Another sphere eversion
JHH 70th birthday conference in Bremen

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(Yet) Another sphere eversion

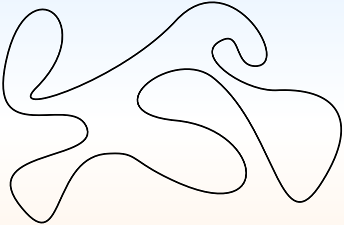
Movie

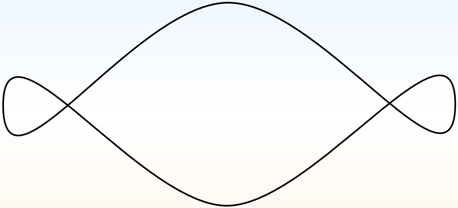
Link to a movie showing a sphere eversion.

Meshes computed by a C++ program by the author, and rendered by Jos Leys using PovRay.

Quick reminder

Embedding vs. immersion





This smooth closed loop is embedded in the plane

This smooth closed loop is immersed in the plane

Beginning of the story

In the late 50's, Smale proved a theorem that has the following consequence:

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~~Wait... This is obviously wrong: think of the degree of the Gauss map.~~
In fact, there is no contradiction: $\mathrm{id}|_{S^2}$ and $-\mathrm{id}|_{S^2}$ have the same Gauss map.

Explicit eversions

Smale's theorem proved the existence of a path but did not give an explicit one. Since, many people have described several ways to perform sphere eversions:

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- More on next page!

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- Ian Atchison, arxiv 2010, close to Shapiro's idea but simpler $\longrightarrow$ (partial) movie Holiverse

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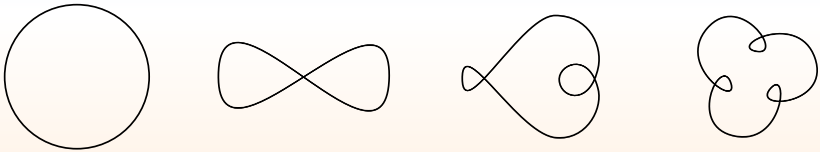
Immersed loops in the plane

Given an immersion $\gamma : S^1 \to \mathbb{R}^2$, denote by $W(\gamma)$ the winding number around 0 of the tangent vector as you follow the curve: $W(\gamma) \in \mathbb{Z}$.

Examples:



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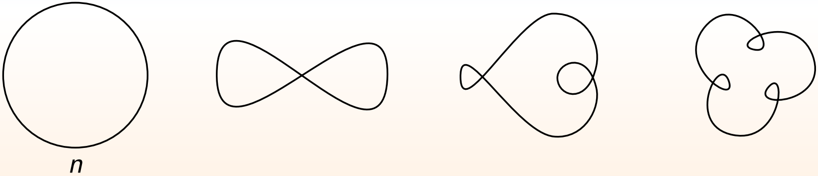

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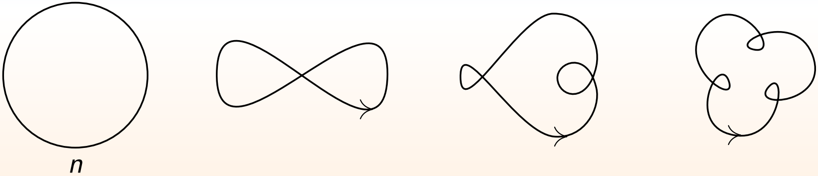

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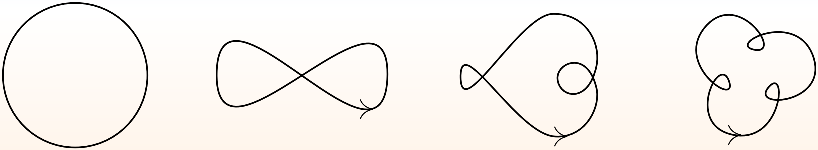

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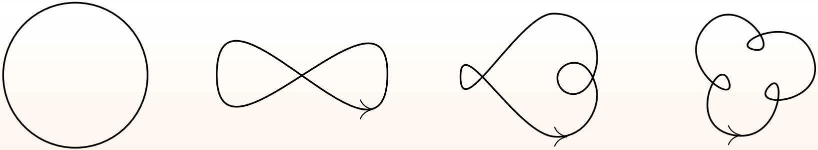

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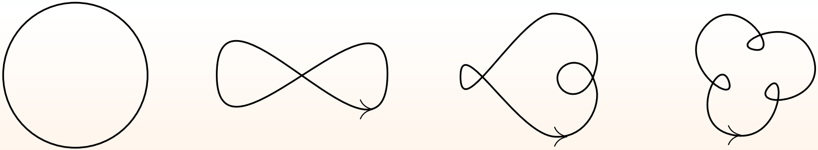

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Immersed loops in the plane

Whitney-Graustein theorem

Let $\mathcal{I}$ denote the set of immersions $S^1 \to \mathbb{R}^2$ and write $\gamma_1 \sim \gamma_2$ if there is a path from γ_1 to γ_2 within $\mathcal{I}$. This is called a *regular homotopy*.

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This is also the case for $n = 0$, but has been proved by someone else (Michor?).

Proof

of the Whitney-Graustein Theorem

Recall $W(\gamma_0) = W(\gamma_1) = n \neq 0$.

Identify $\mathbb{R}^2 \simeq \mathbb{C}$ and $S^1 \simeq [0,1]/(0 \sim 1)$.

For $\gamma \in \mathcal{I}_n$ decompose its derivative γ' in polar coordinates : $\gamma'(s) = r(s)e^{i\theta(s)}$ with $r, \theta : [0,1] \to \mathbb{R}$ continuous. Then

$$\begin{aligned} r(1) &= r(0), \\ \theta(1) &= \theta(0) + 2\pi n. \end{aligned}$$

Proof

of the Whitney-Graustein Theorem

We only explain the case when the speed of both curves is constant and equal to 1: $r(s) = 1$.

Let $\gamma_0'(s) = e^{i\theta_0(s)}$ and $\gamma_1'(s) = e^{i\theta_1(s)}$ and define

$$\begin{aligned}
\theta_t(s) &= (1-t)\theta_0(s) + t\theta_1(s), \\
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Then $\gamma_t'(0) = \gamma_t'(1)$.

Problem!

For $t \neq 0$ or 1, there is no reason for γ_t to be a *closed* loop: typically $\gamma_t(0) \neq \gamma_t(1)$.

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If we choose $\gamma_t'(s) = e^{i\theta_t(s)}$ then γ_t does not (necessarily) loop.

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One computes $a_t = \int_0^1 e^{i\theta_t(s)} ds$. So $|a_t| \leq 1$. Equality $|a_t| = 1$ occurs only when $\theta_t(s)$ is independent of s. If $n \neq 0$ this cannot happen so $|a_t| < 1$. Therefore $\gamma_t'(s) \neq 0$ hence the curve γ_t remains immersed for all t.

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In the case of non-constant speed, we can either adapt the formula with a non-constant $s \mapsto a_t(s)$ or reduce the problem to the case of constant speed.

Improvement

Recall: $\mathcal{I}_n$ is the set of immersions $\gamma : S^1 \to \mathbb{R}^2$ with $W(\gamma) = n$.

Let $\mathcal{I}'_n$ be the set of $\gamma \in \mathcal{I}_n$ such that $\arg \gamma'(0) = 0$.

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Theorem (whom?)

For all $n \neq 0$, $\mathcal{I}'_n$ is contractible (in a strong sense) and $\mathcal{I}_n$ deformation retracts to a subset homeomorphic S^1.*

*: The set of curves that follow the unit circle n times at constant speed. It is paremeterized by the starting point in S^1.

Proof: Fix any $\gamma^* \in \mathcal{I}_n$. The explicit Whitney-Graustein formula that interpolates between γ and γ^* depends continuously (smoothly!) on γ. $\square$

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Case $n = 0$: $\mathcal{I}'_0$ is also contractible; $\mathcal{I}_0$ does not retract on a circle; Kodama and Michor determined the homotopy groups of $\mathcal{I}_0$.

+1D

slicing

For any embedding or immersion $S^2 \to \mathbb{R}^3$ we may try to understand it by considering the slice by a horizontal plane and vary the height z of the plane. Generically we get a finite collection of immersed curves, that changes as z changes.

These curves will likely undergo bifurcations when the plane crosses points of the immersed surface where the tangent plane is horizontal.
There is at least two such points: for the max and min heights.

link to video showing an example

$+1D$
slicing

Let $\mathcal{I}_T$ denote the set of immersions $S^2 \to \mathbb{R}^3$ such that the tangent plane is horizontal only at two points. We call them *transverse* in this talk.

Then for all intermediate height, the intersection with a horizontal plane is a single immersed smooth curve, with $W = \pm 1$, that varies continuously with z. It can be parameterized as a continuous path in $\mathcal{I}_1$.

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For practical reasons, we work with the following variant: $\mathcal{I}_H$ is the set of immersions $S^2 \subset R^3 \to \mathbb{R}^3$ that preserve the height coordinate z.

Note : $\mathcal{I}_H \subset \mathcal{I}_T$. Up to a reparameterization, elements of $\mathcal{I}_H$ correspond to the maps in $\mathcal{I}_T$ for which the height function is Morse.

+1D
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Recall: $\mathcal{I}_T$ denote the set of immersions $S^2 \to \mathbb{R}^3$ such that the tangent plane is horizontal only at two points
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Proposition

The spaces $\mathcal{I}_H$ and $\mathcal{I}_T$ are connected.

Proof: (technicalities under the rug) for $\mathcal{I}_H$, apply a WG-contraction in $\mathcal{I}_n$ ($n = \pm 1$) with limit=the circle, all layers at the same time. This implies the result for $\mathcal{I}_T$. $\square$

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In other words, this proves that you can untie any transversally immersed S^2; moreover this gives an explicit way of doing it, easily programmable.

Remark

Orientation

To be noted : in this de-knotting process of transversally immersed spheres, the caps remain (nearly) unchanged. In particular the color (orientation) of the final sphere will be the same as the the one we see looking at the top cap from above.

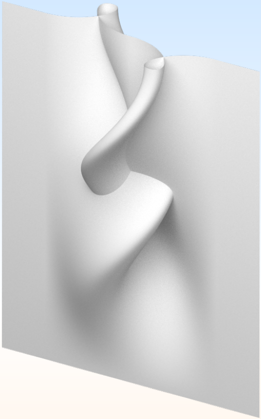
Base shape

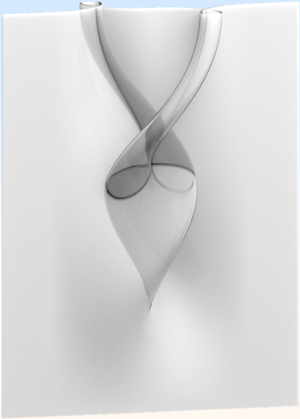
2D

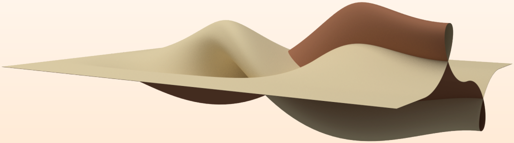
Link to video:

a segment deforms into an open curve with a pair of loops.

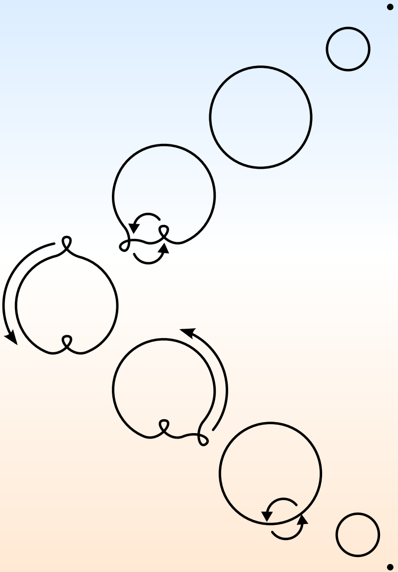
Next slide: the 3D immersed open surface defined by the movie above.







Key shapes



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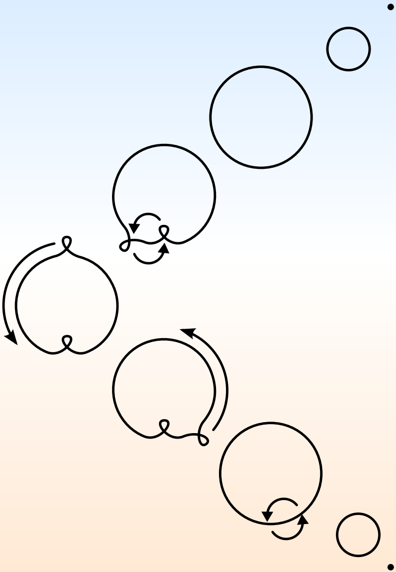

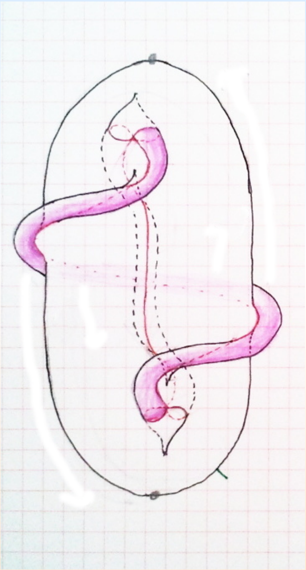

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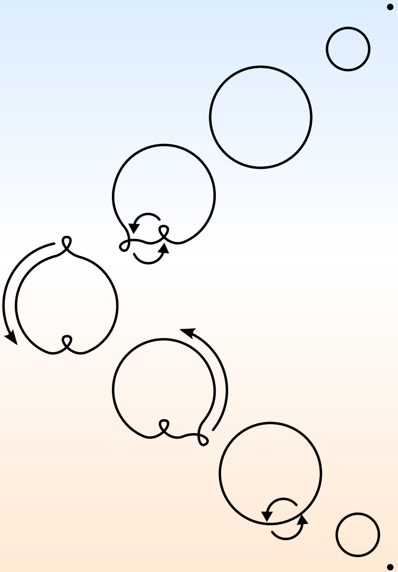

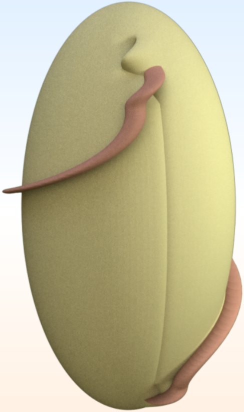

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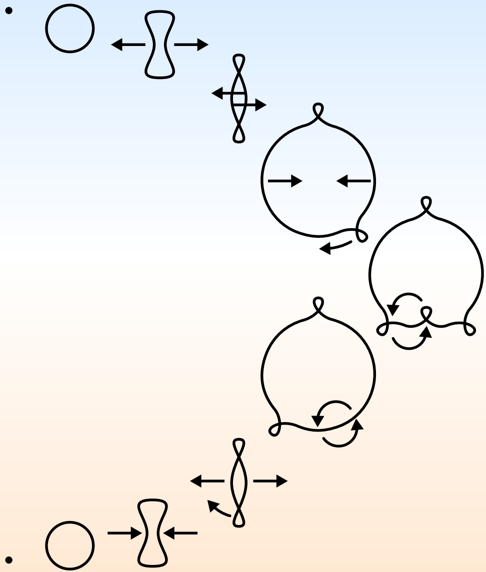

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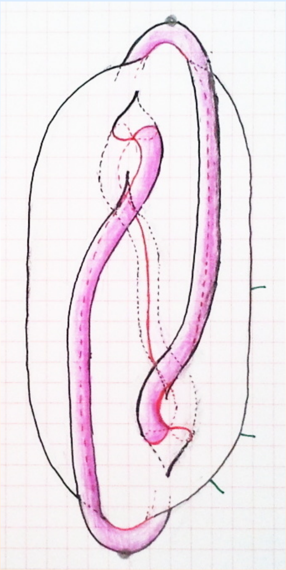
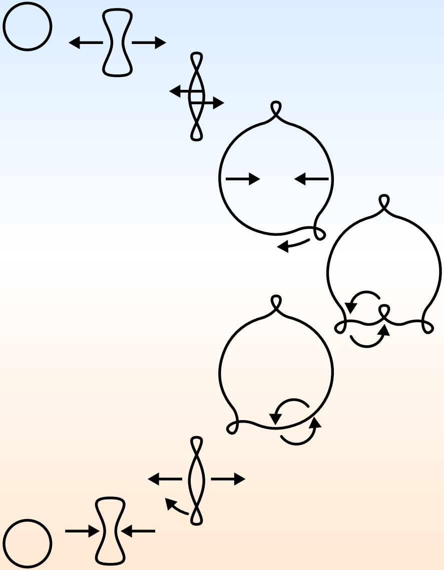


- 

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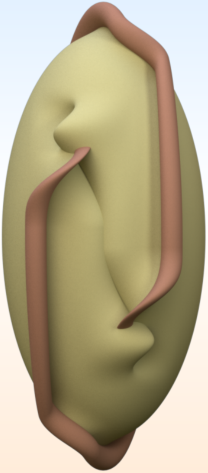
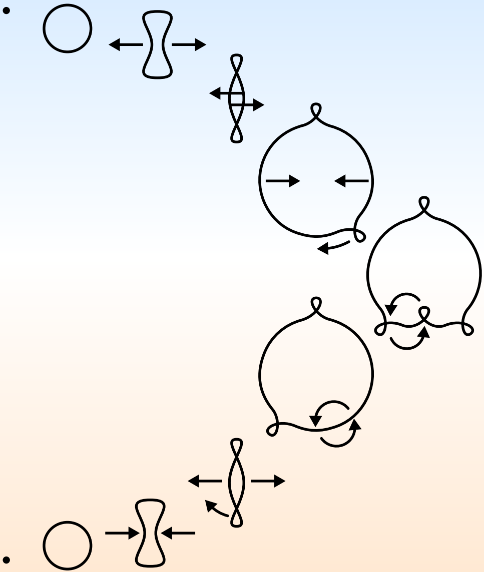
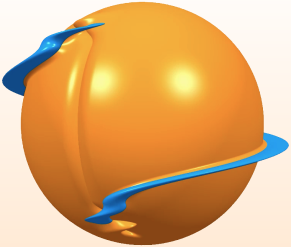
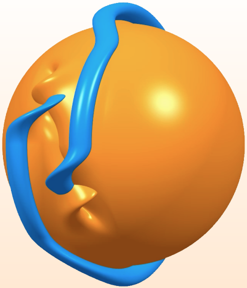
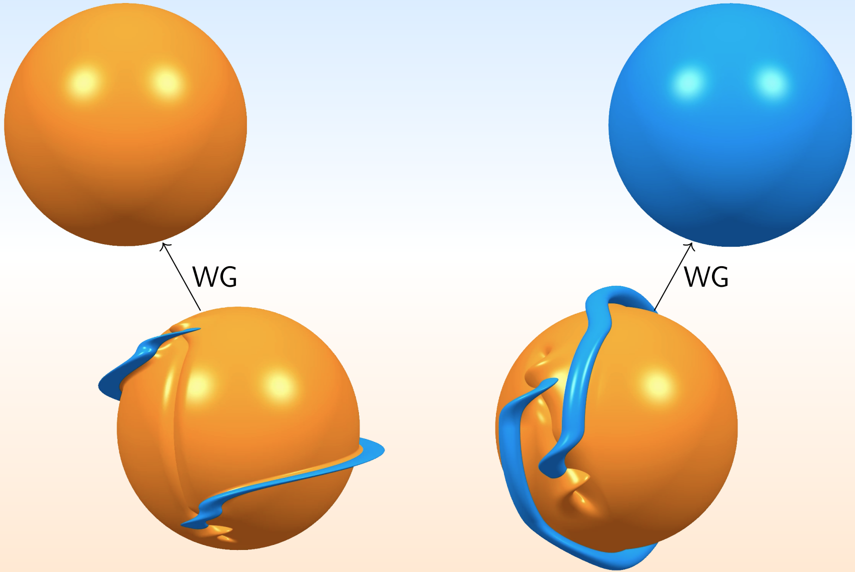
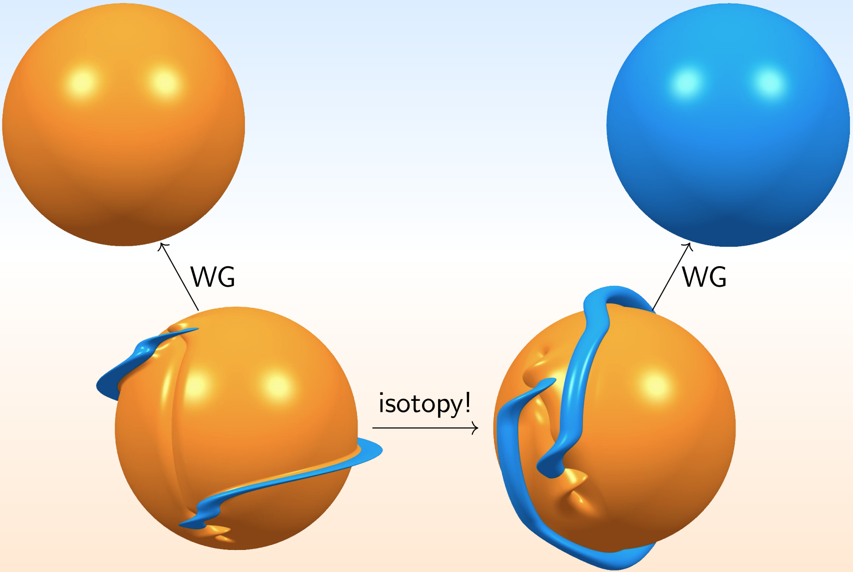








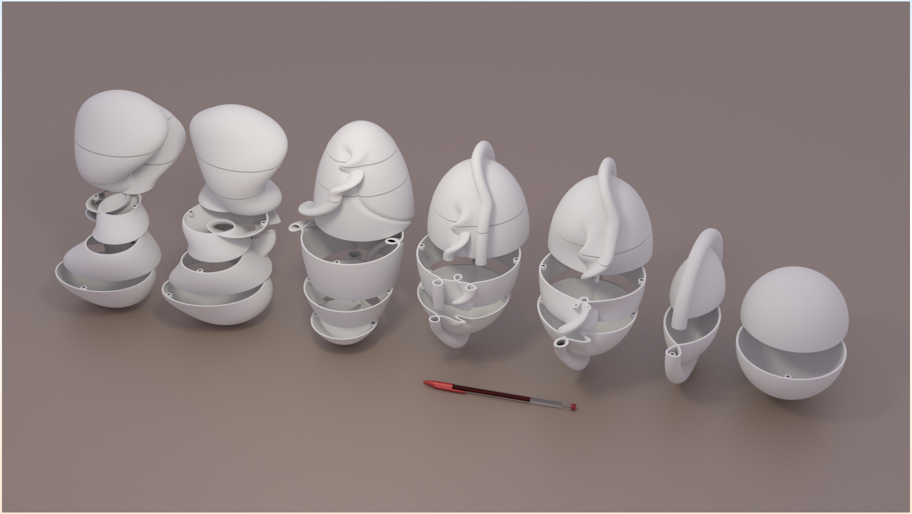

WG
WG


WG
WG
isotopy!

Movie again!

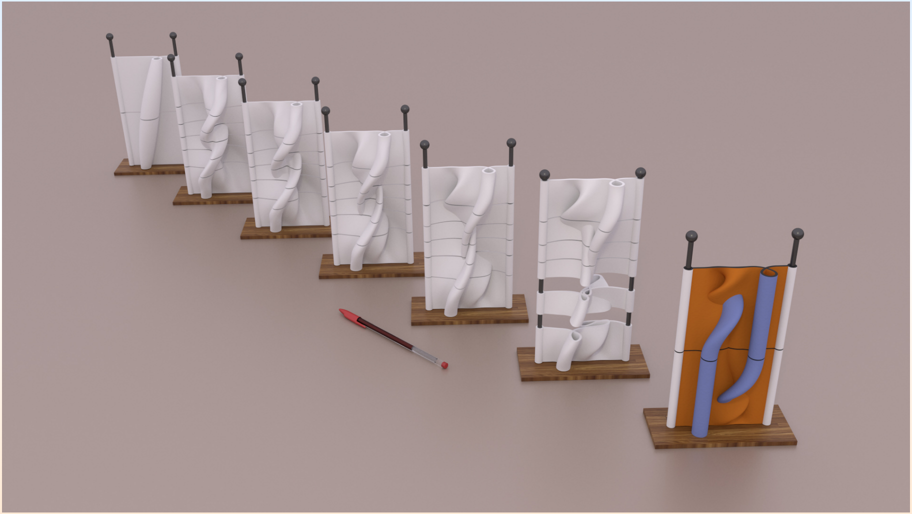
3D printing

The project



3D printing

The project



3D printing!

The objects

Designed by the author, purchased at Shapeways
by Insitut de Mathématiques de Toulouse.

3D printing!

The objects

Work in progress

3D printing!

The objects

3D printing!

The objects