ON LEVEL LINES OF A CONFORMAL ISOMORPHISM OF A DOMAIN ONTO THE DISK

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Let D be a simply connected domain in $\overline{\mathbb{C}}$. By Riemann theorem, there exists a holomorphic function defining a conformal isomorphism of D onto the unit disk. Let us denote the level line of the absolute value of this function corresponding to the value r by γ_r. This note conserns the behavior of the level lines γ_r near the boundary. In some aspects, this behavior may be unexpectable. For example, it is shown in [1] that the number of flex points is not always monotonically increasing. However, in the question studied here, the behavior of the level lines is as one could expect it.

Definition. Let a set A lies outside a Jordan curve γ. Let us call the *exterior distance* from γ to A the quantity

$$\text{ext. dist.}(\gamma, A) = \sup_{z\in\gamma} \inf_{\alpha} |\alpha|,$$

where $|\alpha|$ is the length of α, and inf is taken over all curves connecting the point z with A and lying outside γ.

Let f be a function defining a conformal mapping of the unit disk onto D. Then γ_r is the image of the circle $|z| = r$ under this mapping. Moreover, let us denote the area of the image of the annulus $r \le |z| < 1$ under the mapping f by $S(1-r)$.

Theorem 1. *If the boundary ∂D of D is bounded, then* ext. dist.$(\gamma_r, \partial D) \to 0$ *as $r \to 1$. The limit is uniform in the sense that there exists a function $\Phi(s)$ independent of the choice of D, monotonically decreasing and tending to zero as $s \to 0$, such that* ext. dist.$(\gamma_r, \partial D) \le R\Phi(S(1-r))$*, where R is the radius of a circle containing ∂D.*

This theorem was conjectured by A.G. Vitushkin.

As a corollary, we get a new (and in our opinion, simpler) proof of the following theorem conjectured by V.P. Khavin [2] and proved by A.L. Varfolomeev [3]. Given a number $a > 0$ and a set $K \subset \mathbb{C}$, let us call a function *a-analytic on K*, if it is analytic in some neighbourhood of K, and its radius of convergency at any point $z \in K$ is greater than a.

Theorem. (A.L. Varfolomeev). *Let K be a connected compact subset of $\mathbb{C}$, and $a > 0$. Then there exists an open set V containing K such that any a-analytic function on K can be uniquely extended to V.*

Proof. Without loss of generality we may assume that K is contained in the unit disk. Let $U_1, \dots, U_n$ be the connected components of $\mathbb{C} - K$ whose area is greater than $\Phi^{-1}(a)$. (It follows from the boundedness of K that the number of them is finite.) Let V_j $(j = 1, \dots, n)$ be the interior of that level line of the conformal isomorphism of U_j onto the disk, starting with which the ext. dist. to the boundary is less than a. Then any a-analytic function on K can be extended to $\mathbb{C} - \bigcup V_j$. Indeed, let us consider the minimal level line to whose exterior the function is extendable. Then, if the ext. dist. were less than a, then the function would be extendable to the exterior of a smaller level line.

Theorem 2. *If the area of D is bounded then* ext. dist.$(\gamma_r, \partial D) \to 0$ *as $r \to 1$.*

However, we did not succeed to prove that the limit is uniform when the boundary contains the infinite point.

Remark 1. Theorems 1 and 2 do not hold for real diffeomorphisms of a domain onto the disk.

Theorems 1 and 2 would immediate consequences of the following assertion.

Conjecture. *Let K be a circle lying outside of γ_r which has the tangency of the second order with γ_r. Then K intersects with ∂D.*

In order to prove Theorem 2, we shall need the following lemma.

Lemma. *Let a function f maps conformally the unit disk onto a domain D of a finite area S. Then f can be decomposed into the sum of holomorphic functions f_1 and f_2 such that*

$$|f_1'(z)| \le (1 - |z|)^{-1}, \tag{1}$$

$$|f_2'(z)| \le g'(|z|), \tag{2}$$

where g is a continuous on $[0, 1]$ and differentiable on $[0, 1)$ function such that $g(1) < S$.

Proof. Let $f = \sum a_k z^k$, $K = \{k \in \mathbb{Z}_+ \,:\, |a_k| > 1/k\}$. Set $f_1(z) = \sum_{k \in \mathbb{Z}_+ - K} a_k z^k$, $f_2(z) = \sum_{k \in K} a_k z^k$, $g(r) = \sum_{k \in K} |a_k| r^k$. Then the inequalities (1) and (2) are obvious. Let us prove that g is continuous on $[0, 1]$ and differentiable on $[0, 1)$. Indeed, by the interior area theorem [4, p. 418], S – the area of D – is equal to $\sum_k |a_k|^2 k$. Hence,

$$S > \sum_{k \in K} |a_k|^2 k > \sum_{k \in K} \left(\frac{1}{k}\right)^2 k = \sum_{k \in K} \frac{1}{k}\,.$$

Therefore, by Cauchy-Bunyakovski inequality we have

$$\sum_{k \in K} |a_k| = \sum_{k \in K} \Big(|a_k| k^{1/2}\Big) k^{-1/2} < \left(\Big(\sum_{k \in K} |a_k|^2 k\Big)\Big(\sum_{k \in K} \frac{1}{k}\Big)\right)^{1/2} < S.$$

Remark 2. In the case when the domain is contained in the unit circle, we can set $f_1 = f$, $f_2 = g = 0$, and then the above lemma trivially follows from Schwarz lemma [4, p. 363]. Thus we need this lemma only for the proof of Theorem 2, but it is not actually used in the proof of Theorem 1 (and consequently, in the proof of Varfalomeev's theorem).

Proof of Theorems 1 and 2. From the similarity arguments and the monotonity of Φ, it follows that it suffices to prove Theorem 1 for $R = 1$.

Let z_0 be any point on the circle $\{|z| = r_0\}$. Let us denote: $h = 1 - r_0$, $L(h)$ =ext. dist.$(\gamma_{r_0}, \partial D)$, $S(h)$ = the area of the image of the annulus $\{r_0 \le |z| \le 1\}$ under the mapping f. Let us introduce the polar coordinates $z = r\exp(i\varphi)$. Let $\alpha_\theta : [0, h] \to \mathbb{C}$ be the curve parametrized in the polar coordinates (r, φ) by $\alpha_\theta(s) = (r_0 + s, \varphi_0 + s\theta/h)$, where (r_0, φ_0) are the polar coordinates of z_0 (i.e. α_θ is a segment of the Archimedian spiral connecting z_0 to the point $\exp(i(\varphi_0 + \theta))$ of the unit circle).

Then, by the definition of the ext. dist. we have

$$
\begin{aligned}
L(h) = \text{ext. dist.}(\gamma_{r_0}, \partial D) &\le \inf_{0\le\theta\le h} |f(\alpha_\theta)| \le \frac{1}{h}\int_0^h |f(\alpha_\theta)|\, d\theta \\
&= \frac{1}{h}\int_0^h d\theta \int_0^h |f'(\alpha_\theta(s))| \cdot \left|\frac{d\alpha_\theta}{ds}\right| ds.
\end{aligned}
$$

Let us change the coordinates $s = u$, $\theta = hv/u$. Then $\partial(s, \theta)/\partial(u, v) = h/u$, $\alpha_\theta(s) = z(u, v) = (u + r_0)\exp(i(v + \varphi_0))$. Since $|d\alpha_\theta/ds| = r\sqrt{1 + (\theta/h)^2} < \sqrt{2}$ for $\theta < h$ and $r < 1$, we obtain

$$
\begin{aligned}
L(h) &< \sqrt{2}\int_0^h \frac{du}{u}\int_0^u |f'(z(u, v))|\, dv \\
&< \sqrt{2}\int_0^{h\omega} \frac{du}{u}\int_0^u |f_1'|\, dv + \sqrt{2}\int_{h\omega}^h \frac{du}{u}\int_0^u |f_1'|\, dv + \sqrt{2}\int_0^h \frac{du}{u}\int_0^u |f_2'|\, dv \\
&= \sqrt{2}(I_1 + I_2 + I_3),
\end{aligned}
$$

where $f = f_1 + f_2$ is the decomposition from Lemma (see also Remark 2), and $\omega = \omega(h)$ is some function of h tending to zero as $h \to 0$, which we shall choose later. Let us estimate separately each of the three integrals. By (1) we have $|f_1'| \le 1/(1 - r) = 1/(h - u) \le 1/(h - h\omega)$ for $u \le h\omega$. Hence,

$$
I_1 < \Phi_1 = \omega/(1 - \omega). \tag{3}
$$

By Cauchy-Bunyakovski inequality, we have

$$
I_2 < \left(\int_{h\omega}^h \frac{du}{u^2}\int_0^u dv\right)^{1/2} \left(\int_{h\omega}^h du \int_0^u |f_1'|^2\, dv\right)^{1/2}.
$$

But the expression under the second radical is equal to the area of the image under f of the fugure which is cut from the annulus $r_0 + h\omega \le |z| \le 1$ by the curves α_0 and α_h, hence, it is smaller than $S(h)$. Thus,

$$
I_2 < \Phi_2 = \sqrt{-S(h)\ln\omega}. \tag{4}
$$

To make $\Phi_1 \to 0$ and $\Phi_2 \to 0$ as $h \to 0$, it suffices to set $\omega = \exp(-S(h)^{-1/2})$. Theorem 1 is proved.

Finally, by virtue of (2) we have $I_3 < h \cdot (g(1) - g(r_0))$. Combined with (3) and (4), this proves Theorem 2.

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