# CORRIGENDUM TO THE PAPER "A FLEXIBLE AFFINE $M$-SEXTIC WHICH IS ALGEBRAICALLY UNREALIZABLE"

S. Fiedler–Le Touzé, S. Orevkov, and E. Shustin

Abstract. We prove the algebraic unrealizability of certain isotopy type of plane affine real algebraic $M$-sextic which is pseudoholomorphically realizable. This result completes the classification up to isotopy of real algebraic affine $M$-sextics. The proof of this result given in a previous paper by the first two authors was incorrect.

The main theorem of the paper [4] states that the arrangement $B_2(1,4,5)$ in $\mathbb{RP}^2$ (see Figure 1) is unrealizable by a union of a line and a real smooth algebraic sextic curve. The precise statement is:

**Theorem 1.** *Let $C$ be a real algebraic curve of degree 6 in $\mathbb{RP}^2$ and $L$ a line. Then there does not exist an ambient isotopy of $\mathbb{RP}^2$ which deforms $C$ and $L$ into the curve and the line in Figure 1.*

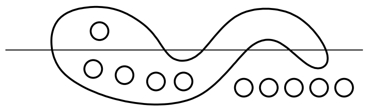

Figure 1. The arrangement $B_2(1,4,5)$

Recently, the first author found a mistake in the final part of the proof of Theorem 1 given in [4] (we discuss this mistake in detail in Section 3 below). However the result is correct and here we give another proof.

An affine smooth irreducible real algebraic curve $A$ in $\mathbb{R}^2$ of degree $d$ is an *affine $M$-curve* if it has maximal possible number of connected components, which is equal to $g+d$ where $g=(d-1)(d-2)/2$ is the genus of the complexification of $A$. This condition is equivalent to the fact that the projective closure of $A$ is an $M$-curve (i.e. it has $g+1$ connected components) and all intersections with the infinite line are real and transverse and sit on the same connected component of the closure of $A$. Thus, if Figure 1 were algebraically realizable, it would provide an affine $M$-sextic in the affine plane $\mathbb{RP}^2 \setminus L$.

A classification of affine $M$-sextics up to isotopy was started in [8, 9] and completed in [13, Theorem 1.1] assuming that [4] is correct. So, here we fill a gap in the proof of this classification as well. Note that a pseudo-holomorphic classification of affine $M$-sextics was previously obtained in [9], and it differs from the algebraic one.

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Three arrangements are realizable pseudo-holomorphically, but not algebraically; see [13]. The arrangement $B_2(1,4,5)$ in Figure 1 is one of them. This is why it is more difficult to exclude it.

We prove Theorem 1 arguing by contradiction and proceed in three steps:

(i) assuming that a smooth sextic curve $C_0$ arranged with respect to the line $L$ as shown in Figure 1 exists, we derive that there exists a real elliptic sextic curve $C_9$ with 9 nodes located with respect to $L$ as shown in Figure 2(a) (see Lemma 1 in Section 1);

(ii) from the existence of a sextic $C_9$ we derive the existence of an elliptic real sextic having 7 nodes (five isolated and two non-isolated) and a singularity $A_3$, and located with respect to $L$ as shown in Figure 2(b) (see Lemma 2 in Section 1);

(iii) we prohibit the existence of the latter real elliptic sextic using a suitable version of cubic resolvent (see Section 2).

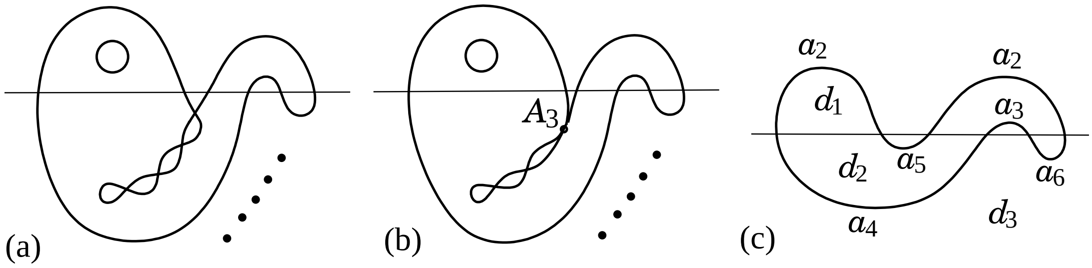


FIGURE 2. (a) See Lemma 1. (b) See Lemma 2. (c) Notation $a_i$, $d_i$.

So, the general scheme of the proof of Theorem 1 is almost the same as for the proof in [13] of algebraic unrealizability of the affine sextic $C_2(1,3,6)$. However, there is a difference in the last step. In [13], the cubic resolvent is algebraically realizable but its mutual position with respect to the axis is pseudo-holomorphically unrealizable. Here the situation is opposite: the mutual position of the resolvent and the axis is pseudo-holomorphically realizable, but the resolvent itself is algebraically unrealizable.

Note also that only the $A_3$ singularity is needed in Step (iii), thus we may undo all the remaining $A_1$ singularities obtained in Step (ii). However, we do not know how to attain the $A_3$ singularity (keeping the required position of the curve with respect to the line $L$) without passing through a genus 1 nodal curve.

## 1. APPLICATION OF HILBERT-ROHN-GUDKOV METHOD

The following notation will be used in the proof. Denote the nonempty oval of $C_0$ by $O^{\mathrm{ne}}$. In what follows, we deform $C_0$ in certain families, and the corresponding non-empty oval will be denoted by $O^{\mathrm{ne}}$ as well. If $O^{\mathrm{ne}}$ degenerates into a loop with a singular point, we continue to use the same notation for this loop. The open disk bounded by $O^{\mathrm{ne}}$ will be denoted by $d_0$. The line $L$ cuts $O^{\mathrm{ne}}$ into six arcs, which we denote $a_1, \dots, a_6$ according to Figure 2(c). We also use the notation $d_1, d_2, d_3$ for the three of the connected components of $\mathbb{RP}^2 \setminus (O^{\mathrm{ne}} \cup L)$ designated in Figure 2(c). By $O^{\mathrm{e}}$ we denote the empty oval in the domain $d_1$ (it will remain oval in all further deformations).

**Lemma 1.** *Suppose that there exists a sextic curve $C_0$ shown in Figure 1. Then there exists a real irreducible sextic curve $C_9$ with 9 nodes located with respect to the line $L$ as shown in Figure 2(a).*

*Proof.* We construct the curve $C_9$ inductively. Abusing notation we denote a plane curve and its defining polynomial by the same symbol.

Start with a pencil of sextic curves $\{C_0^{(t)} = C_0 + \varepsilon t K_0^2,\ \varepsilon = \pm 1,\ t \geq 0\}$, where $K_0$ is a generic real cubic curve passing through a point $p$ chosen on the arc $a_6$. Choose $\varepsilon$ so that the disk $d_0$ contracts as $t$ grows. Furthermore, the oval $O^{\mathrm{ne}}$ always intersects $L$ as shown in Figure 1, the disks bounded by the empty ovals inside $d_0$ grow, while the disks bounded by the empty ovals inside $d_3$ shrink. Note that $t$ cannot tend to $\infty$ without degeneration of $C_0^{(t)}$. Indeed, otherwise the curve $C_0^{(t)}$ would approach the double cubic curve $K_0^2$. This, however, is impossible: we consider the real line through a fixed point inside the oval $O^{\mathrm{e}}$ and a point embraced by one of the empty ovals in the domain $d_2$, and then, on any sufficiently close real line, we will observe a pair of real intersection points with $C_0^{(t)}$ that approach the intersection point with $L$ (see Figure 3, left). This finally would yield that $K_0$ contains a segment of $L$. A contradiction.

Due to the general choice of $K_0$, the first degeneration is a sextic curve $C_1$ with one node (see more detailed arguments in the proof of the induction step below). Note that this node cannot join $O^{\mathrm{ne}}$ with the empty oval in the domain $d_1$, since they form a positive complex oriented injective pair (see [4, Figure 9]).[1]

Further, the node cannot join the arc $a_4$ with an empty oval in the domain $d_2$. Indeed, suppose there exists such a nodal degeneration $C^*$. Let us consider the pencil of lines through a point inside $O^{\mathrm{e}}$. By [4, Lemma 2.2], the lines of this pencil passing through the exterior ovals do not separate the ovals in $d_2$ from each other and from the arc $a_5$. Hence (cf. [4, §6.2] and [11, §4.5]) the braid of $C^*$ with respect to this pencil coincides with that of a curve $C^{**}$ obtained from Figure 2(a) by joining one of the ovals in $d_2$ with $a_4$ through one more node. Thus there exists such a pseudo-holomorphic curve $C^{**}$ which contradicts the genus formula. Hence,

(1) either the node joins a pair of empty ovals in the domain $d_2$,
(2) or the node joins an empty oval with the arc $a_5$,
(3) or the node is an isolated point obtained by shrinking one of the empty ovals in the domain $d_3$.

For the induction step, suppose that $C_k$, $1 \leq k \leq 8$, is a real irreducible sextic curve with $k$ real nodes and such that

(i) there exists a smoothing of $C_k$ into a smooth sextic as shown in Figure 1,
(ii) all nodes are real, each non-isolated node joins either a pair of empty ovals in the domain $d_2$, or an empty oval in the domain $d_2$ with the arc $a_5$, while each isolated node is obtained by shrinking an oval in the domain $d_3$.

By [13, Proposition 2.4(b)], we can suppose that all $k$ nodes and the point $p$ are in general position. Consider a pencil

$$\{C_k^{(t)} = C_k + \varepsilon t K_k^2, \quad \varepsilon = \pm 1, \quad t \geq 0\}\ , \tag{1}$$

[1] We refer to [16, §2.1 and §2.4B] for a definition of the complex orientations and of positive/negative injective pairs of ovals respectively. Also [16] can be used as a general introduction to the subject.

where $K_k$ is a generic real cubic passing through the nodes of $C_k$ and the point $p \in a_6$, and $\varepsilon$ is chosen so that $d_0$ shrinks as $t$ grows. The argument, used for the base of induction, ensures that there must be a degeneration at some $t \in (0, \infty)$. We claim that the first degeneration $C^*$ is a $(k+1)$-nodal curve $C_{k+1}$ possessing the above properties (i) and (ii). We explain this just in the most difficult case of $k = 8$. By [5, Theorem 1] (see also [6]) we can suppose that all node of $C_8$ are in general position. Blowing up the 8 fixed nodes, we obtain curves in the 3-dimensional linear system $|D| = |6L - 2E_1 - \cdots - 2E_8|$ on a general del Pezzo surface $\Sigma$ of degree 1. By [7, Lemma 9(1)], the curves that are not immersed form a set of dimension at most 1 in $|D|$. Fixing the point $p \in \Sigma$, we obtain that the pencils spanned by non-immersed curves and the (unique) double curve passing through $p$, sweep a subset of dimension $\leq 2$ in $|D|$, while the (smooth) blown up curve $C_8$ can be moved to a general position in $|D|$. Hence, the considered pencil of sextics (1) does not contain non-immersed curves (except for the double cubic). We then see that the degenerate curve $C^*$ cannot have an immersed singularity more complicated than a node by the genus formula and the Harnack-Klein bound stating that the number of connected components of the real point set of the normalization of the curve does not exceed genus plus one. For the same reason, an extra singularity cannot be a popping up isolated real node, nor two nodes can appear on the arc $a_5$. Thus a possible position of the new non-isolated node is determined by the rules (i) and (ii), as we have seen in the base induction step. □

**Remark 1.** Statements similar to that of Lemma 1 are contained also in [14, Step (1) in the proof of Lemma 3.3] and [13, Lemma 2.10], where detailed proofs have been skipped. Moreover, Lemma 1 and the above cited statements follow from [6, Theorem 10 (proof in §7-11)]. For the reader's convenience we have provided here a proof with all necessary details that also complete the proofs in [14, Step (1) in the proof of Lemma 3.3] and [13, Lemma 2.10].

**Lemma 2.** *Let $C_9$ be a real nodal sextic as in Lemma 1and $p, q \in C_9$ be as in Figure 3, right. Then there exists a real elliptic sextic $C(A_3)$ with 7 nodes and a singularity $A_3$ located with respect to the line $L$ as shown in Figure 2(b).*

*Proof.* We apply the Hilbert-Rohn-Gudkov method in the form developed in [14, Section 4] and proceed similarly to the lines of the proof of [14, Lemma 5.3].

By [13, Proposition 2.4(b)], we can suppose that

$$\begin{array}{c}\text{the configuration consisting of any prescribed 7 nodes of } C_9,\\ \text{the points } p \text{ and } q, \text{ and of the tangent at } p \text{ is in general position.}\end{array} \tag{2}$$

Let us order the non-isolated nodes $z_1, \ldots, z_4$ of $C_9$ assuming that $z_4 \in O^{\mathrm{ne}}$, and respectively denote by $d'_1, \ldots, d'_4$ the disks inside $d_0$ bounded by the arcs ending at $z_1, \ldots, z_4$ (so, $z_3, z_4 \in d'_4$). Pick a point $p \in a_6$ and a point $q \in O^{\mathrm{e}}$. Consider the germ $\mathcal{M}$ at $C_9$ of the equisingular family of real elliptic sextics which

(i) have nodal singularities at all the isolated nodes of $C_9$ and at $z_1$ and $z_2$,
(ii) have a node in a neighborhood of $z_i$, $i = 3, 4$,
(iii) intersect $C_9$ at $p$ with multiplicity 2,
(iv) pass through the point $q$.

The germ $\mathcal{M}$ is smooth and one-dimensional by [15, Theorem in page 31].

By formula (24) in [14, Lemma 4.2] and formulas (15), (16) in [14, Lemma 4.1], each curve $C' \in \mathcal{M} \backslash \{C_9\}$ intersects $C_9$ with multiplicity 4 at each of the seven fixed

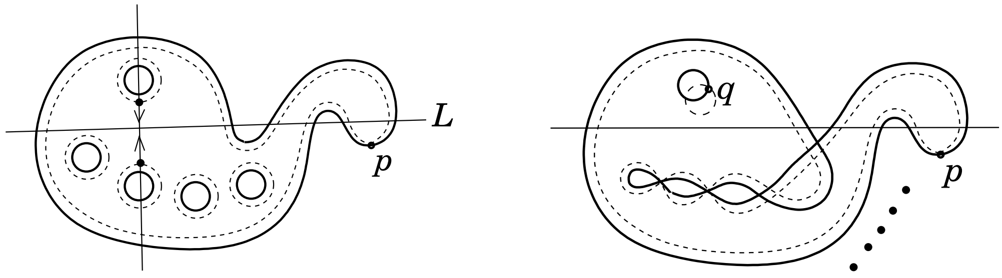


FIGURE 3

nodes, at two real points in a neighborhood of $z_i$, $i = 3, 4$, and with multiplicity 3 at $\{p, q\}$. In total this gives 35, and by the parity argument, one more (real) intersection point of $C'$ with $C_9$ lies on the oval $O^e$. Altogether this yields that, moving along $\mathcal{M}$ in a certain direction, we obtain a deformation of the real point set such that (see the dashed lines in Figure 3, right):

$$\text{the disks } d'_1, d'_2, d'_3 \text{ grow, the disk } d'_4 \text{ and the domain } d_0 \text{ shrink;} \tag{3}$$

cf. [13, Proposition 2.5 and Figure 3].

Extending the germ $\mathcal{M}$ to a global equisingular family subject to conditions (i), (iii), (iv) above, we see that the element of $\mathcal{M}$ moving in the designated direction cannot return to $C_9$ due to the strongly monotone changes (3), and hence must undergo a degeneration. The argument used in the proof of Lemma 1 shows that it is not a double cubic. The general position condition (2) excludes all other splittings of the degenerate curve into three or more components (counting multiplicities). Let us show that no splitting into two distinct components is possible. Indeed, it follows from (3), that the fixed isolated nodes in the domain $d_3$ remain isolated in the degeneration, and that no component of odd degree can split off. Note also that, in case of a splitting, no isolated node can be an intersection point of two components, since otherwise, the components must be complex conjugate, which is impossible. All this leaves the only possibility of a splitting into a conic and a quartic, but such a curve cannot have 5 isolated nodes.

Thus, the appearance of an extra node can only be the contraction of the oval $O^e$ to the point $q$, otherwise, one would encounter a forbidden reducible curve. In this case, we simply ignore the degeneration and continue the movement along our one-dimensional family: the oval $O^e$ pop up again, and the rest of the real part of the current curve deforms as shown in Figure 3(right). Then at some moment we have to encounter another degeneration, and the only possibility left is the shrinking to a point of the disk $d'_4$, thus, giving the required elliptic curve $C(A_3)$. Indeed, the genus formula combined with (3) do not allow any singularity of the form $A_n$, $n \neq 3$. $\square$

## 2. APPLICATION OF CUBIC RESOLVENTS. END OF PROOF OF THEOREM 1

We denote the standard real Hirzebruch surface of degree $n > 0$ (the fiberwise compactification of the line bundle $\mathcal{O}(n)$ over $\mathbb{P}^1$) by $\mathcal{F}_n$. Let $\mathbb{R}\mathcal{F}_n$ be the set of real points of $\mathcal{F}_n$. It is diffeomorphic to a torus or a Klein bottle. In Figures 4 and 5 we represent $\mathbb{R}\mathcal{F}_n$ by a rectangle whose opposite sides are identified. The horizontal sides represent the exceptional section $E$, $E^2 = -n$, and vertical lines (in particular, the vertical sides of the rectangle) represent fibers of the projection

$\mathcal{F}_n \to \mathbb{P}^1$. Let $F$ be one of the fibers. The Picard group of $\mathcal{F}_n$ is generated by $E$ and $F$. A generic section disjoint from $E$ belongs to the linear system $|E + nF|$.

When speaking of a *fiberwise arrangement* of a curve on $\mathbb{R}\mathcal{F}_n$, we mean its arrangement up to isotopies which fix $E$ and send each fiber to a fiber. If it is known that the curve belongs to $|dE + ndF|$, its *almost fiberwise arrangement* is the arrangement up to isotopies fixing $E$ and such that any fiber at any moment intersects the curve at $\le d$ points counting the multiplicities. In particular, the ovals of trigonal curves ($d = 3$) cannot pass one over another during such isotopies.

*Proof of Theorem 1.* Suppose that Figure 1 is realizable. Then, by Lemma 2, there exists a singular sextic curve with an $A_3$ singularity arranged with respect to $L$ as in Figure 2(b). It can be perturbed into a curve $C'$ arranged in one of the two ways shown in Figure 4 (left) with respect to $L$ and the two dashed lines (by rotating $L$ around the common point of the arcs $a_3$ and $a_6$ we can achieve that $L$ passes through $A_3$). The relative position of $C'$ with respect to the (dashed) line through $O^{\mathrm{e}}$ follows from [4, Lemma 2.2]. The position of the two nodes with respect to the tangent line at $A_3$ follows from Bezout theorem for an auxiliary conic tangent to $C'$ at $A_3$ and passing through $O^{\mathrm{e}}$, one empty exterior oval, and one of the nodes.

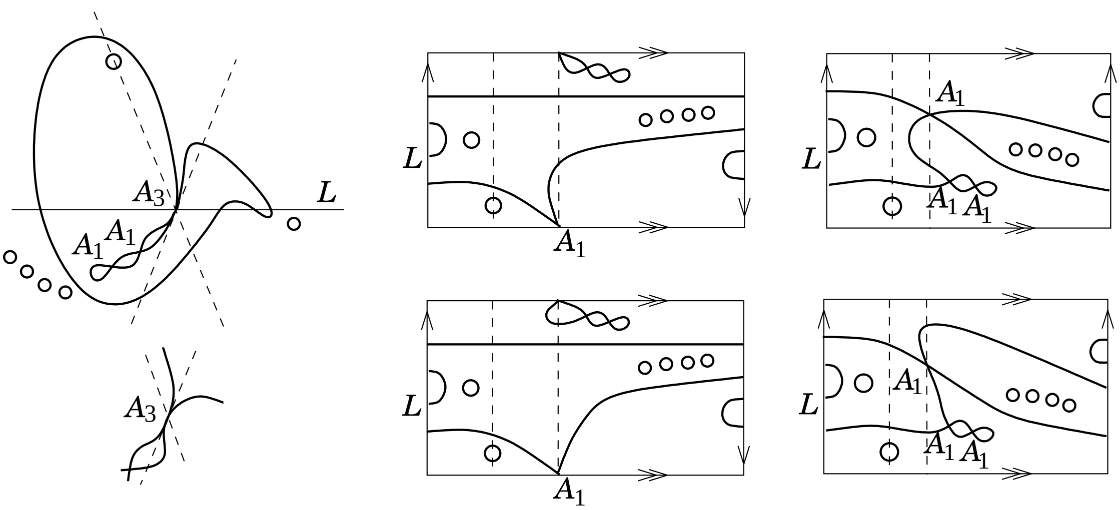


FIGURE 4. $C' \cup L$ and its transforms on $\mathcal{F}_1$ and on $\mathcal{F}_2$

Let us blow up the point $A_3$. We obtain the arrangement in Figure 4 (middle) on $\mathbb{R}\mathcal{F}_1$. Then we blow up the point $A_1$ on $E$ and blow down the strict transform of the fiber passing through it. We obtain a curve in $\mathbb{R}\mathcal{F}_2$ belonging to $|4E + 8F|$ arranged (up to isotopy) with respect to $E$ and the indicated fibers as shown in Figure 4 (right). Its cubic resolvent is a trigonal curve in $\mathcal{F}_4$ arranged with respect to $E$ and the indicated fibers as in Figure 5 (left); see [13, §3].

Using [10], it is an easy exercise to check that the arrangement in Figure 5 (left) is unrealizable by a trigonal algebraic curve on $\mathbb{R}\mathcal{F}_4$ (note that it is realizable by a trigonal pseudoholomorphic curve). To this end one should exclude all its possible fiberwise arrangements, namely, the one depicted in Figure 5 (left) and those obtained from it by inserting a zigzag ∽ or ∼ between some two consecutive ovals (at most one zigzag can be inserted because otherwise we obtain too many vertical tangents). Note that insertion of a zigzag is really necessary because there are unrealizable fiberwise arrangements which become realizable after a zigzag insertion, see [11, Appendix B]; this phenomenon is impossible in pseudoholomorphic context.

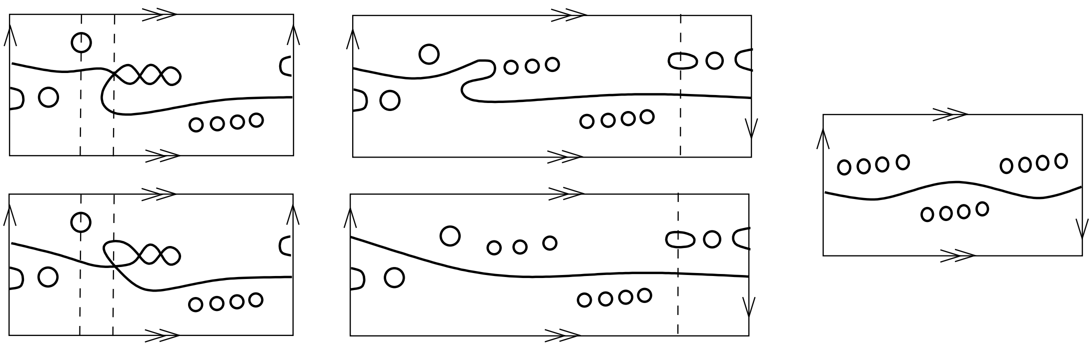

FIGURE 5. The cubic resolvent on $\mathcal{F}_4$; the glued curve on $\mathcal{F}_5$

According to [10], to exclude each of these fiberwise arrangements, it is enough to check that there does not exist a graph in $\mathbb{CP}^1$ satisfying Conditions (1)–(7) at the end of [10, §4] and having a prescribed behavior near $\mathbb{RP}^1$. Indeed, since the number of vertices of the graph and their nature is dictated by these conditions, only a finite number of cases should be considered which can be done by hand in a reasonable time. This fact can be also derived from Erwan Brugallé’s result. He checked in [1, Proof of Proposition 5.6] by this method that the almost fiberwise arrangement in $\mathbb{R}\mathcal{F}_5$ shown in Figure 5 (right) is algebraically unrealizable. Indeed, [1, Proposition 3.6] implies that it is enough to consider zigzag insertions of the form  up to symmetry, thus the almost fiberwise unrealizability of Figure 5 (right) is a consequence of [1, Lemmas 5.4 and 5.5].

The unrealizability of Figure 5 (left) follows from that of Figure 5 (right) because the latter is obtained from the former by gluing it together with an $M$-cubic in $\mathbb{RP}^2$ according to Figure 5 (middle). The gluing can be understood either in the sense of [10] or in the sense of Viro [16]. In the latter case we interpret the two parts of Figure 5 (middle) as charts in the triangles $[(0,0),(12,0),(0,3)]$ and $[(12,0),(15,0),(0,3)]$. $\square$

## 3. THE MISTAKE IN [4]

The idea of the prohibition of the sextic in question realized in [4] was to consider the pencil of real cubics through 8 specific fixed points on the hypothetical sextic, then, using an information on the location of the fixed points with respect to lines and conic, to construct the evolution of cubics along the pencil, and then to show that such a pencil does not satisfy some necessary conditions (does not reveal 8 distinguished cubics, see definition in [4, §4]).

In this section we assume that the reader is familiar with the paper [4] and we use the notation from there. The mistake is in the last step (“From $C^4$ to contradiction”) in [4, §5.4]: the assertion that the arcs 6 and 8 of $C^t$ are separated by $C^4 \cup N$ is erroneous. As a matter of fact, the non-base point of $C^t \cap N$ may escape the loop of $N$ as shown in Figure 6.

One could hope to repair the proof in [4] by continuing the construction of the pencil of cubics and obtaining a contradiction on some further step. Unfortunately, this is not so. In Figure 7 (see also Figure 8) we complete the pencil of cubics without any contradiction to Bezout theorem for the auxiliary curves considered in [4].

In the rest of the section we explain how we construct the pencil, using the tools

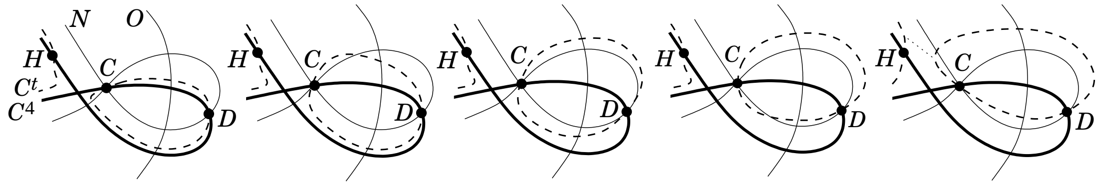


FIGURE 6. The non-base point of $C^t \cap N$ escapes the loop of $N$

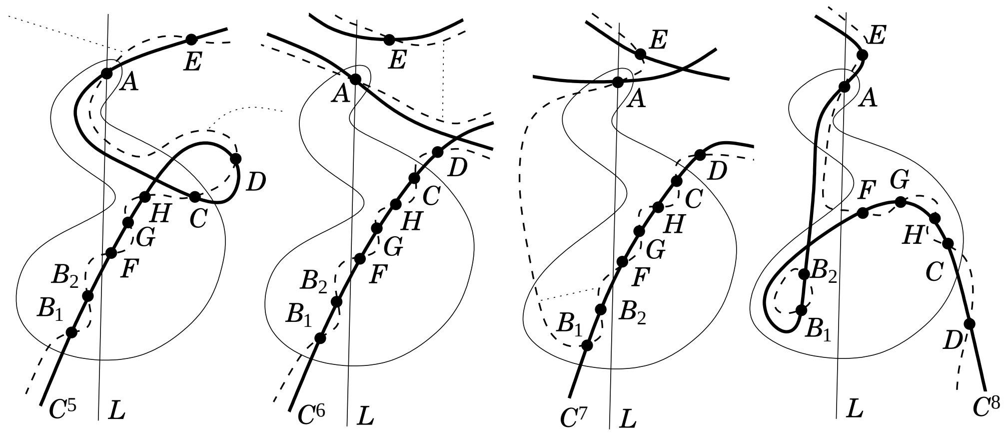


FIGURE 7. Completing the pencil of cubics (cf. [4, Figure 17])

from [2, 3]. The *combinatorial configuration* of $n$ points in the plane is the data describing the mutual position of each point with respect to the lines through two others and the conics through five others. The *combinatorial pencil of cubics* determined by eight points is given by the arrangement of the nine base points on the eight successive *distinguished cubics* (see the definition in [4, §4]). Let us consider eight points $(1,2,3,4,5,6,7,8)$ distributed in the ovals $(A, D, C, H, G, F, B, E)$. Using [4, Lemma 2.2] plus Bezout's theorem between $C_6$ and some auxiliary rational cubics (passing through seven of the points, with node at one of them), we determine the combinatorial configuration $\mathcal{C}$ realized by these eight points. It is formed of five 7-subconfigurations of type $(3,4,0,0)_2$, plus three of type $(7,0,0,0)$ (see [3, §3.1]). To find this pencil determined by $1,\ldots,8$, free the point 7 away from the oval $B$ and move it till it crosses the line $(AE)$. The new configuration $(1,\ldots 8)$ lies in convex position, its combinatorial configuration $\mathcal{C}$ is replaced by $\max(\hat{1} = 8+)$ (see [2, §2.3]). As 7 is close to the line (18), it lies outside of the loops of the cubics $(\hat{7}, 1)$ and $(\hat{7}, 8)$, hence (see [2, §5.1]) the pencil is the first one in [2, Figure 35]. Move 7 back to its initial position in $B$, the combinatorial pencil changes when 7 crosses the line $(AE)$, see upper part of Figure 8. Afterwards, the eight points realize $\mathcal{C}$ for all positions of 7 on the path. The only way to change the pencil would be to let 9 cross another base point $k$: when $9 = k$, the points $1,\ldots,8$ lie on a nodal cubic with node at $k$ (see [2]). But one proves that $\mathcal{C}$ is not realizable by eight points on such a cubic. So, the pencil undergoes no further change. The complete pencil of cubics is shown in the lower part of this Figure 8. (Note that with the notation of [4], $B_1 = 9$ and $B_2 = 7$.)

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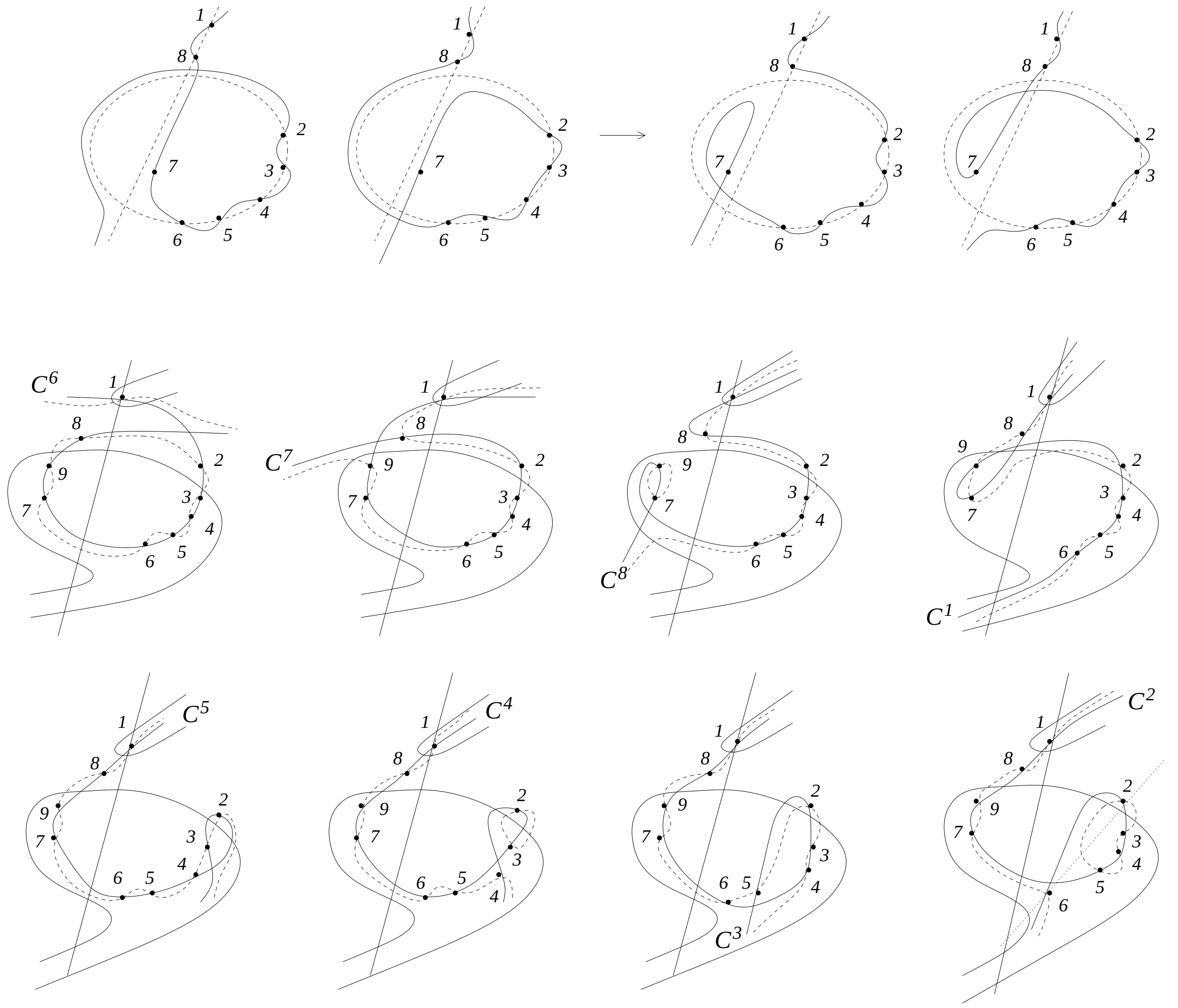


FIGURE 8. Upper part: change of two distinguished cubics induced letting 7 cross the line (18); lower part: the complete pencil of cubics through points $1, 2, 3, 4, 5, 6, 7, 8$ distributed in the ovals $A, D, C, H, G, F, B, E$

## REFERENCES


1. E. Brugallé, *Symmetric plane curves of degree 7: pseudoholomorphic and algebraic classification*, J. Reine Angew. Math. **612** (2007), 129–171.
2. S. Fiedler-Le Touzé, *Pencils of cubics with eight base points lying in convex position in* $\mathbb{R}P^2$, arXiv [math AG] 1012.2679.
3. S. Fiedler-Le Touzé, *Rational pencils of cubics and configurations of six or seven points in* $\mathbb{R}P^2$, arXiv[AG] 1210.7146.
4. S. Fiedler-LeTouzé, S. Orevkov, *A flexible affine $M$-sextic which is algebraically unrealizable*, J. Alg. Geom. **11** (2002), 293–320.
5. D. A. Gudkov, *Variability of simple double points on real plane algebraic curves*, Dokl. Akad. Nauk SSSR **142** (1962), 1233-1235 (Russian); English transl., Sov. Math., Dokl. **3** (1962), 273–275.
6. D. A. Gudkov, *Systems of k points in general position and algebraic curves of different orders*, Nine papers on Hilbert’s 16th problem, Amer. Math. Soc. Transl (2), vol. 112, 1978, pp. 15–45.
7. I. Itenberg, V. Kharlamov, and E. Shustin, *Welschinger invariants revisited*, Analysis Meets Geometry: A Tribute to Mikael Passare. Trends in Math., Birkhäuser, 2017, pp. 239–260.
8. A. B. Korchagin, E. I. Shustin, *Affine curves of degree 6 and smoothings of a non-degenerate sixth order singular point*, Izv. AN SSSR, ser. mat. **52** (1988), no. 6, 1181–1199 (Russian);

English transl., Math. USSR, Izv. **33** (1989), 501–520.

9. S. Yu. Orevkov, *Link theory and oval arrangements of real algebraic curves*, Topology **38** (1999), 779–810.
10. S. Yu. Orevkov, *Riemann existence theorem and construction of real algebraic curves*, Ann. Fac. Sci. de Toulouse. Math. (6), **12** (2003), 517–531.
11. S. Yu. Orevkov, *Arrangements of an $M$-quintic with respect to a conic that maximally intersects its odd branch*, Algebra i Analiz **19** (2007), no. 4, 174–242 (Russian); English transl. St. Petersburg Math. J. **19** (2008), 625–674.
12. S. Yu. Orevkov, E. I. Shustin, *Flexible algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach*, J. für die Reine und Angew. Math. **511** (2002), 145–172.
13. S. Yu. Orevkov, E. I. Shustin, *Pseudoholomorphic algebraically unrealizable curves*, Moscow Math. J. **3** (2003), 1053–1083.
14. S. Yu. Orevkov and E. I. Shustin, *Real algebraic and pseudoholomorphic curves on the quadratic cone and smoothings of singularity $X_{21}$*, Algebra i Analiz **28** (2016), no. 2, 138–186 (Russian); English transl., St. Petersburg Math. J. **28** (2017), 225–257.
15. E. I. Shustin, *On manifolds of singular algebraic curves*, Selecta Math. Sov. **10** (1991), no. 1, 27–37.
16. O. Ya. Viro, *Real algebraic plane curves: constructions with controlled topology*, Algebra i Analiz **1** (1989), no. 5, 1–73 (Russian); English transl., Leningrad J. Math. **1** (1990), 1059–1134.

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