ON FOUR-SHEETED POLYNOMIAL MAPPINGS OF $\mathbb{C}^2$. I. THE CASE OF AN IRREDUCIBLE RAMIFICATION CURVE

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0. Introduction

Let $f : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ — be a polynomial mapping such that

$$\text{(0.1)} \qquad Jacobian(f) = const \neq 0.$$

Here $\widetilde{\mathbb{C}}^2$ and $\mathbb{C}^2$ are two copies of the complex plane. The well-known Jacobian Conjecture states that such f is polynomially invertible (see surveys in [1], [2].)

The mapping f satisfying (0.1) is polynomially invertible if and only if a generic point in $\mathbb{C}^2$ has one preimage. The *topological degree* of a mapping is said to be the number of preimages of a generic point. It is known that there are no polynomial mappings of degree two or three satisfying (0.1). The case of degree two is elementary and the case of degree three was considered in [3].

Theorem. *If the topological degree of a polynomial mapping $f : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is four then the jacobian of f can not be a nonzero constant.*

In this paper we prove a particular case of this theorem, namely when f has one dicritical component. (The definition of a dicritical component is given below.) The complete proof of the theorem will appear in the next paper. Here we apply the approach of [4], [5], [6].

By a finite number of blow-ups at infinity we can extend f to a regular rational mapping $F : \widetilde{X} \to X$, where $\widetilde{X}$, X are nonsingular compact complex varieties containing $\widetilde{\mathbb{C}}^2$ and $\mathbb{C}^2$ respectively as open subsets. The curves $\widetilde{L} := \widetilde{X} \setminus \widetilde{\mathbb{C}}^2$ and $L := X \setminus \mathbb{C}^2$ consist of finitely many nonsingular rational components with transversal and at most pairwise intersections.

An irreducible component $\widetilde{g}$ of $\widetilde{L}$ is called *dicritical* if $F(\widetilde{g}) \not\subset L$ and F is non-constant on $\widetilde{g}$.

The main result of the paper is as follows

Theorem 0.2. *There is no four-sheeted locally invertible polynomial mapping $f : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ with one dicritical component.*

The scheme of the proof is as follows. Blowing up $\widetilde{X}$ and X we can achieve that the image of the dicritical component meets L transversally. The dual weighted graphs of $\widetilde{L}$ and L are unimodal[1] trees. A weighted unimodal tree is determined by

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Partially supported by Grants RFFI-96-01-01218 and DGICYT SAB95-0502

[1] the determinant of the intersection matrix is ± 1.

the topological type of the graph and the determinants of the intersection matrices of the branches at nodal (of valence ≥ 3) vertices (see [7, 8, 9] where these data are called *splice diagram*). Since the degree of F is bounded, there are only finitely many possibilities for the combinatorial structure of the mapping of the splice diagrams. We consider each case separately.

It follows from [5] that weights of the splice diagram of $\widetilde{L}$ (i.e. the determinants of the branches) are determined by the weights of the splice diagram of L and by the branching orders at points of the components corresponding to the nodal vertices. The fact that $\widetilde{L}$ was obtained by blow-ups from the infinite line, imposes rather strong restrictions for the splice diagram of $\widetilde{L}$. Combining these restrictions with the condition $F^*(d\widetilde{x} \wedge d\widetilde{y}) = dx \wedge dy$ written in terms of canonical divisors of X and $\widetilde{X}$ (Lemma 5.1) we obtain a contradiction in each case (the coefficients in the expansion $K_{\widetilde{L}} = \sum_{\tilde{l} \subset \widetilde{L}} k_{\tilde{l}} \tilde{l}$ also can be expressed in terms of splice diagrams — Lemma 1.8)

The example [4] (see also [6]) shows that without involving additional arguments this approach fails for a big topological degree. The topological degree of the mapping constructed in [4] is 36. Modifying the numerical data one can reduce the degree up to 9 and this is the minimal possible degree for examples of this kind. So, it seems to be very probable that using the methods of the present paper one could prove that the topological degree of a potential counter-example to the Jacobian Conjecture must be ≥ 9.

Note that the degrees of polynomials defining the mapping of topological degree 9 should be (48,64). These are exactly the numbers appearing in [10, 11], as the first difficult case. Moreover, all the numerical data of other examples similar to [4] also well correspond to those in [11] (this observation was communicated to the second author by Pierrette Cassou-Nogues). Thus, moving from different directions, we meet the same difficulty.

We are grateful to A.G. Vitushkin and P. Cassou-Nogues for useful discussions.

1. Preliminaries

Dual graphs. Let Y be a nonsingular analytic surface, $L \subset Y$ a reduced curve (i.e. without multiple components) and all its irreducible components are closed non-singular complex curves with transversal and at most pairwise intersections. We associate with L a graph Γ_L (called *the dual graph of* L) whose vertices are irreducible components of L and the edges are the intersection points. Non-compact curves will be depicted as arrowhead vertices.

The pair (Y, L) is called *regular* if Y is a compact algebaic surface and all irreducible components of L are algebraic curves. In this case graph Γ_L is weighted, the weights are the self-intersection numbers. Let A_L be the intersection matrix of the curve L. The *determinant of* Γ_L is defined as $\det \Gamma_L = \det(-A_L)$. It will be convenient for us to suppose that the pair (Y, L) is also regular when $L = \varnothing$ or L is a single point. In these cases $\Gamma_L = \varnothing$ and $\det \Gamma_L = 1$.

If (Y, L) is a regular pair and the curve C meets L transversally then the *dual graph* $\Gamma_{L,C}$ *of* C *near* L is defined as the dual graph of the pair $(U, (L \cup C) \cap U)$ where U is a sufficiently small neighbourhood of L. So the arrowhead vertices correspond to the germs of C at L. Other vertices are weighted. In particular, if $C = \varnothing$ then $\Gamma_{L,C} = \Gamma_L$. If L is an ordinary double point of C then the dual graph of C near L is of the form $\longleftrightarrow$.

All the graphs considered below are the dual graphs of some curves. Therefore, by a *subgraph* of a graph Γ we shall mean a graph Γ' such that

a) vertices of Γ' are some vertices of Γ,
b) if two vertices of Γ' are connected by an edge of Γ then they must be connected by an edge of Γ'.

So, a subgraph is determined by its vertices. Further on we shall denote irreducible components and the corresponding vertices by the same small letters, we shall also denote curves (in general, reducible) and the corresponding subgraphs by the same capital letters. *The valence* $\nu_\Gamma(s)$ of a vertex s in a graph Γ is the number of incident edges. The vertex s is said to be an *end, linear* or *nodal* vertex of Γ if its valence is 1, 2 or > 2 respectively. The connected subgraph C of graph Γ is said to be *linear*, or *a linear chain* if for any its non-end vertex a we have $\nu_\Gamma(a) = \nu_C(a) = 2$. Let v be a vertex of Γ. The *branches of* Γ *at* v are, by definition, the connected components of $\Gamma - v$. The vertex v is said to be *incident* at vertex g if they are connected by an edge. The subgraph Q is said to be *incident* at vertex v if $v \notin Q$ and v is incident at some vertex of Q.

Notation (1.1). Let Γ be a tree. (Recall that a *tree* is a connected graph without cycles). Let a and b be different vertices in Γ. Let us denote:
by $\mathrm{br}_a(b,\Gamma)$ the branch of Γ at a that contains b;
$(ab)_\Gamma = \mathrm{br}_a(b,\Gamma) \cap \mathrm{br}_b(a,\Gamma)$;
$[ab)_\Gamma = (ab)_\Gamma \cup \{a\}$;
$[ab]_\Gamma = [ab)_\Gamma \cup \{b\}$;
$\langle ab\rangle_\Gamma$ is the minimal connected subgraph of Γ that contains a and b;
$\delta(ab)_\Gamma = [ab]_\Gamma - \langle ab\rangle_\Gamma$.

Remark. All the graphs considered below are trees.

We shall omit the index Γ in the above notation whenever the ambient graph Γ is clear from the context. For example $\mathrm{br}_a(b)$, (ab), and so on.

Combining the Cramer rule for the inverse matrix of A_Γ and the Jacobi formula for a minor of the inverse matrix, it is easy to prove (see [12, (1.4)])

Lemma (1.2). *Let* Γ *be a weighted graph, a, b be vertices of* Γ. *Then*

$$\det(ab)\det\Gamma = \det \mathrm{br}_a(b)\det\mathrm{br}_b(a) - \det(\Gamma - [ab])(\det\delta(ab))^2 \tag{1.3}$$

We shall refer to (1.3) as to the "edge" determinant formula. If (ab) is a linear chain and $\det(\Gamma) = \pm 1$ then (1.3) is the edge determinant formula due to Eisenbud-Neumann [7].

Lemma (1.4). [9, Corollary 2.2]. Let Γ be a weighted graph and $\det\Gamma = \pm 1$. Then the determinants of all branches at the same vertex are pairwise coprime. If one of them equals 0 then any other must be equal to 1 or to -1.

Dual graphs with a root and splice diagrams.

Definition (1.5). Let l be a line on $\mathbb{CP}^2$, let (Y, L) be a regular pair obtained from $(\mathbb{CP}^2, l)$ by blow-ups. Then we say that (Y, L) is *contractible to* $\overset{+1}{\circ}$ and the proper transform of l is called the *root* of Γ_L. (Note that in this case Γ_L is a tree, $\det\Gamma_L = -1$).

The *Eisenbud – Neumann splice diagram*[2] of C near L is the graph $\Gamma_{L,C}$ where each linear chain of vertices is replaced with a single edge and the beginning of each edge is marked by the determinant of the branch of Γ_L starting with this edge.

Lemma (1.6). *Let v be a root of Γ and $s \in \Gamma$ any vertex. If $s \neq v$ then the determinants of all the branches of $\Gamma - \mathrm{br}_s(v)$ at s, except at most one, equal 1. The determinants of all branches at v equal 1.*

Lemma (1.7). *Let Γ be a graph contractible to $\overset{+1}{\circ}$ and v be its root; let Q be a subgraph of Γ and $v \notin \Gamma$. Then $\det Q > 0$.*

Proof. The intersection form of any branch at v is negatively definite. $\square$

Let (Y, L) be contractible to $\overset{+1}{\circ}$, l_0 be the root of L and $l_1, \dots, l_n$ other irreducible components of L. Let $K_Y = \sum_{j=0}^{n} k_j l_j$ be the canonical divisor. Writing the adjunction formula for each l_j we obtain a linear system of simultaneous equations for the indeterminates $k_0, \dots, k_n$. Resolving them by Cramer rule one can express k_j in terms of determinants of branches of Γ_L (see, for instance lemmas 1.1 and 2.1 of [12]) as follows. Clearly, without loss of generality it suffices to write the formulas only for k_0 and k_1. We see from (1.6) that the splice diagram of L near the path $\langle l_0 l_1 \rangle$ looks as on Fig. 1. Then we have

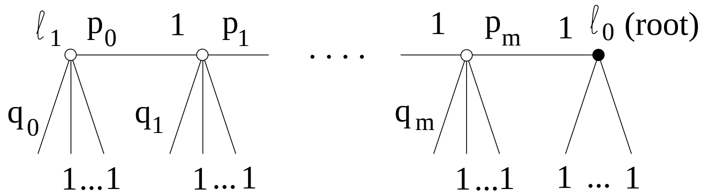

FIG. 1

$$k_0 = -3 \tag{1.8}$$

$$k_1 = -1 - q_0 - p_0 + \sum_{j=1}^{m} q_0 \dots q_{j-1}(q_j - 1)(p_j - 1) \tag{1.9}$$

Remark. The formula (1.9) is related to the formulas for the Milnor number and the Milnor number at infinity in terms of splice diagrams [7, 8]. Using the adjunction formula, they can be easily derived one from another.

Corollary. *If Γ_L is linear then we have*

$$k_1 = -1 - \det B_1 - \det B_2 \tag{1.10}$$

where B_1, B_2 are the branches of Γ at l_1. (Each B_j may be empty.)

[2]This definition is less general than that in [7, 8].

Mappings of algebraic surfaces.

In the rest of §1 we use the following notation: $f : (\widetilde{Y}, \widetilde{L}) \to (Y, L)$ is a regular mapping of regular pairs, $\widetilde{C}$ and C are curves transversal to $\widetilde{L}$ and L respectively, $f^{-1}(L) = \widetilde{L}$, $f^{-1}(C) = \widetilde{C}$ and the jacobian of f does not vanish outside $\widetilde{L} \cup \widetilde{C}$.

We also suppose that f satisfies the *minimality condition.* It means that there is no (-1)-curve $\widetilde{c} \subset \widetilde{Y}$ (i.e. non-singular rational curve with $\widetilde{c}^2 = -1$) such that $f|_{\widetilde{c}}$ is constant. All the mappings considered in the paper satisfy this condition.

Notation (1.11). For a subset $\widetilde{E} \subset \widetilde{Y}$ let us denote $\operatorname{Deg}(\widetilde{E}) = \min_U \deg f \mid_U$ where U runs over the set of all neighborhoods of $\widetilde{E}$ (in the complex topology) such that $f \mid_U$ is proper.

Let $\widetilde{l} \subset \widetilde{Y}$ be an irreducible curve such that f is non-constant on $\widetilde{l}$. Denote by $n(\widetilde{l})$ the order of ramification of f along $\widetilde{l}$ (i.e. $n(\widetilde{l}) = \operatorname{Deg}(x)$ for a generic point $x \in \widetilde{l}$) and denote $\deg f|_{\widetilde{l}}$ by $m(\widetilde{l})$. Thus, if $f(\widetilde{l})$ is nonsingular then one has $\operatorname{Deg}(\widetilde{l}) = m(\widetilde{l})n(\widetilde{l})$. If $p \in \widetilde{l}$ and f is nonsingular near p then denote by $m_p(\widetilde{l})$ the ramification order of $f|_{\widetilde{l}}$ at p.

Let f be non-constant on $\widetilde{l}$, and $l = f(\widetilde{l})$ be non-singular. Let $p \in l$, $\{p_1, \dots p_k\} = f^{-1}(p)$. By definition of $m(\widetilde{l})$ we have

$$m_{p_1}(\widetilde{l}) + \dots m_{p_k}(\widetilde{l}) = m(\widetilde{l}) \tag{1.12}$$

Riemann – Hurwitz Formula. *Let f be non-constant on a non-singular rational curve $\widetilde{l} \subset \widetilde{L}$ (recall that (Y, L) is a regular pair, so $f(l)$ is non-singular). Suppose that $\widetilde{l} \cap (\widetilde{L} + \widetilde{C} - \widetilde{l}) = \{p_1 \dots p_k\}$. Then we have*

$$2(m(\widetilde{l}) - 1) = \sum_{j=1}^{k} (m_{p_j}(\widetilde{l}) - 1). \tag{1.13}$$

Definition (1.14). A subgraph $\widetilde{\Gamma}' \subset \widetilde{\Gamma}$ is called *a subgraph of constant degree* if $\operatorname{Deg}(\widetilde{l}) = \operatorname{Deg}(\widetilde{\Gamma}'))$ for all $\widetilde{l} \in \widetilde{\Gamma}'$. We shall suppose that the edges are also subgraphs of constant degree. If $\widetilde{\Gamma}'$ is an edge connecting two vertices $\widetilde{l}_1$ and $\widetilde{l}_2$ then we put $\operatorname{Deg}(\widetilde{\Gamma}') := \operatorname{Deg} \widetilde{q}$, where $\widetilde{q}$ is the intersection point of the curves $\widetilde{l}_1$ and $\widetilde{l}_2$.

Proposition (1.15). *Suppose that the splice diagram of C near L is of the form $\longleftrightarrow$ and c_1, c_2 are the germs of $\widetilde{C}$ corresponding to the arrows. This means that either L is a double point of C or a linear chain whose end components meet c_1 and c_2 transversally. Then for any connected component $\widetilde{L}_1$ of $\widetilde{L}$ we have:*

a) *The splice diagram of $\widetilde{C}$ near $\widetilde{L}_1$ is of the form $\longleftrightarrow$. Denote by $\widetilde{c}_1$ and $\widetilde{c}_2$ the germs of $\widetilde{C}$ near $\widetilde{L}$.*

b) *$\widetilde{L}_1$ is a subgraph of constant degree.*

c) *Let $\widetilde{p}_i = \widetilde{c}_i \cap \widetilde{L}_1$. Then*

$$\operatorname{Deg}(\widetilde{L}_1) = m_{\widetilde{p}_i}(\widetilde{c}_i) n(\widetilde{c}_i), \qquad i = 1, 2 \tag{1.16}$$

Proposition (1.17). [5] *Suppose*

(1) $\widetilde{L}_1$ *is a connected component of* L.
(2) *The dual graphs* $\Gamma_{\widetilde{L}_1}$, Γ_L *are trees.*
(3) *There exists an irreducible component* $\widetilde{l} \subset \widetilde{L}_1$ *such that* f *is non-constant on* $\widetilde{l}$ *and* $\operatorname{Deg}(\widetilde{l}) = \operatorname{Deg}(\widetilde{L}_1)$.

Let $l = f(\widetilde{l})$. *Then we have*

$$\frac{\det(\widetilde{L}_1 - \widetilde{l})}{\det \widetilde{L}_1} = \frac{n(\widetilde{l})}{m(\widetilde{l})} \cdot \frac{\det(L - l)}{\det L}. \tag{1.18}$$

Moreover, if one of the denominators equals zero, then the other also equals zero.

Lemma (1.19). [5] *Suppose the conditions (1), (2) of (1.17) to be satisfied and* $\widetilde{L}_1$ *to be a subgraph of constant degree.*

a) *If* $\widetilde{C} \cap \widetilde{L}_1 = \{\widetilde{x}\}$, $\widetilde{x} \in \widetilde{c} \subset \widetilde{C}$ *then*

$$\frac{\det \widetilde{L}_1}{\det L} = \frac{1}{m_{\widetilde{x}}(\widetilde{c})} = \frac{n(\widetilde{c})}{\operatorname{Deg} \widetilde{L}_1} \tag{1.20}$$

b) *If* $\widetilde{C} \cap \widetilde{L}_1 = \{\widetilde{x}_1, \widetilde{x}_2\}$, $\widetilde{x}_i \in \widetilde{c}_i \subset \widetilde{C}$ *then*

$$\frac{\det \widetilde{L}_1}{\det L} = \frac{n(\widetilde{c}_1)}{m_{\widetilde{x}_2}(\widetilde{c}_2)} = \frac{n(\widetilde{c}_1)n(\widetilde{c}_2)}{\operatorname{Deg} \widetilde{L}_1}. \tag{1.21}$$

2. The properties of the dicritical component and its resolution at infinity

Now we begin the proof of theorem (0.2). Suppose the contrary: there exists a four-sheeted mapping f with one dicritical component. Let $\widetilde{X}$, X, F, $\widetilde{L}$, L be as in Introduction. Since $\widetilde{X}$ and X are obtained from $\widetilde{\mathbb{C}}^2$ and $\mathbb{C}^2$ by blow-ups, the pairs $(\widetilde{X}, \widetilde{L})$ and (X, L) are contractible to $\overset{+1}{\circ}$.

It follows from [3, lemma 4.2] that the ramification order of F along the dicritical component is 1, 2, or 3. The ramification of order 3 is impossible, see [3, remark after 5.3]. If the ramification order is 1 then F is an unbranched covering over $\mathbb{C}^2$. This is also impossible. So, the only possible order of ramification is 2.

Let us denote the dicritical component of f by $\widetilde{g}$ and put $g = F(\widetilde{g})$ and $\widetilde{L}_\infty = F^{-1}(L)$. It follows from [3, lemma 4.2] that $m(\widetilde{g}) = 1$, $n(\widetilde{g}) = 2$. Resolving, if necessary, the singularities of g at infinity, we can achieve that:

(2a) $\widetilde{g}$ has only one point of intersection with $\widetilde{L}_\infty$ (see [3]). We denote this point by $\widetilde{p}$. Thus, $\widetilde{L}_\infty$ is a branch of $\widetilde{L}$ at $\widetilde{g}$.
(2b) $g \cap L = p := F(\widetilde{p})$, the intersection is transversal, and g is smooth at p.

We may assume that $\widetilde{X}$ and X are minimal possible under these conditions. So,

(2c) F satisfies the minimality condition.

Applying if necessary a polynomial automorphism of $\mathbb{C}^2$ we may assume also that

(2d) In some affine coordinates (x, y) in $\mathbb{C}^2$ neither $\deg_x g$ divides $\deg_y g$ nor $\deg_y g$ divides $\deg_x g$.

This condition is equivalent to the minimality of L in the sense that L contains no (-1)-curve c such that $c\cdot(L-c)\le 2$.

Recall that we use the same notation for the curves $\widetilde{L}$, $\widetilde{L}_\infty$, L and their dual graphs and we also use the same notation for the irreducible components and the corresponding vertices. All graphs to be considered below are subgraphs either of $\widetilde{L}$ or of L. Hence, in the notation for the graphs from definition (1.1) we shall omit the index of ambient graph.

Proposition (2.1). *The Eisenbud – Neumann splice diagram of g near L is as on Fig. 2 For a nodal vertex a let us denote by R_a, L_a, D_a the branches of L at a in the directions respectively Right, Left, Down.*

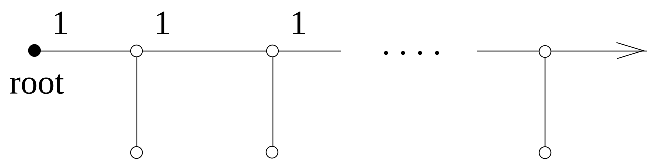

FIG. 2

Then for any nodal a we have:

a) $\det R_a = 1$, $\det D_a > 1$.
b) $\det L_a > 1$.
c) $\det L = -1$.

The only non-trivial assertion of this lemma is 2.1b. In terms of Puiseux characteristic pairs this inequality was proven in [13, 14] (see [7, 8] for the interpretation of characteristic pairs in terms of subgraph determinants). In the notation of [13, 14] $\det L_a = r_i/d_{i+1}$ if a is the i-th (from the left) nodal vertex. Hence, 2.1b is an immediate consequence from the fact that the additive semigroup generated by $r_1,\dots,r_n$ is strictly generated by these numbers (here one needs the condition (2d)). A topological proof of a weaker inequality $\det L_a > 0$ can be found in [8].

Corollary. *a) $\det(ab)$ and $\det D_a$ are coprime; b) $\det L_a$ and $\det D_a$ are coprime. (Here a and b are nodal vertices of L; b is to the left of a)*

Proof. a) Apply (1.4) to R_b. b) Follows from (1.4). $\square$

Lemma (2.2). *$m_{\widetilde{p}}(\widetilde{g}) = 1$ (recall that $\widetilde{p} := \widetilde{g}\cap\widetilde{L}_\infty$). If the subgraph $(\widetilde{a}\widetilde{g})$ is linear for some $\widetilde{a}$ then $\mathrm{Deg}(\widetilde{a}\widetilde{g}) = 2$.*

Proof. We have $n(\widetilde{g}) = 2$, $m(\widetilde{g}) = 1$. Combine this with (2a), (2b), (1.15). $\square$

3. THE STRUCTURE OF $\widetilde{L}_\infty$

Till the end of section 3 we always use tilde in the notation of the vertices of $\widetilde{L}_\infty$. To denote the image of a vertex, we remove the tilde. For example, $a := F(\widetilde{a})$.

Definition (3.1).

a) A vertex $\widetilde{l}\in\widetilde{L}_\infty$ is said to be a *fork* if $m(\widetilde{l}) > 1$.
b) A subgraph of constant degree is said to be *maximal* if it is not contained in another subgraph of constant degree.

The next proposition shows that forks are obstructions for the extension of maximal subgraphs of constant degree. This means that any end vertex of a maximal subgraph of constant degree is either an end vertex of $\widetilde{L}_\infty$, or it is incident at some fork.

Proposition (3.2). *Let $\widetilde{s} \in \widetilde{L}_\infty$ be a nodal vertex. Then the following conditions are equivalent.*

a) $\nu(\widetilde{s}) = \nu(s)$ *(here $\nu(s)$ is the valence of s in $\Gamma_{L,g}$).*

b) $m(s) = 1$.

c) *For any linear chain $\widetilde{C}$ incident at $\widetilde{s}$ one has* $\mathrm{Deg}(\widetilde{C}) = \mathrm{Deg}(\widetilde{s}) = n(\widetilde{s})$.

It follows from (3.2) that after replacing each linear chain with a single edge a subgraph $\widetilde{Q}$ of constant degree becomes isomorphic to $F(\widetilde{Q})$. So, we see from Fig. 2 that at most two end vertices of $\widetilde{Q}$ are not end vertices of $\widetilde{L}$. Therefore, on the pictures below we shall depict such graphs by edges unless the contrary is specified We write over such an "edge" the degree.

The pair of numbers written near a fork $\widetilde{l}$ on all the pictures below denotes $(m(\widetilde{l}), n(\widetilde{l}))$; to the left of $\widetilde{l}$ (respectively to the right of $\widetilde{l}$ or beneath $\widetilde{l}$) we indicate the maximal subgraphs of constant degree whose image under F lies to the left of l (respectively to the right of l or beneath l). We mark the subgraphs whose image under F lies beneath l by a little cross. (It follows from (2c) that these subgraphs are linear.)

For $a \in L$ denote by a' the set of all $\widetilde{a} \in F^{-1}(a)$ such that $F\mid_{\widetilde{a}}$ is non-constant. Since the topological degree of F is four, we have

$$\sum_{\widetilde{a} \in a'} \mathrm{Deg}(\widetilde{a}) = 4 \tag{3.3}$$

Lemma (3.4). *Let $\widetilde{a}$ and $\widetilde{b}$ be forks of $\widetilde{L}_\infty$ such that $(\widetilde{a}\widetilde{b})$ is a subgraph of constant degree, and let c be a vertex of L such that $b \in (ac)$. Then*

$$\sum_{\widetilde{c} \in c' \cap \mathrm{br}_{\widetilde{a}}(\widetilde{b})} \mathrm{Deg}(\widetilde{c}) \geq \mathrm{Deg}(\widetilde{a}\widetilde{b}). \tag{3.5}$$

Suppose $\widetilde{a}$ is a fork of $\widetilde{L}_\infty$. Let $U_{\widetilde{a}} \subset \widetilde{L}$ be a connected subgraph that consists of all maximal subgraphsof constant degree incident at $\widetilde{a}$ and all vertices of L incident at these subgraphs. (we see that any end vertex of $U_{\widetilde{a}}$ is either an end vertex of L, or a fork, or $\widetilde{g}$).

Lemma (3.6). *Let $\widetilde{a}$ be a fork.*

a) *If* $\mathrm{Deg}(\widetilde{a}) = 2$, *then $U_{\widetilde{a}}$ is of the form Fig. 3 (b). Let $\widetilde{b}$ be the left end vertex of $U_{\widetilde{a}}$. Then $n(\widetilde{b}) = 2$, $m(\widetilde{b}) = 2$,* $\mathrm{Deg}(\widetilde{b}) = 4$.

b) *If* $\mathrm{Deg}(\widetilde{a}) = 3$ *then $U_{\widetilde{a}}$ is one of Fig. 4 (e), (f).*

Proof. If $\mathrm{Deg}(\widetilde{a}) = 2$ then $m(\widetilde{a})n(\widetilde{a}) = 2$. The vertex $\widetilde{a}$ is a fork, hence, $m(\widetilde{a}) = 2$, $n(\widetilde{a}) = 1$. Combining (1.12), (1.13) we see that $U_{\widetilde{a}}$ has one of the types depicted in Fig. 3.

Suppose $U_{\widetilde{a}}$ is of the form 3(a). It follows from (1.20) that $\det(\widetilde{a}\widetilde{b}] = \det(\widetilde{a}\widetilde{c}] = \det(ab] \neq 1$. (Here $\widetilde{b}$ and $\widetilde{c}$ are the end vertices depicted on (2a).) But $\det(\widetilde{a}\widetilde{b}]$ and

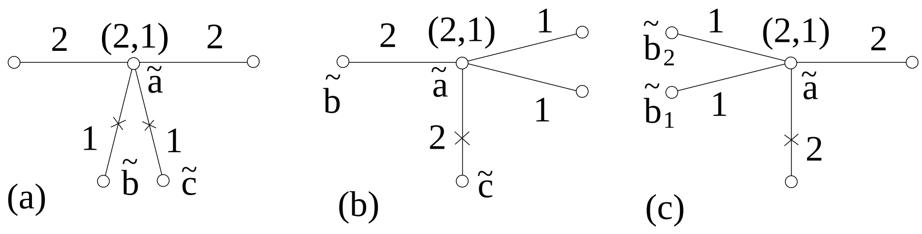

FIG. 3

$\det(\widetilde{a}\widetilde{c}]$ are coprime by (1.4), hence, 3(a) is impossible. Suppose $U_{\widetilde{a}}$ is of the form 3(b). Then (1.20) and (1.21) imply

$$\det(ac] = 2\det(\widetilde{a}\widetilde{c}],$$

$$\det(ab] = 2\det(\widetilde{a}\widetilde{b}] \quad \text{if } \widetilde{b} \text{ is an end vertex of } \widetilde{L},$$

$$n(\widetilde{b})\det(ab) = 2\det(\widetilde{a}\widetilde{b}) \quad \text{if } \widetilde{b} \text{ is a fork.}$$

We see from (1.4) and the corollary of (2.1) that $\det(ab)$ and $\det(ac]$ are coprime. Hence, $\widetilde{b}$ is a fork and $n(\widetilde{b})$ is divisible by 2. But $\operatorname{Deg}(\widetilde{b}) \le 4$ and $m(\widetilde{b}) \ge 2$ because $\widetilde{b}$ is a fork. Thus, $n(\widetilde{b}) = m(\widetilde{b}) = 2$.

Suppose that the both left end vertices of 3(c) (denote them by $\widetilde{b}_1$, $\widetilde{b}_2$) are end vertices of $\widetilde{L}_\infty$. Combining (1.20) and (2.1b) we obtain

$$\det[\widetilde{b}_1\widetilde{a}) = \det[\widetilde{b}_2\widetilde{a}) = \det[b_1a) > 1.$$

We see from (1.4) that this is impossible. So, one of the left end vertices of 3(c) (for example $\widetilde{b}_1$) is a fork. The case 3(c) will be prohibited below, after discussion of the case of degree three.

If $\operatorname{Deg}(\widetilde{a}) = 3$ then $m(\widetilde{a}) = 3$, $n(\widetilde{a}) = 1$. Using (1.12) and (1.13), we obtain the types depicted on Fig. 4.

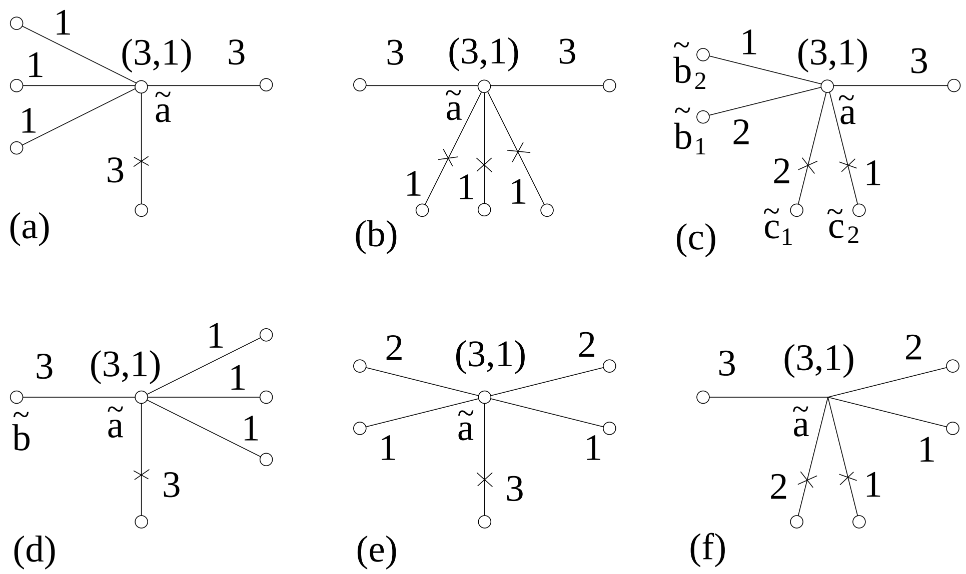

FIG. 4

The case 4(b) is impossible analogously to 3(a).

In the case 4(c) we see analogously to 3(b) that $\widetilde{b}_1$ is a fork and $n(\widetilde{b}_1)$ is divisible by 2. Hence, $\mathrm{Deg}(\widetilde{b}_1) \geq 4$, which contradicts (3.3) and (3.5) for $\mathrm{br}_{\widetilde{a}}(\widetilde{b}_2)$. Thus, 4(c) is impossible.

In the case 4(d), like in 3(b), we see that $\widetilde{b}$ is a fork and $n(\widetilde{b})$ is divisible by 3. Hence, $m(\widetilde{b}) > 1$ and $\mathrm{Deg}(\widetilde{b}) \geq 6$, which contradicts (3.3). Thus, 4(d) is impossible.

Let us show that the case 4(a) is impossible. Analogously to Fig. 3(c) we obtain that one of the left end vertices of this subgraph is a fork $\widetilde{b} \in \widetilde{L}_\infty$. Then (3.3), (3.4) imply $m(\widetilde{b}) = 2$, $n(\widetilde{b}) = 1$, $\mathrm{Deg}(\widetilde{b}) = 2$. Hence $U_{\widetilde{b}}$ is of the form Fig. 3(b). The left end vertex of $U_{\widetilde{b}}$ is a fork of degree 4, so, it follows from (3.3) that all the branches of $\widetilde{L}_\infty$ at $\widetilde{a}$ which are to the left of $\widetilde{a}$ must contain the left end vertex of $U_{\widetilde{b}}$. But this is impossible since $\widetilde{L}_\infty$ is a tree.

Now, consider Fig. 3(c). Recall that $\widetilde{b}_1$ is a fork. Then (3.3) and (3.5) imply $\mathrm{Deg}(\widetilde{b}_1) < 4$. In the same way as for Fig. 4(a) we have $\mathrm{Deg}(\widetilde{b}_1) \neq 2$. So, $\mathrm{Deg}(\widetilde{b}_1) = 3$ and $U_{\widetilde{b}_1}$ has either the type Fig. 4(e) or Fig. 4(f). Pasting the subgraph $U_{\widetilde{b}_1}$ to $U_{\widetilde{a}}$ along $(\widetilde{b}_1\widetilde{a})$, we obtain Fig. 5. Denote by $R_{\widetilde{a}}$ the right branch of $\widetilde{L}_\infty$ at $\widetilde{a}$ and by $R_{\widetilde{b}_1}$ the right branch of $\widetilde{L}_\infty$ at $\widetilde{b}_1$ such that $\widetilde{a} \notin R_{\widetilde{b}_1}$.

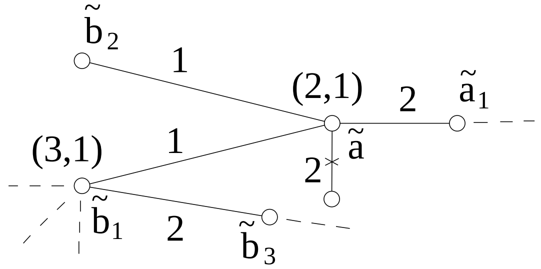

FIG. 5

Let us show that these branches are subgraphs of constant degree. Suppose that $R_{\widetilde{b}_1}$ is not . Denote by $\widetilde{b}_3$ the fork of $R_{\widetilde{b}_1}$ nearest to $\widetilde{b}_1$. If $b_3 \in (b_1 a)$ then (3.3) implies $\mathrm{Deg}\,\widetilde{b}_3 = 2$. If b_3 is to the right of a then combining (3.3) and (3.5) we also have $\mathrm{Deg}\,\widetilde{b}_3 = 2$. Since $\mathrm{Deg}\,\widetilde{b}_3 = 2$ it follows that $U_{\widetilde{b}_3}$ has the type Fig. 3(b), but this contradicts $\mathrm{Deg}\,\widetilde{b}_1 \neq 4$. So, $R_{\widetilde{b}_1}$ is a subgraph of constant degree.

Suppose $R_{\widetilde{a}}$ is not a subgraph of constant degree. Let $\widetilde{a}_1$ be the nearest to $\widetilde{a}$ fork of $R_{\widetilde{a}}$. Since $R_{\widetilde{b}_1}$ is a subgraph of constant degree it follows from (3.3) that $\mathrm{Deg}(\widetilde{a}_1) = 2$. So, $U_{\widetilde{a}_1}$ has the type Fig. 3(b), which contradicts $\mathrm{Deg}\,\widetilde{a} \neq 4$. Hence, $R_{\widetilde{a}}$ is a subgraph of constant degree.

Since $\widetilde{L}$ is a tree, it follows that one of $R_{\widetilde{a}}$ and $R_{\widetilde{b}_1}$ (for example, $R_{\widetilde{a}}$) does not contain $\widetilde{g}$. It follows from (2.1) that $\det R_a = 1$. Using (1.20) we get $\det R_{\widetilde{a}} = \frac{1}{2}$, which is impossible. □

Corollary (3.7). *If* $\mathrm{Deg}\,\widetilde{a} < 4$ *then there are two branches of* $U_{\widetilde{a}}$ *at* $\widetilde{a}$ *lying to the right of* $\widetilde{a}$*.*

Lemma (3.8). *Suppose* $\mathrm{Deg}(\widetilde{a}) = 4$.

a) *Let* $\widetilde{b}$ *be a left end vertex of* $U_{\widetilde{a}}$*. Then* $\widetilde{b}$ *is an end vertex for* $\widetilde{L}_\infty$*.*

b) *The subgraph* $U_{\widetilde{a}}$ *has one of the types Fig. 6(a)-(d),* $m(\widetilde{a}) = 4$*,* $n(\widetilde{a}) = 1$*.*

Proof. Since $\operatorname{Deg}(\widetilde{a}) = 4$, it follows that any connected subgraph of $\widetilde{L}_\infty - \widetilde{a}$ lies either entirely to the left, or to the right, or beneath of $\widetilde{a}$. If $\operatorname{Deg}(\widetilde{b}) < 4$ then proposition a) follows from (3.7).

Consider the proposition b). Two cases are possible.

Case 1) $m(\widetilde{a}) = n(\widetilde{a}) = 2$ is analogous to the cases of degree 2. The only possible subgraph is Fig. 6(g).

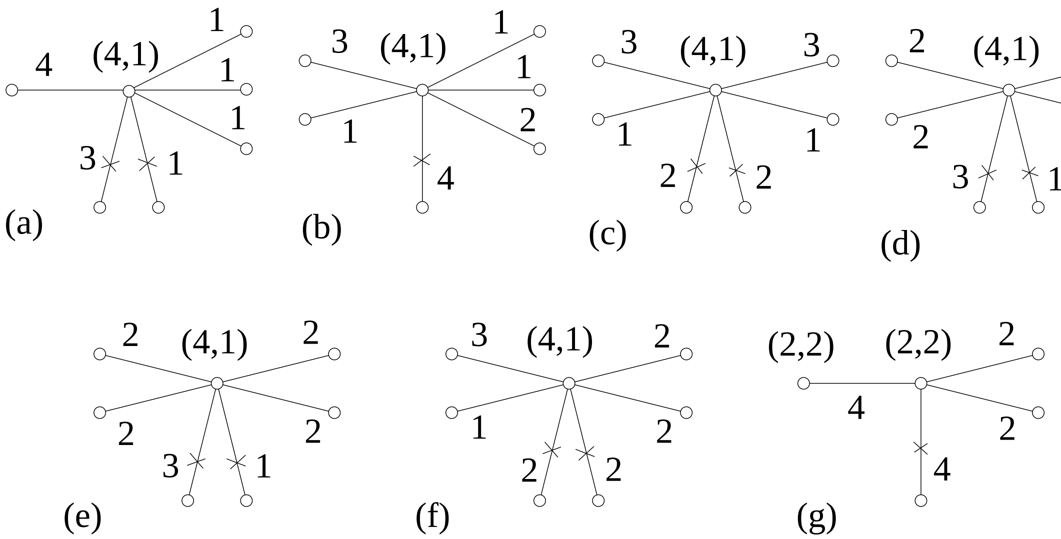

FIG. 6

Let $\widetilde{b}$ be the left end vertex of $U_{\widetilde{a}}$. Then it is a fork, hence, the both subgraphs $U_{\widetilde{a}}$ and $U_{\widetilde{a}}$ must be of the form Fig. 6(g), which is impossible. So, the case 1 is prohibited.

Case 2) $m(\widetilde{a}) = 4$, $n(\widetilde{a}) = 1$. Suppose that $\widetilde{a}$ has only one left branch. The determinants of the branches beneath $\widetilde{a}$ must be coprime. Combining this with (1.12), (1.13), (1.20), (2.1), we see that the only possible subgraph is Fig. 6(a).

Now suppose that $\widetilde{a}$ has at least two left branches. Then (3.8a) is already proved for these cases and these branches are the subgraphs of constant degree. The determinants of the left and the lower branches at $\widetilde{a}$ must be coprime. So, like in the cases $\operatorname{Deg} = 2, 3$, using (1.12), (1.13), (1.20), and (2.1) we obtain that $U_{\widetilde{a}}$ is one of the graphs Fig. 6. (b)-(f).

Let us show that the graphs on Fig. 6 (e), (f) are impossible. As in the prohibition of (2c) in (3.6), we see that the subgraphs lying to the right of $\widetilde{a}$ are of constant degree and one of their determinants equals $1/2$, which is impossible.

The proposition b) is proved.

Consider again the proposition a). If $\operatorname{Deg}(\widetilde{b}) = 4$ then $U_{\widetilde{a}}$ is of the form Fig. 6(a). It follows from Fig. 6 that $\widetilde{b}$ is an end vertex of $\widetilde{L}$. This completes the proof of (3.8). $\square$

Corollary (3.9). *If $\widetilde{a}$ is a fork then* $\operatorname{Deg}\widetilde{a} > 2$.

Proof. Using (3.8b) we obtain that Fig. 3(b) is impossible. $\square$

Thus, we have listed all the possible $U_{\widetilde{a}}$ for any fork $\widetilde{a}$.

Corollary (3.10). *The graph $\widetilde{L}$ must have one of the types depicted on Fig. 7. (The black vertex represents $\widetilde{g}$.)*

Proof. Consider the maximal subgraph of constant degree incident at $\widetilde{g}$. It follows from (2.2) that the degree is two. Combining this with (3.6), (3.8), and (3.9) we get the proof. □

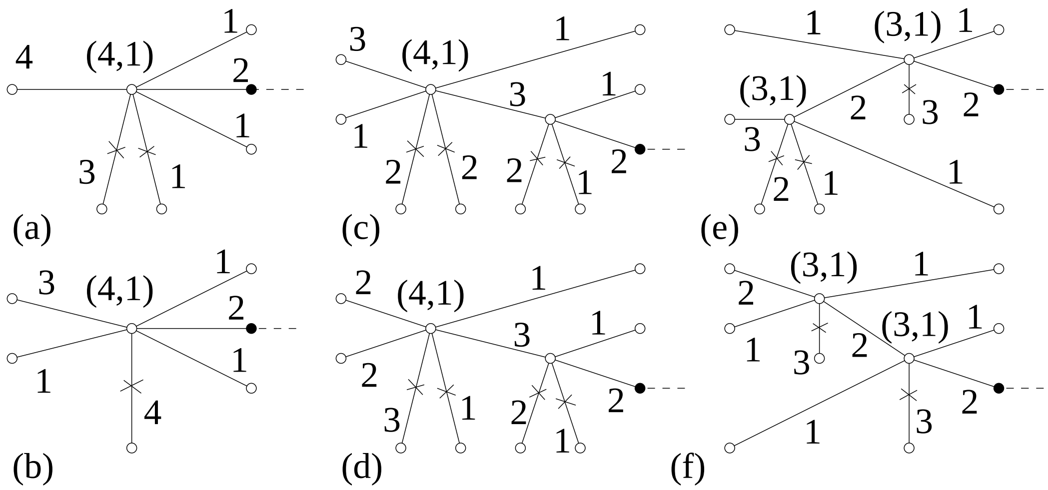

FIG. 7

4. THE PROHIBITION OF GRAPHS FIG. 7(C)-(F)

First, we shall prohibit the graphs of Fig. 7(e), (f). We work out the case of Fig. 7(e) in detail. The case of Fig. 7(f) can be considered analogously.

Lemma (4.1). *The case of Fig. 7(e) is impossible.*

Proof. Assume the contrary.

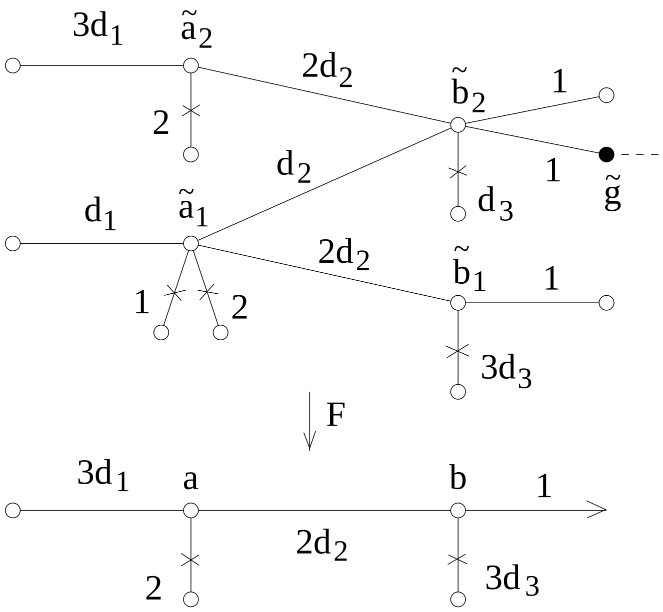

FIG. 8

The graph $\widetilde{L}$ is depicted in the upper part of Fig. 8 and L is below it. As above, the horizontal edges mean subgraphs of constant degree. It follows from (1.20), (1.21) that their determinants are equal to the numbers written nearby.

Denote the root of $\widetilde{L}$ by $\widetilde{v}$. We see on Fig. 8 that the determinants of two branches at $\widetilde{a}_2$ are already greater than 1, hence, by (1.6) we have

$$\det \mathrm{br}_{\widetilde{a}_2}(\widetilde{b}_2) = 1 \qquad \text{and} \qquad \widetilde{v} \notin \mathrm{br}_{\widetilde{a}_2}(\widetilde{b}_2). \tag{4.2}$$

Therefore, $d_1 = 1$ by (1.6) applied to $\widetilde{a}_1$.

Denote $\delta = \det \delta(ab)$ (see (1.1)). It follows from (3.1), (1.20) that $\det \delta(\widetilde{a}_1\widetilde{b}_1) = \det \delta(\widetilde{a}_2\widetilde{b}_2) = \det \delta(\widetilde{a}_1\widetilde{b}_2) = \delta$. Applying (1.20) and the formula (1.3) for (ab) considered as a subgraph of L we obtain

$$\det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_2) \overset{(1.20)}{=} \det \mathrm{br}_b(a) \overset{(1.3)}{=} 18 d_1 d_3 \delta^2 - 2d_2 = 18 d_3 \delta^2 - 2d_2 \tag{4.3}$$

Let $\widetilde{L}'$ be the component of $F^{-1}(\mathrm{br}_a(b))$ containing $\widetilde{b}_2$. Since $\det \mathrm{br}_a(b) = 1$,

$$\det \widetilde{L}' \overset{(1.18)}{=} \frac{3}{1} \cdot \frac{d_3}{3d_3} \cdot \frac{2d_2 \cdot d_2}{2d_2} = d_2 \tag{4.4}$$

Denote $\Delta = \mathrm{br}_{\widetilde{a}_2}(\widetilde{b}_2) \cap \widetilde{L}_\infty$. By (4.2), $\widetilde{v} \notin \Delta$, hence, $\det \Delta > 0$ by (1.7). Using (1.3) for $(\widetilde{a}_1\widetilde{b}_2)$ with Δ as the ambient graph we obtain

$$d_2 \det \Delta \overset{(1.3),(4.4)}{=} d_2 \det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_1) - 4 d_2 d_3 \delta^2, \qquad \text{hence,}$$

$$\det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_1) = \det \Delta + 4 d_3 \delta^2 > 4 \tag{4.5}$$

Applying (1.6) to $\widetilde{b}_2$ we get $d_3 = \det \mathrm{br}_{\widetilde{b}_2}(\widetilde{g}) = 1$. Using formula (1.3) for the determinant of $(\widetilde{a}_2\widetilde{b}_2) \subset \widetilde{L}$, we get

$$\begin{aligned} -2d_2 &= \det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_2) \cdot \det \mathrm{br}_{\widetilde{a}_2}(\widetilde{b}_2) - 6 \det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_1)\delta^2 \\ &= 18\delta^2 - 2d_2 - 6 \det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_1)\delta^2 \qquad \text{by (4.2), (4.3).} \end{aligned}$$

Hence, $\det \mathrm{br}_{\widetilde{b}_2}(\widetilde{a}_1) = 3$ that contradicts (4.5). $\square$

Lemma (4.6). *The graphs of Fig. 7(c), (d) are impossible.*

Proof. Suppose $\widetilde{L}_\infty$ is one of Fig. 7(c), (d). Suppose that $\widetilde{a}$ and $\widetilde{b}$ are forks of $\widetilde{L}$ and $\widetilde{e}$, $\widetilde{q}$, $\widetilde{p}$ are the end vertices, see Fig .9(left) where the numbers written near the edges are their degrees.

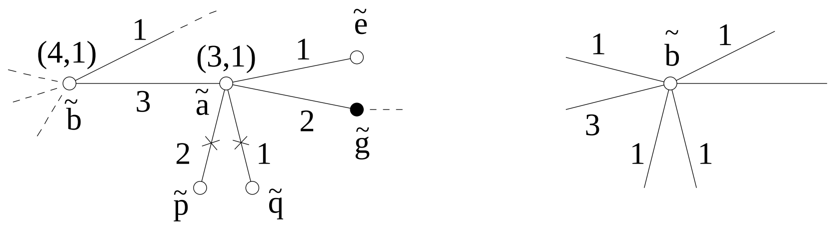

FIG. 9

Let us show that $\det \widetilde{L}_\infty = -2$. Consider, for instance, Fig. 7(c) (the arguments for Fig. 7(d) are the same). It follows from (1.20) and (1.4) that $\det L_b = 3$, $\det D_b = 2$, and the determinants of the branches at $\widetilde{b}$ are as shown on Fig. 9(right). Applying (1.18) to $\widetilde{b}$ we get

$$\frac{\det \widetilde{L}_\infty}{\det L} = \frac{4}{1}\cdot\frac{1\cdot 3}{3}\cdot\frac{1\cdot 1}{2}\cdot\frac{1\cdot\det(\widetilde{b}\widetilde{g})}{\det R_b} \tag{4.7}$$

Recall (see (2.1)) that $\det L = -1$, $\det R_b = 1$. So, (4.7) implies

$$\det \widetilde{L}_\infty = -2\det(\widetilde{b}\widetilde{g}). \tag{4.8}$$

Combining (1.4), (1.20), (1.21), and (2.1) we get $\det D_a = 2$, $\det(\widetilde{a}\widetilde{p}] = 1$, $\det(\widetilde{a}\widetilde{q}] = 2$, $\det(\widetilde{a}\widetilde{e}] = 1$, $\det(\widetilde{a}\widetilde{g}) = 1$, $3\det(\widetilde{b}\widetilde{a}) = \det(ab)$. Applying (1.18) to the vertex $\widetilde{a}$ in the ambient graph $(\widetilde{b}\widetilde{g})$ we get $\det(\widetilde{b}\widetilde{g}) = 1$, so, (4.8) implies $\det \widetilde{L}_\infty = -2$.

Denote by α the determinant of the graph $A := \widetilde{L} - \widetilde{g} - \widetilde{L}_\infty$. The curve A is the subset of $\widetilde{L}$ which is contracted by F to a single point lying on $g \cap \mathbb{C}^2$ (if $A \neq \varnothing$, this point must be a singular point of g).

Using formula (1.3) for the determinant of the subgraph $(\widetilde{a}\widetilde{g}) \subset \widetilde{L}$, we get

$$-1 = -2\det br_{\widetilde{a}}(\widetilde{g}) - 2\alpha\det br_{\widetilde{a}}(\widetilde{b})(\det\delta(\widetilde{a}\widetilde{g}))^2.$$

But this is impossible since the right hand side is divisible by 2. $\square$

5. The prohibition of Fig. 7(a), (b)

Lemma (5.1). *Let $\widetilde{K} = \sum a_{\widetilde{s}}\widetilde{s}$ be the representation of the canonical class of $\widetilde{X}$ as a linear combination of irreducible components $\widetilde{s}$ of the curve $\widetilde{L}$. Then $a_{\widetilde{g}} = 1$.*

Proof. Let K be the canonical divisor of X with the support on $X \setminus \mathbb{C}^2$ and $B = \sum_{\widetilde{s}\in\widetilde{L}} b_{\widetilde{s}}\widetilde{s}$ the ramification divisor of $\widetilde{F}$ (the divisor of the jacobian of $\widetilde{F}$). One has

$$\widetilde{K} = F^*(K) + B.$$

Since $g \not\subset X \setminus \mathbb{C}^2$, we have $g \not\subset K$. Comparing the coefficients of $\widetilde{g}$ in the left hand side and the right hand side, we obtain $a_{\widetilde{g}} = b_{\widetilde{g}} = n(\widetilde{g}) - 1 = 1$. $\square$

Let $\widetilde{v}$ be the root of $\widetilde{L}$. Clearly, $\widetilde{v} \in \widetilde{L}_\infty$.

Lemma (5.2). *Suppose $\Delta \subset \widetilde{L}_\infty$ is a maximal subgraph of constant degree incident at $\widetilde{g}$. (We see that* $\operatorname{Deg}\Delta = 2$.*) Then the subgraph $\Delta \cap (\widetilde{v}\widetilde{g})$ is linear.*

Proof. Suppose that $\Delta \cap (\widetilde{v}\widetilde{g})$ is not linear. Let $\widetilde{v}_1$ be the nearest to $\widetilde{g}$ nodal vertex of Δ and $v_1 = F(\widetilde{v}_1)$. It follows from (3.2) that $m(\widetilde{v}_1) = 1$, $n(\widetilde{v}_1) = 2$. Denote, as in (4.6), $\alpha = \det(\widetilde{L} - \widetilde{L_\infty\widetilde{g}})$. Since $\widetilde{v} \in \widetilde{L}_\infty$, we have $\alpha \geq 1$, $\det \mathrm{br}_{\widetilde{v}_1}(\widetilde{g}) \geq 1$.

Denote $d_{\widetilde{v}} = \det \mathrm{br}_{\widetilde{v}_1}(\widetilde{v})$ and let d be the determinant of the branch at $\widetilde{v}_1$ not containing $\widetilde{g}$, $\widetilde{v}$. The determinant of the lower branch at $\widetilde{v}_1$ equals D_{v_1} by (1.20). We see from (2.1) that $D_{v_1} > 1$. Hence, $\max(d, d_{\widetilde{v}}) > 1$. Combining (1.21) and (2.1a) we get

$$\det(\widetilde{v}_1\widetilde{g}) = 2\det(v_1 g) = 2$$

Consider the case $d > 1$. It follows from (1.7) that $\det br_{\widetilde{v}_1}(\widetilde{g}) = 1$. Using formula (1.3) for the determinant of the subgraph $(\widetilde{v}_1\widetilde{g}) \subset \widetilde{L}$, we obtain

$$-2 = \det \widetilde{L}_\infty - \alpha d d_{\widetilde{v}} \tag{5.3}$$

It follows from (5.1) and (1.9) that

$$1 = k_{\widetilde{g}} = -1 - \alpha - \det \widetilde{L}_\infty + \alpha(d-1)(d_{\widetilde{v}} - 1) + \alpha d_1 \Sigma, \tag{5.4}$$

where Σ is the rest of (1.9). Combining (5.3) and (5.4) we get $0 = -\alpha d - \alpha d_{\widetilde{v}} + \alpha d \Sigma$, hence, $d_{\widetilde{v}}$ must be divisible by $d \neq 1$. This contradicts (1.4).

Consider the case $d = 1$. As it was shown above, the determinant of the lower branch at $\widetilde{v}_1$ is greater than 1. Hence, $\widetilde{v}$ lies below $\widetilde{v}_1$ and $d_{\widetilde{v}} = \det D_{v_1} > 1$. Using formula (1.3) for the determinant of the subgraph $(\widetilde{v}_1\widetilde{g})$ in the graph $\widetilde{L}$, we obtain

$$-2 = \det \widetilde{L}_\infty \det \mathrm{br}_{\widetilde{v}_1}(\widetilde{g}) - \alpha d_{\widetilde{v}},$$

so $\det \widetilde{L}_\infty \geq 0$. It follows from (5.1) and (1.9) that

$$1 = k_{\widetilde{g}} = -1 - \alpha - \det \widetilde{L}_\infty$$

hence, $\det \widetilde{L}_\infty = -2 - \alpha < 0$. Contradiction. $\square$

Lemma (5.5). *Suppose that either* $\det \widetilde{L}_\infty = -1$ *or* $\det \widetilde{L}_\infty = -2$. *Let Δ satisfy the hypothesis of (5.2). Then $\widetilde{v} \notin \Delta$.*

Proof. Suppose $\widetilde{v} \in \Delta$. By (5.2) we obtain that the graph $(\widetilde{v}\widetilde{g})$ is linear. The decomposition formula (1.9) with respect to $\widetilde{g}$ implies $k_{\widetilde{g}} = -\alpha - \det \widetilde{L}_\infty - 1$, but $k_{\widetilde{g}} = 1$, hence, $\det \widetilde{L}_\infty = -\alpha - 2 < -2$. Contradiction. $\square$

Lemma (5.6). *Graphs $\widetilde{L}$ of the form Fig. 7(a,b) are impossible.*

Proof. Let $\widetilde{a}$ be the fork of $\widetilde{L}_\infty$. Then $(\widetilde{a}\widetilde{g})$ is a maximal subgraph of constant degree, $\mathrm{Deg}(\widetilde{a}\widetilde{g}) = 2$, (1.21), and (2.1) imply $\det(\widetilde{a}\widetilde{g}) = 1$. Combining (1.20) and (2.1a) we see that the determinants of the rest two branches of $\widetilde{L}_\infty$ to the right of $\widetilde{a}$ equal one.

Consider Fig. 7(a). Let L_a, $D_a, \ldots$ be as in (2.1) and denote by $L_{\widetilde{a}}$ (resp. $D_{\widetilde{a}}^{(d)}$) the determinant of the left branch at $\widetilde{a}$ (resp. the lower branch of degree d). Then $\det L_{\widetilde{a}} / \det L_a = 1/4$ (by (1.20)) and $\det D_{\widetilde{a}}^{(3)} = 1$, $\det D_a = D_{\widetilde{a}}^{(1)} = 3$ (by (1.20), (1.4)). Thus, we have

$$\det(\widetilde{L}_\infty - [\widetilde{a}\widetilde{g}]) = \det L_{\widetilde{a}} \det D_{\widetilde{a}}^{(3)} \det D_{\widetilde{a}}^{(1)} \geq 3, \tag{5.7}$$

$$\frac{\det \widetilde{L}_\infty}{\det L} \overset{(1.18)}{=} \frac{4}{1} \cdot \frac{1 \cdot 3}{3} \cdot \frac{1}{4} = 1, \qquad \text{hence,} \qquad \det \widetilde{L}_\infty = \det L = -1 \tag{5.8}$$

.

Analogously, one obtains (5.7), (5.8) for Fig. 7(b). Let $\delta = (\widetilde{a}\widetilde{g})$. It follows from (5.5) and (5.2) that Δ is linear and $\widetilde{v} \notin \Delta$. Since $\widetilde{v} \in \widetilde{L}_\infty$ we get $\widetilde{v} \notin \mathrm{br}_{\widetilde{a}}(\widetilde{g})$, hence, $\det \mathrm{br}_{\widetilde{a}}(\widetilde{g}) \geq 1$ by (1.7).

Denote, as in (4.6) and (5.2), $\alpha = \det(\widetilde{L} - \widetilde{L}_\infty - \widetilde{g})$. Using formula (1.3) for the determinant of the subgraph Δ in the ambient graph $\widetilde{L}$, we get

$$-1 = \det \widetilde{L}_\infty \cdot \det \mathrm{br}_{\widetilde{a}}(\widetilde{g}) - \alpha \cdot \det(\widetilde{L}_\infty - [\widetilde{a}\widetilde{g}]).$$

Since $\alpha \geq 1$ and $\det \mathrm{br}_{\widetilde{a}}(\widetilde{g}) \geq 1$, this contradicts (5.7), (5.8). $\square$

Thus, all the graphs depicted on Fig. 7 are impossible. This completes the proof of theorem (0.2).

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