EXAMPLE OF A CONTINUOUS MAPPING $S^2 \to \mathbb{R}^2$ WHOSE SET OF DOMINATING POINTS IS DENSE IN S^2

S.Yu. Orevkov

Abstract. Using the Cantor function, we construct a continuous mapping $f : S^2 \to \mathbb{R}^2$ such that the set $\{p \in S^2 \mid f^{-1}(f(p)) = \{p\}\}$ is dense in S^2 and the image of f is an infinite binary tree. This answers to a question posed to me by Daciberg Gonçalves.

[1]

Let $f : X \to Y$ be a continuous mapping of manifolds. Let us say that a point $p \in X$ is a *dominating point* if $f^{-1}(f(p))$ contains only p and no other points. Let $X = S^2$ be the 2-sphere and let $Y = \mathbb{R}^2$ be the real plane. It is easy to see that if f is differentiable at least once then the set of the dominating points cannot be dense. Daciberg Gonçalves asked me if the set of dominating points can be dense for a continuous mapping $f : S^2 \to \mathbb{R}^2$. In this note I give a positive answer to this question.

Preliminaries. Let $B = \{0,1\}^\infty$ be the set of all binary sequences $(b^{(1)}, b^{(2)}, \dots)$ where $b^{(i)} \in \{0,1\}$, $i = 1,2,\dots$ and only finite number of $b^{(i)}$ are nonzero. For $b \in B$ we define its *length* as $\operatorname{len}(b) = \max\{\, n \mid b^{(n)} = 1 \,\}$ and we set $B_n = \{\, b \in B \mid \operatorname{len}(b) = n \,\}$. If $b = (b^{(1)}, b^{(2)}, \dots)$ is a binary sequence of length n, we shall represent it by a word (without any delimiters) $b^{(1)} \dots b^{(n)}$, i. e., we shall write just 0101 instead of $(0,1,0,1,0,0,\dots)$. Thus, we have $B_0 = \varnothing$, $B_1 = \{1\}$, $B_2 = \{01, 11\}$, $B_3 = \{001, 011, 101, 111\}$, etc., and we have $B = \bigcup_{n=1}^\infty B_n$.

For $b \in B_n$, let $y(b)$ be the binary number

$$y(b) = 0\,.\,b^{(1)}b^{(2)}\cdots = \sum_{k\ge1} b^{(k)}/2^k$$

and let $t(b)$ be the trenary number

$$t(b) = 2 \times 0\,.\,b^{(1)}b^{(2)}\cdots = \sum_{k\ge1} 2b^{(k)}/3^k.$$

Let $F : [0,1] \to [0,1]$ be Cantor function, i.e. the monotone function uniquely determined by the condition that

$$F(t(b)) = F(t(b) - 3^{-n}) = y(b) \qquad \text{for } b \in B_n$$

(see Figure 1). For $b \in B_n$, let I_b be the closed interval

$$I_b = F^{-1}(t(b)) = [t(b) - 3^{-n}\,,\, t(b)]$$

[1]This note is published as an appendix to Daciberg Gonçalves' paper "The size of multiple points of maps between manifolds" (to appear in *Topology Proceedings*).

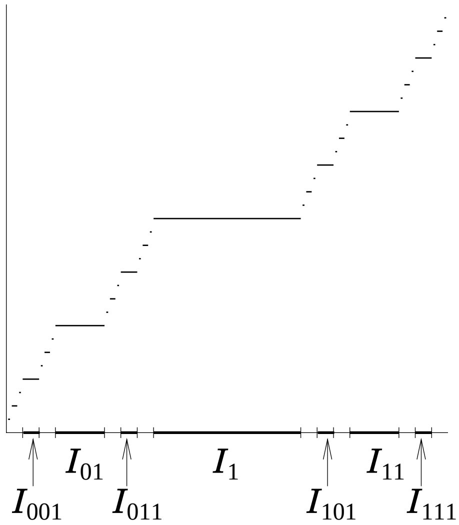

FIGURE 1

(see Figure 1). Let $\mathbf{B} = \bigcup_{m=1}^{\infty} \mathbf{B}_m$ where $\mathbf{B}_m = \{(b_1,\dots,b_m)\,|\,b_i \in B\,\}$. We shall identify $\mathbf{B}_1$ with B. For $\vec{b} = (b_1,\dots,b_m) \in \mathbf{B}_m$, we denote $\vec{b}' = (b_1,\dots,b_{m-1}) \in \mathbf{B}_{m-1}$ and we set $\operatorname{len}(\vec{b}) = \operatorname{len}(b_1)+\dots+\operatorname{len}(b_m)$. We write $\vec{b}_1 \prec \vec{b}_2$ if $\vec{b}_1$ is an initial segment of $\vec{b}_2$, i.e. $\vec{b}_1 = (b_1,\dots,b_{m_1})$ and $\vec{b}_2 = (b_1,\dots,b_{m_1},b_{m_1+1},\dots,b_{m_2})$.

1. Construction of annuli. Let $\mathbb{D}$ be the closed unit disk in $\mathbb{C}$. For $b \in B$, let us denote the annulus $\{\,z \in \mathbb{D}\,|\,F(|z|) \in I_b\}$ by A_b. Let $\mathcal{U} = \{U_1, U_2, \dots\}$ be some countable base of the standard topology in $\mathbb{D}$. For any $\vec{b} = (b_1,\dots,b_m) \in \mathbf{B}$, we define an annulus $A_{\vec{b}}$, a distinguished point $p_{\vec{b}}$ in it, and a mapping $\varphi_{\vec{b}} : \mathbb{D} \to A_{\vec{b}}$. We shall define them inductively. First, for $\vec{b}$ with $\operatorname{len}(\vec{b}) = 1$, then for all $\vec{b}$ with $\operatorname{len}(\vec{b}) = 2$, then for all $\vec{b}$ with $\operatorname{len}(\vec{b}) = 3$, etc.

If $\vec{b} = (b_1) \in \mathbf{B}_1$ then we set $A_{\vec{b}} = A_{b_1}$.

If $A_{\vec{b}}$ is already defined then we choose $p_{\vec{b}}$ as any point in $\operatorname{Int} A_{\vec{b}} \cap U_k$ where where k is the minimal number such that $\operatorname{Int} A_{\vec{b}} \cap U_k$ is non-empty and U_k was not used on previous steps.

If $p_{\vec{b}}$ is already defined then we define $\varphi_{\vec{b}} : \mathbb{D} \to A_{\vec{b}}$ as a continuous map such that

(1) $\varphi_{\vec{b}}(0) = p_{\vec{b}}$,
(2) $\varphi_{\vec{b}}(\mathbb{D}) = A_{\vec{b}}$,
(3) $\varphi_{\vec{b}}$ maps $\operatorname{Int}\mathbb{D}$ homeomorphically onto a dense open subset of $A_{\vec{b}}$.

If $\varphi_{\vec{b}'}$ is already defined then we set $A_{\vec{b}} = \varphi_{\vec{b}'}(A_{b_m})$. We have depicted some of the annuli $A_{\vec{b}}$ in Figure 2.

Let us set $A_m = \bigcup_{\vec{b}\in\mathbf{B}_m} \operatorname{Int} A_{\vec{b}}$, $A = \bigcap_{m=1}^{\infty} A_m$, and $P = \{p_{\vec{b}}\,|\,\vec{b} \in \mathbf{B}\,\}$.

Remarks. **1.** Using conformal mappings we can choose $\varphi_{\vec{b}}$ in a canonical way. Namely, we can set $\varphi = \varphi_1^{-1} \circ \varphi_2$ where φ_1 is the conformal mapping of $\operatorname{Int} A_{\vec{b}}$ onto $A_r = \{\,z\,:\,r < |z| < 1\}$ such that $\varphi_1(p_{\vec{b}}) \in [r,1]$ (r is uniquely determined by $A_{\vec{b}}$) and φ_2 is a conformal mapping of $\operatorname{Int}\mathbb{D}$ onto $A_r \setminus [-1,-r]$ such that $\varphi_2(0) = \varphi_1(p_{\vec{b}})$.

Lemma 1. *A and P are dense in $\mathbb{D}$.*

Proof. The fact that A is dense in $\mathbb{D}$ is an immediate consequence from Baire's theorem.

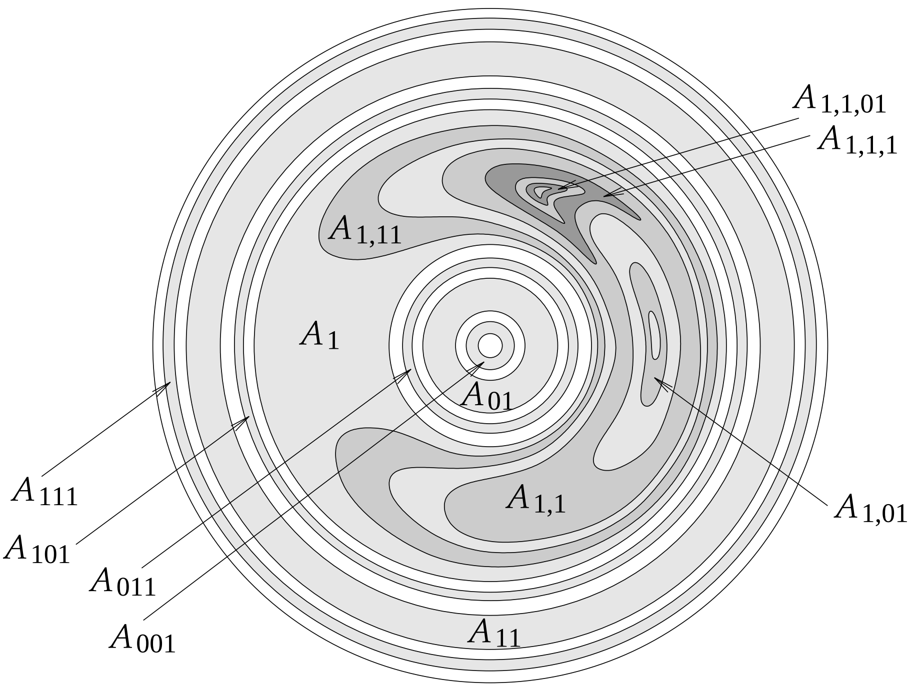

FIGURE 2.

Let us prove by induction that each U_k contains a point of P. Suppose we know already that this is true for $U_1, \dots, U_{k-1}$.

Since A is dense, there exists a point z in $A \cap U_k$. It belongs to each A_m, hence, for any $m = 1, 2, \dots$, there is $\vec{b}_m \in \mathbf{B}_m$ such that $z \in \operatorname{Int} A_{\vec{b}_m}$. Let m be the minimal number such that $p_{\vec{b}_m}$ was not yet defined at the moment when all of $U_1, \dots, U_{k-1}$ had been used. Then $p_{\vec{b}_m}$ must be chosen in $\operatorname{Int} A_{\vec{b}_m} \cap U_k$ because it is non-empty (it contains z). $\square$

3. Construction of a fractal tree. Let us define an infinite tree T embedded into $\mathbb{R}^2$ as follows. Let $I = [0, 1]$ and let $\lambda : I \to \mathbb{R}^2$ be a non-constant linear mapping, say, $\lambda(t) = (t, 0)$. For any $\vec{b} \in \mathbf{B}$, we shall define a linear mapping $\lambda_{\vec{b}}$ inductively as follows. If $m = 0$ (i. e., $\vec{b}$ is empty), we set $\lambda_{\vec{b}} = \lambda$. If $\vec{b} = (b_1, \dots, b_m)$ and $\lambda_{\vec{b}'}$ is already defined then we set $\lambda_{\vec{b}}(t) = (t-1)e_{\vec{b}} + ta_{\vec{b}}$ where $a_{\vec{b}} = \lambda_{\vec{b}'}(y(b_m))$, the segment $\lambda_{\vec{b}}(I) = [a_{\vec{b}}, e_{\vec{b}}]$ is orthogonal to the segment $\lambda_{\vec{b}'}(I)$ (the direction is not so important, we can chose it, for instance, as in Figure 3), and the length of the segment $\lambda_{\vec{b}}(I)$ is $3^{-\operatorname{len}(\vec{b})}$; recall that $\operatorname{len}(\vec{b}) = \operatorname{len}(b_1) + \dots + \operatorname{len}(b_m)$. Let $T = \bigcup_{\vec{b} \in \mathbf{B}} \lambda_{\vec{b}}(I)$ (see Figure 3). We shall call the points $a_{\vec{b}}$ and $e_{\vec{b}}$ the *nodes* and the *ends* of T respectively. Let us denote the branch at $a_{\vec{b}}$ by $T_{\vec{b}}$, i. e., $T_{\vec{b}} = \bigcup_{\vec{b} \prec \vec{b}_1} \lambda_{\vec{b}_1}(I)$. By construction, $T_{\vec{b}} \subset \Delta_{\vec{b}}$ where $\Delta_{\vec{b}}$ is the triangle with vertices $a_{\vec{b}}$, $e_{\vec{b}}$, $e_{\vec{b},1}$. We depicted in Figure 3 the triangles Δ_1 and $\Delta_{1,11}$, i. e., the triangles $\Delta_{\vec{b}}$ for $\vec{b} = (1) \in \mathbf{B}_1$ and for $\vec{b} = (1, 11) \in \mathbf{B}_2$.

One can check that $\Delta_{\vec{b}_1} \supset \Delta_{\vec{b}_2}$ if $\vec{b}_1 \prec \vec{b}_2$ and $\Delta_{\vec{b}_1} \cap \Delta_{\vec{b}_2} = \varnothing$ otherwise. This implies that the segments of T meet each other only at nodes (in particular, the ends cannot lye on other segments).

4. Construction of the mapping. Let us define $f : \mathbb{D} \to T$ as $f = \lim_{m\to\infty} f_m$ where the mappings f_m are inductively constructed as follows.

Let $f_0(z) = \lambda_0(F(|z|))$. Then f_0 is continuous and it contracts each annulus A_b

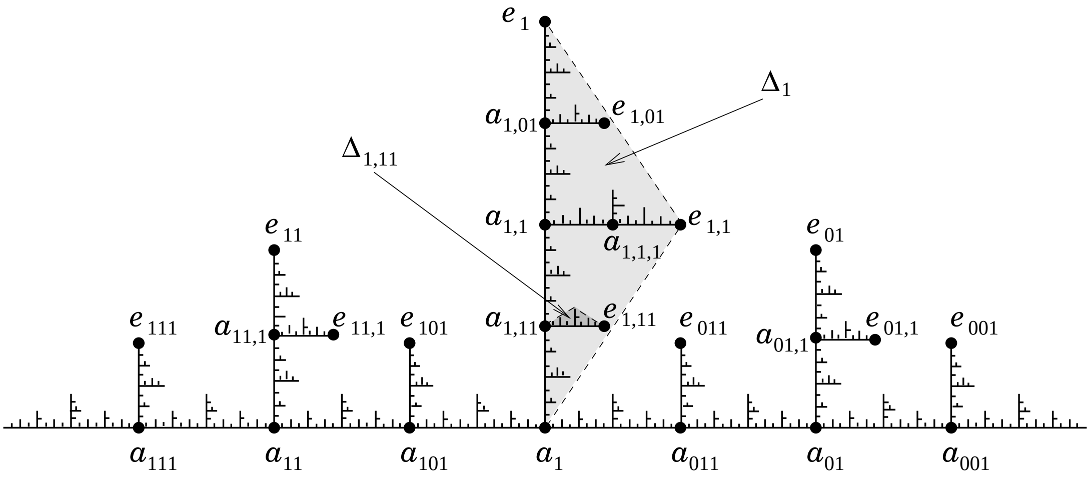

FIGURE 3

into the node e_b. Suppose that f_{m-1} is already constructed. Then we set

$$f_m(z) = \begin{cases} \lambda_{\vec{b}}\big(F\big(|\varphi_{\vec{b}}^{-1}(z)|\big)\big) & \text{if } z \in \operatorname{Int} A_{\vec{b}} \text{ for } \vec{b} \in \mathbf{B}_m \\ f_{m-1}(z) & \text{otherwise} \end{cases}$$

It is clear that if $\vec{b} \in \mathbf{B}_m$ then f_m maps $A_{\vec{b}}$ onto the segment $\lambda_{\vec{b}}(I) = [a_{\vec{b}}, e_{\vec{b}}]$ so that $\partial A_{\vec{b}}$ is mapped to the node $a_{\vec{b}}$, the distinguished point $p_{\vec{b}}$ is mapped to the end $e_{\vec{b}}$, and each annulus $A_{\vec{b},b}$ is contracted to the node $a_{\vec{b},b}$.

Using the fact that $f_m(\partial A_{\vec{b}}) = f_{m-1}(\partial A_{\vec{b}}) = a_{\vec{b}}$ for $\vec{b} \in \mathbf{B}_m$, it is easy to prove by induction that each f_m is continuous.

Let us show that f is continuos. Indeed, when we pass from f_{m_1} to f_{m_2}, we modify f_{m_1} on each $A_{\vec{b}}$, $\vec{b} \in \mathbf{B}_{m_1}$, replacing the value $a_{\vec{b}}$ by values lying in $T_{\vec{b}} \subset \Delta_{\vec{b}}$ and the diameter of $\Delta_{\vec{b}}$ tends to zero as $m \to \infty$. Thus, $\{f_m\}$ is a Cauchy sequence in the metric of uniform convergency.

In fact, f can be characterized as the continuous mapping $\mathbb{D} \to \mathbb{R}^2$ uniquely defined either by the condition $f(\partial A_{\vec{b}}) = a_{\vec{b}}$ for any $\vec{b} \in \mathbf{B}$ or by the condition that $f(A_{\vec{b}}) = T_{\vec{b}}$ for any $\vec{b} \in \mathbf{B}$.

Since f is constant on $\partial\mathbb{D}$, it can be consedered as a continuous mapping of the sphere obtained from $\mathbb{D}$ by contracting the boundary. Let E be the set of ends of T, i.e. $E = \{\, e_{\vec{b}} \,|\, \vec{b} \in \mathbf{B}\,\}$. It is clear that each point of E has only one preimage and $f^{-1}(E) = P$ is dense in the sphere.

IMT, UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31400 TOULOUSE, FRANCE
E-mail address: orevkov@math.ups-tlse.fr