TWISTED MATINGS AND EQUIPOTENTIAL GLUINGS

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Abstract. One crucial tool for studying postcritically finite rational maps is Thurston's topological characterization of rational maps. This theorem is proved by iterating a holomorphic endomorphism on a certain Teichmüller space. The graph of this endomorphism covers a correspondence on the level of moduli space. In favorable cases, this correspondence is the graph of a map, which can be used to study matings. We illustrate this by way of example: we study the mating of the basilica with itself.

Introduction

Our aim in this article is to present a worked example of a family of quadratic matings. The organization is as follows. Section 1 recalls the standard definitions and constructions concerning moduli spaces, Teichmüller spaces, and universal curves over them. We review the formulation of Thurston's Theorem, and describe the mating operation in that context. Following this general discussion, we consider the simplest nontrivial case in much greater detail. In Section 2 we study the self-mating of the unique normalized quadratic polynomial with period 2 critical point, the so-called *basilica* polynomial $z \mapsto z^2 - 1$, and we apply the Bartholdi-Nekrashevych recipe to obtain a skew-product dynamical system on the universal curve over moduli space. While this particular mating happens to be obstructed, equatorial twists of its iterates are realized. We discuss these twisted matings in Section 3, and organize them in terms of the skew-product. Finally, in Section 4 we consider the compactified universal curve, with attention to the dynamics at infinity. We close with a discussion of the equipotential gluing construction, whose implementation via the skew-product is fundamental to the algorithms with which Arnaud Chéritat produces his superb still and moving images.

1. Preliminaries

1.1. **Teichmüller theory.** Let S be an oriented topological 2-sphere and $\mathcal{P} \subset S$ be a finite set containing at least 3 points.

1.1.1. *The moduli space.* The moduli space $\mathcal{M}_{\mathcal{P}}$ is the set of equivalence classes of injective maps $\phi : \mathcal{P} \to \mathbb{P}^1$ under the relation whereby two maps $\phi_1 : \mathcal{P} \to \mathbb{P}^1$ and $\phi_2 : \mathcal{P} \to \mathbb{P}^1$ are equivalent when there is a Möbius transformation $M : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi_2 = M \circ \phi_1$.

Choose three distinct points p_1, p_2 and p_3 in $\mathcal{P}$. In each equivalence class, there is a unique representative $\phi : \mathcal{P} \to \mathbb{P}^1$ such that $\phi(p_1) = 0$, $\phi(p_2) = 1$ and

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$\phi(p_3) = \infty$. It follows that $\mathcal{M}_\mathcal{P}$ may be identified with $(\mathbb{C} - \{0,1\})^\mathcal{Q} - \mathcal{N}_\mathcal{Q}$, where $\mathcal{Q} := \mathcal{P} - \{p_1, p_2, p_3\}$ and $\mathcal{N}_\mathcal{Q} \subset \mathbb{C}^\mathcal{Q}$ consists of the non injective maps.

In this way, the set $\mathcal{M}_\mathcal{P}$ acquires the structure of a smooth irreducible quasiprojective variety of dimension $|\mathcal{P}| - 3$. The structure is easily seen not to depend on the choice of distinct points p_1, p_2 and p_3 in $\mathcal{P}$.

Example. If $|\mathcal{P}| = 4$, then $\mathcal{M}_\mathcal{P}$ is identified with $\mathbb{C} - \{0,1\}$. If $|\mathcal{P}| = 5$, then $\mathcal{M}_\mathcal{P}$ is identified with $\mathbb{C}^2$ minus the five lines defined by the equations $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $x = y$.

1.1.2. *The Teichmüller space.* The Teichmüller space $\mathcal{T}_\mathcal{P}$ is the set of equivalence classes of orientation-preserving homeomorphisms $\phi : S \to \mathbb{P}^1$ under the relation whereby homeomorphisms $\phi_1 : S \to \mathbb{P}^1$ and $\phi_2 : S \to \mathbb{P}^1$ are equivalent when there is a Möbius transformation $M : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\phi_2|_\mathcal{P} = M \circ \phi_1|_\mathcal{P} \quad \text{and} \quad \phi_2 \text{ is isotopic to } M \circ \phi_1 \text{ relative to } \mathcal{P}.$$

The Teichmüller space $\mathcal{T}_\mathcal{P}$ is canonically equipped with the structure of a $\mathbb{C}$-analytic manifold of dimension $|\mathcal{P}| - 3$ so that the restriction $\phi \mapsto \phi|_\mathcal{P}$ induces a universal covering map $\pi : \mathcal{T}_\mathcal{P} \to \mathcal{M}_\mathcal{P}$.

1.1.3. *The universal curve over the moduli space.* There are

- a $\mathbb{C}$-analytic manifold $\mathfrak{M}_\mathcal{P}$,
- an analytic submersion $\mathfrak{m} : \mathfrak{M}_\mathcal{P} \to \mathcal{M}_\mathcal{P}$ whose fibers are Riemann spheres,
- analytic sections $(m_p : \mathcal{M}_\mathcal{P} \to \mathfrak{M}_\mathcal{P})_{p \in \mathcal{P}}$ with pairwise disjoint images,

so that for each $x \in \mathcal{M}_\mathcal{P}$, the injection

$$\mathcal{P} \ni p \mapsto m_p(x) \in \mathfrak{m}^{-1}(x)$$

represents x. The map $\mathfrak{m} : \mathfrak{M}_\mathcal{P} \to \mathcal{M}_\mathcal{P}$ and sections $(m_p : \mathcal{M}_\mathcal{P} \to \mathfrak{M}_\mathcal{P})_{p \in \mathcal{P}}$ are unique up to isomorphism. They are called a universal curve with universal sections over the moduli space. The word universal refers to the following property which in fact gives a characterization up to unique isomorphism: given a proper flat family of curves $\mathfrak{s} : \mathfrak{U}_{P,Q} \to \Delta$ (see [HK] for a definition), with distinct sections $s_p : \Delta \to \mathfrak{U}_{P,Q}$ there exists a unique map $\beta : \Delta \to \mathcal{M}_\mathcal{P}$, such that the pullback by this map is canonically isomorphic to the given family. In particular, there is an analytic map $\alpha : \mathfrak{U}_{P,Q} \to \mathfrak{M}_\mathcal{P}$ such that $\mathfrak{m} \circ \alpha = \beta \circ s$, the restriction $\alpha : \mathfrak{s}^{-1}(x) \to \mathfrak{m}^{-1}\big(\beta(x)\big)$ is an isomorphism for every $x \in \Delta$ and $m_p \circ \beta = \alpha \circ s_p$ for every $p \in \mathcal{P}$.

If $\mathcal{M}_\mathcal{P}$ is identified with $(\mathbb{C} - \{0,1\})^\mathcal{Q} - \mathcal{N}_\mathcal{Q}$ via a choice of three distinct points p_1, p_2 and p_3 in $\mathcal{P}$ as above, one may set $\mathfrak{M}_\mathcal{P} := \mathcal{M}_\mathcal{P} \times \mathbb{P}^1$, let $\mathfrak{m} : \mathfrak{M}_\mathcal{P} \to \mathcal{M}_\mathcal{P}$ be the projection to the first coordinate and consider the sections $m_p : \mathcal{M}_\mathcal{P} \to \mathfrak{M}_\mathcal{P}$ defined at $\phi \in \mathcal{M}_\mathcal{P}$ by

$$m_{p_1}(\phi) = (\phi, 0), \quad m_{p_2}(\phi) = (\phi, 1), \quad m_{p_3}(\phi) = (\phi, \infty)$$

and

$$m_q(\phi) = \big(\phi, \phi(q)\big) \quad \text{for} \quad q \in \mathcal{Q}.$$

Example. If $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ and $\mathcal{M}_\mathcal{P}$ is identified to $\mathbb{C} - \{0,1\}$, the universal curve over moduli space is given by the projection

$$\mathfrak{m} : (\mathbb{C} - \{0,1\}) \times \mathbb{P}^1 \ni (x, z) \mapsto x \in \mathbb{C} - \{0,1\}$$

with sections

$$m_{p_1}(x) = (x, 0), \quad m_{p_2}(x) = (x, 1), \quad m_{p_3}(x) = (x, \infty) \quad \text{and} \quad m_{p_4}(x) = (x, x).$$

1.1.4. *The universal curve over the Teichmüller space.* There is also a universal curve $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \to \mathcal{T}_{\mathcal{P}}$ over the Teichmüller space. This is obtained by considering the pullback of $\pi : \mathcal{T}_{\mathcal{P}} \to \mathcal{M}_{\mathcal{P}}$ along $\mathfrak{m} : \mathfrak{M}_{\mathcal{P}} \to \mathcal{M}_{\mathcal{P}}$, that is

$$\mathfrak{T}_{\mathcal{P}} := \big\{(\tau, \mu) \in \mathcal{T}_{\mathcal{P}} \times \mathfrak{M}_{\mathcal{P}} \mid \pi(\tau) = \mathfrak{m}(\mu)\big\},$$

$\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \to \mathcal{T}_{\mathcal{P}}$ is the projection to the first coordinate and $\varpi : \mathfrak{T}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ is the projection to the second coordinate:

$$\begin{tikzcd}
\mathfrak{T}_{\mathcal{P}} \arrow[r, "\mathfrak{t}"] \arrow[d, "\varpi"'] & \mathcal{T}_{\mathcal{P}} \arrow[d, "\pi"] \\
\mathfrak{M}_{\mathcal{P}} \arrow[r, "\mathfrak{m}"] & \mathcal{M}_{\mathcal{P}}.
\end{tikzcd}$$

The map $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}} \to \mathcal{T}_{\mathcal{P}}$ is a submersion with fiber over $\tau \in \mathcal{T}_{\mathcal{P}}$ isomorphic to $\mathbb{P}^1$. For each $p \in \mathcal{P}$, there is a section $t_p : \mathcal{T}_{\mathcal{P}} \to \mathfrak{T}_{\mathcal{P}}$ given by

$$t_p(\tau) = \big(\tau, m_p \circ \pi(\tau)\big).$$

The injection $\mathcal{P} \ni p \mapsto t_p(\tau) \in \mathfrak{t}^{-1}(\tau)$ represents $\pi(\tau)$ in $\mathcal{M}_{\mathcal{P}}$.

1.2. **Thurston's theory.** We remain in the setting where S is an oriented topological 2-sphere, and we let $f : S \to S$ be an orientation-preserving ramified self-covering map.

1.2.1. *Combinatorial equivalence.* The map f is a *Thurston map* if it is postcritically finite, that is, if all the points in the critical set Ω_f have finite forward orbits, or equivalently if the *postcritical set*

$$\mathcal{P}_f := \bigcup_{n>0} f^{\circ n}(\Omega_f)$$

is finite. Two Thurston maps $f_1 : S_1 \to S_1$ and $f_2 : S_2 \to S_2$ are *combinatorially equivalent* if there are orientation-preserving homeomorphisms $\phi, \psi : S_1 \to S_2$ so that:

- the following diagram commutes,

$$\begin{tikzcd}
(S_1, \mathcal{P}_{f_1}) \arrow[r, "\psi"] \arrow[d, "f_1"'] & (S_2, \mathcal{P}_{f_2}) \arrow[d, "f_2"] \\
(S_1, \mathcal{P}_{f_1}) \arrow[r, "\phi"] & (S_2, \mathcal{P}_{f_2})
\end{tikzcd}$$

- ϕ and ψ agree on $\mathcal{P}_{f_1}$,
- $\phi(\mathcal{P}_{f_1}) = \psi(\mathcal{P}_{f_1}) = \mathcal{P}_{f_2}$ and
- ϕ is isotopic to ψ relative to $\mathcal{P}_{f_1}$.

1.2.2. *Thurston's Theorem.* Thurston's topological characterization of rational maps provides a purely topological criterion under which a Thurston map $f : S \to S$ is combinatorially equivalent to a critically finite rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$. If f is not combinatorially equivalent to a rational map, then f is *obstructed.* Thurston's criterion is formulated in terms of curve systems on $S - \mathcal{P}_f$ which are "invariant" under pullback by f.

Theorem 1 (Thurston)**.** *A Thurston map f with hyperbolic orbifold*[1] *is combinatorially equivalent to a rational map if and only if there are no* obstructing multicurves*. In that case, the rational map is unique up to conjugation by a Möbius transformation.*

A *multicurve* $\Gamma := \{\gamma_1, \gamma_2, \dots, \gamma_n\}$, is a set of simple, closed, disjoint, nonisotopic, nonperipheral curves in $S - \mathcal{P}_f$. A multicurve is *f-stable* if for every $\gamma \in \Gamma$, every nonperipheral component of $f^{-1}(\gamma)$ is isotopic in $S - \mathcal{P}_f$ to a curve in Γ.

An obstructing multicurve is a special kind of f-stable multicurve; we will restrict our attention to obstructing multicurves which are *Levy cycles*. Let Γ be an f-stable multicurve, and let $\Lambda := \{\gamma_0, \dots, \gamma_k = \gamma_0\} \subseteq \Gamma$ be such that for each $i = 0, \dots, k-1$, γ_i is isotopic relative to $\mathcal{P}_f$ to at least one component γ' of $f^{-1}(\gamma_{i+1})$ and $f : \gamma' \to \gamma_{i+1}$ has degree 1. Then Λ is called a Levy cycle. A Thurston map with a Levy cycle is necessarily obstructed, see chapter 9 of [H2].

1.2.3. *The Thurston endomorphism.* A Thurston map $f : S \to S$ with $|\mathcal{P}_f| \geq 3$ induces a holomorphic endomorphism $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ as follows. Choose three distinct points p_1, p_2 and p_3 in $\mathcal{P}_f$. Let $\tau \in \mathcal{T}_{\mathcal{P}_f}$ be represented by a homeomorphism $\phi : S \to \mathbb{P}^1$ sending p_1 to 0, p_2 to 1 and p_3 to ∞. There is a homeomorphism $\psi : S \to \mathbb{P}^1$ sending p_1 to 0, p_2 to 1 and p_3 to ∞ and a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{tikzcd} (S, \mathcal{P}_f) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(\mathcal{P}_f)) \arrow[d, "F"] \\ (S, \mathcal{P}_f) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(\mathcal{P}_f)). \end{tikzcd} \tag{1}$$

The homeomorphism ψ and rational map F are uniquely determined by ϕ. Moreover, since $\mathcal{P}_f$ contains the critical value set of f, if $\phi_1 : S \to \mathbb{P}^1$ is isotopic to $\phi_2 : S \to \mathbb{P}^1$ relative to $\mathcal{P}_f$, then ψ_1 is isotopic to ψ_2 relative to $f^{-1}(\mathcal{P}_f) \supset \mathcal{P}_f$ and $F_1 = F_2$. As a consequence,

- the class of ψ in $\mathcal{T}_{\mathcal{P}_f}$ is uniquely determined by $\tau \in \mathcal{T}_{\mathcal{P}_f}$ (it does not even depend on the choice of points p_1, p_2, p_3) ; we denote by $\sigma_f(\tau)$ the class of ψ in $\mathcal{T}_{\mathcal{P}_f}$;
- the rational map $F_\tau := F$ only depends on τ and the choice of points p_1, p_2, p_3.

The map

$$\mathcal{T}_{\mathcal{P}_f} \ni \tau \mapsto \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$$

is called the *Thurston endomorphism* associated to f. This map is holomorphic on $\mathcal{T}_{\mathcal{P}_f}$, and weakly-contracting for the Teichmüller metric (see [DH] for details). In

[1]For more information about the orbifold associated to a Thurston map, see [DH].

the context of Theorem 1, a Thurston map $f : S \to S$ is combinatorially equivalent to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ if and only if σ_f has a fixed point in $\mathcal{T}_{\mathcal{P}_f}$.

1.2.4. *A map between curves over Teichmüller space.* Recall that $\mathfrak{t} : \mathfrak{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ is the universal curve over Teichmüller space. According to the previous discussion, the construction of the Thurston endomorphism $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ comes with a family of rational map $F_\tau : \mathbb{P}^1 \to \mathbb{P}^1$. The domain of this rational map is marked with $\psi : S \to \mathbb{P}^1$ and the range is marked with $\phi : S \to \mathbb{P}^1$. So, $F_\tau : \mathbb{P}^1 \to \mathbb{P}^1$ yields a holomorphic map

$$\mathfrak{F}_\tau : \mathfrak{S}_\upsilon \to \mathfrak{S}_\tau \quad \text{with} \quad \upsilon := \sigma_f(\tau), \quad \mathfrak{S}_\tau := \mathfrak{t}^{-1}(\tau) \quad \text{and} \quad \mathfrak{S}_\upsilon := \mathfrak{t}^{-1}(\upsilon).$$

Such a family does not necessarily arise from an endomorphism of $\mathfrak{T}_{\mathcal{P}_f}$ because $\mathfrak{S}_\tau$ is the range, not the domain: if $\upsilon = \sigma_f(\tau_1) = \sigma_f(\tau_2)$ with $\tau_1 \neq \tau_2$, there are distinct maps $\mathfrak{F}_{\tau_1} : \mathfrak{S}_\upsilon \to \mathfrak{S}_{\tau_1}$ and $\mathfrak{F}_{\tau_2} : \mathfrak{S}_\upsilon \to \mathfrak{S}_{\tau_2}$. However there is a map $\mathfrak{F} : \mathfrak{U}_{\mathcal{P}_f} \to \mathfrak{T}_{\mathcal{P}_f}$ where

$$\begin{aligned} \mathfrak{U}_{\mathcal{P}_f} &:= \big\{(\tau, \xi) \in \mathcal{T}_{\mathcal{P}_f} \times \mathfrak{T}_{\mathcal{P}_f} \mid \mathfrak{t}(\xi) = \sigma_f(\tau)\big\} \\ &\approx \big\{(\tau, \upsilon, \mu) \in \mathcal{T}_{\mathcal{P}_f} \times \mathcal{T}_{\mathcal{P}_f} \times \mathfrak{M}_{\mathcal{P}_f} \mid \upsilon = \sigma_f(\tau) \text{ and } \mathfrak{m}(\mu) = \pi(\upsilon)\big\} \end{aligned}$$

and $\mathfrak{F}(\tau, \xi) := \mathfrak{F}_\tau(\xi)$.

By construction, we have the following commutative diagram.

$$\begin{tikzcd} \mathfrak{M}_{\mathcal{P}_f} \arrow[d, "\mathfrak{m}"'] & \mathfrak{T}_{\mathcal{P}_f} \arrow[l, "\varpi"'] \arrow[d, "\mathfrak{t}"] & \mathfrak{U}_{\mathcal{P}_f} \arrow[l, "\mathfrak{F}"'] \arrow[r, "\mathfrak{p}_2"] \arrow[d, "\mathfrak{p}_1"] & \mathfrak{T}_{\mathcal{P}_f} \arrow[r, "\varpi"] \arrow[d, "\mathfrak{t}"] & \mathfrak{M}_{\mathcal{P}_f} \arrow[d, "\mathfrak{m}"] \\ \mathcal{M}_{\mathcal{P}_f} & \mathcal{T}_{\mathcal{P}_f} \arrow[l, "\pi"'] & \mathcal{T}_{\mathcal{P}_f} \arrow[l, "\text{id}"'] \arrow[r, "\sigma_f"] & \mathcal{T}_{\mathcal{P}_f} \arrow[r, "\pi"] & \mathcal{M}_{\mathcal{P}_f} \end{tikzcd} \tag{2}$$

where $\mathfrak{p}_1 : \mathfrak{U}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ is the projection $(\tau, \xi) \mapsto \tau$ and $\mathfrak{p}_2 : \mathfrak{U}_{\mathcal{P}_f} \to \mathfrak{T}_{\mathcal{P}_f}$ is the projection $(\tau, \xi) \mapsto \xi$.

In Section 2, we shall further study this diagram in the context of the mating of the basilica polynomial $z \mapsto z^2 - 1$ with itself.

1.3. **Matings.**

1.3.1. *Polynomials.* All polynomials $P : \mathbb{C} \to \mathbb{C}$ considered in this article will be monic of degree $d \geq 2$. They will be postcritically finite and hyperbolic, that is, the orbit of any critical point eventually lands on a superattracting cycle.

The filled-in Julia set of P is the set

$$K(P) := \big\{z \in \mathbb{C} \mid n \mapsto P^{\circ n}(z) \text{ is bounded}\big\}.$$

The Julia set $J(P)$ is the boundary of $K(P)$.

When P is postcritically finite, $K(P)$ and $J(P)$ are connected.[2] In this situation, the complement of $K(P)$ is isomorphic to $\mathbb{C} - \overline{\mathbb{D}}$, and there is an isomorphism $\mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K(P)$ conjugating $z \mapsto z^d$ to P. Since P is monic, there is a unique such *Böttcher coordinate* $\text{böt} : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K(P)$ tangent to the identity at infinity.

If $\theta \in \mathbb{R}/\mathbb{Z}$, the external ray $\mathcal{R}_P(\theta)$ of angle θ is the set of points of the form $\text{böt}(\rho e^{2i\pi\theta})$ with $\rho > 1$. The polynomial P sends the external ray of angle θ to the external ray of angle $d \cdot \theta$.

[2]The filled-in Julia set of a polynomial is connected if and only if all the critical points have bounded orbits.

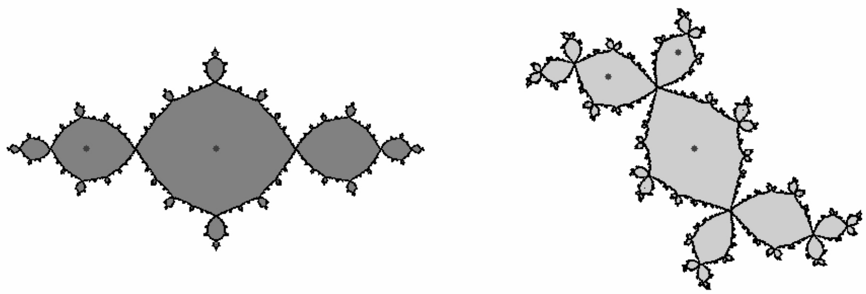

FIGURE 1. Left: the filled-in Julia set of the *basilica* polynomial $z \mapsto z^2 - 1$. The critical point is periodic of period 2. Right: the filled-in Julia set of the *rabbit* polynomial $z \mapsto z^2 + c_0$, where c_0 is chosen so that $\mathrm{Im}(c_0) > 0$ and the critical point is periodic of period 3. The forward orbit of the critical point is marked in both pictures.

1.3.2. *Formal mating.* We add to the complex plane $\mathbb{C}$ the *circle at infinity* which is symbolically denoted $\{\infty \cdot e^{2i\pi\theta} \ ; \ \theta \in \mathbb{R}/\mathbb{Z}\}$. We define

$$\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta} \mid \theta \in \mathbb{R}/\mathbb{Z}\},$$

which is homeomorphic to a closed disk with the standard topology. A monic polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree d extends to $\widetilde{\mathbb{C}}$ by setting

$$P(\infty \cdot e^{2i\pi\theta}) = \infty \cdot e^{2i\pi d\theta}.$$

Let $\widetilde{\mathbb{C}}_P$ and $\widetilde{\mathbb{C}}_Q$ be two copies of $\widetilde{\mathbb{C}}$ and assume $P : \widetilde{\mathbb{C}}_P \to \widetilde{\mathbb{C}}_P$ and $Q : \widetilde{\mathbb{C}}_Q \to \widetilde{\mathbb{C}}_Q$ are two postcritically finite polynomials with common degree $d \geq 2$. Let $\sim$ be the equivalence relation in $\widetilde{\mathbb{C}}_P \sqcup \widetilde{\mathbb{C}}_Q$ (disjoint union) which, for $\theta \in \mathbb{R}/\mathbb{Z}$, identifies $\infty \cdot e^{2i\pi\theta}$ in $\widetilde{\mathbb{C}}_P$ to $\infty \cdot e^{-2i\pi\theta}$ in $\widetilde{\mathbb{C}}_Q$. The quotient by $\sim$ is topologically an oriented 2-sphere. The formal mating $P \uplus Q$ of P and Q is the orientation-preserving branched self-covering which is given by P on the image of $\widetilde{\mathbb{C}}_P$ and by Q on the image of $\widetilde{\mathbb{C}}_Q$.

One can give an equivalent definition as follows (see [M]). Let S be the unit sphere in $\mathbb{C} \times \mathbb{R}$. If $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ are two monic polynomials of the same degree $d \geq 2$, the formal mating of P and Q is the ramified covering $f = P \uplus Q : S \to S$ obtained as follows.

We identify the dynamical plane of P to the upper hemisphere H^+ of S and the dynamical plane of Q to the lower hemisphere H^- of S via the gnomonic projections:

$$\nu_P : \mathbb{C} \to H^+ \quad \text{and} \quad \nu_Q : \mathbb{C} \to H^-$$

given by

$$\nu_P(z) = \frac{(z,1)}{\|(z,1)\|} = \frac{(z,1)}{\sqrt{|z|^2+1}} \quad \text{and} \quad \nu_Q(z) = \frac{(\bar z,-1)}{\|(\bar z,-1)\|} = \frac{(\bar z,-1)}{\sqrt{|z|^2+1}}.$$

Since P and Q are monic polynomials of degree d, the map $\nu_P \circ P \circ \nu_P^{-1}$ defined on the upper hemisphere and $\nu_Q \circ Q \circ \nu_Q^{-1}$ defined in the lower hemisphere extend continuously to the equator of S by

$$(e^{2i\pi\theta}, 0) \mapsto (e^{2i\pi d\theta}, 0).$$

The two maps fit together so as to yield a ramified covering map $f : S \to S$, which is called the formal mating $P \uplus Q$ of P and Q.

1.3.3. *Geometric mating.* Let us now consider the smallest equivalence relation $\sim_{\mathrm{ray}}$ on S such that for all $\theta \in \mathbb{R}/\mathbb{Z}$,

- points in the closure of $\nu_P\bigl(\mathcal{R}_P(\theta)\bigr)$ are in the same equivalence class, and
- points in the closure of $\nu_Q\bigl(\mathcal{R}_Q(\theta)\bigr)$ are in the same equivalence class.

In particular, for all $\theta \in \mathbb{R}/\mathbb{Z}$, $\nu_P\bigl(\mathcal{R}_P(\theta)\bigr)$ and $\nu_Q\bigl(\mathcal{R}_Q(-\theta)\bigr)$ are in the same equivalence class since the closures of these sets intersect at the point $(e^{2i\pi\theta}, 0)$ on the equator of S.

We say that a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ is a geometric mating of P and Q if

(1) the quotient space $S/_{\sim_{\mathrm{ray}}}$ is homeomorphic to S, and
(2) the formal mating $P \uplus Q$ induces a map $S/_{\sim_{\mathrm{ray}}} \to S/_{\sim_{\mathrm{ray}}}$ which is topologically conjugate to $F : \mathbb{P}^1 \to \mathbb{P}^1$.

1.3.4. *The Rees Theorem.* The following theorem of Rees in [R] relates the two mating criteria above to Theorem 1.

Theorem 2 (Rees)**.** *Assume $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ are two postcritically finite hyperbolic polynomials. The formal mating $P \uplus Q$ is combinatorially equivalent to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ if and only if F is a geometric mating of P and Q.*

Remark. A similar result also holds in the case P and Q are postcritically finite polynomials, not necessarily hyperbolic; see [ST] for an appropriate modification of the formal mating.

2. Mating the basilica polynomial with itself

From now on, we will restrict to a particular case: $P = Q$ is the quadratic polynomial $z \mapsto z^2 - 1$ for which the critical point 0 is periodic of period 2, and $f = P \uplus Q$ is the formal mating of this polynomial with itself. The postcritical set $\mathcal{P}_f$ has cardinality 4: there are two critical points p_0 and q_0 (corresponding respectively to the critical point of P and the critical point of Q) and two associated critical values $p_1 = f(p_0)$ and $q_1 = f(q_0)$.

2.1. **Obstructed mating.**

Proposition 1. *No rational map is a geometric mating of P and Q.*

Proof. We propose two independent proofs of this result. The first is algebraic: if there were such a geometric mating, it would be a rational map of degree 2 with two superattracting cycles of period 2, one from each basilica. But a rational map

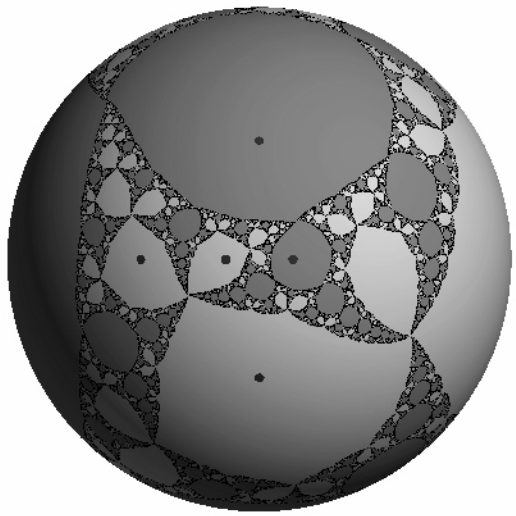

FIGURE 2. The rational map $F : z \mapsto (z^2 - e^{-2i\pi/3})/(z^2-1)$ is a geometric mating of the basilica polynomial and the rabbit polynomial (see Figure 1). The rational map F has a superattracting cycle of period 3 (the basin of which is colored in light grey), and a superattracting cycle of period 2 (the basin of which is colored in dark grey).

of degree 2 is entitled to at most one periodic cycle of period 2, superattracting or otherwise.

The second relies on Theorem 2. If there were a rational map F which was a geometric mating of P with Q, then F would be combinatorially equivalent to f. Consider the curve γ in $S - \mathcal{P}_f$ composed of the arcs

$$\nu_P\big(\mathcal{R}_P(1/3)\big) \cup \nu_P\big(\mathcal{R}_P(2/3)\big) \text{ in } \widetilde{\mathbb{C}}_P \text{ and } \nu_Q\big(\mathcal{R}_Q(2/3)\big) \cup \nu_Q\big(\mathcal{R}_Q(1/3)\big) \text{ in } \widetilde{\mathbb{C}}_Q.$$

It is easy to show that $\Gamma := \{\gamma\}$ is a Levy cycle for f; therefore, f is obstructed and no such F exists. □

2.2. **Maps between universal curves.** Even if the mating is obstructed, we wish to understand better Diagram (2) in that particular case.

Throughout Section 2, we shall assume that the homeomorphisms $\phi : S \to \mathbb{P}^1$ representing points in $\mathcal{T}_{\mathcal{P}_f}$ and the injective maps $\phi : \mathcal{P}_f \to \mathbb{P}^1$ representing points in $\mathcal{M}_{\mathcal{P}_f}$ satisfy the condition

$$\phi(p_0) = 0, \quad \phi(q_0) = \infty, \quad \text{and} \quad \phi(q_1) = 1.$$

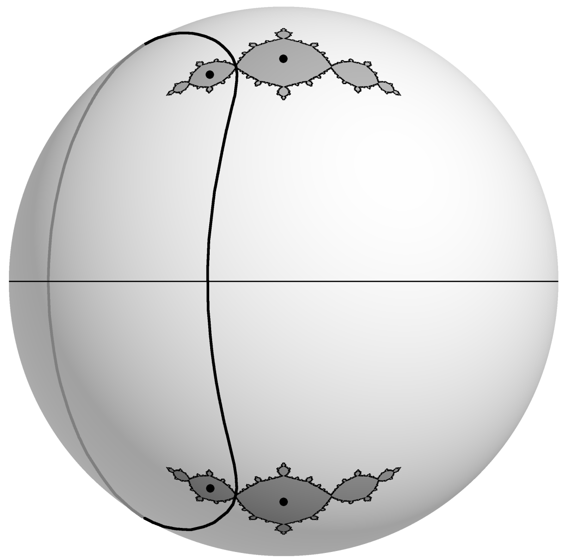

FIGURE 3. The Levy cycle $\Gamma := \{\gamma\}$ is an obstructing multicurve for f, so f is not combinatorially equivalent to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$.

We will refer to such maps as *normalized* maps.

In this way, the moduli space $\mathcal{M}_{\mathcal{P}_f}$ is identified with $\mathbb{C} - \{0,1\}$ via $\phi \mapsto \phi(p_1)$. The Teichmüller space $\mathcal{T}_{\mathcal{P}_f}$ is isomorphic to the unit disc $\mathbb{D}$ and $\pi : \mathcal{T}_{\mathcal{P}_f} \to \mathbb{C} - \{0,1\}$ is a universal cover. The universal curve over moduli space is given by the projection

$$\mathfrak{m} : (\mathbb{C} - \{0,1\}) \times \mathbb{P}^1 \ni (x, z) \mapsto x \in \mathbb{C} - \{0,1\}$$

with sections

$$m_{p_0} : x \mapsto (x, 0), \quad m_{q_0} : x \mapsto (x, \infty), \quad m_{q_1} : x \mapsto (x, 1) \quad \text{and} \quad m_{p_1} : x \mapsto (x, x).$$

- A point in $\mathfrak{M}_{\mathcal{P}_f}$ is identified with a pair $(x, z) \in (\mathbb{C} - \{0,1\}) \times \mathbb{P}^1$.
- A point in $\mathfrak{T}_{\mathcal{P}_f}$ is identified with a triple $(\tau, x, z) \in \mathcal{T}_{\mathcal{P}_f} \times (\mathbb{C} - \{0,1\}) \times \mathbb{P}^1$ satisfying $x = \pi(\tau)$, so that $(x, z) = \varpi(\tau, x, z)$.
- A point in $\mathfrak{U}_{\mathcal{P}_f}$ is identified with a quadruple (τ, υ, x, z) such that $\tau \in \mathcal{T}_{\mathcal{P}_f}$, $\upsilon = \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$, $x = \pi(\upsilon) \in \mathbb{C} - \{0,1\}$ and $z \in \mathbb{P}^1$.

Let (τ, υ, x, z) belong to $\mathfrak{U}_{\mathcal{P}_f}$ and let (τ, y, w) be its image by $\mathfrak{F} : \mathfrak{U}_{\mathcal{P}_f} \to \mathfrak{T}_{\mathcal{P}_f}$. Then, there are a normalized homeomorphism $\phi : S \to \mathbb{P}^1$ representing $\tau \in \mathcal{T}_{\mathcal{P}_f}$, a normalized homeomorphism $\psi : S \to \mathbb{P}^1$ representing $\upsilon = \sigma_f(\tau) \in \mathcal{T}_{\mathcal{P}_f}$ and a rational map $F_\tau : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "f"'] & \mathbb{P}^1 \arrow[d, "F_\tau"] \\ S \arrow[r, "\phi"] & \mathbb{P}^1. \end{tikzcd} \tag{3}$$

In addition,

$$\phi(p_1) = y, \quad \psi(p_1) = x \quad \text{and} \quad F_\tau(z) = w.$$

In particular, $F_\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map of degree 2 with critical points at 0 and ∞ such that

$$F_\tau(\infty) = 1, \quad F_\tau(1) = \infty, \quad F_\tau(x) = 0, \quad \text{and} \quad F_\tau(0) = y.$$

Imposing the first three conditions, we see that

$$w := F_\tau(z) = \frac{z^2 - x^2}{z^2 - 1},$$

and imposing the condition $F_\tau(0) = y$ gives the relation $y = x^2$.

We have just shown that

$$\mathfrak{F}(\tau, \upsilon, x, z) = (\tau, y, w) \quad \text{with} \quad y = x^2 \quad \text{and} \quad w = \frac{z^2 - x^2}{z^2 - 1}.$$

In other words, there are maps $G_f : \mathfrak{M}_{\mathcal{P}_f} \dashrightarrow \mathfrak{M}_{\mathcal{P}_f}$ and $g_f : \mathcal{M}_{\mathcal{P}_f} \dashrightarrow \mathcal{M}_{\mathcal{P}_f}$ so that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{P}_f} \arrow[d, "\mathfrak{m}"'] & \mathfrak{T}_{\mathcal{P}_f} \arrow[l, "\varpi"'] \arrow[d, "\mathfrak{t}"] & \mathfrak{U}_{\mathcal{P}_f} \arrow[l, "\mathfrak{F}"'] \arrow[r, "\mathfrak{p}_2"] \arrow[d, "\mathfrak{p}_1"] & \mathfrak{T}_{\mathcal{P}_f} \arrow[r, "\varpi"] \arrow[d, "\mathfrak{t}"] & \mathfrak{M}_{\mathcal{P}_f} \arrow[d, "\mathfrak{m}"] \arrow[llll, dashed, bend right, "G_f"'] \\
\mathcal{M}_{\mathcal{P}_f} & \mathcal{T}_{\mathcal{P}_f} \arrow[l, "\pi"'] & \mathcal{T}_{\mathcal{P}_f} \arrow[l, "\mathrm{id}"'] \arrow[r, "\sigma_f"] & \mathcal{T}_{\mathcal{P}_f} \arrow[r, "\pi"] & \mathcal{M}_{\mathcal{P}_f} \arrow[llll, dashed, bend left, "g_f"]
\end{tikzcd}$$

The map $g_f : \mathcal{M}_{\mathcal{P}_f} \dashrightarrow \mathcal{M}_{\mathcal{P}_f}$ is defined on $\mathcal{M}_{\mathcal{P}_f} - \{-1\} = \mathbb{C} - \{0, 1, -1\}$ by

$$g_f(x) = x^2.$$

The map $G_f : \mathfrak{M}_{\mathcal{P}_f} \dashrightarrow \mathfrak{M}_{\mathcal{P}_f}$ is defined on $\mathfrak{M}_{\mathcal{P}_f} - \mathfrak{m}^{-1}(-1)$ by

$$G_f(x, z) = \left(x^2, \frac{z^2 - x^2}{z^2 - 1}\right).$$

We have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{P}_f} \arrow[r, dashed, "G_f"] \arrow[d, "\mathfrak{m}"'] & \mathfrak{M}_{\mathcal{P}_f} \arrow[d, "\mathfrak{m}"] \\
\mathcal{M}_{\mathcal{P}_f} \arrow[r, dashed, "g_f"] & \mathcal{M}_{\mathcal{P}_f}
\end{tikzcd} \tag{4}$$

The rest of the article consists of studying the skew-product G_f and understanding its dynamical properties in terms of matings of polynomials.

3. Twisted matings

For $x \in \mathbb{C} - \{0, 1, -1\}$, the restriction of the map G_f to the fiber $\mathfrak{S}_x := \mathfrak{m}^{-1}(x)$ takes its values in the fiber $\mathfrak{S}_{x^2} := \mathfrak{m}^{-1}(x^2)$ and is therefore not a dynamical system in general. However, when x is periodic of period n for g_f, the n-th iterate of the skew-product G_f fixes the fiber $\mathfrak{S}_x := \mathfrak{m}^{-1}(x) \subset \mathcal{M}_{\mathcal{P}_f}$. In that case, it makes sense to consider the rational map $G_f^{\circ n} : \mathfrak{S}_x \to \mathfrak{S}_x$ as a dynamical system. In this section, we show that it is the geometric mating of two polynomials.

Our result is the following.

Proposition 2. *If $x = e^{2\pi ik/(2^n-1)}$ with $k/(2^n-1) \notin \mathbb{Z}$, then $G_f^{\circ n} : \mathfrak{S}_x \to \mathfrak{S}_x$ is the geometric twisted mating of angle $-k/(2^n-1)$ of $P^{\circ n}$ with $Q^{\circ n}$.*

We shall first define the notion of twisted mating, then introduce an intermediate covering space, and finally prove the result.

3.1. **Definition and first property.** If P is a monic polynomial of degree $d \geq 2$, then the polynomial $T(P) : \mathbb{C} \to \mathbb{C}$ defined by

$$T(P)(z) = e^{-2i\pi/(d-1)} P(e^{2i\pi/(d-1)} z)$$

is also monic. The filled Julia set of $T(P)$ is the image of the Julia set of P by the rotation of angle $-1/(d-1)$ turns, centered at 0. The external ray of argument θ of $T(P)$ is the image of the external ray of argument $\theta + 1/(d-1)$ of P.

Definition 1. *If P and Q are monic polynomials of degree d, the* twisted mating *of angle $k/(d-1)$ of P and Q is the mating (formal or geometric depending on the case) of P with $T^{\circ k}(Q)$.*

This corresponds to constructing the formal mating as follows: consider

$$P : \widetilde{\mathbb{C}}_P \to \widetilde{\mathbb{C}}_P \quad \text{and} \quad Q : \widetilde{\mathbb{C}}_Q \to \widetilde{\mathbb{C}}_Q,$$

and glue $\widetilde{\mathbb{C}}_P$ to $\widetilde{\mathbb{C}}_Q$ by identifying $\infty \cdot e^{2i\pi\theta_1}$ in $\widetilde{\mathbb{C}}_P$ to $\infty \cdot e^{2i\pi\theta_2}$ in $\widetilde{\mathbb{C}}_Q$ if and only if $\theta_1 + \theta_2 = -k/(d-1)$.

Proposition 3. *Assume P and Q are two monic polynomials of degree $d \geq 2$, which are critically finite. Then, $P \uplus T(Q)$ is combinatorially equivalent to $D \circ (P \uplus Q)$, where $D : S \to S$ is a Dehn twist around the equator of S relative to the postcritical set of $P \uplus Q$.*

Proof. Set $f := P \uplus Q$ and $g := P \uplus T(Q)$. Choose $1 < r < \infty$, and let $A \subset \widetilde{\mathbb{C}}_P - K(P)$ be the image of the annulus $\{z \in \mathbb{C} \,;\, |z| > r\}$ by the Böttcher coordinate böt of P and $A' \subset A$ be the image of the annulus $\{z \in \mathbb{C} \,;\, |z| > r^d\}$. Let

$$h : [r^d, +\infty] \to [0, 1]$$

be a continuous increasing function such that $h(r^d) = 0$ and $h(+\infty) = 1$. Then, define $\phi_P : A' \to A'$ and $\psi_P : A \to A$ by

$$\phi_P \circ \text{böt}\left(\rho e^{2i\pi\theta}\right) = \text{böt}\left(\rho e^{2i\pi\left(\theta + h(\rho)/(d-1)\right)}\right)$$

for $\rho > r^d$ and $\theta \in \mathbb{R}/\mathbb{Z}$ and by

$$\psi_P \circ \text{böt}(\rho e^{2i\pi\theta}) = \text{böt}\left(\rho e^{2i\pi\left(\theta + h(\rho^d)/(d-1)\right)}\right)$$

for $\rho > r$ and $\theta \in \mathbb{R}/\mathbb{Z}$. The maps ϕ_P and ψ_P extend to self-homeomorphisms of $\widetilde{\mathbb{C}}_P$ by the identity on $\mathbb{C}_P - A'$ and $\mathbb{C}_P - A$ respectively and by

$$\phi_P(\infty \cdot e^{2i\pi\theta}) = \infty \cdot e^{2i\pi\left(\theta + 1/(d-1)\right)}$$

on the circle at infinity. They induce self-homeomorphisms of the upper hemisphere of S via the gnomonic projection ν_P, which themselves extend to self-homeomorphisms of S by

$$\phi \circ \nu_Q(z) = \psi \circ \nu_Q(z) = \nu_Q\left(e^{-2i\pi/(d-1)} z\right)$$

on the lower hemisphere of S. The homeomorphisms ϕ and ψ are isotopic relative to $\mathcal{P}_f$.

Let us now consider the Dehn twist $D : S - \mathcal{P}_f \to S - \mathcal{P}_f$ defined on $\nu_P(A')$ by

$$D \circ \nu_P \circ \text{böt}\left(\rho e^{2i\pi\theta}\right) = \nu_P \circ \text{böt}\left(\rho e^{2i\pi\left(\theta + h(\rho)\right)}\right)$$

for $\rho > r^d$ and $\theta \in \mathbb{R}/\mathbb{Z}$, extended by the identity outside $\nu_P(A')$.

For $z \in \nu_P(A)$, we have $\phi \circ D \circ f = g \circ \psi$. Indeed, a point $z \in \nu_P(A)$ is of the form $z = \nu_P \circ \text{böt}(\rho e^{2i\pi\theta})$. We say that z has Böttcher coordinates (ρ, θ). Then, $f(z)$ has Böttcher coordinates $(\rho^d, d\theta)$. Thus, $D \circ f(z)$ has Böttcher coordinates $\big(\rho^d, d\theta + h(\rho^d)\big)$. Finally, $\phi \circ D \circ f(z)$ has Böttcher coordinates

$$\left(\rho^d, d\theta + h(\rho^d) + \frac{h(\rho^d)}{d-1}\right) = \left(\rho^d, d\theta + \frac{d}{d-1} h(\rho^d)\right).$$

Now, the point $\psi(z)$ has Böttcher coordinates $\big(\rho, \theta + h(\rho^d)/(d-1)\big)$. Thus, the point $g \circ \psi(z)$ has Böttcher coordinates

$$\left(\rho^d, d \cdot \left(\theta + \frac{h(\rho^d)}{d-1}\right)\right) = \left(\rho^d, d\theta + \frac{d}{d-1} h(\rho^d)\right).$$

A similar (and easier) proof shows that the same equality holds outside $\nu_P(A')$: we have the following commutative diagram:

$$\begin{tikzcd}
(S, \mathcal{P}_f) \arrow[r, "f"] \arrow[d, "\psi"'] & (S, \mathcal{P}_f) \arrow[r, "D"] & (S, \mathcal{P}_f) \arrow[d, "\phi"] \\
(S, \mathcal{P}_g) \arrow[rr, "g"'] & & (S, \mathcal{P}_g).
\end{tikzcd}$$

The homeomorphisms ϕ and ψ give a combinatorial equivalence between g and $D \circ f$. □

3.2. **An intermediate covering space.** Before proceeding to the proof of Proposition 2, we prove that the Thurston endomorphism $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ descends to a self-map of an intermediate covering space (see Diagram (5) below). We first need to define the maps involved in this Diagram.

Recall that with our chosen normalizations, $\mathcal{M}_{\mathcal{P}_f}$ is identified with $\mathbb{C} - \{0, 1\}$. Let $\alpha : \mathbb{C} - \mathbb{Z} \to \mathbb{C} - \{0, 1\}$ and $L : \mathbb{C} - \mathbb{Z} \to \mathbb{C} - \frac{1}{2}\mathbb{Z}$ be the covering maps defined by

$$\alpha(X) := e^{2\pi i X} \quad \text{and} \quad L(X) := X/2.$$

We have the following commutative diagram of covering spaces.

$$\begin{tikzcd}
\mathbb{C} - \mathbb{Z} \arrow[r, "L"] \arrow[d, "\alpha"'] & \mathbb{C} - \frac{1}{2}\mathbb{Z} \arrow[d, "\alpha"] \\
\mathbb{C} - \{0, 1\} & \mathbb{C} - \{0, 1, -1\} \arrow[l, "g_f"']
\end{tikzcd}$$

The universal cover $\pi : \mathcal{T}_{\mathcal{P}_f} \to \mathbb{C} - \{0, 1\}$ factors as $\pi = \alpha \circ \beta$ for some (non-unique) universal cover $\beta : \mathcal{T}_{\mathcal{P}_f} \to \mathbb{C} - \mathbb{Z}$. We wish to choose a particular β in a way that we now describe. Let Γ be the topological circle in $S - \mathcal{P}_f$ obtained as the closure of $\nu_P(\mathbb{R}) \cup \nu_Q(\mathbb{R})$, where the gnomonic projections ν_P and ν_Q satisfy

$$\nu_P(\mathbb{R}) \subset H^+ \quad \text{and} \quad \nu_Q(\mathbb{R}) \subset H^-,$$

and $\mathbb{R}$ is the real axis in the respective dynamical planes of the basilicas P and Q (see Figure 4). Note that $p_0, p_1 \in \nu_P(\mathbb{R})$, and $q_0, q_1 \in \nu_Q(\mathbb{R})$. Let $\Gamma' \subset \Gamma$ be the subarc joining the two critical values p_1 and q_1, avoiding the critical points p_0 and q_0, oriented from q_1 to p_1. If $\phi : S \to \mathbb{P}$ is a normalized homeomorphism representing $\tau \in \mathcal{T}_{\mathcal{P}_f}$, then $\phi(\Gamma')$ is a curve joining $\phi(q_1) = 1$ to $\phi(p_1) = \pi(\tau)$ in $\mathbb{C} - \{0\}$ and we set

$$\beta(\tau) := \frac{1}{2\pi i} \int_{\phi(\Gamma')} \frac{dz}{z},$$

where $\phi : S \to \mathbb{P}^1$ is a normalized homeomorphism representing $\tau \in \mathcal{T}_{\mathcal{P}_f}$. By the Cauchy Theorem, the result does not depend on the choice of normalized representative.

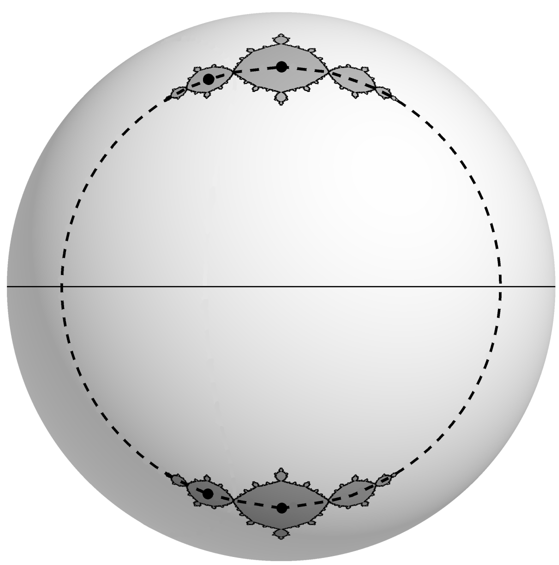

FIGURE 4. The curve drawn with the dashes is Γ; it is a topological circle, which can be decomposed into two subarcs $\Gamma = \Gamma' \cup \Gamma''$, where Γ' is the subarc connecting p_1 and q_1, which does not contain p_0 and q_0, and Γ'' is the subarc connecting p_1 and q_1 which does contain p_0 and q_0.

Proposition 4. *We have the following commutative diagram of covering maps:*

$$
\begin{tikzcd}
\mathcal{T}_{\mathcal{P}_f} \arrow[rrr, "\sigma_f"] \arrow[dd, "\pi"'] \arrow[dr, "\beta"] & & & \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) \arrow[dl, "\beta"'] \arrow[dd, "\pi"] \\
& \mathbb{C} - \mathbb{Z} \arrow[r, "L"] \arrow[dl, "\alpha"] & \mathbb{C} - \frac{1}{2}\mathbb{Z} \arrow[dr, "\alpha"] & \\
\mathbb{C} - \{0,1\} & & & \mathbb{C} - \{0,1,-1\}. \arrow[lll, "g_f"']
\end{tikzcd}
\tag{5}
$$

Proof. What remains is to show that $\sigma_f : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$ is a covering map and that $\beta \circ \sigma_f = L \circ \beta$ on $\mathcal{T}_{\mathcal{P}_f}$.

Since $L \circ \beta : \mathcal{T}_{\mathcal{P}_f} \to \mathbb{C} - \frac{1}{2}\mathbb{Z}$ is a universal cover and $\beta : \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) \to \mathbb{C} - \frac{1}{2}\mathbb{Z}$ is a covering map, if $\tau \in \mathcal{T}_{\mathcal{P}_f}$ and $\upsilon \in \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$ satisfy $\beta(\upsilon) = L \circ \beta(\tau)$, there is a unique covering map $\sigma : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1)$ such that

$$L \circ \beta = \beta \circ \sigma \quad \text{and} \quad \sigma(\tau) = \upsilon.$$

We will show that there is such a pair (τ, υ) with $\upsilon = \sigma_f(\tau)$. Both σ and σ_f are lifts of $\pi : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{M}_{\mathcal{P}_f}$ through $g_f \circ \pi : \mathcal{T}_{\mathcal{P}_f} - \pi^{-1}(-1) \to \mathcal{M}_{\mathcal{P}_f}$ and they coincide at τ. By uniqueness of the lift, σ and σ_f coincide everywhere.

So, let $\tau \in \mathcal{T}_{\mathcal{P}_f}$ be represented by a normalized homeomorphism $\phi : S \to \mathbb{P}^1$, such that $\phi(\Gamma) = \mathbb{R} \cup \{\infty\}$. Set $\upsilon := \sigma_f(\tau)$. We need to prove that $\beta(\upsilon) = L \circ \beta(\tau)$. Let $\psi : S \to \mathbb{P}^1$ be the normalized representative of υ such that

$$F := \phi \circ f \circ \psi^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$$

is rational. Set $y := \phi(p_1)$ and $x := \psi(p_1)$.

Lemma 1. *We have that* $\psi(\Gamma) = \mathbb{R} \cup \{\infty\}$.

Proof. As previously discussed in Section 2.2: $y = x^2$ and

$$F : z \mapsto \frac{z^2 - x^2}{z^2 - 1}$$

with critical values x^2 and 1. Since $\phi(\Gamma) = \mathbb{R} \cup \{\infty\}$, we have that $y \in (0,1)$, so $0 < x^2 < 1$. Note that

$$\phi \circ f(\Gamma) = \phi(\Gamma'') = \mathbb{R} \cup \{\infty\} - (x^2, 1) = F \circ \psi(\Gamma).$$

Since $\Gamma = f^{-1}(\Gamma'')$, we have $\psi(\Gamma) = F^{-1}(\mathbb{R} \cup \{\infty\} - (x^2, 1))$. The map F can be decomposed as $F = M \circ s$, where $s : z \mapsto z^2$, and $M : z \to (z - x^2)/(z - 1)$. Note that for the Möbius transformation M, we have

$$M(x^2) = 0, \quad M(0) = x^2, \quad M(\infty) = 1 \quad \text{and} \quad M(1) = \infty.$$

Furthermore,

$$M^{-1}(\mathbb{R} \cup \{\infty\} - (x^2, 1)) = [0, \infty] \quad \text{and} \quad s^{-1}([0, \infty]) = \mathbb{R} \cup \{\infty\}.$$

It follows that $\psi(\Gamma) = \mathbb{R} \cup \{\infty\}$. □

We conclude the proof of the Proposition as required:

$$\beta(\upsilon) = \frac{1}{2\pi i} \ln y = L\left(\frac{1}{2\pi i} \ln x\right) = L(\beta(\tau)).$$ □

Proposition 5. *Let $D : S \to S$ be a Dehn twist around the equator of S relative to $\mathcal{P}_f$. For any normalized homeomorphism $\phi : S \to \mathbb{P}^1$, we have*

$$\beta\big([\phi \circ D]\big) = \beta\big([\phi]\big) - 1.$$

Proof. Recall that Γ' is oriented from q_1 to p_1. We orient the equator Γ_{eq} of S so that q_1 is on the left of Γ_{eq} with respect to this orientation. The orientations of Γ' and Γ_{eq} induce orientations of the curves $\phi(\Gamma')$ and $\phi(\Gamma_{eq})$:

- if $y := \phi(p_1)$, then the initial point of $\phi(\Gamma')$ is 1, and its terminal point is y, and
- $\phi(\Gamma_{eq}) \subset \mathbb{P}^1$ is a simple closed curve positively-oriented around ∞; that is, ∞ is on the left of $\phi(\Gamma_{eq})$, and 0 is on the right with respect to this orientation.

The chains $D(\Gamma')$ and $\Gamma' + \Gamma_{eq}$ are homologous in $S - \mathcal{P}_f$. So, by Cauchy's theorem

$$\int_{\phi(D(\Gamma'))} \frac{dz}{z} = \int_{\phi(\Gamma')} \frac{dz}{z} + \int_{\phi(\Gamma_{eq})} \frac{dz}{z}.$$

We ultimately have

$$\beta\big([\phi \circ D]\big) = \frac{1}{2\pi i} \int_{\phi(D(\Gamma'))} \frac{dz}{z} = \frac{1}{2\pi i} \left(\int_{\phi(\Gamma')} \frac{dz}{z} - 2\pi i \right) = \beta\big([\phi]\big) - 1. \qquad \square$$

In other words, the Dehn twist D acts on $\mathbb{C} - \mathbb{Z}$ by translation $X \mapsto X - 1$.

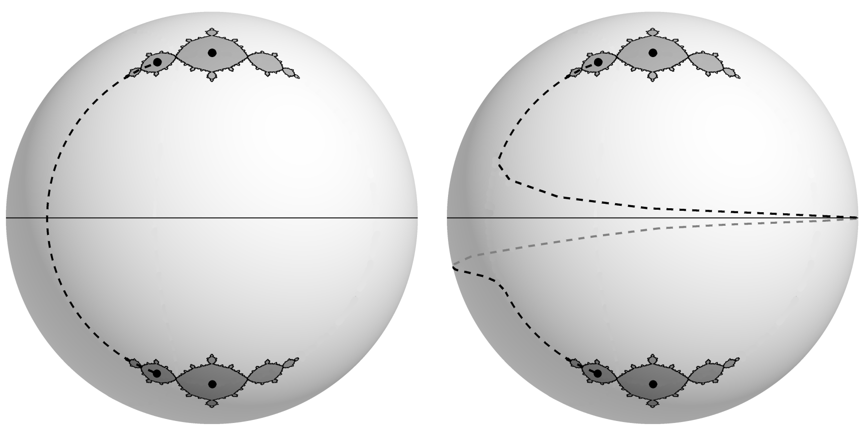

FIGURE 5. The curve Γ' is drawn in dashes on the left sphere, and its image under the Dehn twist D is drawn on the right sphere; its image is wrapped once around the equator, Γ_{eq}.

3.3. **Proof of Proposition 2.** For $n \geq 0$, we define $F_x^n : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$G_f^{\circ n}(x, z) = \big(g_f^{\circ n}(x), F_x^n(z)\big).$$

We must show that if x is a periodic point of g_f of period n, that is, if

$$x = e^{2\pi i k/(2^n - 1)} \quad \text{for} \quad k/(2^n - 1) \notin \mathbb{Z},$$

then the rational map F_x^n is the geometric twisted mating of angle $-k/(2^n - 1)$ of $P^{\circ n}$ with $Q^{\circ n}$. This is a direct consequence of the more precise following result.

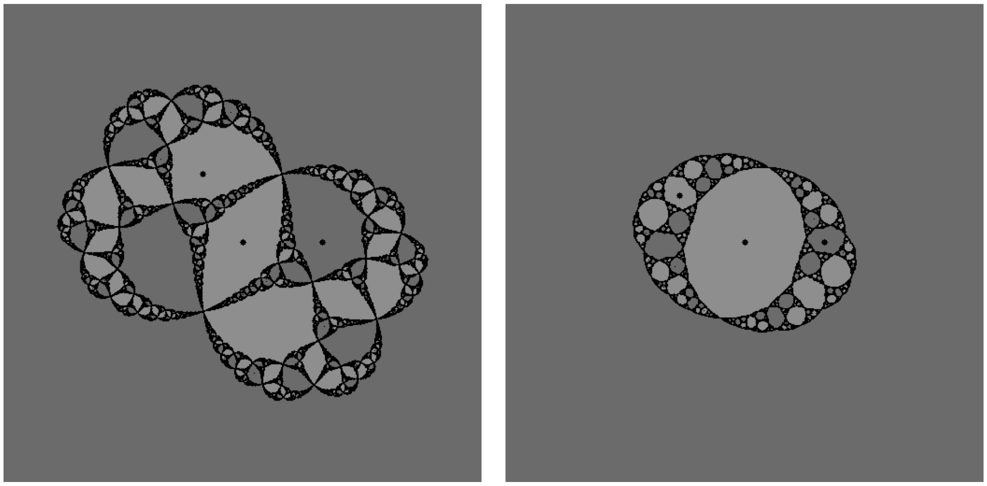

FIGURE 6. Left: the Julia set and Fatou set of F_y^2 for $y = e^{2i\pi/3}$. Right: the Julia set and Fatou set of F_y^4 for $y = e^{2i\pi 2/5}$. Points are colored darker grey if they are in the basin of ∞ and lighter grey if they are in the basin of 0. The Julia set is colored black. The points 0, 1 and y are shown.

Proposition 6. *Fix $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, and let $f_{n,k} : S \to S$ be the twisted mating of angle $-k/(2^n - 1)$ of $P^{\circ n}$ with $Q^{\circ n}$. Then $f_{n,k}$ is combinatorially equivalent to a rational map if and only if*

$$x := e^{2\pi ik/(2^n-1)} \neq 1.$$

In this case, $f_{n,k}$ is combinatorially equivalent to F_x^n.

Proof. By definition, $f_{n,k} = P^{\circ n} \uplus T^{\circ k}(Q^{\circ n})$. According to Proposition 3, $f_{n,k}$ is combinatorially equivalent to

$$D^{\circ k} \circ (P^{\circ n} \uplus Q^{\circ n}) = D^{\circ k} \circ (P \uplus Q)^{\circ n} = D^{\circ k} \circ f^{\circ n}.$$

So, the Thurston endomorphism $\sigma_{f_{n,k}} : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ satisfies

$$\sigma_{f_{n,k}} = \sigma_{D^{\circ k} \circ f^{\circ n}} = \sigma_{f^{\circ n}} \circ \sigma_{D^{\circ k}} = \sigma_f^{\circ n} \circ \sigma_D^{\circ k}.$$

Define $L_{n,k} : \mathbb{C} \to \mathbb{C}$ by

$$L_{n,k}(X) := (X - k)/2^n.$$

According to Proposition 5,

$$\beta \circ \sigma_{f_{n,k}} = L_{n,k} \circ \beta.$$

Fix $\tau_0 \in \mathcal{T}_{\mathcal{P}_f}$, and define the sequence of iterates

$$\tau_m := \sigma_{f_{n,k}}^{\circ m}(\tau_0) \quad \text{for} \quad m \geq 0.$$

- Either $\sigma_{f_{n,k}}$ has a fixed point, in which case the sequence $m \mapsto \tau_m$ converges to it and $f_{n,k}$ is equivalent to a rational map.
- Or $\sigma_{f_{n,k}}$ has no fixed point, in which case $m \mapsto \tau_m$ diverges and $f_{n,k}$ is obstructed.

Let $\gamma_0 : [0,1] \to \mathcal{T}_{\mathcal{P}_f}$ be a path connecting $\gamma_0(0) = \tau_0$ and $\gamma_0(1) = \tau_1$. Let $\kappa_0 : [0,1] \to \mathbb{C} - \mathbb{Z}$ be the path $\kappa_0 = \beta \circ \gamma_0$.

Set $\kappa_m := L_{n,k}^{\circ m} \circ \kappa_0$. Define $X_m := \beta(\tau_m)$; the sequence $m \mapsto X_m$ converges to $X_\star := -k/(2^n - 1)$, the unique fixed point of $L_{n,k}$. Concatenating the κ_m we obtain a path κ that joins X_0 to $X_\star$. The sequence $m \mapsto \kappa_m$ will converge to $X_\star$. Either $X_\star \in \mathbb{Z}$, or it is not. We discuss each case.

If $X_\star \in \mathbb{C} - \mathbb{Z}$, then the path κ lifts uniquely to a path γ connecting τ_0 to a point $\tau_\star \in \beta^{-1}(X_\star) \subset \mathcal{T}_{\mathcal{P}_f}$. By induction on m, we see that the lift of κ_m joins τ_{m-1} to τ_m and therefore coincides with $\gamma_m := \sigma_{f_{n,k}}^{\circ m}(\gamma_0)$. In particular, $\tau_m \to \tau_\star$ as $m \to \infty$, so $\tau_\star$ is a fixed point of the Thurston endomorphism:

$$\sigma_{f_{n,k}}(\tau_\star) = \tau_\star, \quad \beta(\tau) = X_\star.$$

Set $x_\star := \alpha(X_\star) = e^{2\pi i X_\star} \in \mathbb{C} - \{0,1\}$; in particular, $x_\star \neq 1$, and the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "f_{n,k}"'] & \mathbb{P}^1 \arrow[d, "F_{x_\star}^n"] \\ S \arrow[r, "\phi"] & \mathbb{P}^1 \end{tikzcd}$$

where ϕ and ψ are normalized homeomorphisms representing $\tau_\star$. Hence $f_{n,k}$ and $F_{x_\star}^n$ are combinatorially equivalent.

If $X_\star \in \mathbb{Z}$, then $\sigma_{f_{n,k}} : \mathcal{T}_{\mathcal{P}_f} \to \mathcal{T}_{\mathcal{P}_f}$ cannot have a fixed point because

$$\sigma_{f_{n,k}}(\tau_\star) = \tau_\star \implies L_{n,k}(\beta(\tau_\star)) = \beta(\tau_\star),$$

so $\beta(\tau_\star) \in \mathbb{C} - \mathbb{Z}$ would be a fixed point of $L_{n,k}$. However, $L_{n,k} : \mathbb{C} \to \mathbb{C}$ is entitled to exactly one fixed point, which is $X_\star \in \mathbb{Z}$ by hypothesis. So in this case, the map $f_{n,k}$ is obstructed, and $x_\star := \alpha(X_\star) = e^{2\pi i X_\star} = 1$. □

The following corollary follows immediately.

Corollary 1. *If $\theta = k/(2^n - 1) \notin \mathbb{Z}$, the geometric twisted mating of angle $-\theta$ of $P^{\circ n}$ with $Q^{\circ n}$ exists.*

4. Compactifying the universal curve

If $(x,z) \in \mathfrak{M}_{\mathcal{P}_f}$ with $|x| < 1$ or $|x| > 1$, then $G_f^{\circ n}(x,z)$ eventually leaves every compact set of $\mathfrak{M}_{\mathcal{P}_f}$ as $n \to \infty$. Indeed, $g_f^{\circ n}(x) \to 0$ if $|x| < 1$ and $g_f^{\circ n}(x) \to \infty$ if $|x| > 1$. In order to study the asymptotic behaviors, we will compactify $\mathfrak{M}_{\mathcal{P}_f}$ and study how the map G_f extends to the compactified space. Work of Nikita Selinger [S] suggests that when studying Thurston endomorphisms, it might be interesting to consider the Deligne-Mumford compactifications of the moduli space and its universal curve.

4.1. **The Deligne-Mumford compactification.** Let $\mathcal{P}$ be a finite set containing at least three points. A *curve of genus zero marked by $\mathcal{P}$* is an injection $\phi : \mathcal{P} \to \mathcal{C}$ where

- $\mathcal{C}$ is a connected algebraic curve whose only singularities are ordinary double points (called *nodes*), such that
- each irreducible component is isomorphic to $\mathbb{P}^1$
- the graph, whose vertices are the irreducible components and whose edges connect components intersecting at a node, is a tree

- each $\phi(p)$ is a smooth point of $\mathcal{C}$.

The curve is *stable* if the number of marked points and nodes on each irreducible component is at least 3. Two stable marked curves $\phi_1 : \mathcal{P} \to \mathcal{C}_1$ and $\phi_2 : \mathcal{P} \to \mathcal{C}_2$ are isomorphic if there is an isomorphism $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\phi_2 = \alpha \circ \phi_1$.

The set $\overline{\mathcal{M}}_\mathcal{P}$ of stable curves of genus zero marked by $\mathcal{P}$ modulo isomorphism may be naturally regarded as a smooth compact projective variety (see Knudsen [Kn], and Knudsen and Mumford [KM]). It contains $\mathcal{M}_\mathcal{P}$ as a dense open Zariski subset and is called the Deligne-Mumford compactification of $\mathcal{M}_\mathcal{P}$.

If $\mathcal{P} \subset \mathcal{Q}$, there is a map $\overline{\mathcal{M}}_\mathcal{Q} \to \overline{\mathcal{M}}_\mathcal{P}$ which consists of forgetting the points in $\mathcal{Q} - \mathcal{P}$, collapsing only the irreducible components which then contain fewer than three nodes and marked points.

When $\mathcal{Q} = \mathcal{P} \cup \{q\}$ with $q \notin \mathcal{P}$, there are sections $\overline{m}_p : \overline{\mathcal{M}}_\mathcal{P} \to \overline{\mathcal{M}}_\mathcal{Q}$ which may be defined as follows: given a stable curve $\phi : \mathcal{P} \to \mathcal{C}$, one may form a new stable curve $\psi : \mathcal{Q} \to \mathcal{C}'$ where

- $\mathcal{C}'$ is obtained by adjoining to $\mathcal{C}$ a Riemann sphere intersecting $\mathcal{C}$ at $\phi(p)$,
- ψ agrees with ϕ on $\mathcal{P} - \{p\}$, and
- $\psi(p)$ and $\psi(q)$ are distinct points in $\mathcal{C}' - \mathcal{C}$.

We define $\overline{m}_p\big([\phi]\big) := [\psi]$. It turns out that the forgetful map $\overline{\mathfrak{m}} : \overline{\mathcal{M}}_\mathcal{Q} \to \overline{\mathcal{M}}_\mathcal{P}$ with the sections $(\overline{m}_p : \overline{\mathcal{M}}_\mathcal{P} \to \overline{\mathcal{M}}_\mathcal{Q})_{p\in\mathcal{Q}}$ is the universal curve over $\overline{\mathcal{M}}_\mathcal{P}$: for each $x \in \overline{\mathcal{M}}_\mathcal{P}$, the injection

$$\mathcal{P} \ni p \mapsto \overline{m}_p(x) \in \overline{\mathfrak{m}}^{-1}(x)$$

represents x. From now on, we shall use the notations $\overline{\mathfrak{m}} : \overline{\mathfrak{M}}_\mathcal{P} \to \overline{\mathcal{M}}_\mathcal{P}$ and $\overline{m}_p : \overline{\mathcal{M}}_\mathcal{P} \to \overline{\mathfrak{M}}_\mathcal{P}$.

4.2. **Basilica matings.** Let us return to the case where $P(z) = Q(z) = z^2 - 1$ and $f = P \uplus Q$. Recall that $\mathcal{M}_{\mathcal{P}_f}$ is identified with $\mathbb{C} - \{0, 1\}$. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{\mathcal{P}_f}$ is identified with $\mathbb{P}^1$. The three added points $x = 0$, $x = 1$ and $x = \infty$ correspond to marked stable curves $\phi : \mathcal{P}_f \to \mathcal{C}$ where $\mathcal{C}$ is the union of two Riemann spheres intersecting at a single node:

- for $x = 0$, the injection ϕ sends p_0 and p_1 to the same component and q_0 and q_1 to the other component.
- for $x = 1$, the injection ϕ sends p_0 and q_0 to the same component and p_1 and q_1 to the other component.
- for $x = \infty$, the injection ϕ sends p_0 and q_1 to the same component and p_1 and q_0 to the other component.

Describing the universal curve $\overline{\mathfrak{m}} : \overline{\mathfrak{M}}_{\mathcal{P}_f} \to \overline{\mathcal{M}}_{\mathcal{P}_f}$ is slightly more difficult. We first describe $\overline{\mathfrak{M}}_{\mathcal{P}_f}$. Recall that $\mathfrak{M}_{\mathcal{P}_f}$ is identified with $\big(\mathbb{C} - \{0, 1\}\big) \times \mathbb{P}^1$ and one may be tempted to deduce that $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ is identified with $\mathbb{P}^1 \times \mathbb{P}^1$. But this is not the case. In fact, $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ is identified with $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the three points $(0, 0)$, $(1, 1)$ and (∞, ∞) (see [L]). The map $\overline{\mathfrak{m}} : \overline{\mathfrak{M}}_{\mathcal{P}_f} \to \overline{\mathcal{M}}_{\mathcal{P}_f}$ consists of blowing down $\overline{\mathfrak{M}}_{\mathcal{P}_f}$ to $\mathbb{P}^1$ and then projecting to the first coordinate. As expected, the fibers above $\mathbb{C} - \{0, 1\}$ are Riemann spheres, and each of the fibers above 0, 1 and ∞ consists of two Riemann spheres, precisely one of them being a component of the exceptional divisor produced by the blow up. The sections $\overline{m}_p : \overline{\mathcal{M}}_{\mathcal{P}_f} \to \overline{\mathfrak{M}}_{\mathcal{P}_f}$ are the continuous extensions of the sections $m_p : \mathcal{M}_{\mathcal{P}_f} \to \mathfrak{M}_{\mathcal{P}_f}$ (see Figure 7). We will use the notation

$$\mathfrak{S}_x := \overline{\mathfrak{m}}^{-1}(x) \quad \text{for} \quad x \in \overline{\mathcal{M}}_{\mathcal{P}_f}.$$

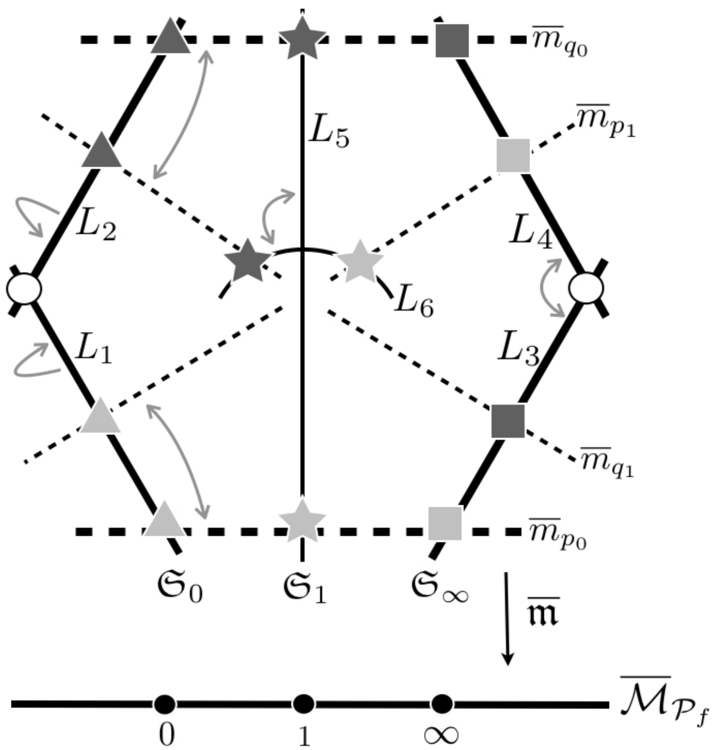

FIGURE 7. The universal curve $\overline{\mathfrak{m}} : \overline{\mathfrak{M}}_{\mathcal{P}_f} \to \overline{\mathcal{M}}_{\mathcal{P}_f}$. The fibers $\mathfrak{S}_0$, $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ are shown. The dashed lines indicate the images of the sections $\overline{m}_{p_0}, \overline{m}_{q_0}, \overline{m}_{p_1}, \overline{m}_{q_1} : \overline{\mathcal{M}}_{\mathcal{P}_f} \to \overline{\mathfrak{M}}_{\mathcal{P}_f}$. The thick lines (continuous and dashed) form the critical set of the rational map $G_f : \overline{\mathfrak{M}}_{\mathcal{P}_f} \dashrightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$. The white circles are superattracting fixed points of G_f. The light and dark grey triangles and squares form superattracting 2-cycles of G_f. The light and dark grey stars form 2-cycles of G_f with eigenvalues 0 and 2.

Now, to the maps

$$g_f : x \mapsto x^2 \quad \text{and} \quad G_f : (x,z) \mapsto \left(x^2, \frac{z^2 - x^2}{z^2 - 1}\right).$$

The first trivial observation is that the map g_f extends to an endomorphism of $\overline{\mathcal{M}}_{\mathcal{P}_f} = \mathbb{P}^1$. On $\mathbb{P}^1 \times \mathbb{P}^1$, the map G_f has five points of indeterminacy: (∞,∞), $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$. The critical locus of G_f on $\mathbb{P}^1 \times \mathbb{P}^1$ consists of the lines of equation $x = 0$, $z = 0$, $x = \infty$ and $z = \infty$. The map G_f has a superattracting fixed point at $(0,0)$.

We blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the indeterminacy points (∞,∞) and $(1,1)$, and at the superattracting fixed point $(0,0)$ to obtain $\overline{\mathfrak{M}}_{\mathcal{P}_f}$. The map G_f may now be regarded as a rational map $G_f : \overline{\mathfrak{M}}_{\mathcal{P}_f} \dashrightarrow \overline{\mathfrak{M}}_{\mathcal{P}_f}$ (see Figure 7).

The map has two symmetries that we wish to emphasize. The first comes from the fact that we mate the basilica polynomial with itself. Let $\iota_1 : \overline{\mathfrak{M}}_{\mathcal{P}_f} \to \overline{\mathfrak{M}}_{\mathcal{P}_f}$ be the involution defined by

$$\iota_1(x,z) := \left(x, \frac{x}{z}\right).$$

Then,

$$\iota_1 \circ G_f(x,z) = \left(x^2, \frac{x^2z^2 - x^2}{z^2-1}\right) = \left(x^2, \frac{x^2 - x^2/z^2}{1 - x^2/z^2}\right) = G_f \circ \iota_1(x,z).$$

The second symmetry is more subtle. It relates the dynamics of G_f over $\mathbb{P}^1 - \overline{\mathbb{D}}$ to the dynamics of G_f over $\mathbb{D}$. Let $\iota_2 : \overline{\mathfrak{M}}_{\mathcal{P}_f} \to \overline{\mathfrak{M}}_{\mathcal{P}_f}$ be the involution defined by

$$\iota_2(x,z) := \left(\frac{1}{x}, \frac{1}{z}\right)$$

which exchanges $\mathfrak{m}^{-1}(\mathbb{D})$ and $\mathfrak{m}^{-1}(\mathbb{P}^1 - \overline{\mathbb{D}})$. Then,

$$\iota_2 \circ \iota_1(x,z) = \left(\frac{1}{x}, \frac{z}{x}\right) = \iota_1 \circ \iota_2(x,z)$$

and

$$\iota_2 \circ G_f(x,z) = \left(\frac{1}{x^2}, \frac{z^2-1}{z^2-x^2}\right) = \left(\frac{1}{x^2}, \frac{z^2/x^2 - 1/x^2}{z^2/x^2 - 1}\right) = G_f \circ \iota_2 \circ \iota_1(x,z).$$

As a consequence,

$$\begin{aligned} \iota_2 \circ G_f^{\circ 2} = (G_f \circ \iota_2 \circ \iota_1) \circ G_f &= G_f \circ \iota_2 \circ G_f \circ \iota_1 \\ &= G_f \circ (G_f \circ \iota_2 \circ \iota_1) \circ \iota_1 = G_f^{\circ 2} \circ \iota_2. \end{aligned}$$

4.3. **The fiber over** 0**.** The fiber $\mathfrak{S}_0$ is the union of two lines L_1 and L_2 which intersect at a node. This node, represented by a white circle, is a superattracting fixed point. The map G_f fixes both of the lines L_1 and L_2. In each line, the map restricts to a rational map of degree 2 which has a superattracting fixed point at the node and a superattracting cycle of period 2:

- the light grey triangles $\overline{m}_{p_0}(0)$ and $\overline{m}_{p_1}(0)$ are exchanged in L_1 and
- the dark grey triangles $\overline{m}_{q_0}(0)$ and $\overline{m}_{q_1}(0)$ are exchanged in L_2.

Hence, the restrictions $G_f : L_1 \to L_1$ and $G_f : L_2 \to L_2$ are conjugate to the basilica polynomial P.

4.4. **The fiber over** ∞**.** The fiber $\mathfrak{S}_\infty$ is the union of two lines L_3 and L_4 which intersect at a node. This node, represented by a white circle, is a superattracting fixed point. The lines L_3 and L_4 are exchanged by G_f:

- the light grey squares $\overline{m}_{q_1}(\infty)$ in L_3 and $\overline{m}_{q_0}(\infty)$ in L_4 are exchanged and form a superattracting 2-cycle and
- the dark grey squares $\overline{m}_{p_0}(\infty)$ in L_3 and $\overline{m}_{p_1}(\infty)$ in L_4 are exchanged and form a superattracting 2-cycle.

The second iterate of the map G_f commutes with ι_2 which sends L_3 to L_2 and L_4 to L_1. So, it fixes L_3 and L_4 and the restrictions $G_f^{\circ 2} : L_3 \to L_3$ and $G_f^{\circ 2} : L_4 \to L_4$ are conjugate to the second iterate $P^{\circ 2}$ of the basilica polynomial.

4.5. **The fiber over** 1. The fiber $\mathfrak{S}_1$ is the union of two lines L_5 and L_6 which intersect at a node. This node is a repelling fixed point of G_f. The derivative of $G_f^{\circ 2}$ at the node is multiplication by 2. The two lines are exchanged by G_f:

- the line L_5, which intersects the critical locus at $m_{p_0}(1)$ and $m_{q_0}(1)$ is mapped with degree 2 to the line L_6,
- the line L_6 is mapped isomorphically to L_5,
- the points $\overline{m}_{p_0}(1) \in L_5$ and $\overline{m}_{p_1}(1) \in L_6$ form a cycle of period 2 with eigenvalues 0 and 2,
- the points $\overline{m}_{q_0}(1) \in L_5$ and $\overline{m}_{q_1}(1) \in L_6$ form a cycle of period 2 with eigenvalues 0 and 2.

The restriction of $G_f^{\circ 2}$ to each line L_5 and L_6 is conjugate to $z \mapsto z^2$ with repelling fixed point at the node and superattracting fixed points at the marked points in $\mathfrak{S}_1$ represented by the grey stars.

4.6. **The fibers over** $\mathbb{D} - \{0\}$**.** If $x \in \mathbb{D} - \{0\}$, the fiber $\mathfrak{S}_x$ gets attracted by $\mathfrak{S}_0$ under iteration of G_f. We may partition $\mathfrak{S}_x$ according to whether the orbit of a point is attracted by

- the superattracting fixed point in $\mathfrak{S}_0$,
- the superattracting 2-cycle $\{\overline{m}_{p_0}(0), \overline{m}_{p_1}(0)\}$,
- the superattracting 2-cycle $\{\overline{m}_{q_0}(0), \overline{m}_{q_1}(0)\}$ or
- none of the above.

Figure 8 shows such a partition in different fibers. The color corresponds to the color of the cycle the orbit gets attracted to and is black otherwise. In Section 4.9, we shall describe this partition in terms of *equipotential gluing*.

4.7. **The fibers over** $\mathbb{C} - \overline{\mathbb{D}}$**.** If $x \in \mathbb{C} - \overline{\mathbb{D}}$, the fiber $\mathfrak{S}_x$ gets attracted by $\mathfrak{S}_\infty$ under iteration of G_f. We may partition $\mathfrak{S}_x$ according to whether the orbit of a point is attracted by

- the superattracting fixed point in $\mathfrak{S}_\infty$,
- the superattracting 2-cycle $\{\overline{m}_{p_0}(\infty), \overline{m}_{p_1}(\infty)\}$,
- the superattracting 2-cycle $\{\overline{m}_{q_0}(\infty), \overline{m}_{q_1}(\infty)\}$ or
- none of the above.

4.8. **Equipotential gluing.** Let $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ be monic polynomials of degree $d \geq 2$ with connected Julia sets (not necessarily postcritically finite). Let $h_P : \mathbb{C} \to [0, +\infty)$ be the Green's function associated to P:

$$h_P(z) := \lim_{n \to +\infty} \frac{1}{d^n} \max\bigl(0, \log |P^{\circ n}(z)|\bigr),$$

and $h_Q : \mathbb{C} \to [0, +\infty]$ the Green's function associated to Q. Let U_P and U_Q be the subsets of $\mathbb{D} \times \mathbb{C}$ defined by

$$U_P := \bigl\{(t, z) \mid h_P(z) < -\log |t|\bigr\} \quad \text{and} \quad U_Q := \bigl\{(t, w) \mid h_Q(w) < -\log |t|\bigr\}.$$

Let $V \subset \mathbb{C}^3$ be the complex surface defined by

$$V := \bigl\{(t, z, w) \in \mathbb{C}^3 \mid zw = t, |t| < 1, |z| < 1, |w| < 1\bigr\}.$$

Let $\mathfrak{U}_{P,Q}$ be the analytic manifold obtained by gluing V, U_P and U_Q, in the following way:

- a point (t, z, w) in V with $z \neq 0$ is identified with the point $\bigl(t, \text{böt}_P(1/z)\bigr)$ in U_P and

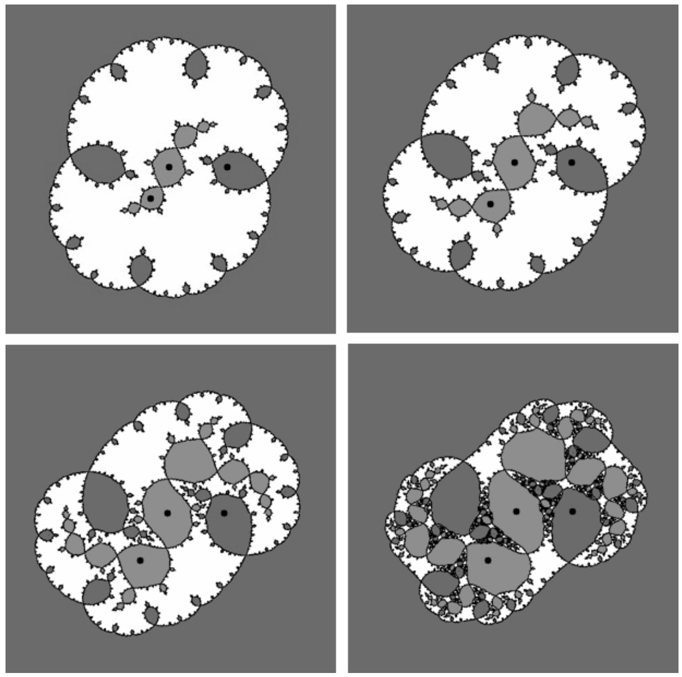

FIGURE 8. The partition in four fibers $\mathfrak{S}_{x_1}$, $\mathfrak{S}_{x_2}$, $\mathfrak{S}_{x_3}$, $\mathfrak{S}_{x_4}$ with $\arg(x_j) = 4\pi/3$ and $0 < |x_1| < |x_2| < |x_3| < |x_4| < 1$. The rational map $G_f^{\circ 2} : \mathfrak{S}_{e^{4\pi i/3}} \to \mathfrak{S}_{e^{4\pi i/3}}$ is the geometric twisted mating of angle $1/3$ of $P^{\circ 2}$ with itself. Points colored white are in the basin of the superattracting fixed point in $\mathfrak{S}_0$, those colored light grey are in the basin of the light grey 2-cycle in L_1, and those colored dark grey are in the basin of the dark grey 2-cycle in L_2.

- a point (t,z,w) in V with $w \neq 0$ is identified with the point $\big(t, \text{böt}_P(1/w)\big)$ in U_Q.

Note that automatically, if $t \neq 0$ and $(t,z,w) \in V$, then the points

$$\big(t, \text{böt}_P(1/z)\big) \in U_P \quad \text{and} \quad \big(t, \text{böt}_Q(1/w)\big) \in U_Q$$

are identified.

Let $\mathfrak{s} : \mathfrak{U}_{P,Q} \to \mathbb{D}$ be the map given by $\mathfrak{s}(t,z) := t$ on U_P, $\mathfrak{s}(t,w) := t$ on U_Q and $\mathfrak{s}(t,z,w) := t$ on V. According to the characterization given in [HK] Definition 3.1, this is a proper flat family of curves. The fiber Σ_t of this map over a point $t \in \mathbb{D} - \{0\}$ is a Riemann sphere, whereas the fiber Σ_0 over 0 is singular: it is the union of two Riemann spheres intersecting at an ordinary double point.

It is important to observe that there is a skew-product $S_{P,Q} : \mathfrak{U}_{P,Q} \to \mathfrak{U}_{P,Q}$ which is defined by

- $(t,z) \mapsto \big(t^d, P(z)\big)$ on U_P,
- $(t,w) \mapsto \big(t^d, Q(w)\big)$ on U_Q and
- $(t,z,w) \mapsto (t^d, z^d, w^d)$ on V.

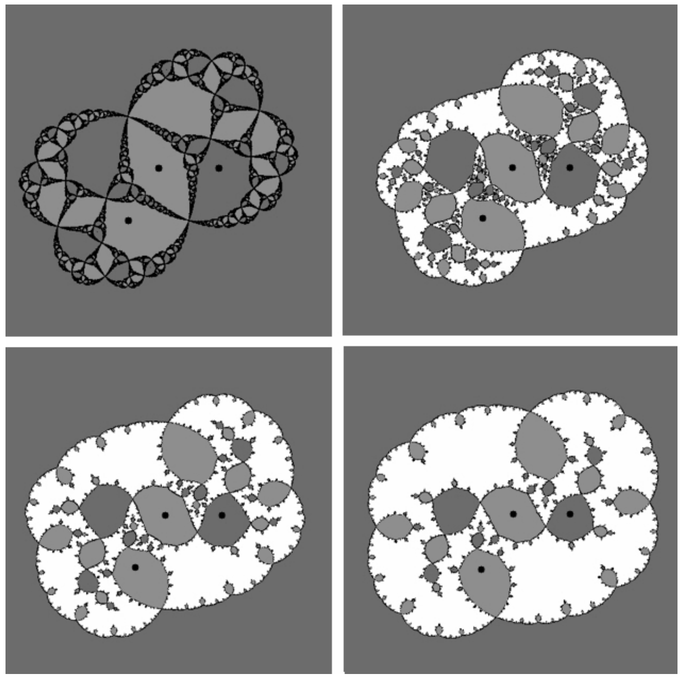

FIGURE 9. The partition in four fibers $\mathfrak{S}_{x_5}$, $\mathfrak{S}_{x_6}$, $\mathfrak{S}_{x_7}$, $\mathfrak{S}_{x_8}$ with $\arg(x_j) = 4\pi/3$ and $1 = |x_5| < |x_6| < |x_7| < |x_8|$. In the upper-left, $G_f^{\circ 2} : \mathfrak{S}_{x_5} \to \mathfrak{S}_{x_5}$ is the geometric twisted mating of angle $1/3$ of $P^{\circ 2}$ with itself. In the other fibers, points colored white are in the basin of the superattracting fixed point in $\mathfrak{S}_\infty$, those colored light grey are in the basin of the light grey 2-cycle in $\mathfrak{S}_\infty$, and those colored dark grey are in the basin of the dark grey 2-cycle in $\mathfrak{S}_\infty$.

and satisfies the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{U}_{P,Q} \arrow[r, "S_{P,Q}"] \arrow[d, "\mathfrak{s}"'] & \mathfrak{U}_{P,Q} \arrow[d, "\mathfrak{s}"] \\
\mathbb{D} \arrow[r, "t\mapsto t^d"] & \mathbb{D}
\end{tikzcd}$$

Each component of Σ_0 is fixed by $S_{P,Q}$. The restriction of $S_{P,Q}$ to one of them is conjugate to P and the restriction to the other is conjugate to Q. Under iteration of $S_{P,Q}$, the orbit of every point in $\mathfrak{U}_{P,Q}$ converges to Σ_0. The fiber Σ_t lying above $t \in \mathbb{D} - \{0\}$ contains an isomorphic copy of K_P and an isomorphic copy of K_Q, separated by an annulus of modulus $-\log|t|$. In Σ_t, points in the external ray $\mathcal{R}_P(\theta_1)$ are glued to points in the external ray $\mathcal{R}_Q(\theta_2)$ if and only if

$$\theta_1 + \theta_2 = -\frac{1}{2\pi}\arg(t).$$

The skew-product $S_{P,Q}$ sends the fiber Σ_t to the fiber Σ_{t^d}. The copy of K_P in Σ_t is mapped to the copy of K_P in Σ_{t^d} and the copy of K_Q in Σ_t is mapped to the copy of K_Q in Σ_{t^d}.

4.9. **The dynamics of G_f over $\mathbb{D}$.** We return to the case where $P = Q$ is the basilica polynomial. As mentioned previously, the fiber $\mathfrak{S}_0$ has a basin of attraction

$$\mathfrak{B}_f := \overline{\mathfrak{m}}^{-1}(\mathbb{D}).$$

Under iteration of G_f, the orbit of every point in $\mathfrak{B}_f$ converges to $\mathfrak{S}_0$. We understand the dynamics of G_f in $\mathfrak{B}_f$ as follows.

Proposition 7. *There is an isomorphism $\alpha : \mathfrak{U}_{P,Q} \to \mathfrak{B}_f$ which conjugates the map $S_{P,Q} : \mathfrak{U}_{P,Q} \to \mathfrak{U}_{P,Q}$ to $G_f : \mathfrak{B}_f \to \mathfrak{B}_f$, that is $\alpha \circ S_{P,Q} = G_f \circ \alpha$ on $\mathfrak{U}_{P,Q}$.*

Proof. There are sections $s_{p_0}, s_{q_0}, s_{p_1}, s_{q_1} : \mathbb{D} \to \mathfrak{U}_{P,Q}$ such that for each $t \in \mathbb{D}$,

- $s_{p_0}(t) \in \Sigma_t$ corresponds to $0 \in U_P$,
- $s_{q_0}(t) \in \Sigma_t$ corresponds to $0 \in U_Q$,
- $s_{p_1}(t) \in \Sigma_t$ corresponds to $-1 \in U_P$ and
- $s_{q_1}(t) \in \Sigma_t$ corresponds to $-1 \in U_Q$.

Now, $\mathfrak{s} : \mathfrak{U}_{P,Q} \to \mathbb{D}$ is a proper flat family of stable curves of genus zero with distinct sections $s_{p_0}, s_{q_0}, s_{p_1}, s_{q_1} : \mathbb{D} \to \mathfrak{U}_{P,Q}$. So, there are analytic maps $\alpha : \mathfrak{U}_{P,Q} \to \overline{\mathfrak{M}}_{\mathcal{P}_f}$ and $\beta : \mathbb{D} \to \mathbb{P}^1$ such that $\mathfrak{m} \circ \alpha = \beta \circ s$, the restriction $\alpha : \Sigma_t \to \mathfrak{S}_{\beta(t)}$ is an isomorphism for every $t \in \mathbb{D}$ and $m_p \circ \beta = \alpha \circ s_p$ for every $p \in \mathcal{P}_f$.

Fix $t \in \mathbb{D}$. Set $x := \beta(t)$ and $y := \beta(t^2)$. Let

- $\alpha_x : \Sigma_t \to \mathbb{P}^1$ be the normalized isomorphism which sends $s_{p_0}(t)$ to 0, $s_{q_0}(t)$ to ∞ and $s_{q_1}(t)$ to 1, and
- $\alpha_y : \Sigma_{t^2} \to \mathbb{P}^1$ be the normalized isomorphism which sends $s_{p_0}(t^2)$ to 0, $s_{q_0}(t^2)$ to ∞ and $s_{q_1}(t^2)$ to 1,

so that α_x sends $s_{p_1}(t)$ to x and α_y sends $s_{p_1}(t^2)$ to y. Then,

$$F := \alpha_y \circ S_{P,Q} \circ \alpha_x^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$$

is a rational map of degree 2 with critical points at 0 and ∞ such that

$$F(\infty) = 1, \quad F(1) = \infty, \quad F(x) = 0 \quad \text{and} \quad F(0) = y.$$

According to Section 2.2, $y = x^2$ and the following diagram commutes:

$$\begin{tikzcd} \Sigma_t \arrow[r, "\alpha"] \arrow[d, "S_{P,Q}"'] & \mathfrak{S}_x \arrow[d, "G_f"] \\ \Sigma_{t^2} \arrow[r, "\alpha"] & \mathfrak{S}_{x^2}. \end{tikzcd}$$

In particular, $\beta(t^2) = \big(\beta(t)\big)^2$ and so, β is a power map: $x = t^k$ for some $k \geq 1$.

Lemma 2. *We have that $k = 1$.*

Proof. Let A be the annulus of maximal modulus mod_t separating $s_{p_0}(t)$ and $s_{p_1}(t)$ from $s_{q_0}(t)$ and $s_{q_1}(t)$ in Σ_t. Its equator separates A in two annuli of equal moduli $\frac{1}{2}\mathrm{mod}_t$ and is invariant by the Möbius transformation which exchanges $s_{p_0}(t)$ and $s_{p_1}(t)$ with $s_{q_0}(t)$ and $s_{q_1}(t)$. Since this Möbius transformation exchanges the two copies of filled-in-Julia set K_P and K_Q, this equator is also that of the annulus separating the two copies of filled-in Julia sets. It follows that there is an annulus of modulus $\frac{1}{2}\mathrm{mod}_t$ separating 0 and -1 from the equipotential of level $-\frac{1}{2}\log|t|$. As $t \to 0$, this equipotential is asymptotic to the circle centered at 0 with radius $\sqrt{|t|}$. We deduce that

$$\frac{1}{2}\mathrm{mod}_t \leq -\frac{1}{2}\log|t| + \mathcal{O}(1).$$

In $\mathfrak{S}_x$ the points 0 and x may be separated from the points ∞ and 1 by a Euclidean annulus of modulus $-\log|x|$. Thus, in Σ_t the points $s_{p_0}(t)$ and $s_{p_1}(t)$ may be separated from the points $s_{q_0}(t)$ and $s_{q_1}(t)$ by an annulus of the same modulus $-k\log|t|$. Therefore, as $t \to 0$,

$$-k\log|t| \leq -\log|t| + \mathcal{O}(1)$$

and $k = 1$ as required. □

So, $x = t$ and $\beta : \mathbb{D} \to \mathbb{D}$ is an isomorphism. Thus, $\alpha : \mathfrak{U}_{P,Q} \to \mathfrak{B}_f$ is an isomorphism. It conjugates $S_{P,Q} : \mathfrak{U}_{P,Q} \to \mathfrak{U}_{P,Q}$ to $G_f : \mathfrak{B}_f \to \mathfrak{B}_f$ according to the preceding diagram. □

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