On Thurston's pullback map

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Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with finite postcritical set P_f. Thurston showed that f induces a holomorphic map $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ of the Teichmüller space to itself. The map σ_f fixes the basepoint corresponding to the identity map $\mathrm{id} : (\mathbb{P}^1, P_f) \to (\mathbb{P}^1, P_f)$. We give examples of such maps f showing that the following cases may occur:

1. the basepoint is an attracting fixed point, the image of σ_f is open and dense in $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \sigma_f\big(\mathrm{Teich}(\mathbb{P}^1, P_f)\big)$ is a covering map,
2. the basepoint is a superattracting fixed point, the image of σ_f is $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is a ramified Galois covering map, or
3. the map σ_f is constant.

Introduction

In this article, Σ is an oriented 2-sphere. All maps $\Sigma \to \Sigma$ are assumed to be orientation-preserving. The map $f : \Sigma \to \Sigma$ is a branched covering of degree $d \geqslant 2$. A particular case of interest is when Σ can be equipped with an invariant complex structure for f. In that case, $f : \Sigma \to \Sigma$ is conjugate to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$.

According to the Riemann-Hurwitz formula, the map f has $2d-2$ critical points, counting multiplicities. We denote Ω_f the set of critical points and $V_f := f(\Omega_f)$ the set of critical values of f. The *postcritical set* of f is the set

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f).$$

The map f is *postcritically finite* if P_f is finite. Following the literature, we refer to such maps simply as *Thurston maps*.

Two Thurston maps $f : \Sigma \to \Sigma$ and $g : \Sigma \to \Sigma$ are *equivalent* if there are homeomorphisms $h_0 : (\Sigma, P_f) \to (\Sigma, P_g)$ and $h_1 : (\Sigma, P_f) \to (\Sigma, P_g)$

for which $h_0 \circ f = g \circ h_1$ and h_0 is isotopic to h_1 through homeomorphisms agreeing on P_f. In particular, we have the following commutative diagram:

$$\begin{array}{ccc} (\Sigma, P_f) & \xrightarrow{h_1} & (\Sigma, P_g) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (\Sigma, P_f) & \xrightarrow{h_0} & (\Sigma, P_g). \end{array}$$

In [DH], Douady and Hubbard, following Thurston, give a complete characterization of equivalence classes of rational maps among those of Thurston maps. The characterization takes the following form.

A branched covering $f : (\Sigma, P_f) \to (\Sigma, P_f)$ induces a holomorphic self-map

$$\sigma_f : \mathrm{Teich}(\Sigma, P_f) \to \mathrm{Teich}(\Sigma, P_f)$$

of Teichmüller space (see Section 1 for the definition). Since it is obtained by lifting complex structures under f, we will refer to σ_f as the *pullback map* induced by f. The map f is equivalent to a rational map if and only if the pullback map σ_f has a fixed point. By a generalization of the Schwarz lemma, σ_f does not increase Teichmüller distances. For most maps f, the pullback map σ_f is a contraction, and so a fixed point, if it exists, is unique.

In this note, we give examples showing that the contracting behavior of σ_f near this fixed point can be rather varied.

Theorem 1 *There exist Thurston maps f for which σ_f is contracting, has a fixed point τ and:*

1. *the derivative of σ_f is invertible at τ, the image of σ_f is open and dense in $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \sigma_f\big(\mathrm{Teich}(\mathbb{P}^1, P_f)\big)$ is a covering map,*
2. *the derivative of σ_f is not invertible at τ, the image of σ_f is equal to $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is a ramified Galois covering map,*[1]

   or
3. *the map σ_f is constant.*

In Section 1, we establish notation, define Teichmüller space and the pullback map σ_f precisely, and develop some preliminary results used in our

[1] A ramified covering is Galois if the group of deck transformations acts transitively on the fibers.

subsequent analysis. In Sections 2, 3, and 4.1, respectively, we give concrete examples which together provide the proof of Theorem 1. We supplement these examples with some partial general results. In Section 2, we state a fairly general sufficient condition on f under which σ_f evenly covers it image. This condition, which can sometimes be checked in practice, is excerpted from [K1] and [K2]. Our example illustrating (2) is highly symmetric and atypical; we are not aware of any reasonable generalization. In Section 4.2, we state three conditions on f equivalent to the condition that σ_f is constant. Unfortunately, each is extremely difficult to verify in concrete examples.

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1 Preliminaries

Recall that a Riemann surface is a connected oriented topological surface together with a *complex structure*: a maximal atlas of charts $\phi : U \to \mathbb{C}$ with holomorphic overlap maps. For a given oriented, compact topological surface X, we denote the set of all complex structures on X by $\mathcal{C}(X)$. It is easily verified that an orientation-preserving branch covering map $f : X \to Y$ induces a map $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$; in particular, for any orientation-preserving homeomorphism $\psi : X \to X$, there is an induced map $\psi^* : \mathcal{C}(X) \to \mathcal{C}(X)$.

Let $A \subset X$ be finite. The Teichmüller space of (X, A) is

$$\mathrm{Teich}(X, A) := \mathcal{C}(X)/{\sim_A}$$

where $c_1 \sim_A c_2$ if and only if $c_1 = \psi^*(c_2)$ for some orientation-preserving homeomorphism $\psi : X \to X$ which is isotopic to the identity relative to A. In view of the homotopy-lifting property, if

- $B \subset Y$ is finite and contains the critical value set V_f of f, and
- $A \subseteq f^{-1}(B)$,

then $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ descends to a well-defined map σ_f between the corresponding Teichmüller spaces:

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{\;f^*\;} & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathrm{Teich}(Y, B) & \xrightarrow{\;\sigma_f\;} & \mathrm{Teich}(X, A). \end{array}$$

This map is known as the *pullback map* induced by f.

In addition if $f : X \to Y$ and $g : Y \to Z$ are orientation-preserving branch covering maps and if $A \subset X$, $B \subset Y$ and $C \subset Z$ are such that

- B contains V_f and C contains V_g,
- $A \subseteq f^{-1}(B)$ and $B \subseteq g^{-1}(C)$,

then C contains the critical values of $g \circ f$ and $A \subseteq (g \circ f)^{-1}(C)$. Thus

$$\sigma_{g \circ f} : \mathrm{Teich}(Z, C) \to \mathrm{Teich}(X, A)$$

can be decomposed as $\sigma_{g \circ f} = \sigma_f \circ \sigma_g$:

$$\mathrm{Teich}(Z, C) \xrightarrow{\sigma_g} \mathrm{Teich}(Y, B) \xrightarrow{\sigma_f} \mathrm{Teich}(X, A), \qquad \mathrm{Teich}(Z, C) \xrightarrow{\sigma_{g \circ f}} \mathrm{Teich}(X, A).$$

For the special case of $\mathrm{Teich}(\mathbb{P}^1, A)$, we may use the Uniformization Theorem to obtain the following description. Given a finite set $A \subset \mathbb{P}^1$ we may regard $\mathrm{Teich}(\mathbb{P}^1, A)$ as the quotient of the space of all orientation-preserving homeomorphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ by the equivalence relation $\sim$ whereby $\phi_1 \sim \phi_2$ if there exists a Möbius transformation μ such that $\mu \circ \phi_1 = \phi_2$ on A, and $\mu \circ \phi_1$ is isotopic to ϕ_2 relative to A. Note that there is a natural basepoint $\circledast$ given by the class of the identity map on $\mathbb{P}^1$. It is well-known that $\mathrm{Teich}(\mathbb{P}^1, A)$ has a natural topology and complex manifold structure (see [H]).

The *moduli space* is the space of all injections $\psi : A \hookrightarrow \mathbb{P}^1$ modulo postcomposition with Möbius transformations. The moduli space will be denoted as $\mathrm{Mod}(\mathbb{P}^1, A)$. If ϕ represents an element of $\mathrm{Teich}(\mathbb{P}^1, A)$, the restriction $[\phi] \mapsto \phi|_A$ induces a universal covering $\pi : \mathrm{Teich}(\mathbb{P}^1, A) \to \mathrm{Mod}(\mathbb{P}^1, A)$ which is a local biholomorphism with respect to the complex structures on $\mathrm{Teich}(\mathbb{P}^1, A)$ and $\mathrm{Mod}(\mathbb{P}^1, A)$.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a Thurston map with $|P_f| \geqslant 3$. For any $\Theta \subseteq P_f$ with $|\Theta| = 3$, there is an obvious identification of $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with an open subset of $(\mathbb{P}^1)^{P_f - \Theta}$. Assume $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$ and let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a homeomorphism representing τ with $\phi|_\Theta = \mathrm{id}|_\Theta$. By the Uniformization Theorem, there exist

- a unique homeomorphism $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ representing $\tau' := \sigma_f(\tau)$ with $\psi|_\Theta = \mathrm{id}|_\Theta$ and
- a unique rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$,

such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{P}^1, P_f) & \xrightarrow{\psi} & \big(\mathbb{P}^1, \psi(P_f)\big) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ (\mathbb{P}^1, P_f) & \xrightarrow{\phi} & \big(\mathbb{P}^1, \phi(P_f)\big). \end{array}$$

Conversely, if we have such a commutative diagram with F holomorphic, then

$$\sigma_f(\tau) = \tau'$$

where $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\tau' \in \mathrm{Teich}(\mathbb{P}^1, P_f)$ are the equivalence classes of ϕ and ψ respectively. In particular, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is rational, then $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ fixes the basepoint $\circledast$.

2 Proof of (1)

In this section, we prove that there are Thurston maps $f : \Sigma \to \Sigma$ such that σ_f

- is contracting,
- has a fixed point $\tau \in \mathrm{Teich}(\Sigma, P_f)$ and
- is a covering map over its image which is open and dense in $\mathrm{Teich}(\Sigma, P_f)$.

In fact, we show that this is the case when $\Sigma = \mathbb{P}^1$ and $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial whose critical points are all periodic. The following is adapted from [K2].

Proposition 1 *If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial of degree $d \geqslant 2$ whose critical points are all periodic, then*

- $\sigma_f\big(\mathrm{Teich}(\mathbb{P}^1, P_f)\big)$ *is open and dense in* $\mathrm{Teich}(\mathbb{P}^1, P_f)$ *and*
- $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \sigma_f\big(\mathrm{Teich}(\mathbb{P}^1, P_f)\big)$ *is a covering map.*

In particular, the derivative $D\sigma_f$ is invertible at the fixed point $\circledast$.

This section is devoted to the proof of this proposition.

Let $n = |P_f| - 3$. We will identify $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with an open subset of $\mathbb{P}^n$ as follows. First enumerate the finite postcritical points as $p_0, \ldots, p_{n+1}$. Any point of $\mathrm{Mod}(\mathbb{P}^1, P_f)$ has a representative $\psi : P_f \hookrightarrow \mathbb{P}^1$ such that

$$\psi(\infty) = \infty \quad \text{and} \quad \psi(p_0) = 0.$$

Two such representatives are equal up to multiplication by a nonzero complex number. We identify the point in $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with the point

$$[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \quad \text{where} \quad x_1 := \psi(p_1) \in \mathbb{C}, \ldots, x_{n+1} := \psi(p_{n+1}) \in \mathbb{C}.$$

In this way, the moduli space $\mathrm{Mod}(\mathbb{P}^1, P_f)$ is identified with $\mathbb{P}^n - \Delta$, where Δ is the *forbidden locus*:

$$\Delta := \big\{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n \ ; \ (\exists i,\ x_i = 0) \text{ or } (\exists i \neq j,\ x_i = x_j)\big\}.$$

The universal cover $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Mod}(\mathbb{P}^1, P_f)$ is then identified with a universal cover $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathbb{P}^n - \Delta$.

Generalizing a result of Bartholdi and Nekrashevych [BN], the thesis [K1] showed that when $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a unicritical polynomial there is an analytic endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$ for which the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Teich}(\mathbb{P}^1, P_f) & \xrightarrow{\ \sigma_f\ } & \mathrm{Teich}(\mathbb{P}^1, P_f) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^n & \xleftarrow{\ g_f\ } & \mathbb{P}^n. \end{array}$$

We show that the same result holds when $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial whose critical points are all periodic.

Proposition 2 *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a polynomial of degree $d \geqslant 2$ whose critical points are all periodic. Set $n := |P_f| - 3$. Then,*

1. *there is an analytic endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$ for which the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Teich}(\mathbb{P}^1, P_f) & \xrightarrow{\ \sigma_f\ } & \mathrm{Teich}(\mathbb{P}^1, P_f) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^n & \xleftarrow{\ g_f\ } & \mathbb{P}^n \end{array}$$

2. *σ_f takes its values in $\mathrm{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L})$ with $\mathcal{L} := g_f^{-1}(\Delta)$,*
3. *$g_f(\Delta) \subseteq \Delta$ and*
4. *the critical point locus and the critical value locus of g_f are contained in Δ.*

Proof of Proposition 1 assuming Proposition 2: Note that $\mathcal{L}$ is a codimension 1 analytic subset of $\mathbb{P}^n$, whence $\pi^{-1}(\mathcal{L})$ is a codimension 1 analytic subset of $\mathrm{Teich}(\mathbb{P}^1, P_f)$. Thus, the complementary open sets are dense and connected. Since $g_f : \mathbb{P}^n - \mathcal{L} \to \mathbb{P}^n - \Delta$ is a covering map, the compostion

$$g_f \circ \pi : \mathrm{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L}) \to \mathbb{P}^n - \Delta$$

is a covering map. Moreover,

$$\pi(\circledast) = g_f \circ \pi \circ \sigma_f(\circledast) = g_f \circ \pi(\circledast).$$

By universality of the covering map $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathbb{P}^n - \Delta$, there is a unique map $\sigma : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L})$ such that

- $\sigma(\circledast) = \circledast$ and
- the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Teich}(\mathbb{P}^1, P_f) & \xrightarrow{\sigma} & \mathrm{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L}) \\ \downarrow \pi & \swarrow g_f \circ \pi & \\ \mathbb{P}^n - \Delta. & & \end{array}$$

Furthermore, $\sigma : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L})$ is a covering map. Finally, by uniqueness we have $\sigma_f = \sigma$. □

Proof of Proposition 2:
(1) We first show the existence of the endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$. We start with the definition of g_f.

The restriction of f to P_f is a permutation which fixes ∞. Denote by $\mu : [0, n+1] \to [0, n+1]$ the permutation defined by:

$$p_{\mu(k)} = f(p_k)$$

and denote by ν the inverse of μ.

For $k \in [0, n+1]$, let m_k be the multiplicity of p_k as a critical point of f (if p_k is not a critical point of f, then $m_k := 0$).

Set $a_0 := 0$ and let $Q \in \mathbb{C}[a_1, \ldots, a_{n+1}, z]$ be the homogeneous polynomial of degree d defined by

$$Q(a_1, \ldots, a_{n+1}, z) := \int_{a_{\nu(0)}}^{z} \left(d \prod_{k=0}^{n+1} (w - a_k)^{m_k} \right) \mathrm{d}w.$$

Given $\mathbf{a} \in \mathbb{C}^{n+1}$, let $F_{\mathbf{a}} \in \mathbb{C}[z]$ be the monic polynomial defined by

$$F_{\mathbf{a}}(z) := Q(a_1, \ldots, a_{n+1}, z).$$

Note that $F_{\mathbf{a}}$ is the unique monic polynomial of degree d which vanishes at $a_{\nu(0)}$ and whose critical points are exactly those points a_k for which $m_k > 0$, counted with multiplicity m_k.

Let $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the homogeneous map of degree d defined by

$$G_f \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} := \begin{pmatrix} F_{\mathbf{a}}(a_{\nu(1)}) \\ \vdots \\ F_{\mathbf{a}}(a_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} Q(a_1, \ldots, a_{n+1}, a_{\nu(1)}) \\ \vdots \\ Q(a_1, \ldots, a_{n+1}, a_{\nu(n+1)}) \end{pmatrix}.$$

We claim that $G_f^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ and thus, $G_f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces an endomorphism $g_f : \mathbb{P}^n \to \mathbb{P}^n$. Indeed, let us consider a point $\mathbf{a} \in \mathbb{C}^{n+1}$. By definition of G_f, if $G_f(\mathbf{a}) = \mathbf{0}$, then the monic polynomial $F_{\mathbf{a}}$ vanishes at $a_0, a_1, \ldots, a_{n+1}$. The critical points of $F_{\mathbf{a}}$ are those points a_k for which $m_k > 0$. They are all mapped to 0 and thus, $F_{\mathbf{a}}$ has only one critical value in $\mathbb{C}$. All the preimages of this critical value must coincide and since $a_0 = 0$, they all coincide at 0. Thus, for all $k \in [0, n+1]$, $a_k = 0$.

Let us now prove that for all $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$, we have

$$\pi(\tau) = g_f \circ \pi \circ \sigma_f(\tau).$$

Let τ be a point in $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and set $\tau' := \sigma_f(\tau)$.

We will show that there is a representative ϕ of τ and a representative ψ of τ' such that $\phi(\infty) = \psi(\infty) = \infty$, $\phi(p_0) = \psi(p_0) = 0$ and

$$G_f\big(\psi(p_1), \ldots, \psi(p_{n+1})\big) = \big(\phi(p_1), \ldots, \phi(p_{n+1})\big). \tag{2.1}$$

It then follows that

$$g_f\big([\psi(p_1) : \ldots : \psi(p_{n+1})]\big) = [\phi(p_1) : \ldots : \phi(p_{n+1})]$$

which concludes the proof since

$$\pi(\tau') = [\psi(p_1) : \ldots : \psi(p_{n+1})] \quad \text{and} \quad \pi(\tau) = [\phi(p_1) : \ldots : \phi(p_{n+1})].$$

To show the existence of ϕ and ψ, we may proceed as follows. Let ϕ be any representative of τ such that $\phi(\infty) = \infty$ and $\phi(p_0) = 0$. Then, there is a representative $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ of τ' and a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1. \end{array}$$

We may normalize ψ so that $\psi(\infty) = \infty$ and $\psi(p_0) = 0$. Then, F is a polynomial of degree d. Multiplying ψ by a nonzero complex number, we may assume that F is a monic polynomial.

We now check that these homeomorphisms ϕ and ψ satisfy the required Property (2.1). For $k \in [0, n+1]$, set

$$x_k := \psi(p_k) \quad \text{and} \quad y_k := \phi(p_k).$$

We must show that

$$G_f(x_1, \ldots, x_{n+1}) = (y_1, \ldots, y_{n+1}).$$

Note that for $k \in [0, n+1]$, we have the following commutative diagram:

$$\begin{array}{ccc} p_{\nu(k)} & \stackrel{\psi}{\longmapsto} & x_{\nu(k)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ p_k & \stackrel{\phi}{\longmapsto} & y_k. \end{array}$$

Consequently, $F(x_{\nu(k)}) = y_k$. In particular $F(x_{\nu(0)}) = 0$. In addition, the critical points of F are exactly those points x_k for which $m_k > 0$, counted with multiplicity m_k. As a consequence, $F = F_{\mathbf{x}}$ and

$$G_f \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} F_{\mathbf{x}}(x_{\nu(1)}) \\ \vdots \\ F_{\mathbf{x}}(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} F(x_{\nu(1)}) \\ \vdots \\ F(x_{\nu(n+1)}) \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

(2) To see that σ_f takes its values in $\text{Teich}(\mathbb{P}^1, P_f) - \pi^{-1}(\mathcal{L})$, we may proceed by contradiction. Assume

$$\tau \in \text{Teich}(\mathbb{P}^1, P_f) \quad \text{and} \quad \tau' := \sigma_f(\tau) \in \pi^{-1}(\mathcal{L}).$$

Then, since $\pi = g_f \circ \pi \circ \sigma_f$, we obtain

$$\pi(\tau) = g_f \circ \pi(\tau') \in \Delta.$$

But if $\tau \in \text{Teich}(\mathbb{P}^1, P_f)$, then $\pi(\tau)$ cannot be in Δ, and we have a contradiction.

(3) To see that $g_f(\Delta) \subseteq \Delta$, assume

$$\mathbf{a} := (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$$

and set $a_0 := 0$. Set

$$(b_0, b_1, \ldots, b_{n+1}) := \big(0, F_{\mathbf{a}}(a_{\nu(1)}), \ldots, F_{\mathbf{a}}(a_{\nu(n+1)})\big).$$

Then,

$$G_f(a_1,\ldots,a_{n+1}) = (b_1,\ldots,b_{n+1}).$$

Note that

$$a_i = a_j \quad\Longrightarrow\quad b_{\mu(i)} = b_{\mu(j)}.$$

In addition $[a_1 : \ldots : a_{n+1}]$ belongs to Δ precisely when there are integers $i \neq j$ in $[0, n+1]$ such that $a_i = a_j$. As a consequence,

$$[a_1 : \ldots : a_{n+1}] \in \Delta \quad\Longrightarrow\quad [b_1 : \ldots : b_{n+1}] \in \Delta.$$

This proves that $g_f(\Delta) \subseteq \Delta$.

(4) To see that the critical point locus of g_f is contained in Δ, we must show that $\mathrm{Jac}\, G_f : \mathbb{C}^{n+1} \to \mathbb{C}$ does not vanish outside Δ. Since $g_f(\Delta) \subseteq \Delta$, we then automatically obtain that the critical value locus of g_f is contained in Δ.

Note that $\mathrm{Jac}\, G_f(a_1,\ldots,a_{n+1})$ is a homogeneous polynomial of degree $(n+1)\cdot(d-1)$ in the variables $a_1,\ldots,a_{n+1}$. Consider the polynomial $J \in \mathbb{C}[a_1,\ldots,a_{n+1}]$ defined by

$$J(a_1,\ldots,a_{n+1}) := \prod_{0\leqslant i<j\leqslant n+1} (a_i - a_j)^{m_i+m_j} \quad \text{with} \quad a_0 := 0.$$

Lemma 1 *The Jacobian* $\mathrm{Jac}\, G_f$ *is divisible by* J.

Proof: Set $a_0 := 0$ and $G_0 := 0$. For $j \in [1, n+1]$, let G_j be the j-th coordinate of $G_f(a_1,\ldots,a_{n+1})$, i.e.

$$G_j := d\int_{a_{\nu(0)}}^{a_{\nu(j)}} \prod_{k=0}^{n+1} (w-a_k)^{m_k}\,\mathrm{d}w.$$

For $0 \leqslant i < j \leqslant n+1$, note that setting $w = a_i + t(a_j - a_i)$, we have

$$\begin{aligned} G_{\mu(j)} - G_{\mu(i)} &= d\int_{a_i}^{a_j} \prod_{k=0}^{n+1} (w-a_k)^{m_k}\mathrm{d}w \\ &= d\int_0^1 \prod_{k=0}^{n+1} (a_i + t(a_j-a_i) - a_k)^{m_k}\cdot (a_j - a_i)\mathrm{d}t \\ &= (a_j - a_i)^{m_i+m_j+1}\cdot H_{i,j} \end{aligned}$$

with

$$H_{i,j} := d\int_0^1 t^{m_i}(t-1)^{m_j} \prod_{\substack{k\in[0,n+1]\\ k\neq i,j}} \big(a_i - a_k + t(a_j - a_i)\big)^{m_k}\mathrm{d}t.$$

In particular, $G_{\mu(j)} - G_{\mu(i)}$ is divisible by $(a_j - a_i)^{m_i+m_j+1}$.

For $k \in [0, n+1]$, let L_k be the row defined as:

$$L_k := \begin{bmatrix} \frac{\partial G_k}{\partial a_1} & \dots & \frac{\partial G_k}{\partial a_{n+1}} \end{bmatrix}.$$

Note that L_0 is the zero row, and for $k \in [1, n+1]$, L_k is the k-th row of the Jacobian matrix of G_f. According to the previous computations, the entries of $L_{\mu(j)} - L_{\mu(i)}$ are the partial derivatives of $(a_j - a_i)^{m_i+m_j+1} \cdot H_{i,j}$. It follows that $L_{\mu(j)} - L_{\mu(i)}$ is divisible by $(a_j - a_i)^{m_i+m_j}$. Indeed, $L_{\mu(j)} - L_{\mu(i)}$ is either the difference of two rows of the Jacobian matrix of G_f, or such a row up to sign, when $\mu(i) = 0$ or $\mu(j) = 0$. As a consequence, Jac G_f is divisible by J. □

Since $\sum m_j = d - 1$, an easy computation shows that the degree of J is $(n+1) \cdot (d-1)$. Since J and Jac G_f are homogeneous polynomials of the same degree and since J divides Jac G_f, they are equal up to multiplication by a nonzero complex number. This shows that Jac G_f vanishes exactly when J vanishes, i.e. on a subset of Δ.

This completes the proof of Proposition 2. □

3 Proof of (2)

In this section we present an example of a Thurston map f such that the pullback map $\sigma_f : \text{Teich}(\mathbb{P}^1, P_f) \to \text{Teich}(\mathbb{P}^1, P_f)$ is a ramified Galois covering and has a fixed critical point.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the rational map defined by:

$$f(z) = \frac{3z^2}{2z^3 + 1}.$$

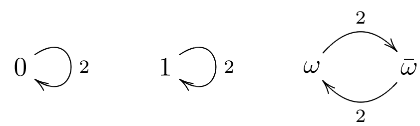
Note that f has critical points at $\Omega_f = \{0, 1, \omega, \bar{\omega}\}$, where

$$\omega := -1/2 + i\sqrt{3}/2 \quad \text{and} \quad \bar{\omega} := -1/2 - i\sqrt{3}/2$$

are cube roots of unity. Notice that

$$f(0) = 0,\ f(1) = 1,\ f(\omega) = \bar{\omega} \text{ and } f(\bar{\omega}) = \omega.$$

So, $P_f = \{0, 1, \omega, \bar{\omega}\}$ and f is a Thurston map. We illustrate the critical dynamics of f with the following *ramification portrait*:



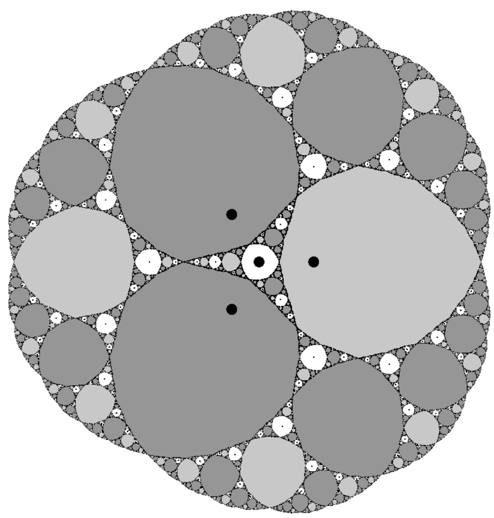


Figure 3.1. The Julia set of the rational map $f : z \mapsto 3z^2/(2z^3+1)$. The basin of 0 is white. The basin of 1 is light grey. The basin of $\{\omega, \bar{\omega}\}$ is dark grey.

Since $|P_f| = 4$, the Teichmüller space $\mathrm{Teich}(\mathbb{P}^1, P_f)$ has complex dimension 1.

Set $\Theta := \{1, \omega, \bar{\omega}\} \subset P_f$. We identify the moduli space $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with $\mathbb{P}^1 - \Theta$. More precisely, if $\phi : P_f \hookrightarrow \mathbb{P}^1$ represents a point in $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with $\phi|_\Theta = \mathrm{id}|_\Theta$, we identify the class of ϕ in $\mathrm{Mod}(\mathbb{P}^1, P_f)$ with the point $\phi(0)$ in $\mathbb{P}^1 - \Theta$. The universal covering $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Mod}(\mathbb{P}^1, P_f)$ is identified with a universal covering $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathbb{P}^1 - \Theta$ and $\pi(\circledast)$ is identified with 0.

Assume $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$ and let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a homeomorphism representing τ with $\phi|_\Theta = \mathrm{id}|_\Theta$. There exists a unique homeomorphism

$\psi : \mathbb{P}^1 \to \mathbb{P}^1$ representing $\tau' := \sigma_f(\tau)$ and a unique cubic rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ such that

- $\psi|_\Theta = \mathrm{id}|_\Theta$ and
- the following diagram commutes

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1. \end{array}$$

We set

$$y := \phi(0) = \pi(\tau) \quad \text{and} \quad x := \psi(0) = \pi(\tau').$$

The rational map F has the following properties:

(P1) 1, ω and $\bar{\omega}$ are critical points of F, $F(1) = 1$, $F(\omega) = \bar{\omega}$, $F(\bar{\omega}) = \omega$ and

(P2) $x \in \mathbb{P}^1 - \Theta$ is a critical point of F and $y = F(x) \in \mathbb{P}^1 - \Theta$ is the corresponding critical value.

For $\alpha = [a : b] \in \mathbb{P}^1$, let F_α be the rational map defined by

$$F_\alpha(z) := \frac{az^3 + 3bz^2 + 2a}{2bz^3 + 3az + b}.$$

Note that $f = F_0$.

We first show that $F = F_\alpha$ for some $\alpha \in \mathbb{P}^1$. For this purpose, we may write $F = P/Q$ with P and Q polynomials of degree $\leqslant 3$. Note that if $\widehat{F} = \widehat{P}/\widehat{Q}$ is another rational map of degree 3 satisfying Property (P1), then $F - \widehat{F}$ and $(F - \widehat{F})'$ vanish at 1, ω and $\bar{\omega}$. Since

$$F - \widehat{F} = \frac{P\widehat{Q} - Q\widehat{P}}{Q\widehat{Q}}$$

and since $P\widehat{Q} - Q\widehat{P}$ has degree $\leqslant 6$, we see that $P\widehat{Q} - Q\widehat{P}$ is equal to $(z^3 - 1)^2$ up to multiplication by a complex number.

A computation shows that F_0 and F_∞ satisfy Property (P1). We may write $F_0 = P_0/Q_0$ and $F_\infty = P_\infty/Q_\infty$ with

$$P_0(z) = 3z^2, \quad Q_0(z) = 2z^3 + 1, \quad P_\infty(z) = z^3 + 2 \quad \text{and} \quad Q_\infty(z) = 3z.$$

The previous observation shows that $PQ_0 - QP_0$ and $PQ_\infty - QP_\infty$ are both scalar multiples of $(z^3-1)^2$, and thus, we can find complex numbers a and b such that

$$a \cdot (PQ_\infty - QP_\infty) + b \cdot (PQ_0 - QP_0) = 0$$

whence

$$P \cdot (aQ_\infty + bQ_0) = Q \cdot (aP_\infty + bP_0).$$

This implies that

$$F = \frac{P}{Q} = \frac{aP_\infty + bP_0}{aQ_\infty + bQ_0} = F_\alpha \quad \text{with} \quad \alpha = [a:b] \in \mathbb{P}^1.$$

We now study how $\alpha \in \mathbb{P}^1$ depends on $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$. The critical points of F_α are 1, ω, $\bar{\omega}$ and α^2. We therefore have

$$x = \alpha^2 \quad \text{and} \quad y = F_\alpha(\alpha^2) = \frac{\alpha(\alpha^3+2)}{2\alpha^3+1} = \frac{x^2+2\alpha}{2x\alpha+1}.$$

In particular,

$$\alpha = \frac{x^2-y}{2xy-2}.$$

Consider now the holomorphic maps $X : \mathbb{P}^1 \to \mathbb{P}^1$, $Y : \mathbb{P}^1 \to \mathbb{P}^1$ and $A : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathbb{P}^1$ defined by

$$X(\alpha) := \alpha^2, \quad Y(\alpha) := \frac{\alpha(\alpha^3+2)}{2\alpha^3+1}$$

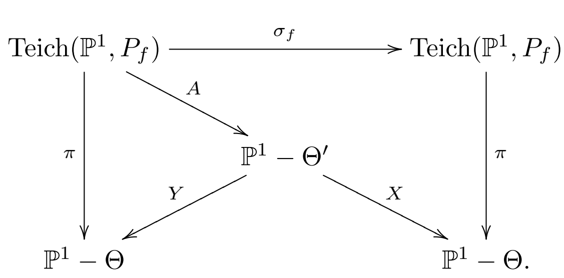
and

$$A(\tau) := \frac{x^2-y}{2xy-2} \quad \text{with} \quad y = \pi(\tau) \quad \text{and} \quad x = \pi \circ \sigma_f(\tau).$$

Observe that

$$X^{-1}\big(\{1,\omega,\bar{\omega}\}\big) = Y^{-1}\big(\{1,\omega,\bar{\omega}\}\big) = \Theta' := \{1,\omega,\bar{\omega},-1,-\omega,-\bar{\omega}\}.$$

Thus, we have the following commutative diagram,



In this paragraph, we show that σ_f has local degree two at the fixed basepoint. Since $f = F_0$, we have $A(\circledast) = 0$. In addition, $\pi(\circledast) = \pi \circ \sigma_f(\circledast) = 0$. Since $Y(\alpha) = 2\alpha + \mathcal{O}(\alpha^2)$, the germ $Y : (\mathbb{P}^1, 0) \to (\mathbb{P}^1, 0)$ is locally invertible at 0. Since $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Mod}(\mathbb{P}^1, P_f)$ is a universal covering, the germ $\pi : \big(\mathrm{Teich}(\mathbb{P}^1, P_f), \circledast\big) \to \big(\mathrm{Teich}(\mathbb{P}^1, P_f), \circledast\big)$ is also locally invertible at 0. Since $X(\alpha) = \alpha^2$, the germ $X : (\mathbb{P}^1, 0) \to (\mathbb{P}^1, 0)$ has degree 2 at 0. It follows that σ_f has degree 2 at $\circledast$ as required.

Finally, we prove that σ_f is a surjective Galois orbifold covering. First, note that the critical value set of Y is Θ whence $Y : \mathbb{P}^1 - \Theta' \to \mathbb{P}^1 - \Theta$ is a covering map. Since $\pi = Y \circ A$ and since $\pi : \mathrm{Teich}(\mathbb{P}^1, P_f) \to P^1 - \Theta$ is a universal covering map, we see that $A : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathbb{P}^1 - \Theta'$ is a covering map (hence a universal covering map).

Second, note that $X : \mathbb{P}^1 - \Theta' \to \mathbb{P}^1 - \Theta$ is a ramified Galois covering of degree 2, ramified above 0 and ∞ with local degree 2. Let M be the orbifold whose underlying surface is $\mathbb{P}^1 - \Theta$ and whose weight function takes the value 1 everywhere except at 0 and ∞ where it takes the value 2. Then, $X : \mathbb{P}^1 - \Theta' \to M$ is a covering of orbifolds and $X \circ A : \mathrm{Teich}(\mathbb{P}^1, P_f) \to M$ is a universal covering of orbifolds.

Third, let T be the orbifold whose underlying surface is $\mathrm{Teich}(\mathbb{P}^1, P_f)$ and whose weight function takes the value 1 everywhere except at points in $\pi^{-1}\big(\{0, \infty\}\big)$ where it takes the value 2. Then $\pi : T \to M$ is a covering of orbifolds. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Teich}(\mathbb{P}^1, P_f) & \xrightarrow{\sigma_f} & T \\ & \searrow^{X \circ A} & \downarrow \pi \\ & & M. \end{array}$$

It follows that $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to T$ is a covering of orbifolds (thus a universal covering). Equivalently, the map $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is a ramified Galois covering, ramified above points in $\pi^{-1}\big(\{0, \infty\}\big)$ with local degree 2.

Figure 3.2 illustrates the behavior of the map σ_f.

4 Proof of (3)

4.1 Examples

Here, we give examples of Thurston maps f such that

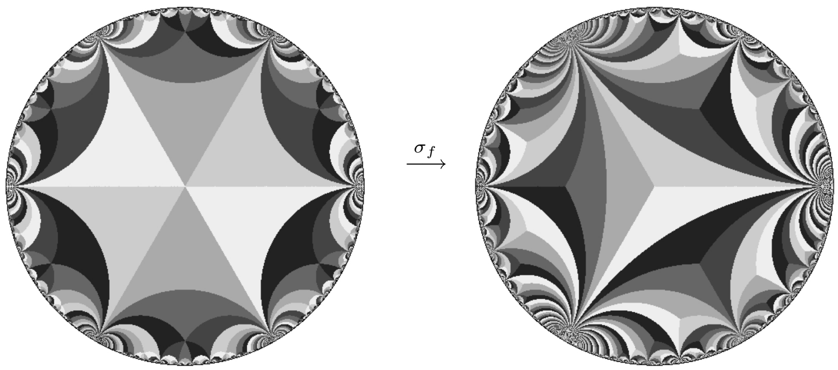


Figure 3.2. For $f(z) = 3z^2/(2z^3+1)$, the pullback map σ_f fixes $0 = \circledast$. It sends hexagons to triangles. There is a critical point with local degree 2 at the center of each hexagon and a corresponding critical value at the center of the image triangle. The map $X \circ A$ sends light grey hexagons to the unit disk in $\mathbb{P}^1 - \Theta$ and dark grey hexagons to the complement of the unit disk in $\mathbb{P}^1 - \Theta$. The map π sends light grey triangles to the unit disk in $\mathbb{P}^1 - \Theta$ and dark grey triangles to the complement of the unit disk in $\mathbb{P}^1 - \Theta$.

- P_f contains at least 4 points, so $\mathrm{Teich}(\mathbb{P}^1, P_f)$ is not reduced to a point, and
- $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is constant.

The main result, essentially due to McMullen, is the following.

Proposition 3 *Let $s : \mathbb{P}^1 \to \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ be rational maps with critical value sets V_s and V_g. Let $A \subset \mathbb{P}^1$ be finite. Assume $V_s \subseteq A$ and $V_g \cup g(A) \subseteq s^{-1}(A)$. Then*

- *$f := g \circ s$ is a Thurston map,*
- *$V_g \cup g(V_s) \subseteq P_f \subseteq V_g \cup g(A)$ and*
- *the dimension of the image of $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is at most $|A| - 3$.*

Remark 1 If $|A| = 3$ the pullback map σ_f is constant.

Proof: Set $B := V_g \cup g(A)$. The set of critical values of f is the set

$$V_f = V_g \cup g(V_s) \subseteq B.$$

By assumption,

$$f(B) = g \circ s(B) \subseteq g(A) \subseteq B.$$

So, the map f is a Thurston map and $V_g \cup g(V_s) \subseteq P_f \subseteq B$.

Note that $P_f \subseteq s^{-1}(A)$ and $A \subseteq g^{-1}(P_f)$. According to the discussion at the beginning of Section 1, the rational maps s and g induce pullback maps

$$\sigma_s : \mathrm{Teich}(\mathbb{P}^1, A) \to \mathrm{Teich}(\mathbb{P}^1, P_f) \quad \text{and} \quad \sigma_g : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, A).$$

In addition,

$$\sigma_f = \sigma_s \circ \sigma_g.$$

The dimension of the Teichmüller space $\mathrm{Teich}(\mathbb{P}^1, A)$ is $|A| - 3$. Thus, the rank of $D\sigma_g$, and so that of $D\sigma_f$, at a generic point in $\mathrm{Teich}(\mathbb{P}^1, A)$ is at most $|A| - 3$. This completes the proof of the proposition. □

Let us now illustrate this proposition with some examples.

Example 1 We are not aware of any rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 or 3 for which $|P_f| \geqslant 4$ and $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is constant. We have an example in degree 4: the polynomial f defined by

$$f(z) = 2i \left(z^2 - \frac{1+i}{2} \right)^2.$$

This polynomial can be decomposed as $f = g \circ s$ with

$$s(z) = z^2 \quad \text{and} \quad g(z) = 2i \left(z - \frac{1+i}{2} \right)^2.$$

The critical value set of s is

$$V_s = \{0, \infty\} \subset A := \{0, 1, \infty\}.$$

The critical value set of g is

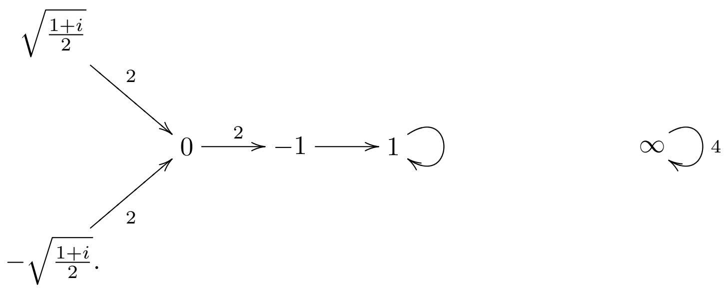
$$V_g = \{0, \infty\} \subset \{0, \infty, -1, 1\} = s^{-1}(A).$$

In addition, $g(0) = -1$, $g(1) = 1$ and $g(\infty) = \infty$, so

$$g(A) = \{-1, 1, \infty\} \subset s^{-1}(A).$$

According to the previous proposition, $f = g \circ s$ is a Thurston map and since $|A| = 3$, the map $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is constant.

Note that $V_f = \{0, -1, \infty\}$ and $P_f = \{0, 1, -1, \infty\}$. The ramification portrait for f is:



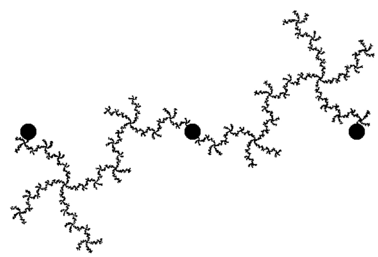


Figure 4.3. The Julia set of the degree 4 polynomial $f : z \mapsto 2i\left(z^2 - \frac{1+i}{2}\right)^2$ is a dendrite. There is a fixed critical point at ∞. Its basin is white. The point $z = 1$ is a repelling fixed point. All critical points are in the backward orbit of 1.

Example 2 We also have examples of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ for which $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is constant and $|P_f| \geqslant 4$ is an arbitrary integer. Assume $n \geqslant 2$ and consider $s : \mathbb{P}^1 \to \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ the polynomials defined by

$$s(z) = z^n \quad \text{and} \quad g(z) = \frac{(n+1)z - z^{n+1}}{n}.$$

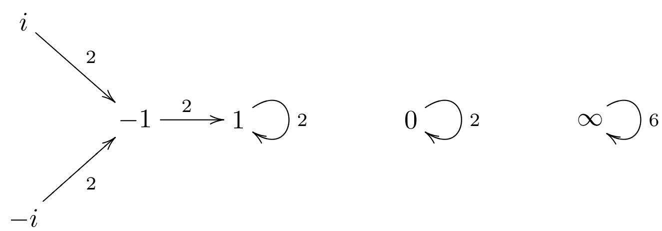
Set $A := \{0, 1, \infty\}$. The critical value set of s is $V_s = \{0, \infty\} \subset A$.

The critical points of g are the n-th roots of unity and g fixes those points; the critical values of g are the n-th roots of unity. In addition, $g(0) = 0$. Thus

$$V_g \cup g(V_s) = V_g \cup g(A) = s^{-1}(A).$$

According to Proposition 3, $P_f = s^{-1}(A)$ and the pullback map σ_f is constant. In particular, $|P_f| = n+2$.

For $n = 2$, f has the following ramification portrait:



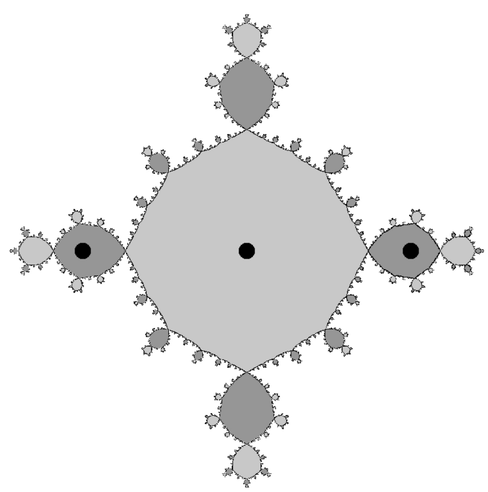


Figure 4.4. The Julia set of the degree 6 polynomial $f : z \mapsto z^2(3 - z^4)/2$. There are superattracting fixed points at $z = 0$, $z = 1$ and $z = \infty$. All other critical points are in the backward orbit of 1. The basin of ∞ is white. The basin of 0 is light grey. The basin of 1 is dark grey.

Example 3 Proposition 3 can be further exploited to produce examples of Thurston maps f where σ_f has a *skinny image*, which is not just a point.

For $n \geqslant 2$, let A_n be the union of $\{0, \infty\}$ and the set of n-th roots of unity. Let $s_n : \mathbb{P}^1 \to \mathbb{P}^1$ and $g_n : \mathbb{P}^1 \to \mathbb{P}^1$ be the polynomials defined by

$$s_n(z) = z^n \quad \text{and} \quad g_n(z) = \frac{(n+1)z - z^{n+1}}{n}.$$

The critical points of g_n are the n-th roots of unity and g_n fixes those points; the critical values of g_n are the n-th roots of unity. In particular, $V_{g_n} \subset A_n$. In addition, $g_n(0) = 0$, and so,

$$g_n(A_n) = A_n.$$

Assume $n \geqslant 2$ and $m \geqslant 1$ are integers with m dividing n, let's say $n = km$. Note that

$$V_{s_k} \subset A_m \quad \text{and} \quad V_{g_n} \cup g_n(A_n) = A_n = s_k^{-1}(A_m).$$

It follows that the polynomial $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$f := g_n \circ s_k$$

is a Thurston map and

$$A_n = V_{g_n} \cup g_n(V_{s_k}) \subseteq P_f \subseteq V_{g_n} \cup g_n(A_n) = A_n \quad \text{so,} \quad P_f = A_n.$$

In particular, the dimension of the Teichmüller space $\text{Teich}(\mathbb{P}^1, P_f)$ is $n-1$.

Claim *The dimension of the image of* $\sigma_f : \text{Teich}(\mathbb{P}^1, P_f) \to \text{Teich}(\mathbb{P}^1, P_f)$ *is* $m - 1$. *Thus, its codimension is* $(k-1)m$.

Proof: On the one hand, since g_n is a polynomial whose critical points are all fixed, Proposition 1 implies that $\sigma_{g_n} : \text{Teich}(\mathbb{P}^1, A_n) \to \text{Teich}(\mathbb{P}^1, A_n)$ has open image. Composing with the forgetful projection

$$\text{Teich}(\mathbb{P}^1, A_n) \to \text{Teich}(\mathbb{P}^1, A_m),$$

we deduce that $\sigma_{g_n} : \text{Teich}(\mathbb{P}^1, A_n) \to \text{Teich}(\mathbb{P}^1, A_m)$ has open image.

On the other hand, since $s_k : \mathbb{P}^1 - A_n \to \mathbb{P}^1 - A_m$ is a covering map, it follows from general principle that $\sigma_{s_k} : \text{Teich}(\mathbb{P}^1, A_m) \to \text{Teich}(\mathbb{P}^1, A_n)$ is a holomorphic embedding with everywhere injective derivative. □

Question If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a Thurston map such that the pullback map $\sigma_f : \text{Teich}(\mathbb{P}^1, P_f) \to \text{Teich}(\mathbb{P}^1, P_f)$ is constant, then is it necessarily of the form described above? In particular, is there a Thurston map $f : \mathbb{P}^1 \to \mathbb{P}^1$ with constant $\sigma_f : \text{Teich}(\mathbb{P}^1, P_f) \to \text{Teich}(\mathbb{P}^1, P_f)$, such that $\deg(f)$ is prime?

4.2 Characterizing when σ_f is constant.

Suppose f is a Thurston map with $|P_f| \geqslant 4$.

Thurston linear transformation. Let $\mathcal{S}$ denote the set of free homotopy classes of simple, closed, unoriented, essential, nonperipheral curves in $\Sigma - P_f$. Let $\mathbb{R}[\mathcal{S}]$ denote the free $\mathbb{R}$-module generated by $\mathcal{S}$. Given $[\gamma]$ and $[\widetilde{\gamma}]$ in $\mathcal{S}$, define the *pullback relation* on $\mathcal{S}$, denoted $\underset{f}{\leftarrow}$, by defining $[\gamma]\underset{f}{\leftarrow}[\widetilde{\gamma}]$ if and only if there is a component δ of $f^{-1}(\gamma)$ which, as a curve in $\Sigma - P_f$, is homotopic to $\widetilde{\gamma}$.

The *Thurston linear map*

$$\lambda_f : \mathbb{R}[\mathcal{S}] \to \mathbb{R}[\mathcal{S}]$$

is defined by specifying the image of basis elements $[\gamma] \in \mathcal{S}$ as follows:

$$\lambda_f\big([\gamma]\big) = \sum_{[\gamma]\underset{f}{\leftarrow}[\gamma_i]} d_i[\gamma_i].$$

Here, the sum ranges over all $[\gamma_i]$ for which $[\gamma]\underset{f}{\leftarrow}[\gamma_i]$, and

$$d_i = \sum_{f^{-1}(\gamma)\supset\delta\simeq\gamma_i} \frac{1}{|\deg(\delta \to \gamma)|},$$

where the sum ranges over components δ of $f^{-1}(\gamma)$ homotopic to γ_i.

Virtual endomorphism. Let $\mathrm{PMCG}(\mathbb{P}^1, P_f)$ denote the pure mapping class group of $(\mathbb{P}^1, P_f)$—that is, the quotient of the group of orientation-preserving homeomorphisms fixing P_f pointwise by the subgroup of such maps isotopic to the identity relative to P_f. Thus,

$$\mathrm{Mod}(\mathbb{P}^1, P_f) = \mathrm{Teich}(\mathbb{P}^1, P_f)/\mathrm{PMCG}(\mathbb{P}^1, P_f).$$

Elementary covering space theory and homotopy-lifting facts imply that there is a finite-index subgroup $H_f < \mathrm{PMCG}(\mathbb{P}^1, P_f)$ consisting of those classes represented by homeomorphisms h lifting under f to a homeomorphism $\tilde{h}$ which fixes P_f pointwise. This yields a homomorphism

$$\varphi_f : H_f \to \mathrm{PMCG}(\mathbb{P}^1, P_f)$$

defined by

$$\phi_f\big([h]\big) = [\tilde{h}] \quad \text{with} \quad h \circ f = f \circ \tilde{h}.$$

Following [BN] we refer to the homomorphism φ_f as the *virtual endomorphism of* $\mathrm{PMCG}(\mathbb{P}^1, P_f)$ associated to f.

Theorem 2 *The following are equivalent:*

1. $\underset{f}{\leftarrow}$ *is empty*
2. $\lambda_f \equiv 0$
3. $\varphi_f \equiv 1$
4. σ_f *is constant*

Proof: We will show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4), and that failure of (1) implies failure of (4).

That (1) $\iff$ (2) is an immediate consequence of the definitions.

Suppose (2) holds. Since $\mathrm{PMCG}(\mathbb{P}^1, P_f)$ is generated by Dehn twists, it suffices to show $\varphi_f(g) = 1$ for every Dehn twist g. We may represent g by a homeomorphism h which is the identity off an annulus A. By (2), the set $f^{-1}(A)$ is contained in a disjoint union of disks D_i, each of which contains at most one point of P_f. There is a lift $\tilde{h}$ of h under f which is the identity outside $\Sigma - \cup_i D_i$. Since the disks D_i are peripheral, $\tilde{h}$ represents the trivial element of $\mathrm{PMCG}(\mathbb{P}^1, P_f)$.

Suppose (3) holds. Then $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ descends to a holomorphic map

$$\overline{\sigma}_f : \mathrm{Teich}(\mathbb{P}^1, P_f)/H_f \to \mathrm{Teich}(\mathbb{P}^1, P_f).$$

We now use a classical fact pointed out to us by M. Bainbridge: the domain of $\overline{\sigma}_f$ is a finite cover of $\mathrm{Mod}(\mathbb{P}^1, P_f)$, hence is a quasiprojective variety. Since $\mathrm{Teich}(\mathbb{P}^1, P_f)$ is biholomorphic to a bounded domain in $\mathbb{C}^n$, the induced map $\overline{\sigma}_f$ is a bounded analytic function on a quasiprojective variety, and is therefore constant, by a generalization of the classical Liouville theorem.

Suppose (1) fails. For $\tau \in \mathrm{Teich}(\mathbb{P}^1, P_f)$ and $\delta \in \mathcal{S}$ let $\ell_\tau(\delta)$ denote the length of the unique simple closed geodesic representing δ on $\mathbb{P}^1 - P_f$, equipped with a hyperbolic metric corresponding to a complex structure representing τ. It is known (see [DH]) that if $l_\tau(\gamma)$ is sufficiently small, then whenever $[\gamma]\underset{f}{\leftarrow}[\widetilde{\gamma}]$, one has $l_{\sigma_f(\tau)}(\widetilde{\gamma}) \leqslant c \cdot l_\tau(\gamma)$ where c depends only on $|P_f|$ and on $\deg f$. In particular, $l_{\sigma_f(\tau)}(\widetilde{\gamma}) \to 0$ as $l_\tau(\gamma) \to 0$. Hence σ_f cannot be constant. $\square$

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