is θ-monotone and the projection of some neighborhood of $L \cup b(I^2)$ to the (θ, z)-plane looks as in the middle panel of Figure 1.

An isotopy $\{L_t\}_{t\in I}$ is called *θ-monotone* if each link L_t is θ-monotone.

2.2. The construction. Let L be a θ-monotone link which is the braid closure of a quasipositive (maybe, trivial) braid, and let $\mathbf{b}$ be a collection of pairwise disjoint bands attached to L. Suppose that there exists a sequence of links $L = L_0, L_1, \dots, L_n = L'$ and a sequence of collections of bands $\mathbf{b} = \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n = \mathbf{b}'$

($\mathbf{b}_i$ is attached to L_i) such that for each $i = 1, \dots, n$, one of the following two conditions holds:

(1) $L_i \cup \mathbf{b}_i$ is obtained from $L_{i-1} \cup \mathbf{b}_{i-1}$ by an isotopy which restricts to a θ-monotone isotopy between L_{i-1} and L_i or
(2) $L_i = (L_{i-1})_{c_i}$ and $\mathbf{b}_i = \mathbf{b}_{i-1}$ where c_i is a positive θ-monotone band attached to L_{i-1} (see Figure 1) disjoint from $\mathbf{b}_{i-1}$.

Proposition 2.1. *If* $\mathbf{b}'$ *is trivial, then* $L_{\mathbf{b}}$ *is a* $\mathbb{C}$*-boundary.*

Proof. By construction, L' is the braid closure of a quasipositive braid. Hence, by Rudolph's theorem [13], $(\mathbb{S}^3, L')$ is diffeomorphic to $(\partial B', A \cap \partial B')$, $B' = \Delta' \times \Delta''$, where Δ' and Δ'' are some disks in $\mathbb{C}$, and A is a complex algebraic curve in $\mathbb{C}^2$ disjoint from $\Delta' \times \partial\Delta''$. Moreover, the arguments in [13] imply that there exist embedded disks $\Delta = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_n = \Delta'$ (with $\Delta_{i-1} \subset \operatorname{Int}\Delta_i$) such that $(B_i, A \cap \partial B_i) \approx (\mathbb{S}^3, L_i)$, $i = 0, \dots, n$, where $B_i = \Delta_i \times \Delta''$, and each time when L_i is obtained from L_{i-1} by a positive θ-monotone band-move, the annulus $\Delta_i \setminus \Delta_{i-1}$ contains a ramification point of $\pi|_{A\cap B'}$ where $\pi : B' = \Delta' \times \Delta'' \to \Delta'$ is the projection onto the first factor.

We include the disk boundaries $\partial\Delta_0, \dots, \partial\Delta_n$ into a continuous family. Namely, consider a diffeomorphism $f : \mathbb{S}^1 \times I \to \Delta' \setminus \operatorname{Int}\Delta$, such that $\partial\Delta_i = f_{t_i}(\mathbb{S}^1)$ for some numbers t_i in the range $0 = t_0 < t_1 < \cdots < t_n = 1$ (here $f_t : \mathbb{S}^1 \to \Delta'$ is defined by $f_t(p) = f(p, t)$).

For each band b from $\mathbf{b}$, the isotopies between $\mathbf{b}_{i-1}$ and $\mathbf{b}_i$ (which are identical when $\mathbf{b}_i = (\mathbf{b}_{i-1})_{c_i}$) can be glued together into a single isotopy $\{b_t : I^2 \to B'\}_{t\in I}$ such that $\pi(b_t(I^2)) \subset f_t(\mathbb{S}^1)$. Let $\{h_t : I^3 \to B'\}_{t\in I}$ be an isotopy obtained from $\{b_t\}$ by replacing each band $b_t : I^2 \to \pi^{-1}(f_t(\mathbb{S}^1))$ with it thickening $h_t : I^3 \to \pi^{-1}(f_t(\mathbb{S}^1))$ as shown in Figure 2. Finally, we modify this isotopy near $t = 1$ as shown in Figure 3 (still keeping the notation h_t for it).

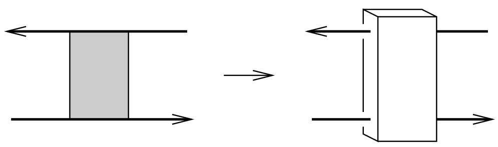

FIGURE 2. A thickened band.

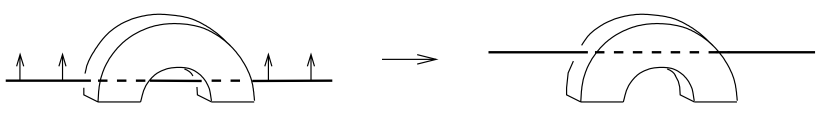

FIGURE 3. The final isotopy of a thickened trivial band.

We obtain an embedding of the 4-cube $h = h^{(b)} : I^4 = I^3 \times I \to B'$, $h(p, t) = h_t(p)$. The intersection of $h(I^4)$ with $B = \Delta \times \Delta''$ is the image of one of the eight faces of I^4, that is $h(I^4) \cap B = h(I^3 \times 0)$. Thus the following set $\hat{B}$ (appropriately smoothed) is an embedded 4-ball:

$$\hat{B} = B \cup \bigcup_{b\in\mathbf{b}} h^{(b)}(I^4).$$

Let us show that $(\partial\hat{B}, A\cap\partial\hat{B}) \approx (\mathbb{S}^3, L_{\mathbf{b}})$. Indeed, when the 4-cube $h^{(b)}(I^4)$ is attached to B, the following surgery is applied to the boundary link: the 3-ball $h^{(b)}(I^4\times 0)$, which is a face of the attached 4-ball, is replaced by the union of the seven other faces. One can see in Figure 4 that this operation coincides with the band-move on L along b (cf. Figure 1). □

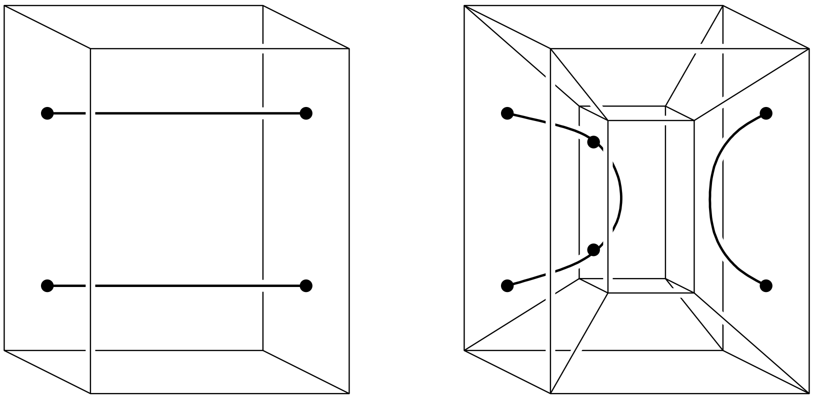

FIGURE 4. Replacement of one face of I^4 by seven other faces.

Corollary 2.2. *If $\mathbf{b}' = \mathbf{b}'_0 \cup \mathbf{b}'_1$ where $\mathbf{b}'_0$ is a trivial subset of $\mathbf{b}'$ and each band in $\mathbf{b}'_1$ is positive θ-monotone, then $L_{\mathbf{b}}$ is a $\mathbb{C}$-boundary.*

Proof. We continue the sequences $(L_1,\dots)$ and $(\mathbf{b}_1,\dots)$ as shown in Figure 5. Then each positive θ-monotone band is transformed into a trivial band. □

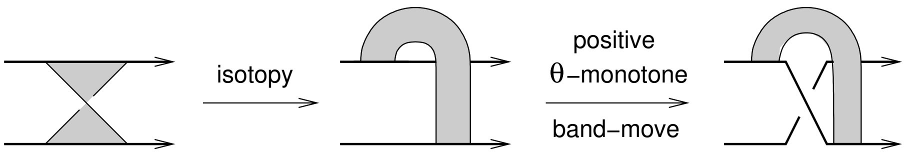

FIGURE 5. From a positive θ-monotone band to a trivial band.

Example 2.3. It is almost evident (see [7, §2]) that $L \sqcup -L^*$ and $L \mathbin{\#} -L^*$ are $\mathbb{C}$-boundaries, if L is a quasipositive link (in $L \mathbin{\#} -L^*$, some component of L is supposed to be joint with its mirror image). This fact can be also obtained as a simplest application of Proposition 2.1. Indeed, let β be a quasipositive braid whose braid closure is L. We start with the geometric braid with bands shown in Figure 6, and apply positive θ-monotone band-moves to the β^{-1} half so that it is transformed into a trivial braid.

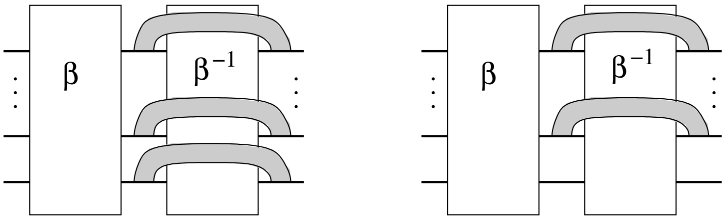

FIGURE 6. $\mathbb{C}$-boundary realization of $L \sqcup -L^*$ (on the left) and $L \mathbin{\#} -L^*$ (on the right) based on Proposition 2.1.

Example 2.4. Similarly to Example 2.3, we obtain a $\mathbb{C}$-boundary realization of $L \sqcup -L_1^*$ and $L \# -L_1^*$ where L and L_1 are the braid closures of $\beta = \beta_1\beta_2$ and β_1 respectively for quasipositive braids β_1 and β_2: just replace β^{-1} by β_1^{-1} in Figure 6.

Example 2.5. Similarly to the previous examples, we obtain a $\mathbb{C}$-boundary realization of $L \#_{(K,-K^*)} -L^*$ where L is the braid closure of an n-braid β belonging to the monoid M generated by $\{\sigma_1, \dots, \sigma_{k-1}, \sigma_k^{-1}, \dots, \sigma_{n-1}^{-1}\}$ for some $k \le n$, and K is the component of L containing the k-th string of β. To this end, we start with a configuration as in Figure 6 but with bands attached to all the strings except the k-th one. Then, by a sequence of positive θ-monotone band-moves we eliminate all the negative crossings and obtain a trivial collection of bands attached to a positive braid. In particular $4_1\#4_1$ is a $\mathbb{C}$-boundary: 4_1 is the braid closure of $(\sigma_1\sigma_2^{-1})^2$.

The same construction (and its generalization in the spirit of Example 2.4) also applies if M is replaced by the monoid generated by

$$\{a\sigma_1 a^{-1} \mid a \in \iota_0(B_k)\} \cup \{a\sigma_k^{-1}a^{-1} \mid a \in \iota_{k-1}(B_{n-k+1})\}$$

where $\iota_p : B_m \to B_n$ (for $m+p \le n$) is the homomorhism of the braid groups which takes σ_i of B_m to σ_{i+p} of B_n.

2.3. When the obtained $\mathbb{C}$-boundary is strong? Let the setting be as in §2.2. Suppose that $\mathbf{b}'$ is trivial, and hence $L_{\mathbf{b}}$ is a $\mathbb{C}$-boundary by Proposition 2.1. Let us explain how to check if it is strong or not.

Let G be the graph whose vertices correspond to all components (which are open intervals or circles) of $L_i \setminus \mathbf{b}_i$ for all $i = 0, \dots, n$, and the edges are defined as follows. If $L_i \cup \mathbf{b}_i$ is obtained from $L_{i-1} \cup \mathbf{b}_{i-1}$ by a θ-monotone isotopy, then the corresponding pairs of vertices are connected by an edge of G. If $L_i = (L_{i-1})_{c_i}$ and $\mathbf{b}_i = \mathbf{b}_{i-1}$, then all the four vertices corresponding the components of $L_{i-1} \setminus \mathbf{b}_{i-1}$ and $L_i \setminus \mathbf{b}_i$ adjacent to c_i are connected to each other by edges of G.

Proposition 2.6. *If each vertex of G corresponding to a vanishing arc of $(L', \mathbf{b}')$ is connected by a path with a vertex corresponding to a non-vanishing arc of $(L', \mathbf{b}')$, then the link $L_{\mathbf{b}}$ is a strong $\mathbb{C}$-boundary.*

Note that the hypothesis of Proposition 2.6 is equivalent to the fact that the sequences $L_1, \dots, L_n$ and $\mathbf{b}_1, \dots, \mathbf{b}_n$ can be extended so that the graph G becomes connected. One can check that the hypothesis of Proposition 2.6 does not hold for $L \sqcup -L^*$ in Example 2.3 (cf. Question 4.5 below).

Example 2.7. (A new $\mathbb{C}$-boundary realization of 5_1^2.) In [7, Prop. 6.3] we proved that the link 5_1^2 (the mirror image of L5a1 in [9]) is a strong $\mathbb{C}$-boundary. Corollary 2.2 provides another proof which seems to be simpler (see Figure 7).

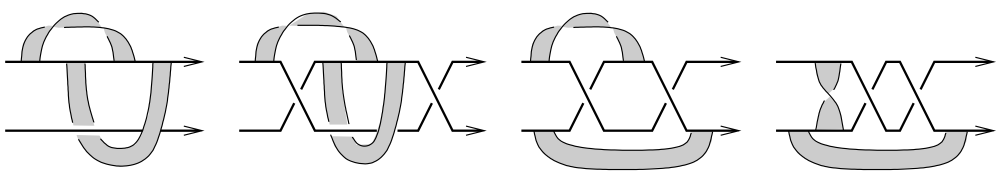

FIGURE 7. A new strong $\mathbb{C}$-boundary realization of 5_1^2.

3. Links with six crossings

As in [7], we denote the class of quasipositive links, the class of $\mathbb{C}$-boundaries, and the class of strong $\mathbb{C}$-boundaries by $\mathcal{Q}$, $\mathcal{B}$, and $\mathcal{SB}$ respectively. In this section, for each link with crossing number 6, we determine if it belongs to these classes (with two exceptions for the class $\mathcal{SB}$). In Tables 1–3 we give the answers for all such links which do not have an unknot split component, however, the answers for links of the form $L \sqcup O \sqcup \cdots \sqcup O$ with 6 crossings easily follow (see [7, §6]). More explanations about Tables 1–3 are given in §3.3.

Table 1. Prime links.

L	braid	sgn lk(,)	$L \in \mathcal{Q}$	$L \in \mathcal{SB}$	$L \in \mathcal{B}$	$\chi_s(L)$	$\chi_s^-(L)$	§3.3
6_1	$112\bar1\bar32\bar3$		no	yes	yes			F
6_1^*			no	yes	yes			F
6_2	$111\bar21\bar2$		no	yes	yes			F
6_2^*			no	no	no	-1	1	B
6_3	$11\bar21\bar2\bar2$		no	no	no	-1	1	B
L6a1(0)		$-$	no	no	no	-2	2	B
L6a1(0)*	$\bar12\bar32\bar1232$	$+$	no	no	no	-2	0	B
L6a1(1)	$112\bar1211$	$+$	yes	yes	yes			
L6a1(1)*		$-$	no	no	no	-2	2	B
L6a2(0)		$-$	no	no	no	-2	2	B
L6a2(0)*	$1\bar212211$	$+$	yes	yes	yes			
L6a3(0)		$-$	no	no	no	-4	2	B
L6a3(0)*	111111	$+$	yes	yes	yes			
L6a3(1)	$2\bar32\bar12\bar32331$	$+$	yes	yes	yes			
L6a3(1)*		$-$	no	no	no	0	2	B
L6a4	$1\bar21\bar21\bar2$	0 0 0	no	yes	yes			F
L6a5(0,0)		$- - -$	no	no	no	-1	3	C
L6a5(0,0)*	$\bar1\bar2321213\bar232$	$+ + +$	yes	yes	yes			
L6a5(0,1)	$1\bar211\bar21$	$+ + -$	no	yes	yes			F
L6a5(0,1)*		$- - +$	no	no	no	-1	1	C
L6n1(0,0)	$\bar1\bar12112$	$+ + -$	yes	yes	yes			
L6n1(0,0)*		$- - +$	no	?	yes			[7]
L6n1(0,1)		$- - -$	no	no	no	-3	3	C
L6n1(0,1)*	121212	$+ + +$	yes	yes	yes			

3.1. The links $2_1\#2_1\#2_1$, $2_1\#2_1\#2_1^*$, etc. Among the links with 6 crossings, almost all connected sums are uniquely determined by their summands. The only exceptions are connected sums of three copies of the Hopf link 2_1 and/or its mirror image 2_1^*. Such a sum is determined by the following graph with signed edges. Its

Table 2. Composite non-split links.

L	braid	$L \in \mathcal{Q}$	$L \in \mathcal{SB}$	$L \in \mathcal{B}$	$\chi_s(L)$	$\chi_s^-(L)$	see §3.3
$3_1\#3_1$	111222	yes	yes	yes			
$3_1\#3_1^*$		no	yes	yes			E
$3_1^*\#3_1^*$		no	no	no	−3	1	C
$4_1\#2_1$	$1\bar{2}1\bar{2}33$	no	yes	yes			F
$4_1\#2_1^*$	$1\bar{2}1\bar{2}\bar{3}\bar{3}$	no	no	no	0	2	B
L4a1(0)$\#2_1$		no	no	no	−1	1	B
L4a1(0)$^*\#2_1$	$2\bar{1}21133$	yes	yes	yes			
L4a1(0)$\#2_1^*$		no	no	no	−1	3	B
L4a1(0)$^*\#2_1^*$	$2\bar{1}211\bar{3}\bar{3}$	no	no	no	−1	1	B
L4a1(1)$\#2_1$	111122	yes	yes	yes			
L4a1(1)$^*\#2_1$		no	no	no	−1	1	B
L4a1(1)$\#2_1^*$	$1111\bar{2}\bar{2}$	no	yes	yes			E
L4a1(1)$^*\#2_1^*$		no	no	no	−1	3	B
I_{+++}	112233	yes	yes	yes			
I_{++-}	$1122\bar{3}\bar{3}$	no	yes	yes			A
I_{+-+}	$11\bar{2}\bar{2}33$	no	yes	yes			A,F
I_{+--}	$11\bar{2}\bar{2}\bar{3}\bar{3}$	no	no	no	0	2	C
I_{-+-}	$\bar{1}\bar{1}22\bar{3}\bar{3}$	no	no	no	0	2	C
I_{---}	$\bar{1}\bar{1}\bar{2}\bar{2}\bar{3}\bar{3}$	no	no	no	−2	4	B,C
Y_{+++}	$1122322\bar{3}$	yes	yes	yes			
Y_{++-}	$11223\bar{2}\bar{2}\bar{3}$	no	yes	yes	0	2	A
Y_{+--}	$11\bar{2}\bar{2}3\bar{2}\bar{2}\bar{3}$	no	no	no	0	2	C
Y_{---}	$\bar{1}\bar{1}\bar{2}\bar{2}3\bar{2}\bar{2}\bar{3}$	no	no	no	−2	4	B,C

vertices correspond to the link components. Two vertices are connected with an edge if the corresponding components are linked, and this edge is labeled by the linking number, which is always ± 1. This graph is a tree. There are only two 4-vertex trees: an I-shaped tree (a 4-path) and a Y-shaped tree (with a vertex of degree 3). We denote the corresponding links by $I_{\pm\pm\pm}$ and $Y_{\pm\pm\pm}$ respectively. The signs in the notation $I_{\pm\pm\pm}$ appear in the same order as in the graph.

Any connected sum of two strong $\mathbb{C}$-boundaries is a strong $\mathbb{C}$-boundary, and $2_1\#2_1^*$ is a strong $\mathbb{C}$-boundary (see Example 2.3). Hence all links of the form $2_1\#2_1\#2_1^*$ (i.e., I_{++-}, I_{+-+}, and Y_{++-}) are strong $\mathbb{C}$-boundaries. Note that I_{++-} and Y_{++-} are strong $\mathbb{C}$-boundaries by [7, Theorem 2.1], and we give another strong $\mathbb{C}$-boundary realization of I_{+-+} in Figure 10.

3.2. Restrictions on $\mathbb{C}$-boundaries. Our proofs that some links are not (strong) $\mathbb{C}$-boundaries are based on Kronheimer-Mrowka's Theorem [6] (former Thom Conjecture) and its version for immersed 2-surfaces in $\mathbb{C}P^2$ with negative double points (see [4], [10, §2]), which was actually proven in [6] but was not explicitly formulated.

Table 3. Split links.

L	braid	$L\in\mathcal{Q}$	$L\in\mathcal{SB}$	$L\in\mathcal{B}$	$\chi_s(L)$	$\chi_s^-(L)$	see §3.3
$3_1\sqcup 3_1$	$111\,333$	yes	yes	yes			
$3_1\sqcup 3_1^*$		no	?	yes			E
$3_1^*\sqcup 3_1^*$		no	no	no	-2	2	C
$4_1\sqcup 2_1$	$1\bar{2}1\bar{2}\,44$	no	no	no	-1	1	D
$4_1\sqcup 2_1^*$	$1\bar{2}1\bar{2}\,\bar{4}\bar{4}$	no	no	no	-1	3	D
$L4a1(0)\sqcup 2_1$		no	no	no	0	2	D
$L4a1(0)^*\sqcup 2_1$	$2\bar{1}211\,44$	yes	yes	yes			
$L4a1(0)\sqcup 2_1^*$		no	no	no	0	4	C
$L4a1(0)^*\sqcup 2_1^*$	$2\bar{1}211\,\bar{4}\bar{4}$	no	no	no	0	2	D
$L4a1(1)\sqcup 2_1$	$1111\,33$	yes	yes	yes			
$L4a1(1)^*\sqcup 2_1$		no	no	no	-2	0	D
$L4a1(1)\sqcup 2_1^*$	$1111\,\bar{3}\bar{3}$	no	no	yes	-2	0	E
$L4a1(1)^*\sqcup 2_1^*$		no	no	no	-2	2	C
$2_1\#2_1\sqcup 2_1$	$1122\,44$	yes	yes	yes			
$2_1\#2_1^*\sqcup 2_1$	$11\bar{2}\bar{2}\,44$	no	yes	yes			A
$2_1^*\#2_1^*\sqcup 2_1$	$\bar{1}\bar{1}\bar{2}\bar{2}\,44$	no	no	no	-1	3	D
$2_1\#2_1\sqcup 2_1^*$	$1122\,\bar{4}\bar{4}$	no	no	yes	-1	1	A
$2_1\#2_1^*\sqcup 2_1^*$	$11\bar{2}\bar{2}\,\bar{4}\bar{4}$	no	no	no	1	3	C
$2_1^*\#2_1^*\sqcup 2_1^*$	$\bar{1}\bar{1}\bar{2}\bar{2}\,\bar{4}\bar{4}$	no	no	no	-1	5	C
$2_1\sqcup 2_1\sqcup 2_1$	$11\,33\,55$	yes	yes	yes			
$2_1\sqcup 2_1\sqcup 2_1^*$	$11\,33\,\bar{5}\bar{5}$	no	no	yes	0	2	A
$2_1\sqcup 2_1^*\sqcup 2_1^*$	$11\,\bar{3}\bar{3}\,\bar{5}\bar{5}$	no	no	no	0	4	D
$2_1^*\sqcup 2_1^*\sqcup 2_1^*$	$\bar{1}\bar{1}\,\bar{3}\bar{3}\,\bar{5}\bar{5}$	no	no	no	0	6	C

Theorem 3.1. (The Immersed Thom Conjecture.) *Let Σ be a connected oriented closed surface of genus g and $j:\Sigma\to\mathbb{C}P^2$ be an immersion which has only negative ordinary double points as self-crossings. Let $j_*([\Sigma])=d[\mathbb{C}P^1]\in H_2(\mathbb{C}P^2)$ with $d>0$. Then g is bounded below by the genus of a smooth algebraic curve of degree d, that is, $g\ge(d-1)(d-2)/2$.*

Given a link L in $S^3=\partial B^4$, we define the *slice Euler characteristic* of L by $\chi_s(L)=\max_\Sigma\chi(\Sigma)$ where the maximum is taken over all embedded smooth oriented surfaces Σ without closed components, such that $\partial\Sigma=L$. Similarly, we define the *slice negatively immersed Euler characteristic* of L by $\chi_s^-(L)=\max_{(\Sigma,j)}\chi(\Sigma)$ where the maximum is taken over all immersions $j:(\Sigma,\partial\Sigma)\to(B^4,S^3)$ of oriented surfaces Σ without closed components such that $j(\Sigma)$ has only negative double points and $j(\partial\Sigma)=L$. Theorem 3.1 easily implies the following two statements.

Proposition 3.2. (See [7, Prop. 3.2 and Lem. 3.8].) *If $\chi_s(L)<\chi_s^-(L)$, then $L\notin\mathcal{SB}$. If, moreover, L does not have a proper sublink which has zero linking number with its complement (for example, if L is a knot), then $L\notin\mathcal{B}$.*

We see in [7, Table 1] and in Tables 1–3 here that for any link with ≤ 6 crossings with a single exception $3_1^*\#2_1$, we have either $L \in \mathcal{SB}$ or $\chi_s(L) < \chi_s^-(L)$.

Proposition 3.3. (See [7, Prop. 3.7]; cf. [2, Cor. 1.6].) *If L is a strong $\mathbb{C}$-boundary and $-L^*$ is a (not necessarily strong) $\mathbb{C}$-boundary, then $\chi_s(L) = \chi_s^-(L) \ge 1$.*

Now let us treat the cases which are not covered by Propositions 3.2, 3.3.

Proposition 3.4. *The links marked by "D" in the last column of Table 3 are not $\mathbb{C}$-boundaries.*

Proof. Case 1. The links $4_1 \sqcup 2_1$ and $4_1 \sqcup 2_1^*$. Suppose that there exists a $\mathbb{C}$-boundary realization $L = A \cap B$ of one of these links. Let $L = L_4 \cup L_2$ where L_4 is 4_1 and L_2 is 2_1 or 2_1^*. L is not a strong $\mathbb{C}$-boundary by Proposition 3.2. Hence $A \setminus B$ is a disjoint union $A_2 \cup A_4$ with $\partial A_k = -L_k$, where either A_4 or A_2 is bounded. The interior piece $A_0 = A \cap B$ of A is connected because otherwise A would be disconnected (here we use the fact that there is only one partition of L into sublinks with zero linking number). Hence $\chi(A_0) \le -1$. Let us replace A_0 by $A_2' \sqcup A_4'$, $\partial A_k' = L_k$, where A_2' is an embedded annulus and A_4' is an immersed disk with a negative self-crossing. Recall that A_2 or A_4 is bounded. If it is A_2, then $A_4 \cup A_4'$ contradicts Theorem 3.1; if it is A_4, then so does $A_2 \cup A_2'$ because in this case $\chi(A_4) \le -1$ (4_1 is not slice), and hence the replacement $A_0 \cup A_4 \to A_2'$ decreases χ_s^- (see Figure 8).

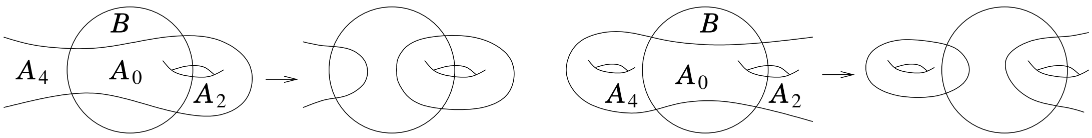

FIGURE 8. Proof that $4_1 \sqcup 2_1$ and $4_1 \sqcup 2_1^*$ are not $\mathbb{C}$-boundaries.

Case 2. The links L4a1(0) $\sqcup\, 2_1$, L4a1(0)$^* \sqcup 2_1^*$, *and* L4a1(1)$^* \sqcup 2_1$. The same proof as in the previous case; see Figure 9.

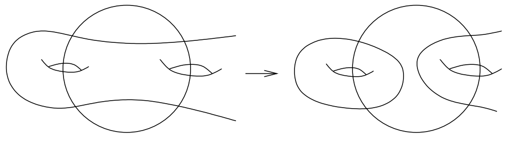

FIGURE 9. Proof that L4a1(0) $\sqcup\, 2_1$, L4a1(0)$^* \sqcup 2_1^*$, L4a1(1)$^* \sqcup 2_1 \notin \mathcal{B}$.

Case 3. The link $L = 2_1^*\#2_1^* \sqcup 2_1$. Suppose that $L = A \cap B$ is a $\mathbb{C}$-boundary realization of L. We have $\chi_s(L) = -1$ and $\chi_s^-(L) = 3$. Thus, if we replace $A \cap B$ with a negatively self-crossing surface A_0' which realizes $\chi_s^-(L) = 3$, we obtain a contradiction with Theorem 3.1 unless at least two spheres split out.

Note that if we glue together several surfaces along their boundaries, we may obtain a sphere only if at least two of the glued surfaces are disks. The link L does not have any component which has zero linking number with its complement, hence no component of $A \setminus B$ is a disk. The negatively self-crossing surface A_0' has at most three disks, hence at most one sphere can split out.

Case 4. The link $L = 2_1 \sqcup 2_1^* \sqcup 2_1^*$. Suppose that $L = A \cap B$ is a $\mathbb{C}$-boundary realization of L. We have $\chi_s(L) = 0$ and $\chi_s^-(L) = 4$. Thus, as in Case 3, if we replace $A \cap B$ with a negatively self-crossing surface A_0', at least two spheres must split out. Again as in Case 3, no component of $A \setminus B$ is a disk but this time A_0' may have four disks (bounded by the components of $2_1^* \sqcup 2_1^*$). Thus all these disks are present in A_0', all of them make part of the two spheres which split out, and $\chi(A \cap B) = 0$ (if $\chi(A \cap B)$ were negative, then one more sphere should split out). This may happen only when $A \setminus B = A_1^- \cup A_2^- \cup A^+$ where A_j^- are cylinders, A^+ is the unbounded component, each ∂A_j^- is 2_1^*, and ∂A^+ is 2_1.

The condition $\chi(A \cap B) = 0$ implies that $A \cap B$ is a union of three cylinders. Since all the linking numbers between their boundaries are zero, one of them must be bounded by 2_1^* (while a priori the boundaries of the other two may realize any $(2,2)$-partition of the four components of $2_1 \sqcup 2_1^*$). This fact contradicts the connectedness of A. □

3.3. Comments on Tables 1–3. The tables are organized similarly to Table 1 in [7]. The links are named according to [8], [9] except the two-component Hopf links 2_1, 2_1^*, and their connected sums, whose notation is explained in §3.1. The braid notation and the signs of linking numbers help to identify the links. The invariants $\chi_s(L)$ and $\chi_s^-(L)$ are introduced in §3.2. They are computed by standard methods (some of them are mentioned in [7, §6]) and we omit the details. The same for the quasipositivity: in all the considered cases, if $L \in \mathcal{Q}$, this fact is clear from the braid given in the tables, and if $L \notin \mathcal{Q}$, this fact follows either from [12], or from the Franks-Williams-Morton Inequality [5], [11] in the form given in [2, Thm. 3.2] (see [7, Thm. 6.1]). In all the cases when $L \in \mathcal{B} \setminus \mathcal{SB}$, the fact that $L \notin \mathcal{SB}$ follows from the inequality $\chi_s(L) < \chi_s^-(L)$ (see Proposition 3.2).

The letters A–F in the last column of each table mean the following:

A. $L \in \mathcal{B}$ or $L \in \mathcal{SB}$ by the additivity of (strong) $\mathbb{C}$-boundaries under split or connected sums [7, Prop. 3.6] (see also §3.1).
B. $L \notin \mathcal{B}$ by Proposition 3.2.
C. $L \notin \mathcal{B}$ by Proposition 3.3.
D. $L \notin \mathcal{B}$ by Proposition 3.4.
E. See Examples 2.3 and 2.4.
F. See Figure 10.

4. Concluding remarks

4.1. Squeezed knots. The referee drew my attention to a very interesting paper [3] by Feller, Lewark, and Lobb, where they introduce *squeezed knots*. These are knots which appear as a slice of a genus-minimizing oriented connected smooth cobordism between a positive torus knot and a negative torus knot.

Proposition 4.1. *Any $\mathbb{C}$-boundary knot is squeezed.*

Proof. Let $A \cap B$, $A = \{f(z,w) = 0\}$, be a $\mathbb{C}$-boundary realization of a knot K. Let $n = \deg f$. Without loss of generality we may assume that A is smooth, $f(0,0) = 0$, and $\deg_z f = n$. Let $A_\varepsilon = \{f(z,w) = \varepsilon w^{n+1}\}$. If $|\varepsilon|$ is sufficiently small, then $K_\varepsilon = A_\varepsilon \cap B$ is isotopic to K and $A_\varepsilon \cap (B_R \setminus B_r)$, $0 < r \ll 1 \ll R$, is a cobordism between the positive torus knot $T(n, n+1)$ and the unknot. This cobordism is genus-minimizing by Kronheimer-Mrowka's theorem. □

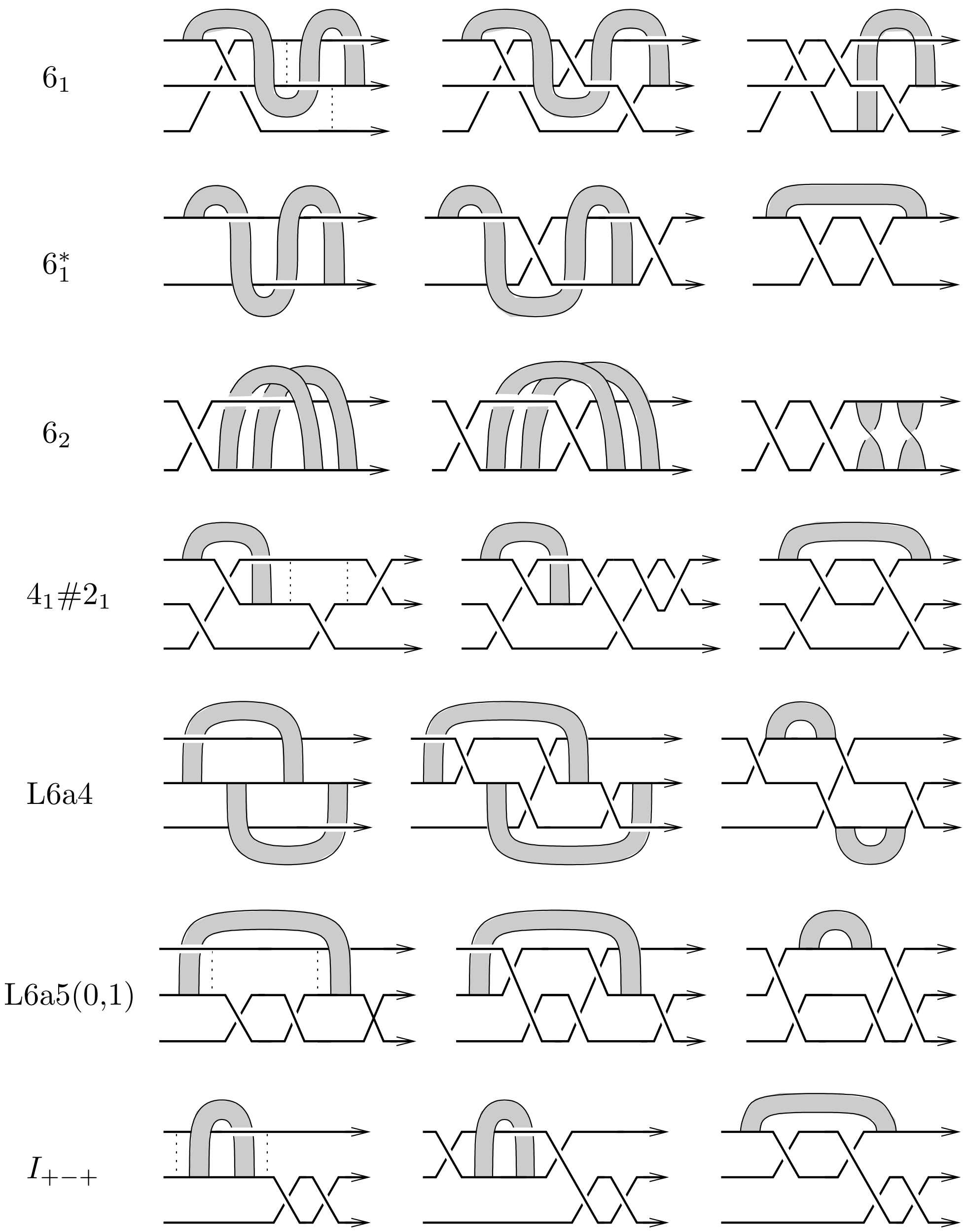

FIGURE 10. A strong ℂ-boundary realization of the knots 6_1, 6_1^*, 6_2, $4_1\#2_1$ and the links L6a4 (Borromeo rings), L6a5(0,1), and I_{+-+}.

As proven in [3], the knots 9_{42}, 10_{125}, 10_{132}, and 10_{136} are not squeezed. Hence they and their mirrors are not ℂ-boundaries. For these knots themselves, this fact follows from Proposition 3.2, because they are not slice (see [8]) but $\chi_s^- = 1$. The latter fact can be checked using the braids (see the beginning of §6 in [7]):

$$\begin{aligned}
\chi_s^-(9_{42}) &= \chi_s^-(\bar{1}\bar{1}\bar{1}211\bar{3}2\bar{3}) \geq \chi_s(\bar{1}(\bar{1}\bar{1}211)(\bar{3}23)) = 1,\\
\chi_s^-(10_{125}) &= \chi_s^-(\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}21112) \geq \chi_s((\bar{1}\bar{1}\bar{1}2111)2) = 1,\\
\chi_s^-(10_{132}) &= \chi_s^-(111\bar{2}\bar{1}\bar{1}\bar{2}\bar{3}2\bar{3}\bar{3}) \geq \chi_s(1(11\bar{2}\bar{1}\bar{1})(\bar{2}\bar{3}2)) = 1,\\
\chi_s^-(10_{136}) &= \chi_s^-(\bar{1}\bar{1}\bar{2}3\bar{2}1\bar{2}\bar{2}322) \geq \chi_s((23\bar{2})1(\bar{2}\bar{2}322)) = 1.
\end{aligned}$$

In contrary, the methods of our paper are not sufficient to prove that the mirror images of these four knots are not ℂ-boundaries. In the first arxiv version of this paper I asked whether $\chi_s^-(K) = \chi_s(K)$ implies that K is $\mathbb{C}$-boundary. The knots 9_{42}^*, $10_{125}*$, 10_{136}^* provide a negative answer. Indeed, if K is one of them, then the signature of $K\#2_1$ equals 3 which implies that $\chi_s^-(K) < 0$, hence $\chi_s^-(K) = \chi_s(K) = -1$.

Notice that a negative answer to the same question for links is provided by the link $L = 3_1^* \sqcup 2_1$. We have $\chi_s(L) = \chi_s^-(L) = 0$ whereas $L \notin \mathcal{B}$ by Proposition 3.3 (see [7]). It also seems plausible that $\chi_s^-(K) = \chi_s(K) = -1$ for $K = 8_{18}$ (the braid closure of $(\sigma_1\sigma_2^{-1})^4$), whereas $K = K^*$ and hence K is not a $\mathbb{C}$-boundary by [2, Cor. 1.6].

4.2. Slice knots. Up to now, all known proofs that a certain knot is not $\mathbb{C}$-boundary, automatically gives that all concordant knot are not $\mathbb{C}$-boundaries neither. Thus there are no tools to prove that a slice knot is not a $\mathbb{C}$-boundary.

There are 15 slice knots with ≤ 9 crossings: 6_1, 8_8, 8_9, 8_{20}, 9_{27}, 9_{41}, 9_{46}, their mirror images (8_9 is amphicheiral), $3_1\# 3_1^*$, and $4_1\# 4_1$. Among them, the following ten are realized as $\mathbb{C}$-boundaries: 6_1, 6_1^* (Figure 10); 8_8, 8_9, 9_{27}^*, 9_{41}^* (Figure 11); 8_{20}^*, 9_{46}^* (quasipositive knots); $3_1\# 3_1^*$ and $4_1\# 4_1$ (Examples 2.3 and 2.5). The realizability of 8_8^*, 8_{20}, 9_{27}, 9_{41}, and 9_{46} is unknown.

FIGURE 11. A $\mathbb{C}$-boundary realization of the knots 8_8, $8_9 = 8_9^*$, 9_{27}^*, 9_{41}^*.

4.3. Some open questions. The study of $\mathbb{C}$-boundary links is at the very beginning. So, most questions on this subject are open. However, I would like to emphasize some of them.

Question 4.2. Can any $\mathbb{C}$-boundary be realized by the construction from §2?

Question 4.3. Do there exist slice knots which are not $\mathbb{C}$-boundaries? In particular, does there exist a knot K such that $K \# -K^*$ is not a $\mathbb{C}$-boundary?

As we pointed out in §4.2, if the answer is affirmative, then essentially new ideas should be evoked to prove this fact.

Question 4.4. Is it true that for any slice knot K, either K or K^* is $\mathbb{C}$-boundary?

The answer is affirmative for knots with at most 9 crossings (see §4.2).

Question 4.5. Is it true that $K \sqcup -K^*$ is not a strong $\mathbb{C}$-boundary for any non-trivial quasipositive knot K?

Question 4.6. (Cancellation of unknot in a split sum.) If $L \sqcup O$ is a (strong) $\mathbb{C}$-boundary, is L a (strong) $\mathbb{C}$-boundary?

Question 4.7. Do there exist non-split non-strong $\mathbb{C}$-boundaries?

The link L6n1(0,0)* (see [7, §4]) is a candidate for an affirmative answer.

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