Critical points on the boundary of Siegel disks

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Theorem (Arnol'd)

For all diophantine θ, all analytic diffeomorphism f of $\mathbb{R}/\mathbb{Z}$ with rotation number θ and close enough to the rotation $x \mapsto x + \theta$ are analytically linearizable.

The rotation numbers for which this holds have been subsequently characterized by Yoccoz and Perez-Marco : these are the Brjuno numbers.

Theorem (Herman)

For all dipohantine θ, the hypothesis "close enough to the rotation" is superfluous.

Yoccoz characterized the set $\mathcal{H}$ of rotation numbers for which all analytic diffeomorphisms are analytically linearizable. He called them *Herman numbers*.

Theorem (Ghys)

If P is a polynomial with a Siegel disk Δ with rotation number in $\mathcal{H}$ and if $\partial\Delta$ is a Jordan curve then $\partial\Delta$ contains a critical point.

Generalized as follows :

Theorem (Herman)

Let f be holomorphic on Ω. If it has a rotation domain U with rotation number in $\mathcal{H}$ and if U has a boundary component X compactly contained in Ω, then f is not injective in any neighborhood of X.

Note : f is injective in no neighborhood of $X \iff$ (f has a critical point on X) or (the restriction of f to X is non-injective).

Theorem (Herman)

If P is a unicritical polynomial $(z^d + v)$ and if Δ is a Siegel disk with rotation number in $\mathcal{H}$ then the critical point is on the boundary of Δ.

However it is not know whether the restriction of f to $\partial\Delta$ is injective or not.

Proof : Let $\widehat{\Delta}$ be the "filled-in" of $\overline{\Delta}$. Let c be the critical point and v the critical value. 3 cases :

1. $v \notin \widehat{\Delta}$ then f is injective on $\partial\Delta$: contradiction with $\theta \in \mathcal{H}$.
2. $v \in \widehat{\Delta} \setminus \partial\Delta$: contradiction with a theorem of Fatou : $\partial\Delta \subset \omega(c)$ + hidden components are Fatou components + classification of Fatou components/Sullivan
3. $v \in \partial\Delta$.

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Theorem (C, R)

If P is a polynomial with two critical points and if Δ is a Siegel disk with rotation number in $\mathcal{H}$ then at least one of the critical points is on the boundary of Δ.

Proof : Let $\widetilde{\Delta}$ be the component of $P^{-1}(\widehat{\Delta})$ containing $\widehat{\Delta}$. Let n_0 be the number of critical points in $\widetilde{\Delta}$. Let U be a simply connected neighborhood of $\widehat{\Delta}$ and denote V the connected component of $P^{-1}(\Delta)$ that contains $\widetilde{\Delta}$. If U is sufficiently small then $P : V \to U$ is a ramified cover with n_0 critical points. 3 cases :

1. $n_0 = 0$: then P is injective on $\partial\Delta$.
2. $n_0 = 2$: we conclude as in the unicritical case, but using Mañé's theorem : there exists a *recurrent* critical point c such that $\partial\Delta \subset \omega(c)$.
3. $n_0 = 1$: the new case.

Case $n_0 = 1$. Sketch of the proof :

- Assume by contradiction that there is no critical point on $\partial\Delta$.
- (A) Prove that $\widetilde{\Delta} = \widehat{\Delta}$, i.e. that $\widehat{\Delta}$ is locally totally invariant.
- Conjugate P by the conformal map from the complement of $\widehat{\Delta}$ to the complement of $\overline{\mathbb{D}}$ in $\mathbb{C}$.
- Obtain par Schwarz reflection a local diffeomorphism and analytic cover from S^1 to S^1 : ϕ.
- (B) Use another theorem of Mañé to show that ϕ is expanding.
- This means that P is a polynomial-like map in a neighborhood of $\widehat{\Delta}$.
- So we are in the already solved unicritical case, which yields a contradiction.

The bounded connected components of $\mathbb{C} \setminus \overline{\Delta}$ are Fatou components. These components are preperiodic. The Kiwi, Poirier, Goldberg-Milnor separation theorem implies that these components eventually fall in Δ.

The ramified cover $P : V \to U$ is equivalent to $z \mapsto z^d$ with $d > 1$.

Recall that we are assuming, by contradiction, that there is no critical point on $\partial\Delta$. Therefore $v \notin \partial\Delta$ (it has only one preimage in $\widetilde{\Delta}$ and at least one on $\partial\Delta$). Thus v is in a component hidden by $\overline{\Delta}$.
The forward iterates of this components are also hidden, and eventually fall in Δ. Just before, it is one of the components of $P^{(-1)}(\Delta)$: $\overline{\Delta}$ *hides a component of* $P^{(-1)}(\Delta)$.
By symmetry (the automorphism group of the cover) and a little bit of topology (if $d \geq 3$ and not a prime number), we conclude that all components of $V \cap P^{(-1)}(\Delta)$ hide each other (like Wada lakes).