Exploration of Douady's conjecture

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Douady's conjecture

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $p_n/q_n \longrightarrow \theta$ its continued fraction convergents. Let $\mathcal{B}$ be the set of θ such that $B(\theta) < +\infty$ where $B(\theta) = \sum \log(q_{n+1})/q_n$. Let f be a holomorphic map with $f(0) = 0$ and $f'(0) = e^{i2\pi\theta}$.

Theorem (Brjuno)

If $\theta \in \mathcal{B}$ then the fixed point is linearizable.

Theorem (Yoccoz)

If $\theta \notin \mathcal{B}$ and f is a degree 2 polynomial then the fixed point is not linearizable.

(it is also true for periodic points of degree 2 polynomials)

Conjecture (Douady)

It also holds for polynomials of higher degree.

Proof by the speaker of Yoccoz's thm

Let $Q_\theta(z) = e^{i2\pi\theta} z + z^2$.

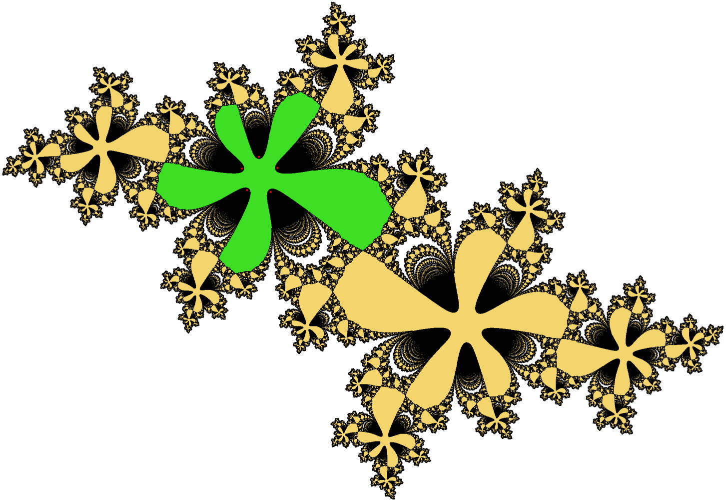
The Brjuno sum $B(\theta)$ is comparable to

$$\sum_n \frac{\log|\theta - p_n/q_n|}{q_n} = \log \prod_n |\theta - p_n/q_n|^{1/q_n}$$

The quantity $|\theta - p_n/q_n|^{1/q_n}$ controls the distance at which the parabolic cycle of Q_{p_n/q_n} explodes.

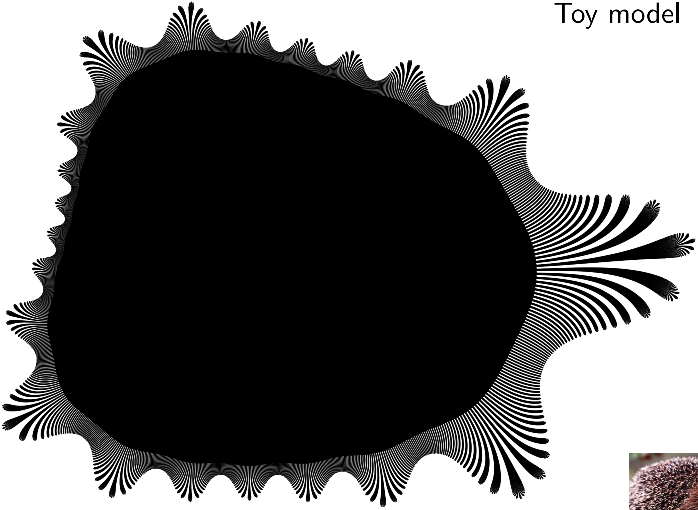
The product comes from the fact that these cycles can not collide.

[Explain on the board if there is one]



$\theta = [0; 2, 2, 2000, 1, 1, 1, \ldots] \approx 2/5$

Toy model



Degree 3

Setting

A degree 3 polynomial with a fixed point of multiplier ρ is affine-conjugated to

$$f_{\rho,a}(z) = \rho\,(z + az^2 + z^3)$$

In this family, the only affine conjugacies fixing the origin are

$$f_{\rho,a} \sim f_{\rho,-a}$$

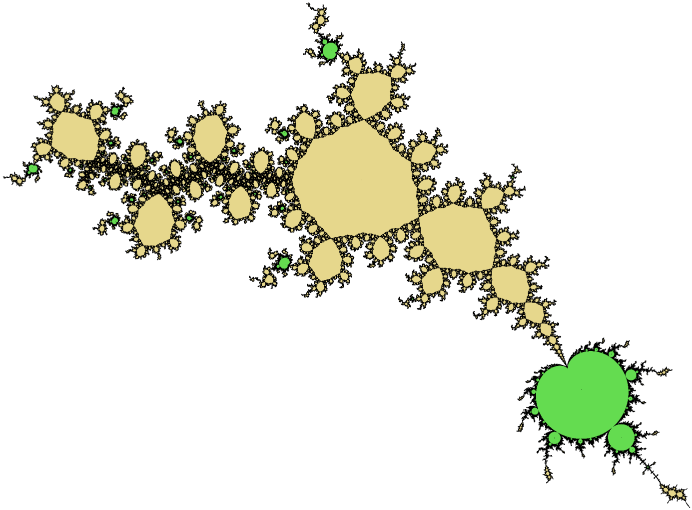
Slices

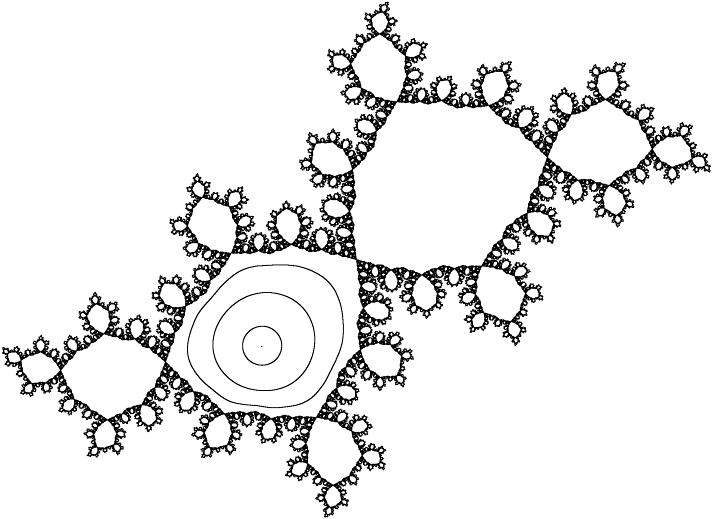
Fixing ρ, what does the bifurcation locus of the family $(f_{\rho,a})_{a\in\mathbb{C}}$ look like?

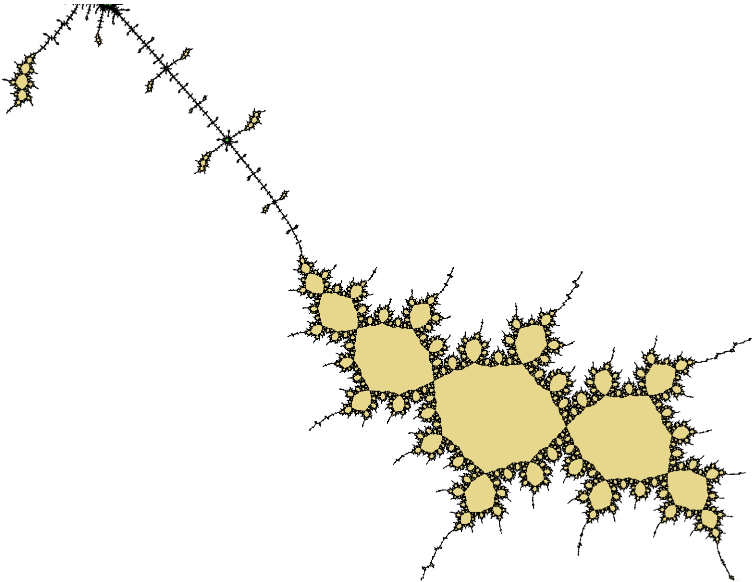
It has the symmetry $a \mapsto -a$ so we may fold it by mapping a to a^2. Let

$$b = a^2$$

The next slide shows the bifurcation locus for $\rho = e^{i2\pi\theta}$, with $\theta =$ the golden mean, viewed in the b-plane.







Another parameterization

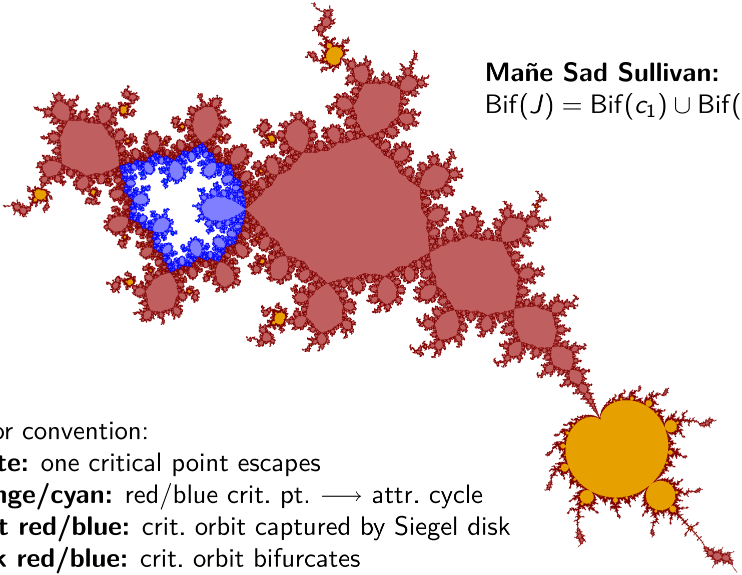
Allowing to label the critical points:

$$f_c(z) = \rho \left(z - \frac{1 + c^{-1}}{2} z^2 + \frac{1}{3c} z^3 \right)$$

Its critical points c_1, c_2 are:

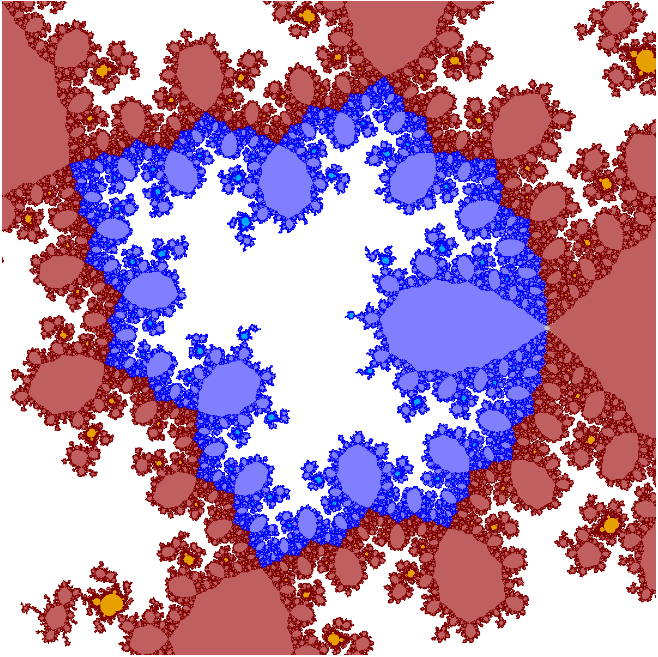
- $c_1 = 1$ (the blue critical point),
- $c_2 = c$ (the red critical point).

In the c-coordinate, the previous bifurcation locus looks like this:



Mañe Sad Sullivan:
$\mathrm{Bif}(J) = \mathrm{Bif}(c_1) \cup \mathrm{Bif}(c_2)$

Color convention:
white: one critical point escapes
orange/cyan: red/blue crit. pt. $\longrightarrow$ attr. cycle
light red/blue: crit. orbit captured by Siegel disk
dark red/blue: crit. orbit bifurcates



The boundary of the Siegel disk as a function of c

When $\theta \in \mathcal{B}$, the Siegel disk moves continuously for the Caratheodory topology.

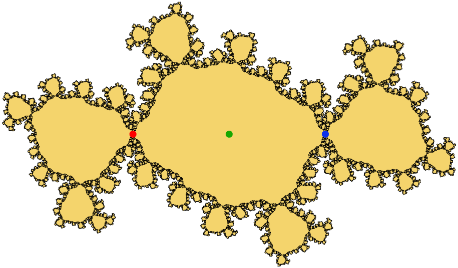
When θ has bounded type, we have $K(\theta)$-quasicircles with $K(\theta)$ independent of c (Shishikura).

[Show Java Applet Golden]

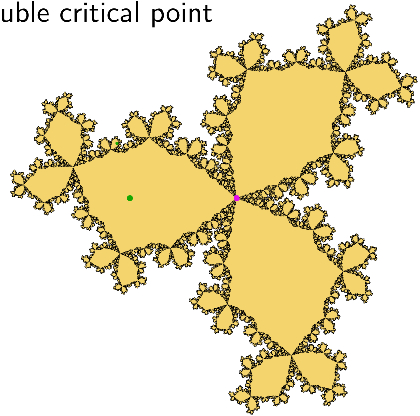
Theorem (Zakeri)

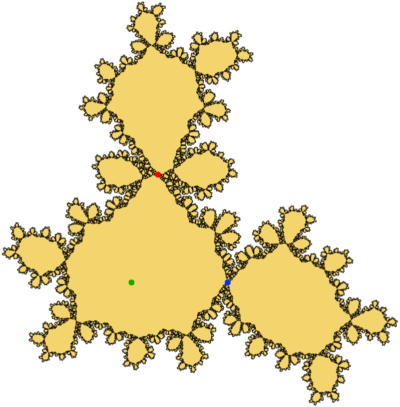
If θ has bounded type then the set of parameters c for which both critical points belong to the boundary of the Siegel disk, is a Jordan curve.

$c = -1,\ a = 0,\ b = 0$
order 2 symmetry around the indifferent fixed point

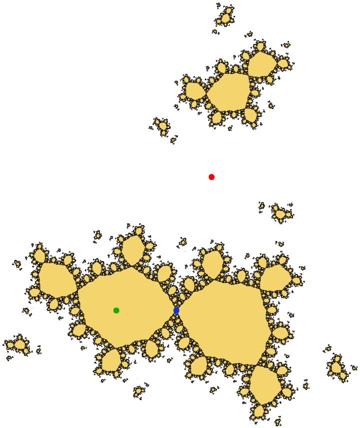


$c = 1,\ a = \pm\sqrt{3},\ b = 3$
order 3 symmetry around the double critical point



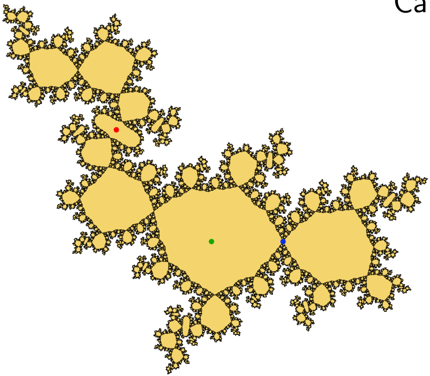


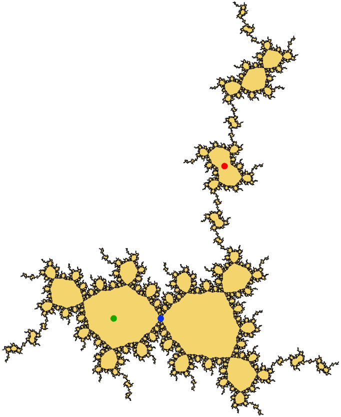
both critical points on the boundary

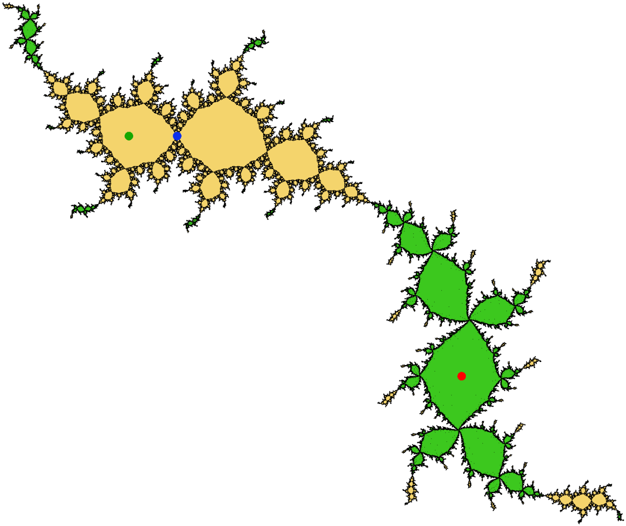


red critical point escapes

Captures







The size of Siegel disks

Brjuno's theorem has a more precise version by Yoccoz: If f is univalent in $B(0,r)$ then its Siegel disk contains $B(0, re^{-B(\theta)-C})$.

Yoccoz also proved that this lower bound is optimal, in that there exists for all θ an f univalent in $B(0,r)$ such that its Siegel disk does not contain $B(0, re^{-B(\theta)+C})$.

Yoccoz almost proved (this was finished by Buff and :) that this is the case for $f = Q_\theta : z \mapsto e^{i2\pi\theta} z + z^2$.

The function $B(\theta)$ is positive, highly discontinuous and takes arbitrarily high values (aside ∞).

Abnormally big Siegel disks

An example via semi-conjugacy

This optimality cannot hold anymore for higher degree polynomials:

Let $\rho = e^{i2\pi\theta}$. The polynomial $\rho(z+z^d)$ is semi-conjugated to $\rho^{d-1}u(1+u)^{d-1}$ by $u = z^{d-1}$.

Lukas Geyer generalized Yoccoz's theorem of non-linearizability to a class of polynomials called *saturated*: the number of infinite critical orbit tails within the Julia set is equal to the number of indifferent cycles

Buff and I generalized the upper bound on the inner size of the Siegel disk to a slightly smaller class: the number of infinite critical orbit tails is equal to the number of indifferent cycles.

This class contains $\rho^{d-1}u(1+u)^{d-1}$.

Hence for $\rho^{d-1}u(1+u)^{d-1}$, the inner radius of the Siegel disk is comparable to $B((d-1)\theta)$, its rotation number being $(d-1)\theta$.

Hence for $\rho(z+z^d)$, the inner radius of the Siegel disk is comparable to $\dfrac{B((d-1)\theta)}{d-1}$. But its rotation number is θ and not $(d-1)\theta$.

Arithmetical lemma: (C depends on d)

$$B(\theta) - C \leq B((d-1)\theta) \leq (d-1)B(\theta) + C$$

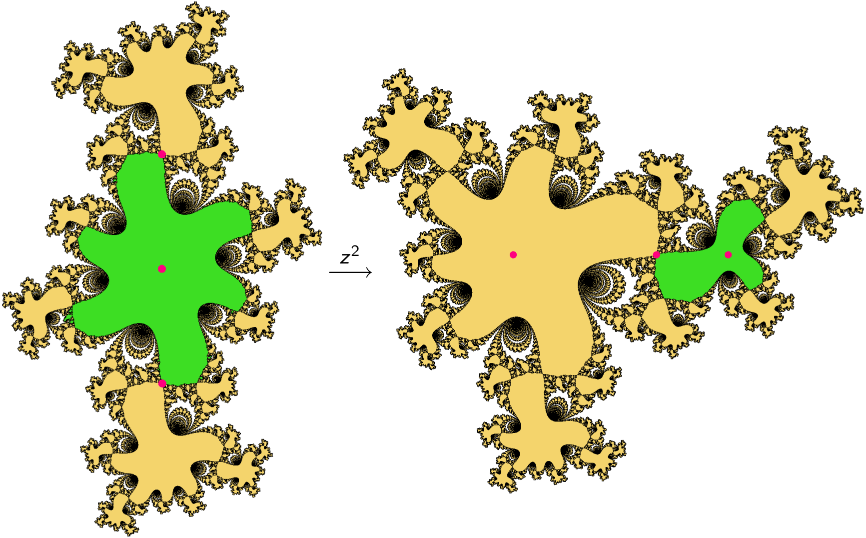
- There are values of θ for which $B(\theta)$ is big and for which $B((d-1)\theta)/B(\theta)$ is close to 1.
- There are values of θ for which $B(\theta)$ is big and for which $B((d-1)\theta)/B(\theta)$ is close to $d-1$.

The rational p/q is a good approximant of θ if and only if $(d-1)p/q$ is a good approximant of $(d-1)\theta$.

$$B(\theta) \approx \sum_n \frac{\ln|\theta - p_n/q_n|}{q_n}$$

$$B((d-1)\theta) \approx \sum_n \frac{\ln|\theta - p_n/q_n|}{q'_n} \text{ with } q'_n = \frac{q_n}{q_n \wedge (d-1)}$$

Roughly, $B((d-1)\theta)/B(\theta)$ is close to $d-1$ for those θ that have enough good approximants p/q for which $d-1|q$.



$\theta = \theta'/2 \approx 1/6$ ou $2/3$ $\qquad$ $\theta' = [0; 3, 100, 1, 1, 1, \ldots] \approx 1/3$

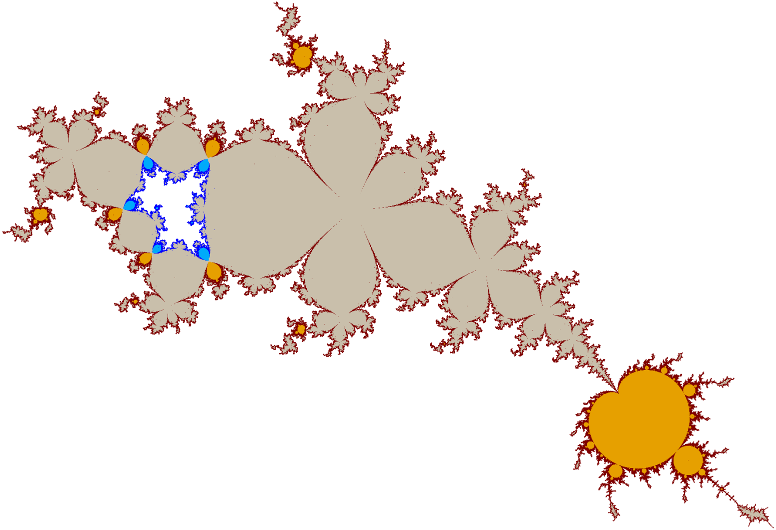
Explosion

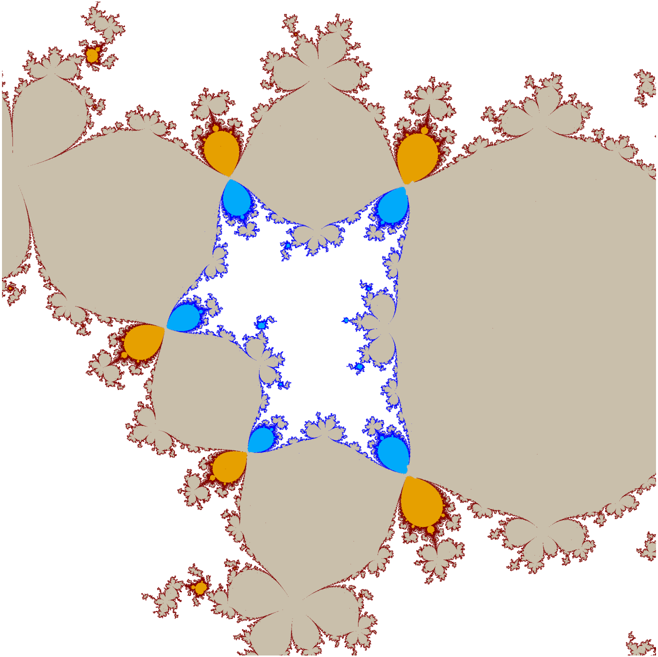
Where does the proof via explosion of parabolic points fail?

Fatou's theorem $\implies$ a parabolic point of a degree d polynomial can have up to $d-1$ cycles of petals.

When there is $k > 1$ cycles of petals, then the size of the exploded cycle is asymptotically controlled by $|\theta - p_n/q_n|^{1/kq_n} > |\theta - p_n/q_n|^{1/q_n}$. When does this happen?

Next slide: slice of cubic polynomials in the c-coordinate for $\rho = e^{i2\pi p/q}$ and $p/q = 3/5$.





When there is $k > 1$ cycles of petals, then the size of the exploded cycle is asymptotically controlled by $|\theta - p_n/q_n|^{1/kq_n} > |\theta - p_n/q_n|^{1/q_n}$.

The idea is that this is better than just asymptotical and that this is the worst case:

Conjecture (Buff)

There exists a constant $C = C(d)$ such that for all polynomial of f of degree d with an indifferent fixed point at the origin,

$$\log R(f) \leq -\frac{Y(\theta)}{d-1} + \log \min |c_i| + C$$

where the c_i are the critical points of f and θ is the rotation number at the origin.

This can be viewed as a refinement over Douady's conjecture.

More modest objectives

Understand the behaviour of perturbations of $\theta = p/q$ from a parameter c that is close to one of the roots of the lemniscates.

This should yield toy models of hedgehogs in higher degree.

[Show Java Applet Other]

From this, one should be able to guess the inner size of the Siegel disk for parameters on the Zakeri curve, as a function of the inner angle between the two critical points.

Cremer points

What happens for very liouvillian values of θ?

Theorem (Okuyama)

If $\limsup_n \dfrac{\log q_{n+1}}{d^{q_n}} > 0$ *and* f *is a rational map of degree* d *then it is not linearizable.*

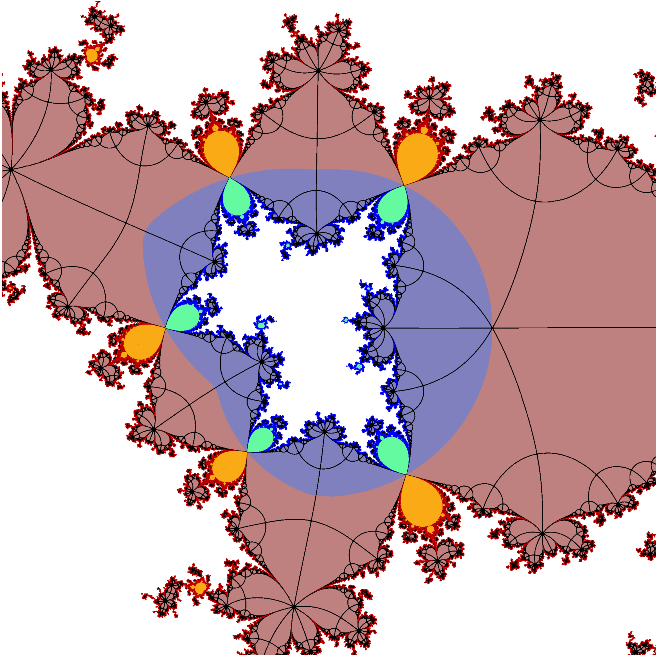
Is there an analog of the blue and red regions? Of the Zakeri arc? Is there also a hairy arc?

Natural idea: approach the parameter slice of $\rho = e^{i2\pi\theta}$ by slices of $\rho = e^{i2\pi\theta_n}$ where $\theta_n = p_n/q_n = [a_0; a_1, \ldots, a_n]$ or $\theta_n = [a_0; a_1, \ldots, a_n, 1, 1, 1, \ldots]$ and figure out how the arc changes from θ_n to θ_n, whether there is a limit, and what that limit means.

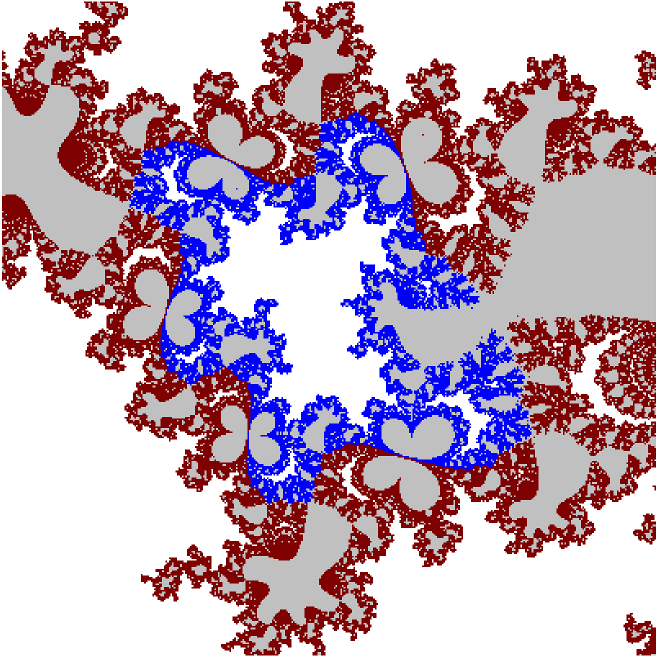
What is the analog of the arc for parabolic points?

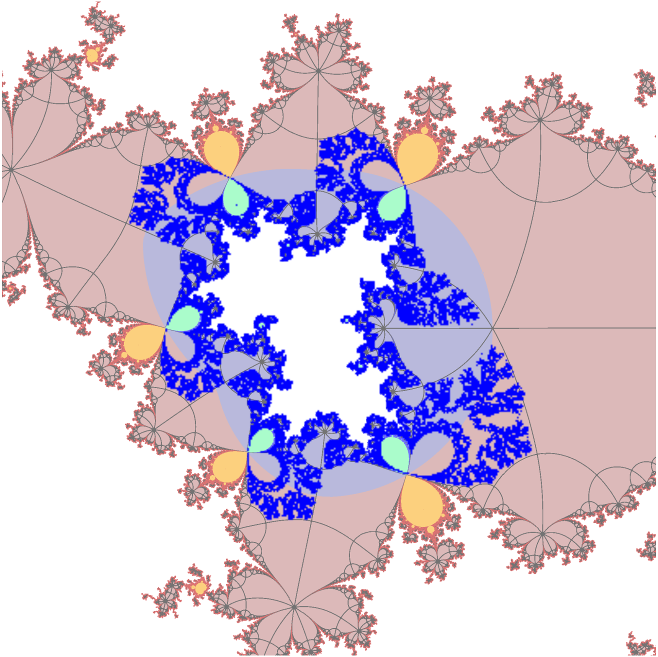
Introduce also the parabolic checkerboard (chessboard).

Picture on next slide.



We get something interesting if we turn $\theta = 3/5 = [0; 1, 1, 1, 1]$ into $\theta = [0; 1, 1, 1, 1, 100, 1, 1, \ldots]$.





Challenge

Explain why it follows so closely the checkerboard.

Renormalization

This hints at the persitence of Zakeri's Jordan curve even for liouvillian θ.

[use board if there is one]

The parabolic renormalization operator now seems to have a one dimensional repelling direction. So the near parabolic renormalization of Inou and Shishikura should have two repelling directions for cubic polynomials, instead of one for quadratic polynomials.

Potential theory approach

The conformal radius of the Siegel disk Δ is $\operatorname{conf}(f) = |\phi'(0)|$ where $\phi : (\mathbb{D}, 0) \to (\Delta, 0)$ is a conformal map.

$$1 \leq \frac{\text{conformal radius}}{\text{inner radius}} \leq 4$$

For a fixed Brjuno number θ, the function

$$V : b \mapsto -\log \operatorname{conf}(f_{\rho,b})$$

is subharmonic, continuous. Its laplacian $\mu = \Delta V$ is therefore a measure. The mass of μ is one. The function V is harmonic wherever the critical point accumulating the boundary is not bifurcating, for instance for b big.

Prove that the support of the measure is the Zakeri arc for bounded type θ.

[Show McShane's movie]

Brjuno numbers

Assume $\theta \in \mathcal{B}$.

Lemma: (Folk. + Avila) V is harmonic for c in some open set $U \iff$ the boundary of the Siegel disk undergoes a holomorphic motion when c varies in U.

Lemma: $\mathrm{supp}(\mu) \subset \mathrm{Bif}(c_1) \cap \mathrm{Bif}(c_2)$. In particular it has empty interior.

What about the converse: $\mathrm{Bif}(c_1) \cap \mathrm{Bif}(c_2) \subset \mathrm{supp}(\mu)$?

Mañe: for all $\theta \in \mathbb{R}$ and c, there is a *recurrent* critical point whose ω-limit set contains the boundary of the Siegel disk, the Cremer point, or the parabolic point.

Other Irrationals

Natural idea: approximate irrational θ by a sequence $\theta_n \longrightarrow \theta$ of bounded type numbers. By compactness of the set of measures, one can take weak limits, even for very Liouvillian θ. What do they mean? What are their properties?

For b big we have a quadratic-like map and the size of the Siegel disk is approximately $1/|b|$ times the size of the quadratic Siegel disk r: $V(b) = \log|b| - \log(r) + o(1)$. So the idea is to consider the limit of $V_{\theta_n} + \log r(\theta_m)$.

When θ is rational, the analog of the conformal radius is the asymptotic size L (it has been proved that they are related, by taking limits). For $\theta = p/q$, write $f^q_{\rho,a} = z + P(a)z^{q+1} + \dots$. Then P is a polynomial and

$$L(a) = \left|\frac{1}{qP(a)}\right|^{1/q}$$

so $-\log L(a) = \frac{1}{q}\log q + \frac{1}{q}\log|P(a)|$ is also harmonic, and its laplacian is a set of dirac masses at the roots of P.