The near parabolic renormalization of Inou and Shishikura

Arnaud Chéritat

Institut de Mathématiques de Toulouse

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About renormalization

- Powerful
- Mysterious (for the speaker)
- No unified notion

One kind of renormalization used in discrete dynamics (very roughly):

- take a dynamical system $f\colon X \longrightarrow X$,
- replace f by one of its iterates f^n,
- restrict f^n to a subset U of X,
- rescale your new dynamical system $f^n|_U$ so as to have it satisfy some normalization.

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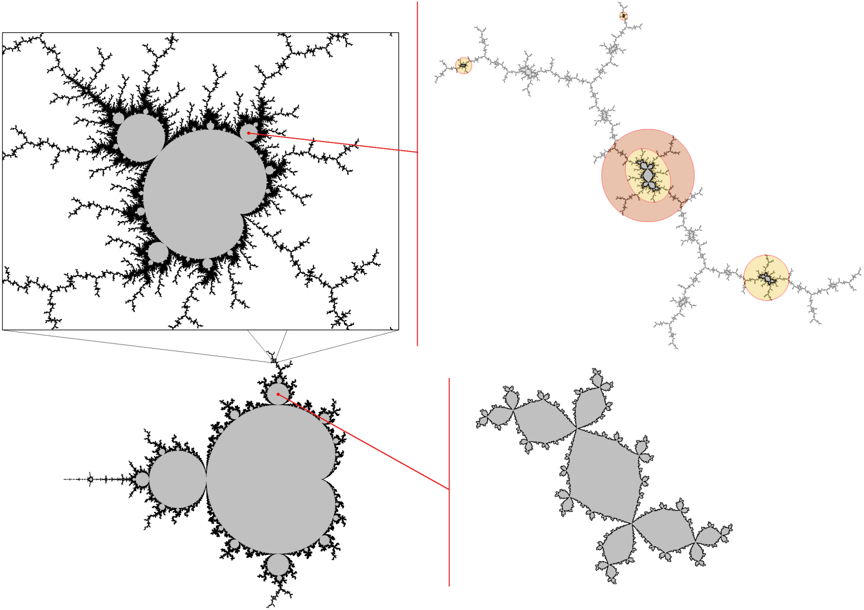

The old and the new dynamical systems do not need to be defined everywhere, the iterate $x \mapsto f^k(x)$ may have its order k that depends on $x \in U$: $k = k(x)$, the rescaling may be replaced by a more general conjugacy.

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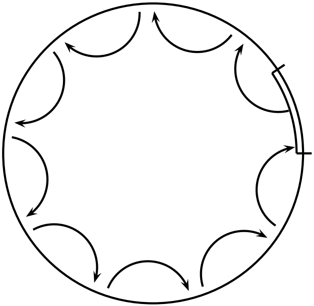

Renormalization operator

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Usually X is infinite dimensional and $\mathcal{R}$ is analytic.

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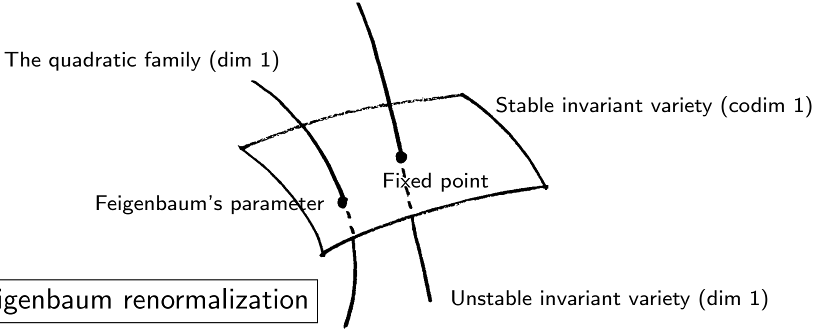

Feigenbaum renormalization

The renormalization operator associated to Feigenbaum's bifurcation cascade has a fixed point. It is hyperbolic at this point. This hyperbolicity proves several experimental findings.

The big picture

Heuristics: renormalization fixed points are universal.

Example: Feigenbaum's universal constant $\delta = 4.669\dots$ is the biggest eigenvalue of the previous operator.

Beyond fixed points, a more global picture is conjectured (Lanford's programme) for several renormalization operators, and proved for a few: there is an invariant compact set, a Cantor or a Solenoid, on which the operator is hyperbolic.

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Definition of the parabolic renormalization

Start from an analytic map fixing 0 with multiplier 1:

$$f(z) = z + a_2 z^2 + \dots$$

Assume $a_2 \neq 0$, i.e. there is only one repelling and one attracting petal in a Leau flower for the parbolic point at the origin.

To this, are associated Fatou coordinates, Φ_{rep} on the repelling petal and Φ_{att} on the attracting petal, that conjugate f to the translation T_1 on "big" domains near infinity. We may assume that the petals are big enough so as to have their image by Φ contain an upper and a lower half plane (otherwise, use the relation $\Phi \circ f = T_1 \circ \Phi$ to extend Φ).

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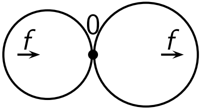

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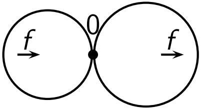

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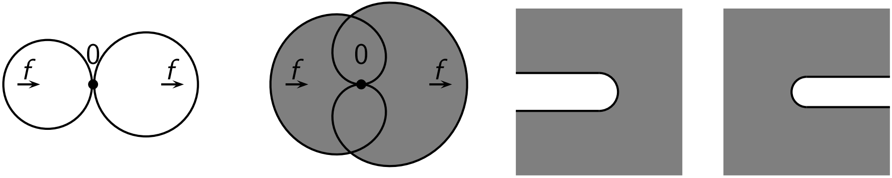

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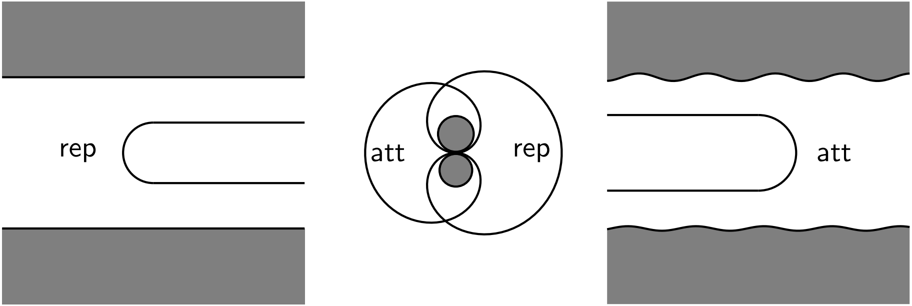
The horn map is defined by

$$h_\sigma = T_\sigma \circ \Phi_{att} \circ \Phi_{\text{rep}}^{-1}$$

$$\begin{matrix}\text{rep. Fatou} \\ \text{coords}\end{matrix} \xrightarrow{\Phi_{\text{rep}}^{-1}} \text{dyn. plane} \xrightarrow{\Phi_{\text{att}}} \begin{matrix}\text{att. Fatou} \\ \text{coords}\end{matrix} \xrightarrow{T_\sigma} \begin{matrix}\text{back to rep.} \\ \text{Fatou coords}\end{matrix}$$

where σ is a parameter (the phase), and commutes with T_1 on its domain of definition, which contains an upper and a lower half plane.

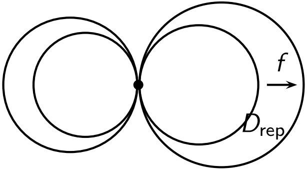


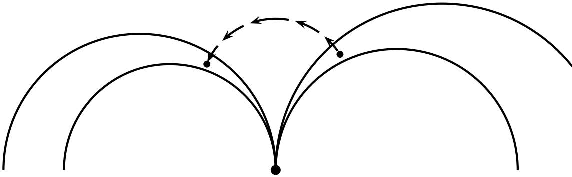
There is a well defined quotient map h_σ mod $\mathbb{Z}$ acting on the cylinder $\mathbb{C}/\mathbb{Z}$.

Another point of view on the same object

Equivalent definition of the horn map on the cylinder (without extending the petals)

The quotient of a petal by the equivalence relation $z \sim f(z)$ is isomorphic, via the Fatou coordinates, to the cylinder $\mathbb{C}/\mathbb{Z}$. This quotient is refferred to as the *attracting/repelling cylinder*.
Now take a fundamental domain D_{rep} in the repeling petal. Take a point in the repelling cylinder. Consider the corresponding point w in D_{rep}. Iterate w until it falls in the attracting petal. To such an iterate corresponds a uniquely defined point in the attracting cylinder. Last, use an identification, of the form T_σ in Fatou coordinates, to go from the attracting cylinder back to the repelling cylinder.





The map $e^{2i\pi z}$ induces an isomorphism from $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C}^*$. Conjugating h_σ mod $\mathbb{Z}$ by this map yields an analytic map g_σ, defined in a neighborhood of 0 and ∞ and fixing both, with multipliers $\neq 0$. Since $g_\sigma = e^{2i\pi\sigma} g_0$, there is a unique value of σ such that $g'_\sigma(0) = 1$. For this σ we get the *the parabolic renormalization* of f:

$$\mathcal{R}(f) \stackrel{\text{def}}{=} g_\sigma$$

Note that this puts the emphasis on the upper end of the cylinder. If one prefers the lower end, replace the conjugacy $z \mapsto e^{2i\pi z}$ by $z \mapsto e^{-2i\pi z}$.

How well-defined is this map?

First, recall that Fatou coordinates are unique only up to addition of a constant. Consequence: $\mathcal{R}(f)$ is unique only up to conjugacy by a linear map. Fortunately, conjugating f itself by a linear map does not change g, so we let $\mathcal{R}$ act on the set of maps f taken up to linear conjugacy. One may choose a canonical representative in each class (=normalization).

The set of definition of g_σ is not clearly well defined either, even if we fix a normalization. To solve that problem, we can work with germs instead of maps.

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An invariant class

There is an invariant class which has been known since around 1990 (Shishikura). It consists in all holomorphic functions $f: U \to \mathbb{C}$ with:

- U is a connected open set
- $0 \in U$ and $f(z) = z + a_2 z^2 + \ldots$ with $a_2 \neq 0$,
- f is a ramified covering from $U \setminus \{0\}$ to $\mathbb{C}^*$,
- all critical points have local degree 2,
- there is exactly one critical value.

For instance, the polynomial $z + z^2$ belongs to this family.

Let us call $\mathcal{C}_0$ the set of maps satisfying these conditions and *normalized* as follows: the critical value is equal to $-1/4$ (same as for $z + z^2$). Then we consider the (well-defined) renormalization operator

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A common covering structure for their horn maps

This $\mathcal{R}: \mathcal{C}_0 \to \mathcal{C}_0$ is not surjective.

Its image is a class $\mathcal{C}_1$ with the following property: any two maps $f_1, f_2 \in \mathcal{C}_1$ are equivalent covers over $\mathbb{C}$, i.e. $\exists \phi$ an isomorphism between their sets of definition such that $f_2 = f_1 \circ \phi$.

Why? Because for all map in $\mathcal{C}_0$, the immediate parabolic basin U contains exactly one critical point and moreover, f is conjugated on U to a universal map: the degree 2 Blachke product $\frac{3z^2+1}{3+z^2}$ on $\mathbb{D}$.

Let's look at my preferred member of $\mathcal{C}_1$, namely $\mathcal{R}(z+z^2)$. —>

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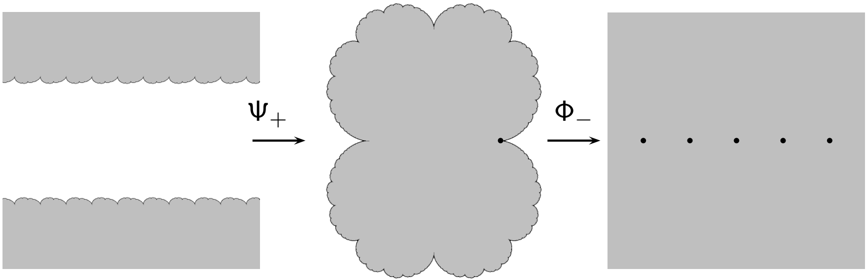

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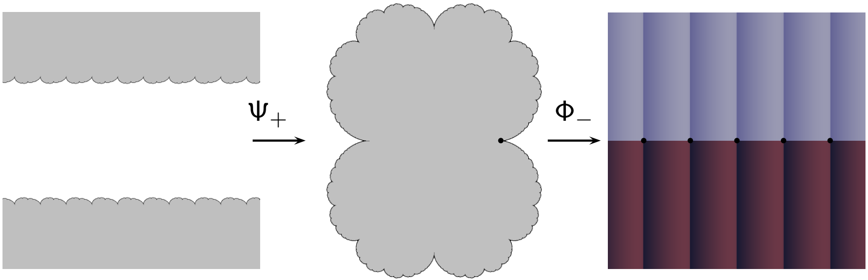
$\Psi_+ : \ \mathbb{C} \to \mathbb{C}$ repelling Fatou parameterization, extended



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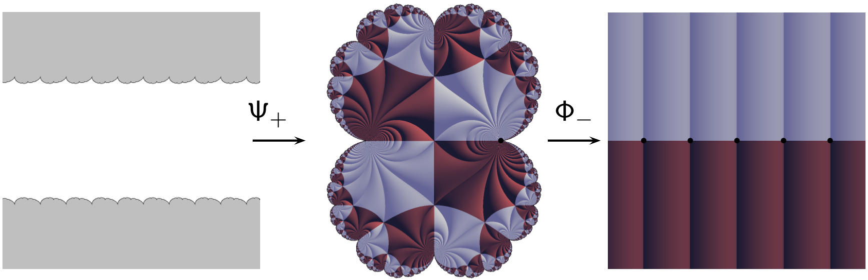

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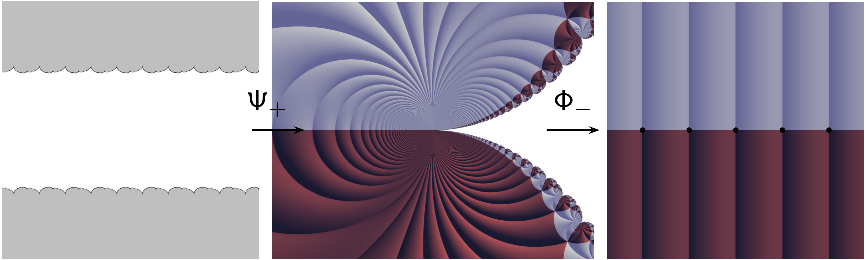

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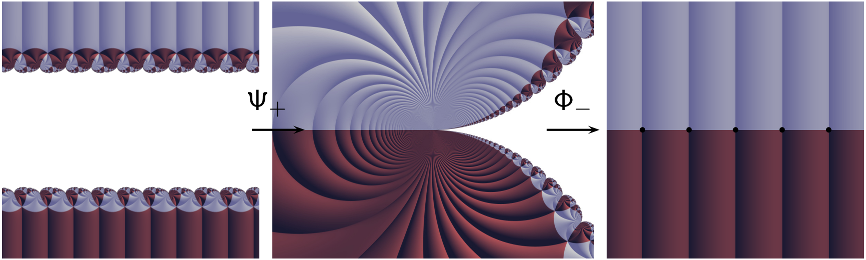

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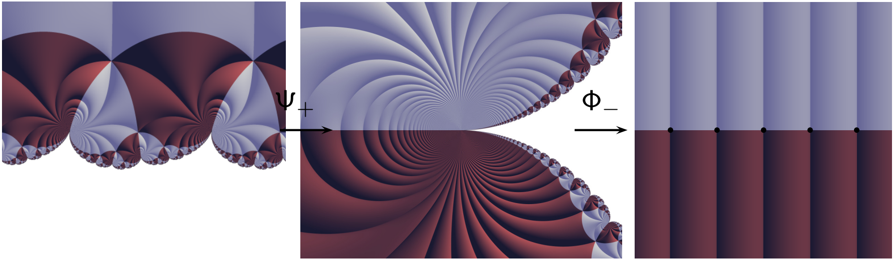

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Introducing some flexibility

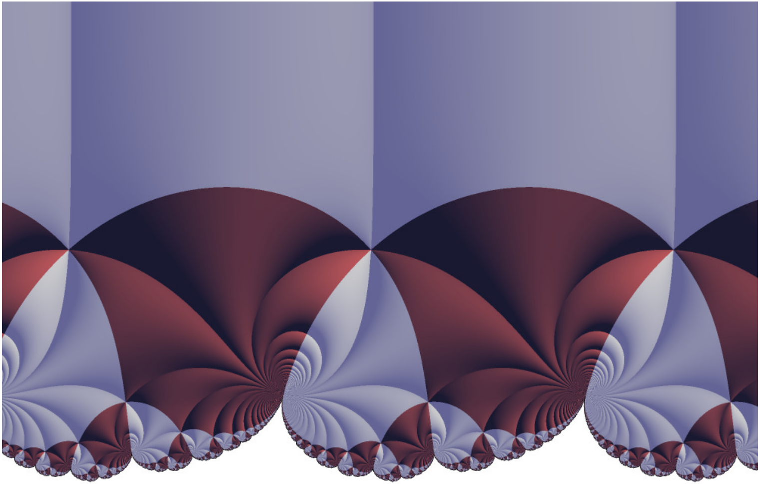
The class $\mathcal{C}_1$ is of the form

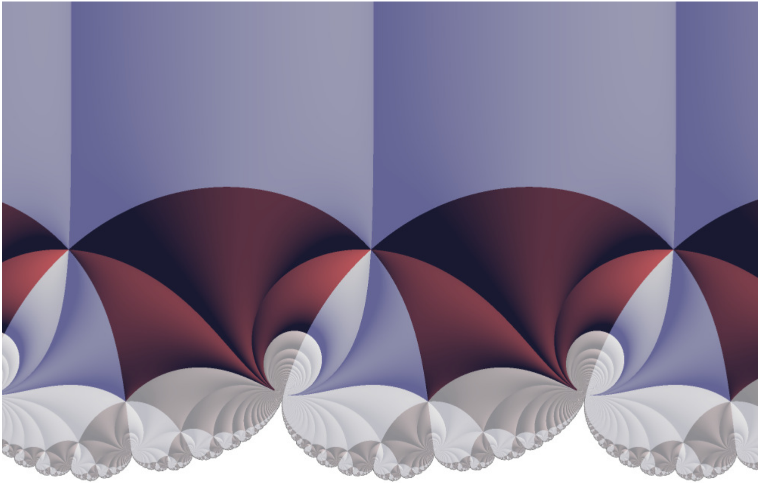
$$\mathcal{C}_1 = \left\{ f_0 \circ \phi^{-1} \,\middle|\, \begin{array}{l} \phi : \ \mathrm{Def}(f_0) \to \mathbb{C} \text{ is a univalent analytic} \\ \text{map with } \phi(0) = 1, \ \phi'(0) = 1 \end{array} \right\}.$$

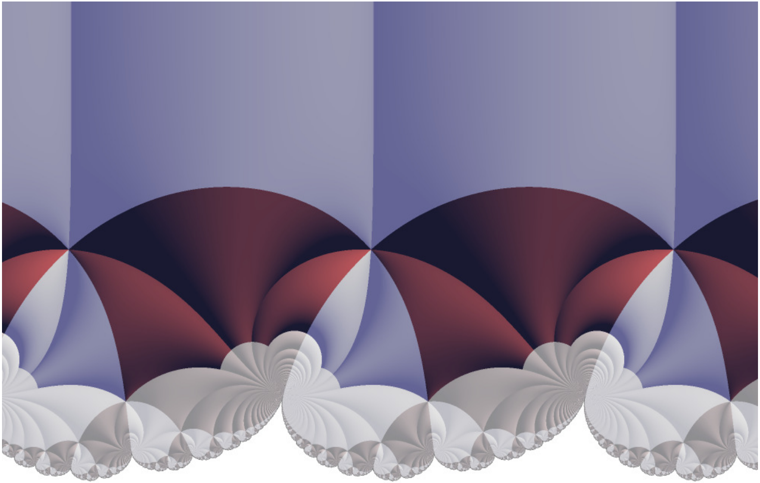
We view this as a ramified covering over $\mathbb{C}$, with a given "Covering structure", which is a mix of topological data (homotopy) and analytic data (moduli).

Since $\mathcal{R}$ maps $\mathcal{C}_1$ to a strict subset of $\mathcal{C}_1$, it is tempting to deduce from this a non-expansion statement, like in Schwarz's lemma; or even better, a strict contraction and the existence of a unique fixed point of $\mathcal{R}$ in $\mathcal{C}_1$.

However, ot is not obvious how to put a complex structure on the space of univalent maps.







The loosened invariant class

Fix f_0 in $\mathcal{C}_1$ and let

$$\mathcal{C}_1(V) = \left\{ f_0 \circ \phi^{-1} \;\middle|\; \begin{array}{l} \phi: \ V \to \mathbb{C} \text{ is a univalent analytic} \\ \text{map with } \phi(0) = 1, \ \phi'(0) = 1 \\ \text{and } \phi(V) \text{ is a quasidisk} \end{array} \right\},$$

Thus "$V' \subset V \implies \mathcal{C}_1(V) \subset \mathcal{C}_1(V')$".

Theorem (Inou , Shishikura): *There exists some $\varepsilon > 0$ such that: for the domain V corresponding to what was illustrated in the previous slide and for some domain $V' \subset\subset V$, one can still define a parabolic renormalization $\mathcal{R}$ (which agrees with the previously defined $\mathcal{R}$ at the level of germs) such that $\mathcal{R}(\mathcal{C}_1(V')) \subset \mathcal{C}_1(V)$.*

In particular $\mathcal{R}(\mathcal{C}_1(V')) \subset \mathcal{C}_1(V')$.

The benefits of leaving some flexibility are manifold:

- Contraction can be proved (c.f. Inou and Shishikura, using the Teichmüller distance between quasidisks).
- Perturbations can be done, easily.

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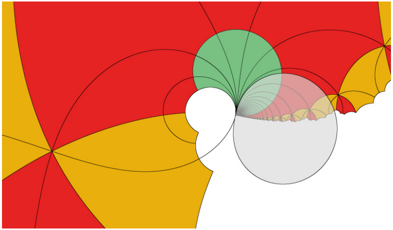
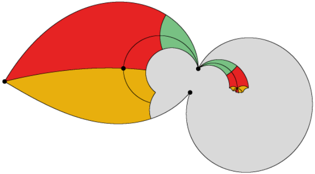
Shorthand: $\mathcal{C}_2 \stackrel{\text{def}}{=} \mathcal{C}_1(V')$.

Theorem (稲生, 宍倉): *If $f = e^{2i\pi\alpha} g$ with $g \in \mathcal{C}_2$ then one can define a (cylinder/near-parabolic) renormalization of f, $\mathcal{R}(f)$ which still belongs to $\mathcal{C}_2$ provided $\alpha \in]0, \varepsilon[$, and corresponds to (sort of) a return map.*

Since the set of univalent maps is compact, ε can be taken independent of g ($\varepsilon = 1/23$ seems to work, c.f. numerical experiments by Inou).





The renormalization picture

Note that if $$f = e^{2i\pi\alpha} g$$

with $\alpha \in]0, \varepsilon[$ and $g \in \mathcal{C}_2$ then $$\mathcal{R}(f) = e^{2i\pi\beta} h$$

with $h \in \mathcal{C}_2$ and $$\beta = \frac{-1}{\alpha} \bmod \mathbb{Z}.$$