ON A THEOREM OF JAKOBSON

XAVIER BUFF AND JOHN H. HUBBARD

Abstract. A theorem of Jakobson asserts that the set of parameters $c \in \mathbb{R}$, for which the quadratic polynomial $f_c(z) = z^2 + c$ admits an invariant measure absolutely continuous with respect to the Lebesgue measure, has positive length. We present a proof based on tableaux and Yoccoz's puzzles.

1. Introduction

In this article, we study dynamical properties of quadratic polynomials

$$f_c(z) = z^2 + c \quad \text{with} \quad c \in \mathbb{C}.$$

The orbit of a point $z_0 \in \mathbb{C}$ is the sequence $\{z_n\}_{n\geq 0}$ defined for $n \geq 1$ by

$$z_n := f_c(z_{n-1}) = f_c^{\circ n}(z_0).$$

The *filled-in Julia set* $\mathcal{K}_c$ is the set of points with bounded orbit. The critical point $c_0 := 0$ plays a special role. For example, $\mathcal{K}_c$ is connected if and only if the critical orbit

$$\big\{c_n := f_c^{\circ n}(0)\big\}_{n\geq 0}$$

is bounded.

When $c \in [-2, -1]$, the interval

$$J_c := [c_1, c_2] = [c, c^2 + c]$$

contains 0 and is invariant: $f_c(J_c) \subseteq J_c$. The object of this paper is to present a (new) proof of the Jakobson Theorem.

Theorem 1.1 (Jakobson). *The set $\mathscr{J}$ of parameters $c \in [-2, -1]$, for which f_c admits an absolutely continuous invariant measure supported on J_c, has positive Lebesgue measure. In addition, -2 is a point of density of $\mathscr{J}$.*

A measure μ on $\mathbb{R}$ is *absolutely continuous* with respect to the Lebesgue measure Leb on $\mathbb{R}$ if there is a function $h \in L^1(\mathbb{R})$ such that $\mu = h \cdot \text{Leb}$. It is *invariant* by $f := f_c$ if $f_*\mu = \mu$, i.e. for all continuous function ϕ

$$\int_{\mathbb{R}} \phi \circ f \ \mathrm{d}\mu = \int_{\mathbb{R}} \phi \ \mathrm{d}\mu$$

or equivalently, if for any Borel set A, we have that

$$\mu\big(f^{-1}(A)\big) = \mu(A).$$

We shall use the abbreviation *a.c.i.m.* for *absolutely continuous invariant measure.*

Our approach is largely inspired by notes of Yoccoz [Y] and by [BH], [H] and [M]. The argument depends on the recurrence properties of the critical point: we need to show that if the critical point is *weakly recurrent*, an appropriate measure exists, and that such weakly recurrent polynomials form a set of positive length. Yoccoz puzzles and Branner-Hubbard tableaux are a tool developed to control such recurrence, and will be the frame of our proof.

2. Sketch of the proof

2.1. **Puzzle pieces.** The Green function $\mathfrak{g}_c : \mathbb{C} \to [0, +\infty)$ is defined by

$$\mathfrak{g}_c(z) = \lim_{n\to+\infty} \frac{1}{2^n} \log\big(1 + |f_c^{\circ n}(z)|\big).$$

If $c \in [-2, -1]$, the filled-in Julia set $\mathcal{K}_c$ is connected and there is an isomorphism $\phi_c : \mathbb{C}\smallsetminus\mathcal{K}_c \to \mathbb{C}\smallsetminus\overline{\mathbb{D}}$ which conjugates f_c to f_0: $\phi_c \circ f_c = f_0 \circ \phi_c$. The Green function satisfies $\mathfrak{g}_c = \log|\phi_c|$. For $t \in \mathbb{R}/\mathbb{Z}$, the curve

$$\mathcal{R}_c(t) := \phi_c^{-1}\big\{r\,\mathrm{e}^{\mathrm{i}2\pi t} \mid r > 1\big\}$$

is called the *external ray* of angle t. The external rays $\mathcal{R}_c(\pm 1/3)$ are exchanged by f_c and land at a common fixed point $\alpha_c \in J_c$, i.e.

$$\overline{\mathcal{R}_c}(\pm 1/3) = \mathcal{R}_c(\pm 1/3) \cup \{\alpha_c\}.$$

The polynomial f_c has a second fixed point $\beta_c \neq \alpha_c$. Set

$$\mathcal{U}_c := \big\{\mathfrak{g}_c < 1\big\}\smallsetminus\big(\mathcal{R}_c(1/3) \cup \mathcal{R}_c(-1/3) \cup \{\alpha_c\}\big).$$

Definition 2.1 (Puzzle pieces)**.** *The puzzle pieces of depth $m \geq -2$ are the connected components of $f_c^{-(m+2)}(\mathcal{U}_c)$.*

We shift the depths by -2 compared to the usual definition for later simplification. In particular, the critical piece $\mathcal{C}_0$ of depth 0 is the domain bounded by the external rays of angle $\pm 1/3$, the preimage rays of angle $\pm 1/6$ and the equipotential of level $1/4$.

We define the enlarged critical piece $\widehat{\mathcal{C}}_0$ as the domain bounded by the external rays of angle $\pm 1/12$, the external rays of angle $\pm 5/12$ and the equipotential of level $1/2$. Then, $\mathcal{C}_0$ is compactly contained in $\widehat{\mathcal{C}}_0$ (see Figure 1).

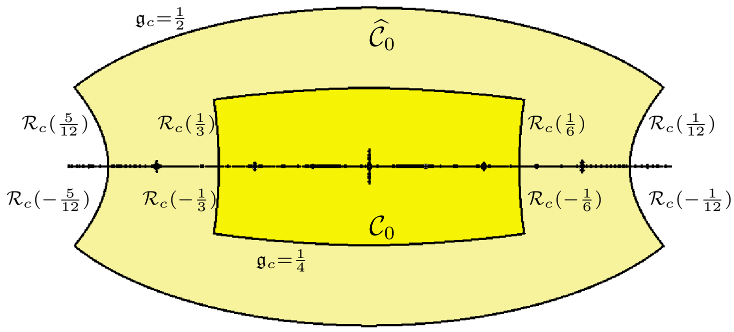


Figure 1: The filled-in Julia set $\mathcal{K}_c$ for $c = z^2 - 1.98$, together with the critical piece $\mathcal{C}_0$ (yellow), compactly contained in the enlarged piece $\widehat{\mathcal{C}}_0$ (light yellow).

2.2. **Regular points.** A regular piece $\mathcal{P}$ of depth $m \geq 1$ is a component of $f_c^{-m}(\mathcal{C}_0)$ which has a neighborhood $\widehat{\mathcal{P}}$ such that $f_c^{\circ m} : \widehat{\mathcal{P}} \to \widehat{\mathcal{C}}_0$ is an isomorphism (see Figure 2). A point $x \in J_c$ is regular if it belongs to a regular piece. Denote by $X_c \subset J_c$ the set of regular points. Define $\boldsymbol{n}_c : J_c \to \mathbb{N} \cup \{+\infty\}$ by

$$\boldsymbol{n}_c(x) := \inf\big\{n \geq 1 \mid x \text{ is contained in a regular piece of depth } n\big\}$$

with the convention $\inf \varnothing := +\infty$, so that $\boldsymbol{n}_c(x)$ is finite if and only if $x \in X_c$. Finally, set $I_c := (\alpha_c, -\alpha_c) = \mathcal{C}_0 \cap \mathbb{R}$ and define $T_c : X_c \to I_c$ by

$$T_c(x) := f_c^{\circ \boldsymbol{n}_c(x)}(x).$$

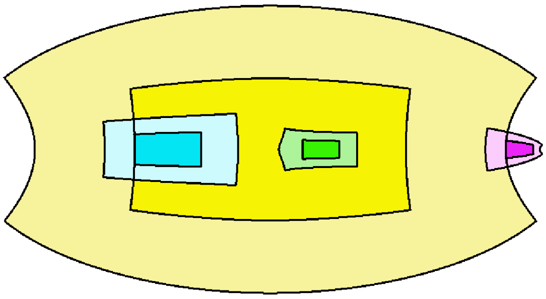


Figure 2: Three regular pieces are indicated (pink and blue of depth 2, green of depth 3). The corresponding neighborhoods mapping isomorphically to $\widehat{\mathcal{C}}_0$ are indicated with lighter coloring.

The first step in the proof is the following dynamical result, proved in §4.

Theorem 2.2. *If $\boldsymbol{n}_c \in L^1(I_c)$, then f_c admits an a.c.i.m. supported on J_c.*

The proof consists in first proving that when $\boldsymbol{n}_c$ is finite almost everywhere, then T_c admits an invariant measure supported on I_c, with $\mathbb{R}$-analytic density. This follows from the fact that the domain of T_c has full measure in I_c and that all iterates of T_c have uniformly bounded distortion. Using the fact that $\boldsymbol{n}_c$ is integrable on I_c, we then promote this T_c-invariant measure on I_c to an f_c-invariant measure on J_c.

Remark 2.3. If f_c is renormalizable, i.e. if there is an interval $I \subseteq I_c$ containing 0 and an integer $p \geq 2$ such that $f^{\circ p}(I) = I$, then $\boldsymbol{n}_c \equiv +\infty$ on I and so, $\boldsymbol{n}_c$ is not integrable. Note that even in this situation, f_c may admit an a.c.i.m. supported on J_c.

Figure 3 illustrates the case of a polynomial f_c for which the critical orbit is finite:

$$c_0 = 0 \mapsto c_1 = c \mapsto c_2 = c^2 + c \mapsto c_3 = -\alpha_c \mapsto c_4 = \alpha_c \mapsto \alpha_c.$$

In that case, $\boldsymbol{n}_c$ is integrable and there is an a.c.i.m. whose density is displayed.

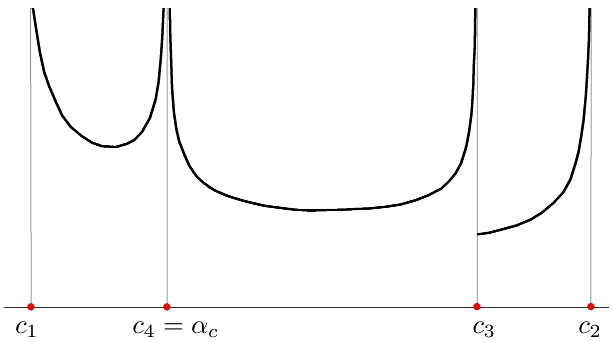


Figure 3: The density of the a.c.i.m. of a polynomial whose postcritical set is finite. The density is unbounded on the postcritical set. When the postcritical set is dense in J_c, the density of the a.c.i.m. is unbounded on every interval.

2.3. **Plowing in the dynamical space.** We need a criterion for $\boldsymbol{n}_c$ to be integrable. This criterion will depend on the behavior of $\boldsymbol{n}_c$ along the critical orbit. In particular, we shall require that $\boldsymbol{n}_c$ is finite on the critical orbit, so that $T_c{}^{\circ j}(c)$ is well-defined for all $j \geq 1$.

Definition 2.4 (Regularity)**.** *The polynomial f_c is* regular *if $\boldsymbol{n}_c$ is finite along the critical orbit. In that case, for $k \geq 1$, we set*

$$\boldsymbol{N}_k(c) := \boldsymbol{n}_c\big(T_c{}^{\circ k}(c)\big).$$

Definition 2.5 (Strong regularity)**.** *The polynomial f_c is* strongly regular *if it is regular and if for all $k \geq 1$,*

$$\frac{1}{k}\sum_{j=1}^{k} \max\big(0, \boldsymbol{N}_j(c) - \boldsymbol{N}_0(c)\big) \leq \frac{1}{4}.$$

Proposition 2.6. *There exist $C > 0$, $\rho < 1$ and $N \geq 1$ such that if f_c is strongly regular with $\boldsymbol{N}_0(c) \geq N$, then*

$$\forall n \geq 1, \quad \frac{\mathrm{Leb}\big\{x \in I_c \mid \boldsymbol{n}_c(x) \geq n\big\}}{\mathrm{Leb}(I_c)} \leq C\rho^n.$$

The proof of Proposition 2.6, given in §11, involves tableaux and bounding lengths of intervals via moduli of annuli. As a corollary, we have the following result.

Theorem 2.7. *If f_c is strongly regular with $c \in [-2, -1]$ close enough to -2, then f_c admits an a.c.i.m. supported on J_c.*

2.4. **Harvesting in the parameter space.** Consider the nested sequence of subsets

$$[-2, -1] \supset \mathscr{I}_0 \supset \mathscr{I}_1 \supset \cdots,$$

defined recursively by:

- $\mathscr{I}_0 := \big\{c \in [-2, -1] \mid c \text{ is regular for } f_c\big\}$ and
- for $\ell \geq 1$, $\mathscr{I}_\ell := \big\{c \in \mathscr{I}_{\ell-1} \mid T_c{}^{\circ \ell}(c) \text{ is regular for } f_c\big\}$.

A parameter $c \in \mathscr{I}_\ell$ is said to be *regular of order ℓ*.

Define $\boldsymbol{N}_\ell : [-2, -1] \to \mathbb{N} \cup \{+\infty\}$ by

$$\boldsymbol{N}_\ell(c) := \begin{cases} \boldsymbol{n}_c\big(T_c{}^{\circ \ell}(c)\big) & \text{if } c \in \mathscr{I}_\ell \\ +\infty & \text{otherwise.} \end{cases}$$

The functions $\boldsymbol{N}_0, \ldots \boldsymbol{N}_\ell$ are locally constant on $\mathscr{I}_\ell$. The depth of a component I of $\mathscr{I}_\ell$ is

$$\mathrm{depth}(I) := \boldsymbol{N}_0(I) + \cdots + \boldsymbol{N}_\ell(I).$$

The component I is strongly regular if for all $k \in [1, \ell]$,

$$\frac{1}{k}\sum_{j=1}^{k} \max\big(0, \boldsymbol{N}_j(I) - \boldsymbol{N}_0(I)\big) \leq \frac{1}{4}.$$

In §13, we prove the following counterpart of Proposition 2.6 in parameter space.

Proposition 2.8. *There exist $K > 0$, $\sigma < 1$ and $N \geq 1$ such that if I is a strongly regular component of $\mathscr{I}_\ell$ with $\boldsymbol{N}_0(I) \geq N$, then*

$$\forall n \in \big[1, \mathrm{depth}(I)\big], \quad \frac{\mathrm{Leb}\big\{c \in I \mid \boldsymbol{N}_{\ell+1}(c) \geq n\big\}}{\mathrm{Leb}(I)} \leq K\sigma^n.$$

2.5. **A probabilistic argument.** To conclude the proof of Theorem 1.1 assuming Propositions 2.6 and 2.8, we shall use a probabilistic argument. Assume (Y, p) is a probability space and let $(M_k : Y \to \mathbb{N})_{k\geq 1}$ be random variables. Given $(m_1, \ldots, m_k) \in \mathbb{N}^k$, set

$$Y(m_1, \ldots, m_k) := \{y \in Y \mid M_1(y) = m_1, \ldots, M_k(y) = m_k\}$$

with the convention $Y(m_1, \ldots, m_k) = Y$ for $k = 0$. On $Y(m_1, \ldots, m_k)$, consider the conditional probabilities

$$p(M_{k+1} = m \mid m_1, \ldots, m_k) := \frac{p\big(Y(m_1, \ldots, m_k, m)\big)}{p\big(Y(m_1, \ldots, m_k)\big)},$$

the conditional expectations

$$E(m_1, \ldots, m_k) := \sum_{m\geq 1} m \cdot p(M_{k+1} = m \mid m_1, \ldots, m_k)$$

and the conditional variances

$$V(m_1, \ldots, m_k) := \sum_{m\geq 1} m^2 \cdot p(M_{k+1} = m \mid m_1, \ldots, m_k) - \big(E(m_1, \ldots, m_k)\big)^2.$$

Set

$$E := \sup_{(m_1,\ldots,m_k)} E(m_1, \ldots, m_k) \quad \text{and} \quad V := \sup_{(m_1,\ldots,m_k)} V(m_1, \ldots, m_k).$$

Lemma 2.9. *For all $\varepsilon > 0$ and $\eta > 0$, if E and V are sufficiently small, then*

$$p\big\{y \in Y \mid \forall k \geq 1,\ M_1(y) + \cdots + M_k(y) \leq k\varepsilon\big\} \geq 1 - \eta.$$

2.6. **Proof of the Jakobson Theorem.** According to Theorem 2.7, if N is large enough and c is stronlgy regular with $\boldsymbol{N}_0(c) \geq N$, then f_c admits an a.c.i.m. supported on J_c. So, it is enough to show that the set of such parameters c has positive Lebesgue measure and that -2 is a Lebesgue density point.

For $k \geq 1$, let $S_k : [-2 - 1] \to \mathbb{N} \cup \{+\infty\}$ be the function defined by:

$$S_k := \begin{cases} \displaystyle\sum_{j=1}^{k} \max(0, \boldsymbol{N}_j - \boldsymbol{N}_0) & \text{on } \mathscr{I}_k \\ +\infty & \text{outside } \mathscr{I}_k. \end{cases}$$

Then, $c \in [-2, -1]$ is strongly regular if and only if $S_k(c) \leq k/4$ for all $k \geq 1$.

Given $n_0 \geq 1$, set

$$Y := \big\{c \in [-2, 1] \mid \boldsymbol{N}_0(c) = n_0\big\} \quad \text{and} \quad p := \frac{\text{Leb}}{\text{Leb}(Y)}.$$

The sets

$$Y_k := \big\{c \in Y \cap \mathscr{I}_k \mid S_j(c) \leq j/4 \text{ for } 1 \leq j \leq k\big\}$$

form a nested sequence.

Consider the functions $(M_k : Y \to \mathbb{N})_{k\geq 0}$ defined recursively by

$$M_0 = 0 \quad \text{and} \quad M_{k+1}(c) := \begin{cases} 0 & \text{if } c \notin Y_k \\ \big\lfloor (k + 1)/4\big\rfloor + 1 - S_k(c) & \text{if } c \in Y_k \smallsetminus Y_{k+1} \\ S_{k+1}(c) - S_k(c) & \text{if } c \in Y_{k+1}. \end{cases}$$

Then, on the one hand, if $c \in Y_k$, then $S_k(c) = M_1(c) + \cdots + M_k(c)$, so that

$$\bigcap_{k\geq 1} Y_k = \big\{c \in Y \mid \forall k \geq 1,\ M_1(c) + \cdots + M_k(c) \leq k/4\big\}.$$

On the other hand, if $M_{k+1}(c) = m \geq 1$, then c belongs to some strongly regular component I of $\mathscr{I}_k$ with $\boldsymbol{N}_0(I) = n_0$. In addition,

$$m \leq \lfloor (k+1)/4 \rfloor + 1 \leq k \leq \boldsymbol{N}_1(I) + \cdots + \boldsymbol{N}_k(I) = \operatorname{depth}(I) - n_0$$

and

$$m = M_{k+1}(c) \leq S_{k+1}(c) - S_k(c) \leq \boldsymbol{N}_{k+1}(c) - \boldsymbol{N}_0(c) = \boldsymbol{N}_{k+1}(c) - n_0.$$

Thus, Proposition 2.8 implies that there are constants $K > 0$, $\sigma < 1$ and $N \geq 1$ such that if $n_0 \geq N$ and $m \geq 1$

$$\frac{\operatorname{Leb}\{c \in I \mid M_{k+1}(c) = m\}}{\operatorname{Leb}(I)} \leq \frac{\operatorname{Leb}\{c \in I \mid \boldsymbol{N}_{k+1}(c) \geq m + n_0\}}{\operatorname{Leb}(I)} \leq K\sigma^{m+n_0}.$$

Since on I, the functions $M_0, \ldots, M_k$ are constant, we deduce that for $m \geq 1$

$$p(M_{k+1} = m \mid m_1, \ldots, m_k) \leq K\sigma^{m+n_0}.$$

It follows that

$$\sup_{(m_1,\ldots,m_k)} E(m_1, \ldots, m_k) = \mathcal{O}(\sigma^{n_0}) \quad \text{and} \quad \sup_{(m_1,\ldots,m_k)} V(m_1, \ldots, m_k) = \mathcal{O}(\sigma^{n_0}).$$

The Jakobson Theorem now follows from the probabilistic Lemma 2.9.

3. The Yoccoz puzzle

Until §12, $c \in [-2, -1]$. When the context is clear, we omit the index c: $f := f_c$, $\alpha := \alpha_c$, ... Puzzle pieces and their depths have been defined in the introduction (see Definition 2.1). We shall use the following notation.

Definition 3.1. *If $x \in \mathcal{K}$ and $f^{\circ(m+2)}(x) \neq \alpha$, we denote by $\mathcal{P}_m(x)$ the puzzle piece of depth m containing x. If $f^{\circ(m+2)}(0) \neq \alpha$, we denote by $\mathcal{C}_m := \mathcal{P}_m(0)$ the* critical piece *of depth m.*

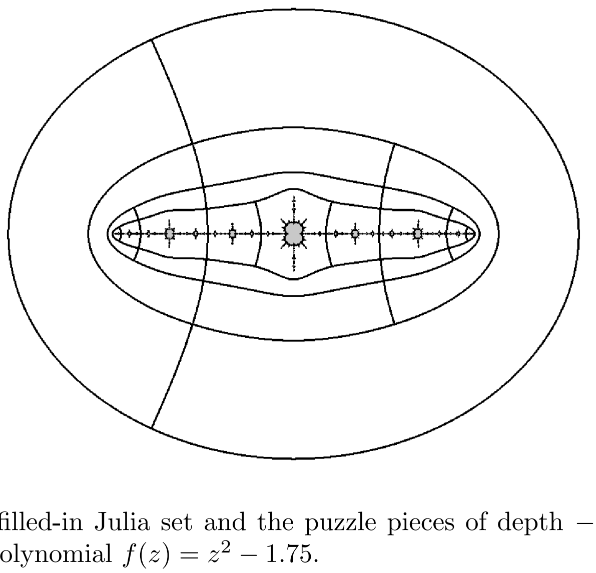


Figure 4: The filled-in Julia set and the puzzle pieces of depth -2 through 1 for the quadratic polynomial $f(z) = z^2 - 1.75$.

Definition 3.2 (Good pieces)**.** *A piece $\mathcal{P}$ is a* good piece *if it maps to $\mathcal{C}_{-2}$ by an iterate of f.*

The preimage of the critical piece $\mathcal{C}_{-2}$ has two components (see Figure 5):

$$\mathcal{S}^+ := \mathcal{P}_{-1}(\beta) \Subset \mathcal{C}_{-2} \quad \text{and} \quad \mathcal{S}^- := \mathcal{P}_{-1}(-\beta) \Subset -\mathcal{C}_{-2},$$

where $\beta \neq \alpha$ is the second fixed point of f. Any good piece is an iterated preimage of $\mathcal{S}^{\pm}$.

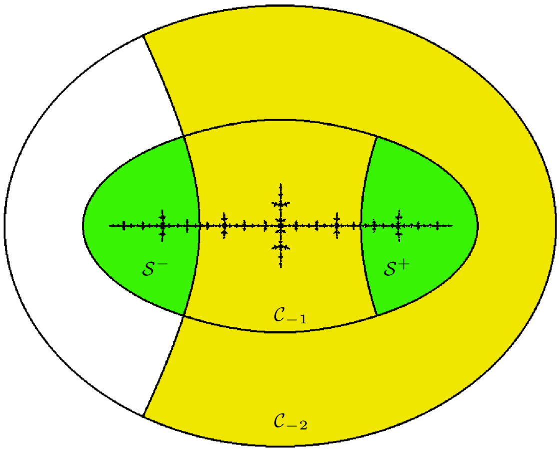


Figure 5: The pieces $\mathcal{S}^{\pm}$ are colored green. Note that $\mathcal{S}^+$ is compactly contained in $\mathcal{C}_{-2}$ which is colored yellow.

Definition 3.3 (Enlarged pieces)**.** *If $\mathcal{P}$ is a good piece of depth $m \geq -1$ with $f^{\circ(m+1)} = \mathcal{S}^{\pm}$, the enlarged piece $\widehat{\mathcal{P}}$ is the connected component of $f^{-(m+1)}(\pm\mathcal{C}_{-2})$ which contains $\mathcal{P}$.*

A key property is that when $\mathcal{P} \subsetneq \mathcal{Q}$ are good pieces, then $\widehat{\mathcal{P}} \Subset \widehat{\mathcal{Q}}$. For later purposes, we will prove a stronger result.

Definition 3.4 (Thickened pieces)**.** *We denote by $\widetilde{\mathcal{C}}_{-2}$ the component of*

$$\{\mathfrak{g} < 1\} \smallsetminus \big(\overline{\mathcal{R}}(5/12) \cup \overline{\mathcal{R}}(-5/12)\big)$$

which contains $\mathcal{C}_{-2}$. If $\mathcal{P}$ is a good piece of depth $m \geq -2$, the thickened piece *$\widetilde{\mathcal{P}}$ is the component of $f^{-(m+2)}(\widetilde{\mathcal{C}}_{-2})$ which contains $\mathcal{P}$.*

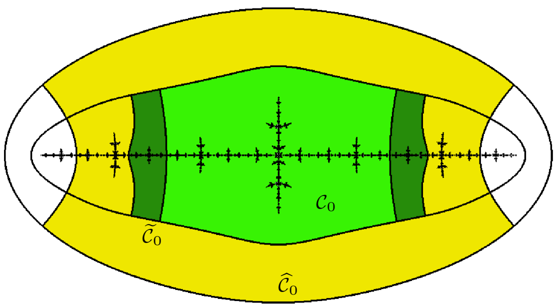


Figure 6: The critical piece $\mathcal{C}_0$ (light green) contained in the enlarged $\widehat{\mathcal{C}}_0$ (yellow) for the polynomial $f(z) = z^2 - 1.67$. The thickened piece $\widetilde{\mathcal{C}}_0$ is obtained by adjoining to $\mathcal{C}_0$ the two dark green pieces on each side. Note that $\widetilde{\mathcal{C}}_0$ is compactly contained in $\widehat{\mathcal{C}}_0$.

From now on, we assume that the critical value c is not contained in $\widetilde{\mathcal{C}}_{-2}$. This precisely occurs when $f(c) = f^{\circ 2}(0) > -\alpha$. In that case, the dynamical rays $\mathcal{R}(5/24)$ and $\mathcal{R}(-5/24)$ land at a common preimage of $-\alpha$.

Lemma 3.5. *If $\mathcal{Q}$ is a good piece of depth $m \geq -1$, then $\widetilde{Q} \Subset \widehat{Q}$.*

Proof. By definition, $\widetilde{\mathcal{S}}^+$ is the component of $f^{-1}(\widetilde{\mathcal{C}}_{-2})$ which contains $\mathcal{S}^+$. Since the critical value c is not contained in $\widetilde{\mathcal{C}}_{-2}$, the thickened piece $\widetilde{\mathcal{S}}^+$ avoid 0 and thus, is compactly contained in $\widehat{\mathcal{S}}^+ = \mathcal{C}_{-2}$ (see Figure 7). By symmetry, we also have $\widetilde{\mathcal{S}}^- \Subset \widehat{\mathcal{S}}^-$. Pulling back via iterates of f, we see that $\widetilde{\mathcal{Q}} \Subset \widehat{\mathcal{Q}}$ for any good piece $\mathcal{Q}$ of depth $m \geq -1$. □

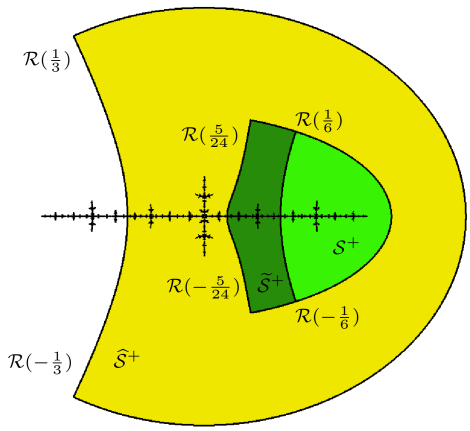


Figure 7: The pieces $\mathcal{S}^+$ (light green), $\widetilde{\mathcal{S}}^+$ (green) and $\widehat{\mathcal{S}}^+$ (yellow). We have the inclusions $\mathcal{S}^+ \subset \widetilde{\mathcal{S}}^+ \Subset \widehat{\mathcal{S}}^+$.

Lemma 3.6. *If $\mathcal{P} \subsetneq \mathcal{Q}$ are good pieces, then $\widehat{\mathcal{P}} \subset \widetilde{\mathcal{Q}}$.*

Proof. Let us say that a good piece $\mathcal{P}$ is consecutive to a good piece $\mathcal{Q}$ when $\mathcal{P} \subsetneq \mathcal{Q}$ and there is no good piece $\mathcal{P}'$ with $\mathcal{P} \subsetneq \mathcal{P}' \subsetneq \mathcal{Q}$. Clearly, it is enough to prove the property when $\mathcal{P}$ is consecutive to $\mathcal{Q}$. We prove it by induction on the depth $m \geq 0$ of $\mathcal{Q}$. For $m = 0$, note that there are only two good pieces consecutive to $\mathcal{C}_{-2}$: the piece $\mathcal{S}^+$ and the critical piece $\mathcal{C}_0$. On the one hand, by definition,

$$\widehat{\mathcal{S}}^+ = \mathcal{C}_{-2} \subset \widetilde{\mathcal{C}}_{-2}.$$

On the other hand (see Figure 8),

$$\widehat{\mathcal{C}}_0 = f^{-1}(-\mathcal{C}_{-2}) \subset \widetilde{\mathcal{C}}_{-2}.$$

This proves that the property holds for $m = 0$. Now, if the property holds for some $m \geq 0$, pulling back via f shows that it holds for $m + 1$. □

4. Regular points

In this section, we prove Theorem 2.2. We first recall the definitions given in the introduction.

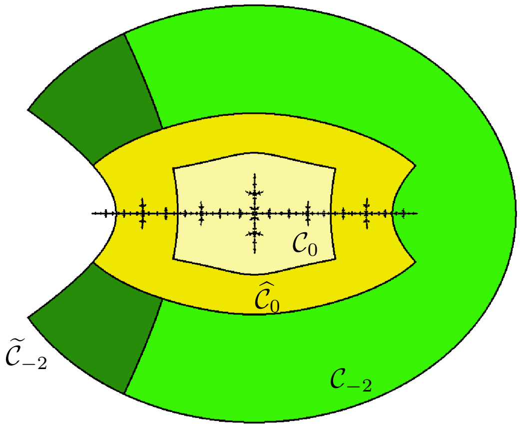


Figure 8: The enlarged piece $\widehat{\mathcal{C}}_0$ (yellow) containing $\mathcal{C}_0$ (light yellow) is contained in the enlarged piece $\widetilde{\mathcal{C}}_{-2}$ (green) containing $\mathcal{C}_{-2}$ (light green).

Definition 4.1 (Regular pieces and points). *A piece $\mathcal{P}$ of depth $m \geq 1$ is* regular *if*

$$f^{\circ m}(\mathcal{P}) = \mathcal{C}_0 \quad \text{and} \quad f^{\circ m} : \widehat{\mathcal{P}} \to \widehat{\mathcal{C}}_0 \text{ is an isomorphism.}$$

The piece is maximally regular *if it is not contained in a regular piece of smaller depth. A point $x \in J := [c, c^2 + c]$ is* regular *if it belongs to some regular piece. We denote by X the set of regular points.*

The function $\boldsymbol{n} : J \to \mathbb{N}$ is defined by

$$\boldsymbol{n}(x) := \inf\bigl\{m \geq 1 \mid x \text{ is contained in a regular piece of depth } m\bigr\}$$

with the convention $\inf \varnothing = +\infty$. Recall that $I := (\alpha, -\alpha)$. The map $T : X \to I$ is defined by

$$T(x) := f^{\boldsymbol{n}(x)}(x).$$

We now prove Theorem 2.2 which asserts that if $\boldsymbol{n} \in L^1(I)$, then f admits an a.c.i.m. supported on J.

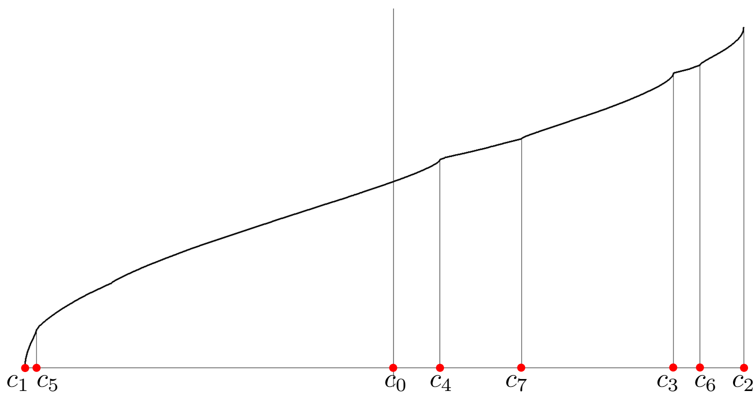


Figure 9: The graph of the function $J \ni x \mapsto \mu\bigl([c, x]\bigr) \in [0, 1]$ for $c \simeq -1.95$. The graph has vertical tangents on one side at each point of the postcritical set. The first seven iterates of the critical point are marked in red.

Proof of Theorem 2.2. First, if $\mathcal{P}$ is a regular piece of depth m, then $f^{\circ m} : \widehat{\mathcal{P}} \to \widehat{\mathcal{C}}_0$ is an isomorphism and the inverse branch $g_{\mathcal{P}} : \widehat{\mathcal{C}}_0 \to \widehat{\mathcal{P}}$ has uniformly bounded distortion on $\mathcal{C}_0$.

Second, since two puzzle pieces are either nested or disjoint, the maximally regular pieces are pairwise disjoint. Let $\mathfrak{R}$ be the set of maximally regular pieces contained in $\mathcal{C}_0$ and set

$$\mathcal{V} := \bigsqcup_{\mathcal{P}\in\mathfrak{R}} \mathcal{P}.$$

Note that $\mathcal{V} \cap I$ is the set of regular points in I. By assumption, $\boldsymbol{n}$ is integrable on I, and so, the complement of $\mathcal{V}$ in I is a set of Lebesgue measure zero. In addition, the function $\boldsymbol{n}$ is constant on each interval $\mathcal{P} \cap I$. We set $n_{\mathcal{P}} := \boldsymbol{n}(\mathcal{P} \cap I)$ and consider the dynamical system $T : \mathcal{V} \to \mathcal{C}_0$ defined by

$$T|_{\mathcal{P}} := f^{\circ n_{\mathcal{P}}} \quad \text{for} \quad \mathcal{P} \in \mathfrak{R}.$$

Third, we control the distortion of iterates of T as follows.

Lemma 4.2. *There exists a constant C such that for any $k \geq 1$ and each component $\mathcal{P}$ of $T^{-k}(\mathcal{C}_0)$, we have that*

$$\sup_{z\in\mathcal{P}} \big|(T^{\circ k})'(z)\big| \leq C \cdot \inf_{z\in\mathcal{P}} \big|(T^{\circ k})'(z)\big| \quad \text{and} \quad \sup_{z\in\mathcal{P}} \frac{1}{\big|(T^{\circ k})'(z)\big|} \leq C \cdot \frac{\mathrm{Leb}(\mathcal{P}\cap\mathbb{R})}{\mathrm{Leb}(I)}.$$

Proof. We first prove by induction on k that any component $\mathcal{P}$ of $T^{-k}(\mathcal{C}_0)$ is a regular piece. This is true for $k = 1$ by definition. Assume $k \geq 2$ and $\mathcal{P}$ is a component of $T^{-k}(\mathcal{C}_0)$. Then it is a good piece of depth $m \geq 1$. It is contained in the domain of T, i.e. in a regular piece $\mathcal{Q} \in \mathfrak{R}$ of depth $n \geq 1$. According to Lemmas 3.5 and 3.6, $\widehat{\mathcal{P}} \Subset \widehat{\mathcal{Q}}$. So, $f^{\circ n}$ is univalent on $\widehat{\mathcal{P}}$. In addition, $T(\mathcal{P}) = f^{\circ n}(\mathcal{P})$ is regular by induction hypothesis, so that $f^{\circ(m-n)}$ is univalent on $f^{\circ n}(\widehat{\mathcal{P}})$. Thus, $f^{\circ m}$ is univalent on $\widehat{\mathcal{P}}$, which shows that $\mathcal{P}$ is regular.

As a consequence, $T^{\circ k} : \mathcal{P} \to \mathcal{C}_0$ is an isomorphism whose inverse extends univalently to $\widehat{\mathcal{C}}_0$. In particular the distortion of $T^{\circ k}$ on $\mathcal{P}$ is uniformly bounded on $\mathcal{P}$ by a constant C which only depends on the modulus of the annulus $\widehat{\mathcal{C}}_0 - \overline{\mathcal{C}}_0$. This proves the first inequality. The second inequality follows immediately. □

Fourth, we can define a sequence of measures ν_k supported on I as follows. Given $k \geq 1$ and $z \in T^{-k}(\mathcal{C}_0)$, let $\mathcal{P}(z)$ be the component of $T^{-k}(\mathcal{C}_0)$ containing z and let $\varepsilon_k(z) \in \{-1, +1\}$ be the sign of the derivative of $T^{\circ k}$ on $\mathcal{P}(z) \cap \mathbb{R}$. According to the previous lemma, there is a constant C such that for all $k \geq 1$, the series

$$h_k(y) := \sum_{x\in T^{-k}(y)} \frac{\varepsilon_k(x)}{(T^{\circ k})'(x)}$$

is uniformly convergent and bounded by C for $y \in \mathcal{C}_0$. In particular, it defines a function h_k which is holomorphic on $\mathcal{C}_0$. The restriction of this function to I is the density of the measure ν_k defined on I by

$$\nu_k = (T^{\circ k})_* \,\mathrm{Leb}\,.$$

Since $\boldsymbol{n}$ is integrable, it is finite almost everywhere and so, $\mathcal{V} \cap I$ has full measure in I. The points which escapes from $\mathcal{V} \cap I$ under iteration of T form a countable union of sets of measure zero, thus a set of measure zero. It follows that $T^{-k}(\mathcal{C}_0) \cap I$ has full measure in I and the total mass of ν_k is

$$\|\nu_k\| = \mathrm{Leb}\big(T^{-k}(\mathcal{C}_0) \cap I\big) = \mathrm{Leb}(I).$$

Now, since the functions h_k are uniformly bounded by C on $\mathcal{C}_0$, any weak limit ν of the Cesaro average

$$\bar{\nu}_k := \frac{1}{k}\sum_{j=1}^{k}\nu_j$$

is a measure with analytic density. It is clearly invariant by T since

$$T_*\bar{\nu}_k + \frac{\nu_1}{k} = \bar{\nu}_k + \frac{\nu_{k+1}}{k}.$$

We finally promote this measure ν which is invariant by T, into a measure μ which is invariant by f. We shall define μ by its action on continuous functions. Any continuous function ϕ on J, may be pulled back to a function $S(\phi)$ defined on I by

$$S(\phi)(x) = \sum_{n=0}^{\boldsymbol{n}(x)-1} \phi\big(f^{\circ n}(x)\big).$$

This function is integrable on I since $\boldsymbol{n}$ is integrable on I and since for $x \in I$,

$$\big|S(\phi)(x)\big| \le \boldsymbol{n}(x)\cdot \|\phi\|_\infty.$$

We define μ by

$$\int_J \phi \ \mathrm{d}\mu := \int_I S(\phi) \ \mathrm{d}\nu.$$

This defines a positive linear functional on the continuous functions on J and thus a measure on J. The formula defining μ extends to characteristic functions of Borel sets, which shows that it is absolutely continuous with respect to the Lebesgue measure: if A has Lebesgue measure zero, then $S(\mathbf{1}_A)$ vanishes outside a set of Lebesgue measure zero, and so,

$$\int_J \mathbf{1}_A \ \mathrm{d}\mu = \int_I S(\mathbf{1}_A) \ \mathrm{d}\nu = 0.$$

The total mass of μ is

$$\int_J \mathbf{1} \ \mathrm{d}\mu = \int_I \boldsymbol{n} \ \mathrm{d}\nu.$$

It is invariant by f since

$$S(\phi\circ f) = S(\phi) + \phi\circ T - \phi \quad \text{and} \quad \nu \text{ is invariant by } T. \qquad \square$$

5. Tableaux

To encode and represent the dynamics between puzzle pieces, we will use the notion of tableaux.

Definition 5.1 (Critical, semi-critical and off-critical pieces)**.** *A good piece $\mathcal{P}$ is*

$$\begin{cases} \textit{critical} & \textit{if } 0 \in \mathcal{P}, \\ \textit{semi-critical} & \textit{if } 0 \in \widehat{\mathcal{P}}\smallsetminus\mathcal{P} \textit{ and} \\ \textit{off-critical} & \textit{if } 0 \notin \widehat{\mathcal{P}}. \end{cases}$$

Definition 5.2 (Tableaux)**.** *The tableau of a point $x \in \mathcal{K}_c$ is an array with one column associated with each $x_n := f^{\circ n}(x)$ and one row associated to each depth m in the Yoccoz puzzle. The position (m,n) on the m-th row downwards and the n-th column to the right, is marked if $\mathcal{P}_m(x_n)$ is a good piece and unmarked otherwise. It is critical (respectively semi-critical, off-critical) if $\mathcal{P}_m(x_n)$ is critical (respectively semi-critical, off-critical). The k-th diagonal in a tableau is the set of positions (m,n) such that $m+n=k$.*

A position is represented by

- a black disk if it is critical,
- a grey disk if it is semi-critical,
- a circle if it is off-critical,
- a dot if it is unmarked.

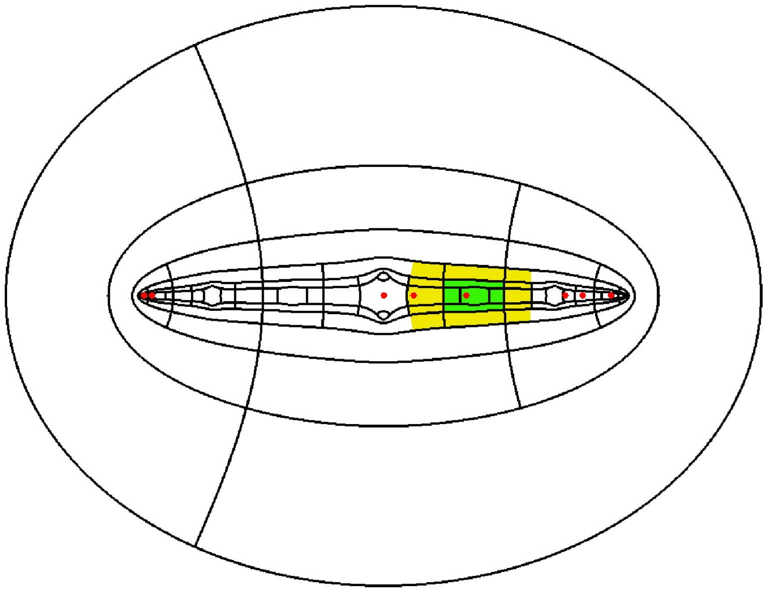


Figure 10: The puzzle pieces of depth -2 through 3 for $f(z) = z^2 - 1.95$. From left to right, we marked in red c_1, c_5, $c_0 = 0$, c_4, c_7, c_3, c_6 and c_2, where $c_k := f^{\circ k}(0)$. The puzzle piece $\mathcal{P}_2(c_7)$ is regular. It is colored green and the enlarged piece $\widehat{\mathcal{P}}_2(c_7)$ is colored yellow.

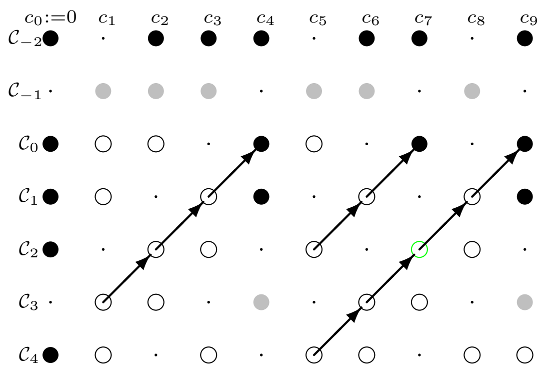


Figure 11: The tableau of the critical point for $c = -1.95$. The pieces at the origin of each arrow are regular. For example, this is the case of the piece $\mathcal{P}_2(c_7)$ which is colored green on Figure 10.

A position (m, n) in the tableau is regular if and only if all positions on the diagonal starting at this position (m, n) and going north-east are off-critical, except the position $(0, m + n)$ which is critical.

The tableaux satisfy some rules. Since we are considering marked and unmarked positions, critical, semi-critical and off-critical positions, those rules are not exactly the same as the classical rules.

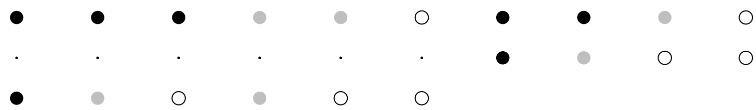
Lemma 5.3 (Rule for unmarked positions)**.** *In a diagonal of a tableau, either all positions are marked, or all positions are unmarked. The k-th diagonal of a tableau is unmarked if and only if the $(k+1)$-th diagonal is marked with position $(0, k+1)$ critical.*

Proof. To see that in a diagonal either all positions are marked, or all positions are unmarked, observe that a piece $\mathcal{P}_m(x)$ is a good piece if and only if $\mathcal{P}_{m-1}\big(f(x)\big)$ is a good piece. Thus the position (m, n) is marked if and only if the position $(m-1, n+1)$ is marked.

Now, the k-th diagonal of the tableau is unmarked if and only if the position $(-1, k+1)$ corresponds to the unique unmarked piece of depth -1, i.e. $\mathcal{C}_{-1}$. Since $\mathcal{C}_{-1} \cap \mathcal{K} = \mathcal{C}_0 \cap \mathcal{K}$, this is the case if and only if the position $(0, k+1)$ corresponds to $\mathcal{C}_0$. □

According to this rule, we see that going downwards in the column of a tableau, we cannot meet two consecutive unmarked positions. The following rule establishes the possible configuration between marked positions in the same column. They may or may not be separated by an unmarked position.

Lemma 5.4 (Rule for marked positions)**.** *In a column, the only possible configurations between consecutive marked pieces are the following:*



In particular, if there are two semi-critical positions in the same column, the difference of their depths is even and semi-critical positions alternate with unmarked positions.

In other words, the forbidden configurations are the following:

Proof. Every piece containing a critical piece is itself critical and every good piece contained in an off-critical piece is itself off-critical.

If $\mathcal{P} \subset \mathcal{Q}$ are good pieces of respective depths m and $m-1$, then $f^{\circ(m+1)}(\mathcal{Q}) = \mathcal{C}_{-2}$ and $f^{\circ(m+1)}(\mathcal{P})$ is the only good piece of depth -1 contained in $\mathcal{C}_{-2}$, i.e. $f^{\circ(m+1)}(\mathcal{P}) = \mathcal{S}^+$. Since $\widehat{\mathcal{P}}$ is the component of $f^{-(m+1)}(\mathcal{C}_{-2})$ containing $\mathcal{P}$, we have $\widehat{\mathcal{P}} = \mathcal{Q}$. Thus, if $\mathcal{Q}$ is critical, then $\widehat{\mathcal{P}} = \mathcal{Q}$ contains the critical point and $\mathcal{P}$ cannot be off-critical; and if $\mathcal{Q}$ is semi-critical, then $\widehat{\mathcal{P}} = \mathcal{Q}$ does not contain the critical point and $\mathcal{P}$ cannot be semi-critical.

The possibility of two consecutive unmarked positions is ruled out by the previous lemma. □

The third tableau rule relates an arbitrary tableau, to the tableau of the critical point (see Figure 12).

Lemma 5.5 (Rule for critical positions)**.** *If in the tableau the position (m, n) is critical, then for $-2 \le m' \le m$ and $0 \le n' \le m'+2$, the position $(m'-n', n+n')$ in the tableau has the same nature as the position $(m'-n', n')$ in the tableau of the critical point $c_0 = 0$.*

Proof. At each depth, there is a unique critical piece: $\mathcal{C}_m$. □

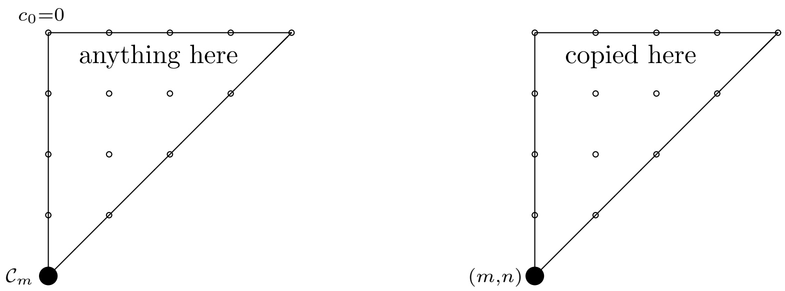


Figure 12: Illustration of the rule for copying.

6. The return time $n_0 := \boldsymbol{n}(c)$

Given $c \in [-2, -1]$, we define recursively a sequence $(\mathcal{B}_m^{\pm})_{m\geq 0}$ of good pieces as follows (see Figure 13):

- $\mathcal{B}_0^{\pm} := \mathcal{B}_0 := \mathcal{C}_0 \subset \mathcal{P}_{-2}(\beta)$;
- for $m \geq 1$, $\mathcal{B}_m^{\pm}$ is the component of $f^{-1}(\mathcal{B}_{m-1}^{+})$ contained in $\mathcal{P}_{m-2}(\pm\beta)$.

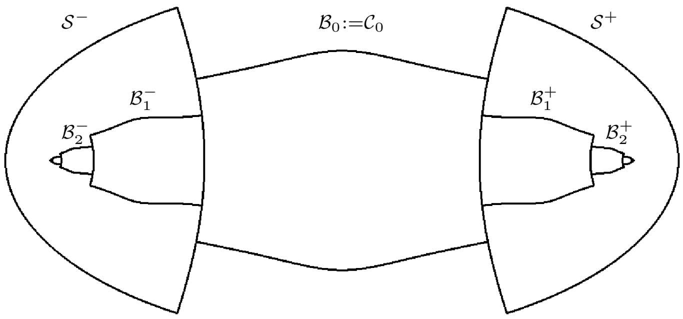


Figure 13: For $m \geq 1$, the puzzle piece $\mathcal{B}_m^{\pm}$ are mapped by f to $\mathcal{B}_{m-1}^{+}$.

Lemma 6.1. *For all $m \geq 1$, the pieces $\mathcal{B}_m^{\pm}$ are regular.*

Proof. On the one hand, $\mathcal{B}_m^{+} \subset \mathcal{P}_{m-2}(\beta) \subset \mathcal{S}^{+}$. So, according to Lemma 3.5, $\widehat{\mathcal{B}}_m^{+} \subset \widetilde{\mathcal{S}}^{+}$. On the other hand, the map $f : \widetilde{\mathcal{S}}^{+} \to \widetilde{\mathcal{C}}_{-2}$ is an isomorphism. It follows that $f : \widehat{\mathcal{B}}_m^{+} \to \widehat{\mathcal{B}}_{m-1}^{+}$ is an isomorphism. By induction on $m \geq 1$, we deduce that $f^{\circ m} : \widehat{\mathcal{B}}_m^{+} \to \widehat{\mathcal{B}}_0^{+} = \widehat{\mathcal{C}}_0$ is an isomorphism, thus $\mathcal{B}_m^{+}$ is regular. By symmetry, $\mathcal{B}_m^{-} = -\mathcal{B}_m^{+}$ is also regular. □

From now on, we assume that

$$c \in \mathcal{B}_{n_0}^{-} \quad \text{for some integer } n_0 \geq 3, \quad \text{so that} \quad \boldsymbol{n}(c) = n_0.$$

For $m \in [2, n_0 + 1]$, we let $\mathcal{D}_m^{-}$ be the component of $f^{-1}(\mathcal{B}_{m-1}^{-})$ intersecting $(\alpha, 0)$ and $\mathcal{D}_m^{+}$ be the component intersecting $(0, -\alpha)$ (see Figure 14 for $c = -1.98$; in that case $n_0 = 4$). For $m = n_0 + 1$, we have

$$\mathcal{D}_{n_0+1}^{+} = \mathcal{D}_{n_0+1}^{-} = \mathcal{C}_{n_0+1}.$$

The $2n_0 - 1$ puzzle pieces $\mathcal{D}_m^{\pm}$ cover I, up to finitely many iterated preimages of α.

Remark 6.2. Note that $\widehat{\mathcal{B}}_0 = \widehat{\mathcal{C}}_0$ contains $\mathcal{B}_1^{+}$ but avoids $\mathcal{B}_k^{+}$ for $k \geq 2$.

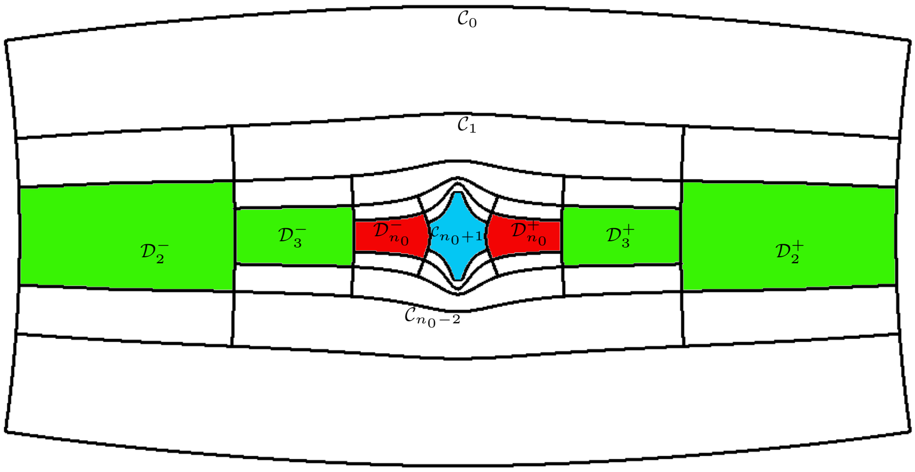


Figure 14: For the polynomial $f(z) = z^2 - 1.98$, $n_0 = 4$. There are 4 maximally regular pieces $\mathcal{D}_2^\pm$ (of depth 2) and $\mathcal{D}_3^\pm$ (of depth 3) colored green, 2 semi-critical pieces $\mathcal{D}_4^\pm$ (of depth 4) colored red and 1 critical piece $\mathcal{C}_5$ (of depth 5) colored blue.

Lemma 6.3. *Assume $m \in [2, n_0 + 1]$. Then, $f(\mathcal{D}_m^\pm)$ is regular and $\mathcal{D}_m^\pm \subset \mathcal{C}_{m-2}$. If $m \le n_0 - 1$, the pieces $\mathcal{D}_m^\pm$ are regular. The pieces $\mathcal{D}_{n_0}^\pm$ are semi-critical. The piece $\mathcal{C}_{n_0+1}$ is critical.*

Proof. By definition, $f(\mathcal{D}_m^\pm) = \mathcal{B}_{m-1}^-$ which is regular according to the previous lemma. In addition, $c \in \mathcal{B}_{n_0}^- \subset \mathcal{P}_{n_0-2}(-\beta) \subseteq \mathcal{P}_{m-3}(-\beta)$. As a consequence, $f(\mathcal{C}_{m-2}) = \mathcal{P}_{m-3}(-\beta)$ contains $f(\mathcal{D}_m^\pm) = \mathcal{B}_{m-1}^-$. It follows that $\mathcal{D}_m^\pm \subset \mathcal{C}_{m-2}$.

Since $\widehat{\mathcal{B}}_0 = \widehat{\mathcal{C}}_0$ contains $\mathcal{B}_1^+$ but avoids $\mathcal{B}_k^+$ for $k \ge 2$, pulling back by the isomorphism $f^{\circ m} : \widetilde{\mathcal{P}}_{m-2}(\beta) \to \widetilde{\mathcal{C}}_{-2}$, we see that $\widehat{\mathcal{B}}_m^+$ contains $\widehat{\mathcal{B}}_{m+1}^+$ but avoids $\mathcal{B}_{m+k}^+$ for $k \ge 2$. As a consequence, for $m \le n_0 - 1$, $f(\widehat{\mathcal{D}}_m^\pm) = \widehat{\mathcal{B}}_{m-1}^-$ avoids the critical value which is contained in $\mathcal{B}_{n_0}^-$ and $f : \widehat{\mathcal{D}}_m^\pm \to \widehat{\mathcal{B}}_{m-1}^-$ is an isomorphism. This shows that for $m \in [2, n_0 - 1]$, the piece $\mathcal{D}_m^\pm$ are regular.

In addition, $f(\widehat{\mathcal{D}}_{n_0}^\pm) = \widehat{\mathcal{B}}_{n_0-1}^-$ contains the critical value, so that $\widehat{\mathcal{D}}_{n_0}^\pm$ contains the critical point and $\mathcal{D}_{n_0}^\pm$ is semi-critical. Finally, $\mathcal{C}_{n_0+1}$ is critical. □

Figures 15 shows restricted tableaux of the pieces $\mathcal{D}_m^\pm$ with $0 \le m \le n_0 + 1$. We only show the positions (i, j) with $i + j \le m$, $i \ge -1$ and $j \ge 0$.

The following lemma will be used to compare the orbits of critical and semi-critical pieces.

Lemma 6.4. *The enlarged piece $\widehat{\mathcal{D}}_2^\pm$ is contained in $\widehat{\mathcal{S}}^\pm = \pm\mathcal{C}_{-2}$.*

Proof. Since $\mathcal{D}_2^\pm$ is regular, the enlarged piece $\widehat{\mathcal{D}}_2^\pm$ avoids the critical point 0 and $\widehat{\mathcal{D}}_2^+ \cap \widehat{\mathcal{D}}_2^- = \varnothing$. The piece $\mathcal{D}_2^-$ contains α in its boundary, so $\alpha \in \widehat{\mathcal{D}}_2^-$ and $-\alpha \in \widehat{\mathcal{D}}_2^+$. In particular, $\widehat{\mathcal{D}}_2^-$ avoids $-\alpha$.

By construction, if an enlarged puzzle piece intersects an external ray, it contains the landing point of the external ray. If $\widehat{\mathcal{D}}_2^-$ were not contained in $-\mathcal{C}_{-2}$, it would intersect its boundary, thus the external rays of angle $\pm 1/6$ which land at $-\alpha$. This contradicts the previous discussion. So, $\widehat{\mathcal{D}}_2^- \subset \widehat{\mathcal{S}}^-$, and $\widehat{\mathcal{D}}_2^+ \subset \widehat{\mathcal{S}}^+$ by symmetry. □

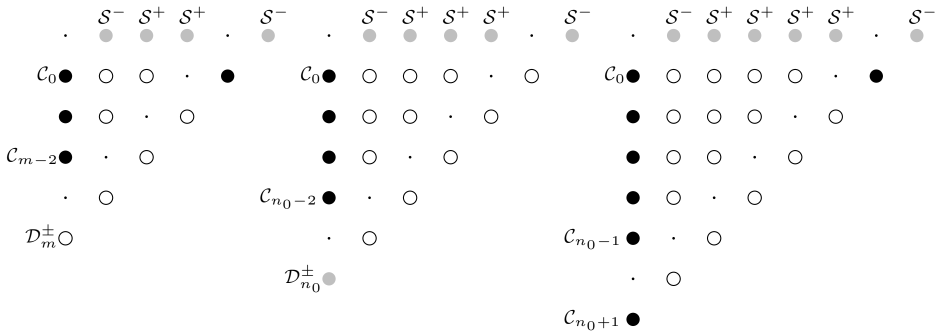


Figure 15: Left: the restricted tableau of a regular piece $\mathcal{D}_m^{\pm}$ with $m \leq n_0 - 1$. Middle: the restricted tableau of a semi-critical piece $\mathcal{D}_{n_0}^{\pm}$. Right : the restricted tableau of the critical piece $\mathcal{C}_{n_0+1}$.

7. Children

We now introduce the notion of child. In this article, we only need to consider children of the critical piece $\mathcal{C}_0$.

Definition 7.1 (Children)**.** *A (semi-)critical* child *of* $\mathcal{C}_0$ *is a (semi-)critical piece* $\mathcal{P} \subset \mathcal{C}_0$ *of depth* $m \geq 1$ *such that* $f^{\circ m}(\mathcal{P}) = \mathcal{C}_0$ *and* $f(\mathcal{P})$ *is regular.*

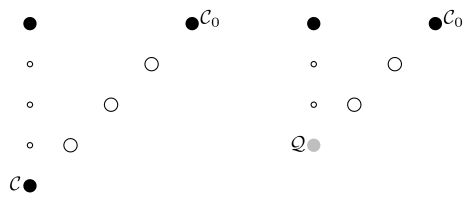


Figure 16: Left: the critical piece $\mathcal{C}$ is a critical child of $\mathcal{C}_0$. Right: the semi-critical piece $\mathcal{Q}$ is a semi-critical child of $\mathcal{C}_0$.

Lemma 7.2. *For a given depth* $n \geq 1$*, either*

- *there are no children of depth* n*, or*
- *there is a unique child of depth* n*, this child is critical and there are no semi-critical pieces of depth* n*, or*
- *there are exactly two children of depth* n*, those children are semi-critical and the critical piece of depth* n *is unmarked.*

Proof. For each depth $n \geq 1$, there is at most one component $\widehat{\mathcal{V}}$ of $f^{-n}(\widehat{\mathcal{C}}_0)$ which contains the critical point (see Figure 17). If there is a child at depth n, then $f^{\circ n} : \widehat{\mathcal{V}} \to \widehat{\mathcal{C}}_0$ is a ramified covering of degree 2. Note that $\mathcal{C}_0$ is the unique piece of depth 0 contained in $\widehat{\mathcal{C}}_0$.

Either the critical value of $f^{\circ n} : \widehat{\mathcal{V}} \to \widehat{\mathcal{C}}_0$ belongs to $\mathcal{C}_0$, in which case $f^{-n}(\mathcal{C}_0)$ is the unique piece of depth n is $\widehat{\mathcal{V}}$; in that case, there is a unique child at depth n, this child is critical, and there are no semi-critical pieces of depth n.

Or the critical value of $f^{\circ n} : \widehat{\mathcal{V}} \to \widehat{\mathcal{C}}_0$ belongs to $\mathcal{B}_1^{\pm}$, in which case $f^{-n}(\mathcal{C}_0)$ has exactly two components in $\mathcal{V}$; those are the only pieces of depth n contained in $\widehat{\mathcal{V}}$; in that case, there are two children at depth n, those children are semi-critical and the critical piece of depth n is unmarked. □

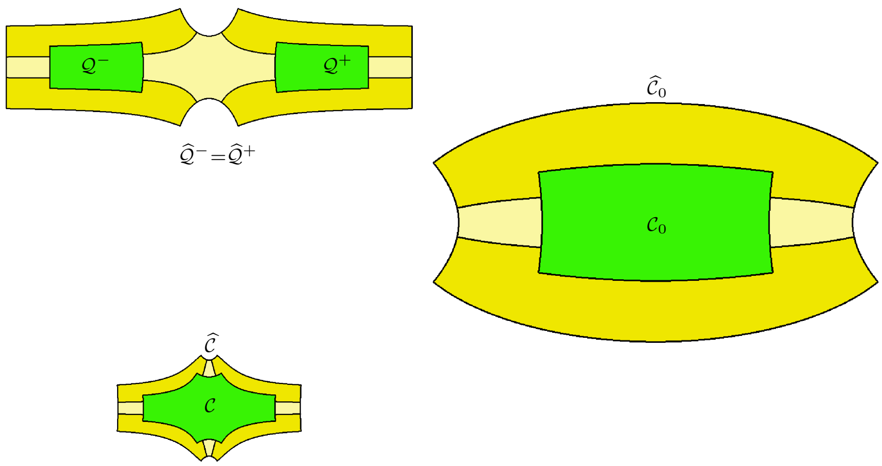
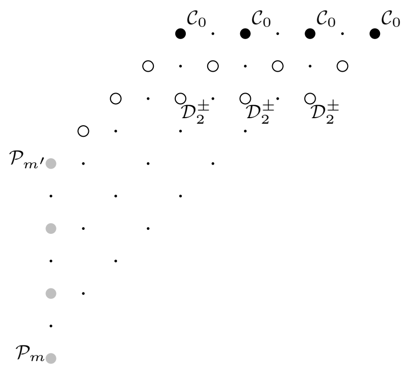


Figure 17: Right: the critical piece $\mathcal{C}_0$ (green) surrounded by $\widehat{\mathcal{C}}_0$ (yellow). The pieces $\mathcal{B}_1^\pm$ are indicated (light yellow). Top left: two semi-critical children $\mathcal{Q}^\pm$ of $\mathcal{C}_0$ of depth $m = 4$ (green) surrounded by the appropriate component of $f^{-m}(\widehat{\mathcal{C}}_0)$ (yellow). The preimages of $\mathcal{B}_1^\pm$ are indicated (light yellow). One of them contains the critical point. Bottom left: a critical child $\mathcal{C}$ of $\mathcal{C}_0$ of depth $m + 1$ (green) surrounded by the appropriate component of $f^{-(m+1)}(\widehat{\mathcal{C}}_0)$ (yellow). The preimages of $\mathcal{B}_1^\pm$ are indicated (light yellow).

Lemma 7.3. *Let $(\mathcal{P}_k)_{1\leq n\leq m}$ be a nested sequence of pieces of depth n. Assume $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ are semi-critical with $m' < m$. Then, $m - m'$ is even and if $\mathcal{P}_{m'}$ is a semi-critical child of $\mathcal{C}_0$, then for $n \in [0, m - m']$ even, $\mathcal{P}_{m'+n}$ is a semi-critical child of $\mathcal{C}_0$.*

Proof. Consider the restricted tableau of $\mathcal{P}_m$. By assumption, the positions $(m, 0)$ and $(m', 0)$ are semi-critical. According to the rule for marked positions (Lemma 5.4), $m-m'$ is even, positions $(m'+n, 0)$ with n even are semi-critical, and positions $(m'+n, 0)$ with n odd are unmarked. According to the rule for unmarked positions (Lemma 5.3), when n is odd the position $(0, m' + n)$ is unmarked and the position $(0, m'+n+1)$ is critical. In particular, if $n \in [2, m-m']$ is even, $f^{\circ(m'+n)}(\mathcal{P}_{m'+n}) = \mathcal{C}_0$ and $f^{\circ(m'+n-2)}(\mathcal{P}_{m'+n})$ is a piece contained in $\mathcal{C}_0$ which maps by $f^{\circ 2}$ to $\mathcal{C}_0$. Such a piece is necessarily $\mathcal{D}_2^\pm$. So, we have the following tableau:

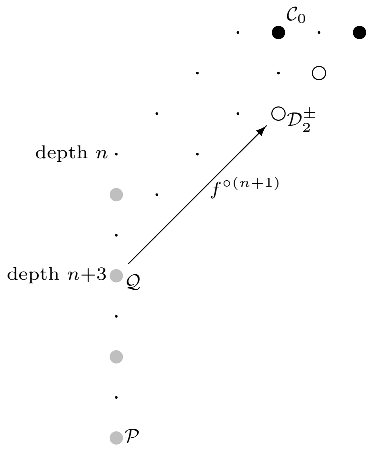


For $n \in [0, m - m']$ even, $\mathcal{P}_{m'+n}$ is a semi-critical child of $\mathcal{C}_0$. $\square$

Lemma 7.4. *Assume $\mathcal{C}_n$ is a critical child of $\mathcal{C}_0$, and $\mathcal{P} \not\subset \mathcal{C}_n$ is semi-critical of depth $m \geq n+3$. Then, the piece of depth $n+3$ containing $\mathcal{P}$ is a semi-critical child of $\mathcal{C}_0$.*

Proof. First, note that the piece of depth n containing $\mathcal{P}$ is not critical by assumption. According to Lemma 7.2, it cannot be semi-critical since there is a critical child of $\mathcal{C}_0$ at depth n. So, it is unmarked.

As in the proof of the previous lemma, it follows from the tableaux rules for unmarked and for marked positions (Lemmas 5.3 and 5.4) that $m-n$ is odd and we have the following restricted tableau:



In particular, if $\mathcal{Q}$ is the semi-critical piece of depth $n+3$ containing $\mathcal{P}$, then $f^{\circ(n+1)}(\mathcal{Q}) = \mathcal{D}_2^{\pm}$. By assumption, $\widehat{\mathcal{Q}} \supset \widehat{\mathcal{P}}$ contains the critical point, thus $\widehat{\mathcal{Q}}$ intersects $\mathcal{C}_n$. As a consequence, $f^{\circ(n+1)}(\widehat{\mathcal{Q}})$ intersects $\mathcal{S}^- = f^{\circ(n+1)}(\mathcal{C}_n)$. According to Lemma 6.4, $\widehat{\mathcal{D}}_2^{\pm} \subset \widehat{\mathcal{S}}^{\pm}$. Thus, $\widehat{\mathcal{D}}_2^{+} \subset \widehat{\mathcal{S}}^{+} = \mathcal{C}_{-2}$ avoids $\mathcal{S}^-$ and

$$f^{\circ(n+1)}(\widehat{\mathcal{Q}}) = \widehat{\mathcal{D}}_2^{-} \subset \widehat{\mathcal{S}}^{-} = f^{\circ(n+1)}(\widehat{\mathcal{C}}_n).$$

In addition, $\widehat{\mathcal{Q}}$ is the component of $f^{-(n+1)}(\widehat{\mathcal{D}}_2^{-})$ containing 0, so $\widehat{\mathcal{Q}} \subset \widehat{\mathcal{C}}_n$. In particular, $f^{\circ(n+1)} : \widehat{\mathcal{Q}} \to \widehat{\mathcal{D}}_2^{-}$ is a ramified covering of degree 2 and $\mathcal{Q}$ is a semi-critical child of $\mathcal{C}_0$. □

8. Singular pieces

Definition 8.1 (Singular pieces)**.** *A piece $\mathcal{P}$ is* singular *if $\mathcal{P} \subseteq \mathcal{C}_0$ is not contained in a regular piece. We denote by $\mathfrak{S}_m$ the set of singular pieces of depth $m \geq 0$.*

Note that $\mathfrak{S}_0 = \{\mathcal{C}_0\}$. In addition, if $\mathcal{P} \subset \mathcal{Q}$ are puzzle pieces and $\mathcal{P}$ is singular, then $\mathcal{Q}$ is also singular. The converse is not true: $\mathcal{Q}$ may be singular and $\mathcal{P}$ regular.

Also, note that for $x \in I$, we have $\boldsymbol{n}(x) \geq m$ precisely when x is an iterated preimage of α or when $x \in \mathcal{P} \in \mathfrak{S}_{m-1}$. Since the set of iterated preimages of α is countable, therefore of measure 0, we have

$$\int_I \boldsymbol{n} = \sum_{m\geq 0} \sum_{\mathcal{P}\in\mathfrak{S}_m} \mathrm{Leb}(\mathcal{P} \cap \mathbb{R})$$

and to prove that $\boldsymbol{n}$ is integrable, we must prove that this series is convergent.

In this section, we show that for $n_0 \geq 2$,

$$\mathrm{card}(\mathfrak{S}_m) \leq 3 \cdot (2n_0)^{m/n_0}.$$

Later, we will control $\mathrm{Leb}(\mathcal{P} \cap \mathbb{R})$ for $\mathcal{P} \in \mathfrak{S}_m$.

Proposition 8.2. *For $n_0 \geq 2$, the number S_m of singular pieces of depth m is bounded from above by $3 \cdot (2n_0)^{m/n_0}$.*

Proof. The proof goes by induction on $m \geq 1$. We first prove that the property holds for $m \leq n_0$. Since $\mathfrak{S}_0 = \{\mathcal{C}_0\}$, $S_0 = 1$.

Lemma 8.3. *For $m \in [1, n_0]$, $S_m = 3$. The three singular pieces of depth m are contained in $\mathcal{C}_{m-1}$. One of them is $\mathcal{C}_m$. The other two are the pieces of depth m containing $\mathcal{D}^{\pm}_{m+1}$.*

Proof. A singular piece of depth $m \in [1, n_0 - 1]$ cannot be contained in the regular pieces $\mathcal{D}^{\pm}_k$ with $k \in [2, m]$. So, it is contained in $\mathcal{C}_{m-1}$ (see Figure 14). Since $\mathcal{C}_m$ and $\mathcal{D}^{\pm}_{m+1}$ cover $\mathcal{C}_{m-1} \cap I$ up to 2 points, there are exactly three pieces of depth m contained in $\mathcal{C}_{m-1}$: $\mathcal{C}_m$ and the two pieces of depth m containing $\mathcal{D}^{\pm}_{m+1}$.

Similarly, a singular piece of depth n_0 cannot be contained in the regular pieces $\mathcal{D}^{\pm}_k$ with $k \in [2, n_0 - 1]$. So, there are three singular pieces of depth n_0: $\mathcal{C}_{n_0}$ and the two pieces $\mathcal{D}^{\pm}_{n_0}$. □

This shows that there are exactly $S_m = 3 \leq 3 \cdot (2n_0)^{m/n_0}$ singular pieces of depth $m \in [1, n_0]$.

Lemma 8.4. *Every singular piece of depth $m \geq n_0 + 1$ is contained in $\mathcal{C}_{n_0+1} \cup \mathcal{D}^{\pm}_{n_0}$.*

Proof. The pieces $\mathcal{D}^{\pm}_k$ with $k \in [2, n_0]$ cover I up to finitely many preimages of α. For $k \leq n_0 - 1$, they are regular pieces. So, if $\mathcal{P}$ is a singular piece of depth $m \geq n_0 + 1$, then $\mathcal{P}$ is contained in $\mathcal{D}^{\pm}_{n_0}$ or in $\mathcal{D}^{\pm}_{n_0+1} = \mathcal{C}_{n_0+1}$. □

We now proceed by induction on $m \geq n_0 + 1$. This requires the following lemma. If $\mathcal{P}$ is contained in a child of $\mathcal{C}_0$, the deepest child containing $\mathcal{P}$ is the one with largest depth.

Lemma 8.5. *Assume $\mathcal{P}$ is a singular piece of depth $m \geq n_0$. Let $n \geq n_0$ be the depth of the deepest child of $\mathcal{C}_0$ containing $\mathcal{P}$. Then,*

- *either $m = n$ and $\mathcal{P}$ is a child of $\mathcal{C}_0$,*
- *or $f^{\circ n}(\mathcal{P}) \subset \mathcal{C}_0$ is a singular piece of depth $m - n \geq 1$.*

Proof. If $m = n$ or $f^{\circ n}(\mathcal{P})$ is singular, there is nothing to prove. So, let us assume that $m > n > 2$ and that $f^{\circ n}(\mathcal{P})$ is not singular. Then, $f^{\circ n}(\mathcal{P})$ is contained in a regular piece $\mathcal{P}'$ of depth $k \in [1, m - n]$. Let $\mathcal{Q}$ be the piece of depth n containing $\mathcal{P}$ and let $\mathcal{Q}' \subset \mathcal{Q}$ be the piece of depth $n + k$ containing $\mathcal{P}$. Then $f(\mathcal{Q}')$ is regular (see Figure 18). However,

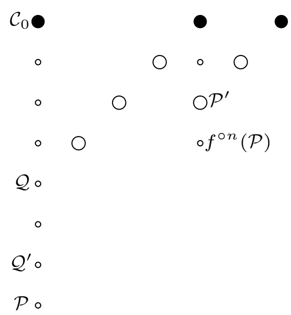


Figure 18: Illustration for the proof of Lemma 8.5

- $\mathcal{Q}'$ cannot be off-critical since it would be regular, contradicting the fact that $\mathcal{P}$ is singular;

- $\mathcal{Q}'$ cannot be critical or semi-critical since it would be a child of $\mathcal{C}_0$ containing $\mathcal{P}$, contradicting the fact that $\mathcal{Q}$ is the deepest such child. □

If $\mathcal{P} \in \mathfrak{S}_m$ with $m \geq n_0 + 1$, then $\mathcal{P}$ is contained either in the semi-critical children $\mathcal{D}^{\pm}_{n_0}$ or in the critical child $\mathcal{C}_{n_0+1}$. Thus, either $\mathcal{P}$ is a child of $\mathcal{C}_0$, or there is a deepest child $\mathcal{Q}$ of depth $n \in [n_0, m]$ containing $\mathcal{P}$ and $f^{\circ n}(\mathcal{P}) \in \mathfrak{S}_{m-n}$. For each $n \geq n_0$, there is either

- no child of depth n, or
- one critical child $\mathcal{Q}$ and each piece of depth $m-n$ has at most two preimages in $\mathcal{Q}$ by $f^{\circ(m-n)}$, or
- two semi-critical children $\mathcal{Q}^{\pm}$ and each piece of depth $m-n$ has one preimage in $\mathcal{Q}^+$ and one preimage in $\mathcal{Q}^-$ by $f^{\circ(m-n)}$.

It follows that

$$S_m \leq \sum_{n=n_0}^{m} 2S_{n-m}.$$

By induction hypothesis, we have

$$\begin{aligned} S_m \leq \sum_{n=n_0}^{m} 3 \cdot 2 \cdot (2n_0)^{(m-n)/n_0} &= 3 \cdot (2n_0)^{m/n_0} \cdot \sum_{n=n_0}^{+\infty} 2 \cdot (2n_0)^{-n/n_0} \\ &= 3 \cdot (2n_0)^{m/n_0} \cdot \frac{1}{n_0 \cdot \big(1 - (2n_0)^{-1/n_0}\big)} \\ &\leq 3 \cdot (2n_0)^{m/n_0} \end{aligned}$$

as soon as $n_0 \geq 2$. □

9. Weights, heights and lengths

We will need to bound the diameters of puzzle pieces from above. This will be done using arguments based on moduli of annuli. More precisely, according to [BDH], for any piece $\mathcal{P}$ contained in $\mathcal{C}_0$,

$$\frac{\mathrm{Leb}(\mathcal{P} \cap \mathbb{R})}{\mathrm{Leb}(\mathcal{C}_0 \cap \mathbb{R})} < \exp\left(-2\pi\left(\mathrm{modulus}(\mathcal{C}_0 \setminus \overline{\mathcal{P}}) - \frac{1}{2}\right)\right).$$

Thus, our goal is to bound from below the modulus of the annulus $\mathcal{C}_0 \setminus \overline{\mathcal{P}}$. We will introduce the notion of height of a piece, with the property that

$$\mathrm{modulus}(\mathcal{C}_0 \setminus \overline{\mathcal{P}}) \geq \mathrm{height}(\mathcal{P}) \cdot \mathrm{modulus}\big(\widehat{\mathcal{S}}^{\pm} \setminus \mathrm{closure}(\widetilde{\mathcal{S}}^{\pm})\big).$$

Our work will consist in finding lower bounds for heights of pieces. More precisely, under appropriate assumptions, the height of singular pieces increases linearly with respect to their depth, so that their diameter decreases geometrically with respect to their depth.

Definition 9.1 (Surrounding annulus)**.** *If $\mathcal{P}$ is a good piece, we set*

$$\mathcal{A}(\mathcal{P}) := \widehat{\mathcal{P}} \setminus \mathrm{closure}(\widetilde{\mathcal{P}}).$$

According to Lemma 3.5, if $\mathcal{P}$ is a good piece, then $\mathcal{A}(\mathcal{P})$ is an annulus. In addition, if $(\mathcal{P}_n)_{1 \leq n \leq m}$ are good pieces with $\mathcal{P}_1 \Subset \mathcal{C}_0$, $\mathcal{P}_m = \mathcal{P}$ and $\mathcal{P}_{n+1} \subsetneq \mathcal{P}_n$, then the annuli $\mathcal{A}(\mathcal{P}_n)$ are disjoint and essentially embedded in $\mathcal{C}_0 \setminus \mathcal{P}$. It follows from the Grötzsch Inequality that

$$\mathrm{modulus}\left(\mathcal{C}_0 \setminus \overline{\mathcal{P}}\right) \geq \sum_{n=1}^{m} \mathrm{modulus}\big(\mathcal{A}(\mathcal{P}_n)\big).$$

Recall that if $\mathcal{P}$ is a good piece of depth $m \geq 0$, then $f^{\circ(m+1)}(\mathcal{P}) = \mathcal{S}^{\pm}$. Note that $\mathcal{A}(\mathcal{S}^-) = -\mathcal{A}(\mathcal{S}^+)$, so that both annuli have the same modulus

$$\mathfrak{m} := \text{modulus}\big(\mathcal{A}(\mathcal{S}^{\pm})\big).$$

Also note that $\mathcal{A}(\mathcal{S}^{\pm})$ contains no critical value of $f^{\circ(m+1)} : \widehat{\mathcal{P}} \to \widehat{\mathcal{S}}^{\pm}$ if and only if $f^{\circ(m+1)} : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{S}^{\pm})$ is a covering map. In this case

$$\text{modulus}\big(\mathcal{A}(\mathcal{P})\big) = \frac{\mathfrak{m}}{d},$$

where d is the degree of $f^{\circ(m+1)} : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{S}^{\pm})$.

This discussion motivates the following definitions.

Definition 9.2 (Excellent pieces)**.** *A piece $\mathcal{P}$ of depth m is* excellent *if it is a good piece and if $f^{\circ(m+1)} : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(\mathcal{S}^{\pm})$ is a covering map. In that case,* $\deg(\mathcal{P})$ *is the degree of this covering map.*

Note that the degree of an excellent piece is always a power of 2.

Definition 9.3 (Weight)**.** *Let $\mathcal{P}$ be a piece of depth m. We set*

$$\text{weight}(\mathcal{P}) := \begin{cases} 1/\deg(\mathcal{P}) & \textit{if } \mathcal{P} \textit{ is an excellent piece,} \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 9.4 (Height)**.** *If $\mathcal{P} \subset \mathcal{C}_0$ is a piece of depth $m \geq 0$, we set*

$$\text{height}(\mathcal{P}) := \sum_{n=1}^{m} \text{weight}(\mathcal{P}_n), \quad \textit{where } \mathcal{P}_n \textit{ is the piece of depth } n \textit{ containing } \mathcal{P}.$$

As mentioned previously, it follows from the Grötzsch Inequality that for any piece $\mathcal{P} \Subset \mathcal{C}_0$,

$$\text{modulus}(\mathcal{C}_0 \setminus \overline{\mathcal{P}}) \geq \mathfrak{m} \cdot \text{height}(\mathcal{P})$$

and so,

$$\frac{\text{Leb}(\mathcal{P} \cap \mathbb{R})}{\text{Leb}(\mathcal{C}_0 \cap \mathbb{R})} \leq \mathrm{e}^{\pi} \cdot \lambda^{\text{height}(\mathcal{P})} \quad \text{with} \quad \lambda := \mathrm{e}^{-2\pi\mathfrak{m}}. \tag{1}$$

We will now control the heights of the critical pieces $\mathcal{C}_m$ with $m \leq n_0 + 1$ and $\mathcal{D}_m^{\pm}$ with $m \leq n_0$ (see Figure 19).

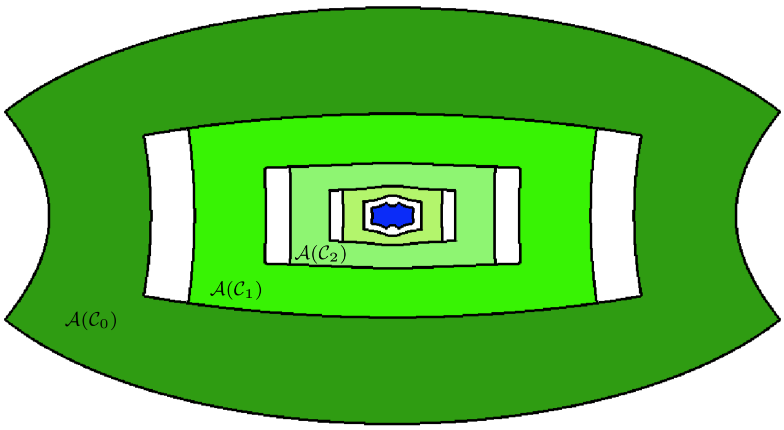


Figure 19: For the polynomial $f(z) = z^2 - 1.99$, $n_0 = 4$ and for $0 \leq m \leq 3$ the annuli $\mathcal{A}(\mathcal{C}_m)$ cover $\mathcal{A}(\mathcal{S}^{\pm})$ with degree 2. They all surround the critical piece $\mathcal{C}_{n_0+1}$ (blue).

Lemma 9.5. *For $m \in [0, n_0+1]$, we have the following weights and heights:*

m	0	1	$[2, n_0-1]$	n_0	n_0+1
weight($\mathcal{C}_m$)	1/2	1/2	1/2	0	1/4
height($\mathcal{C}_m$)	0	1/2	$m/2$	0	$n_0/2 - 1/4$
weight($\mathcal{D}_m^\pm$)			1/2	0	
height($\mathcal{D}_m^\pm$)			$m/2 - 1/2$	$n_0/2 - 1$	

Proof. See Figure 20 (compare with Figure 15). □

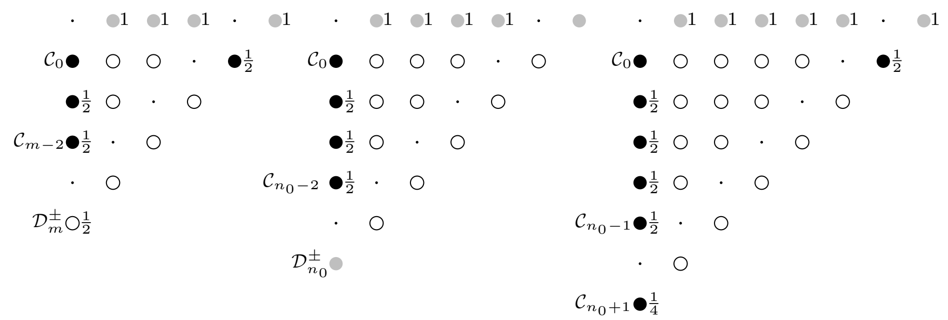


Figure 20: Left: the restricted tableau of the pieces $\mathcal{D}_m^\pm$ for $m \le n_0 - 1$. Middle: the restricted tableau of the semi-critical pieces $\mathcal{D}_{n_0}^\pm$. Right: the restricted tableau of the critical piece $\mathcal{C}_{n_0+1}$. The relevant weights are indicated to the right of the positions.

10. Regular maps

Definition 10.1 (Regular map)**.** *A map f is* regular of order $\ell \ge 1$ *if $\mathcal{C}_0$ has at least ℓ critical children.*

Assume f is regular of order ℓ, and let $m_1 < m_2 < \cdots < m_\ell$ be the depths of the critical children of $\mathcal{C}_0$. Then, $m_1 = n_0 + 1$ and for $k \in [1, \ell-1]$, the point $c_{m_k} := f^{\circ m_k}(0)$ belongs to a maximally regular piece of depth $n_k := m_{k+1} - m_k$.

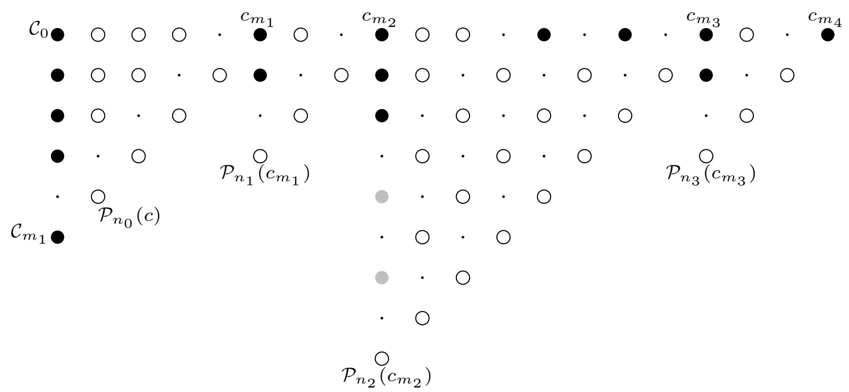


Figure 21: An example of tableau for a regular map of order 4. The pieces $\mathcal{P}_{n_j}(c_{m_j})$ are regular pieces.

The critical piece $\mathcal{C}_0$ does not contain regular pieces of depth n_0 or n_0+1. Thus, for $k \ge 1$,

- either $2 \le n_k \le n_0 - 1$,
- or $n_k \ge n_0 + 2$.

The parameters c we shall consider are parameters for which f is regular to any order $\ell \geq 1$ and for which the sequence of depths $\{n_k\}_{k\geq 0}$ does not increase too fast in the following sense.

Definition 10.2. *A map is* strongly regular of order $\ell \geq 1$ *if it is regular of order* ℓ *and if*

$$\text{for all } k \in [1,\ell], \quad \sum_{j=1}^{k} \max(0, n_j - n_0) \leq \frac{k}{4}.$$

The map is strongly regular *if it is strongly regular of any order* $\ell \geq 1$.

The proof of the following key estimate is rather technical.

Proposition 10.3. *If f is strongly regular of order $\ell \geq 1$ with $n_0 \geq 7$, then*

$$\text{for all } \mathcal{P} \in \mathfrak{S}_m \text{ with } m \in [1, m_\ell], \quad \operatorname{height}(\mathcal{P}) \geq \frac{m-1}{16}.$$

Proof. We shall first control the heights of critical children.

Lemma 10.4. *For all $k \in [1,\ell]$,*

$$\operatorname{height}(\mathcal{C}_{m_k}) \geq \frac{n_0+3}{4} + \frac{3m_k}{32}.$$

Since $3/32 > 1/16$, this shows that

$$\operatorname{height}(\mathcal{C}_{m_k}) \geq \frac{m_k}{16} \geq \frac{m_k - 1}{16}.$$

But we will need the better estimate of the lemma for the height of critical children in order to control the height of other singular pieces.

Proof. According to Lemma 9.5 and since $m_1 = n_0 + 1$,

$$\operatorname{height}(\mathcal{C}_{m_1}) = \frac{n_0}{2} - \frac{1}{4} = \frac{n_0+3}{4} + \frac{3m_1}{32} + \underbrace{\frac{5n_0 - 35}{32}}_{\geq 0} \geq \frac{n_0+3}{4} + \frac{3m_1}{32}.$$

For $k \in [1, \ell-1]$, according to Lemma 9.5 and as indicated in Figure 22:

- either $n_k \leq n_0 - 1$, for $n \in [m_k+1, m_{k+1}-2]$, $f^{\circ m_k}(\mathcal{C}_n)$ is a critical piece of weight $1/2$ and $\mathcal{C}_n$ has weight $1/4$,
- or $n_k \geq n_0 + 2$, for $n \in [m_k+1, m_k+n_0-2]$, $f^{\circ m_k}(\mathcal{C}_n)$ is a critical piece of weight $1/2$ and $\mathcal{C}_n$ has weight $1/4$.

In addition, $\mathcal{C}_{m_{k+1}}$ has weight $1/4$.

As a consequence,

$$\begin{aligned}\operatorname{height}(\mathcal{C}_{m_{k+1}}) &\geq \operatorname{height}(\mathcal{C}_{m_k}) + \begin{cases} (n_k - 1)/4 & \text{if } n_k \leq n_0 \\ (n_0-1)/4 & \text{if } n_k \geq n_0. \end{cases} \\ &= \operatorname{height}(\mathcal{C}_{m_k}) + \frac{n_k - 1}{4} - \frac{\max(0, n_k - n_0)}{4}.\end{aligned}$$

Thus, for all $k \in [1, \ell-1]$,

$$\begin{aligned}\operatorname{height}(\mathcal{C}_{m_{k+1}}) &\geq \operatorname{height}(\mathcal{C}_{m_1}) + \sum_{1\leq j\leq k} \frac{n_j - 1}{4} - \sum_{1\leq j\leq k} \frac{\max(0, n_k - n_0)}{4} \\ &\geq \frac{n_0}{2} - \frac{1}{4} + \frac{m_{k+1} - m_1}{4} - \frac{5k}{16}.\end{aligned}$$

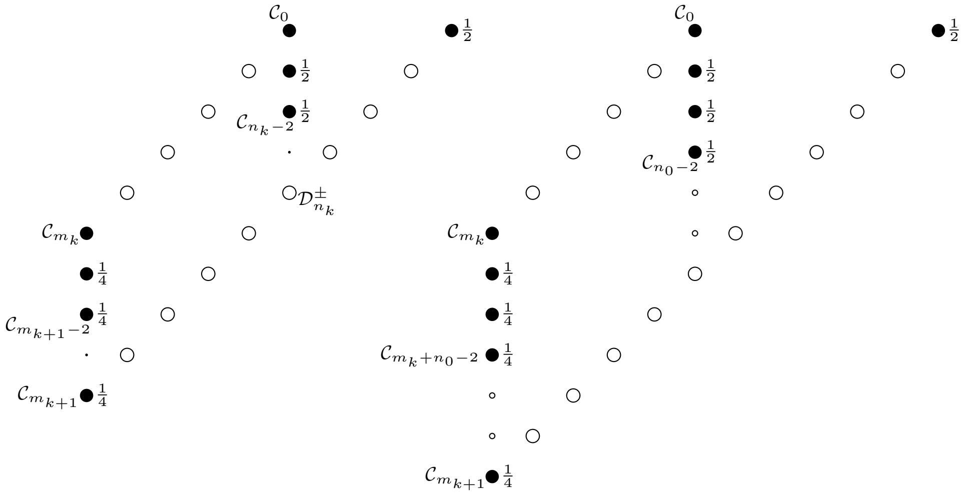


Figure 22: The critical pieces $\mathcal{C}_{m_k}$ and $\mathcal{C}_{m_{k+1}}$ are consecutive critical children. Left: $n_k \leq n_0 - 1$. Right: $n_k \geq n_0 + 2$. The relevant weights are indicated to the right of each position.

Since $n_k \geq 2$, we have that $m_{k+1} - m_1 \geq 2k$, and so

$$\begin{aligned}
\text{height}(\mathcal{C}_{m_{k+1}}) &\geq \frac{n_0}{2} - \frac{1}{4} + \frac{m_{k+1} - m_1}{4} - \frac{5(m_{k+1} - m_1)}{32} \\
&= \frac{n_0}{2} - \frac{1}{4} + \frac{3(m_{k+1} - n_0 - 1)}{32} \\
&= \frac{n_0 + 3}{4} + \frac{3m_{k+1}}{32} + \underbrace{\frac{5n_0 - 35}{32}}_{\geq 0} \geq \frac{n_0 + 3}{4} + \frac{3m_{k+1}}{32}. \qquad \square
\end{aligned}$$

We may now prove by induction on $m \in [1, m_\ell]$ that $\text{height}(\mathcal{P}) \geq (m-1)/16$ for $\mathcal{P} \in \mathfrak{S}_m$. The induction hypothesis trivially holds for $m = 1$. So, let us assume that $\mathcal{P} \in \mathfrak{S}_m$ with $m \in [2, m_\ell]$. For $n \in [1, m]$, let $\mathcal{P}_n$ be the piece of depth n containing $\mathcal{P}$.

Case 1. $\mathcal{P} \not\subset \mathcal{C}_{n_0+1} \cup \mathcal{D}^{\pm}_{n_0}$, then $m \in [2, n_0]$ and according to Lemma 8.3, $\mathcal{P} \subset \mathcal{C}_{m-1}$. According to Lemma 9.5,

$$\text{height}(\mathcal{P}) \geq \text{height}(\mathcal{C}_{m-1}) = \frac{m-1}{2} \geq \frac{m-1}{16}.$$

Case 2. $\mathcal{P} \subset \mathcal{D}^{\pm}_{n_0}$. Let $m_\star \in [n_0, m]$ be the largest integer such that $\mathcal{P}_{m_\star}$ is semi-critical. For $n \in [1, n_0 - 2]$, all pieces are critical with weight $1/2$. According to Lemma 7.3, $m_\star - n_0$ is even and if $n \in [n_0, m_\star]$ is such that $n - n_0$ is even, then the piece $\mathcal{P}_n$ is a semi-critical child of $\mathcal{C}_0$. Except $\mathcal{P}_{m_\star}$ which may have weight 0, those semi-critical children have weight $1/4$. If $n \in [m_\star + 1, m]$, the piece $\mathcal{P}_n$ is either unmarked or off-critical. In both cases $\text{weight}(\mathcal{P}_n) = \text{weight}(\mathcal{P}'_{n-m_\star})$, where $\mathcal{P}'_{n-m_\star} := f^{\circ m_\star}(\mathcal{P}_n)$ is the piece of depth $n - m_\star$ containing $\mathcal{P}' := f^{\circ m_\star}(\mathcal{P})$. The situation is illustrated on Figure 23. It follows by induction that

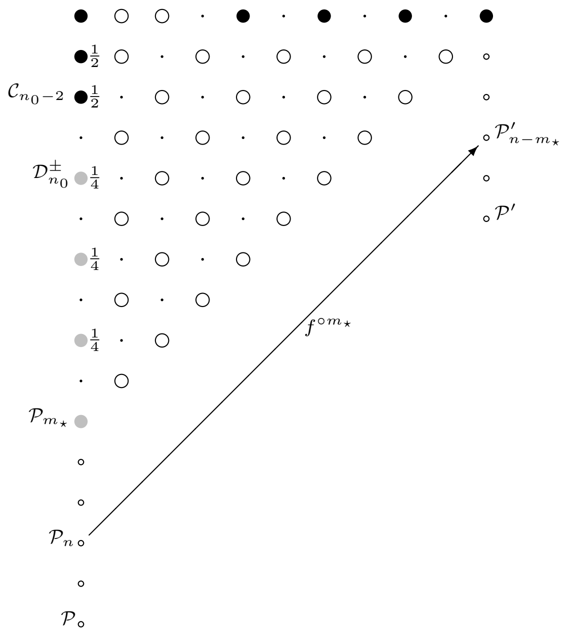


Figure 23: The situation when $\mathcal{P} \in \mathcal{D}^{\pm}_{n_0}$.

$$\begin{aligned} \text{height}(\mathcal{P}) &\geq \frac{n_0-2}{2} + \frac{m_\star - n_0}{8} + \text{height}(\mathcal{P}') \\ &\geq \frac{n_0-2}{2} + \frac{m_\star - n_0}{8} + \frac{m - m_\star - 1}{16} \\ &= \frac{m-1}{16} + \underbrace{\frac{m_\star}{16} + \frac{6n_0-9}{16}}_{>0} \geq \frac{m-1}{16}. \end{aligned}$$

Case 3. $\mathcal{P} \in \mathcal{C}_{n_0+1}$. Either $\mathcal{P}$ is a critical child of $\mathcal{C}_0$ and the result is already proven by Lemma 10.4. Or there is a largest integer $k \in [1, \ell-1]$ such that $\mathcal{P} \in \mathcal{C}_{m_k} \smallsetminus \mathcal{C}_{m_{k+1}}$. For $n \in [1, m]$, let $\mathcal{P}_n$ be the piece of depth n containing $\mathcal{P}$. Let $m_\star \in [m_k, m]$ be the largest integer such that $\mathcal{P}_{m_\star}$ is critical or semi-critical. Our study will depend on the value of $m_\star$.

Case 3.a. $m_\star \leq m_{k+1} + 2$. Fix $n \in [m_k+1, m]$ and let $\mathcal{P}'_{n-m_k} := f^{\circ m_k}(\mathcal{P}_n)$ be the piece of depth $n - m_k$ containing $\mathcal{P}' := f^{\circ m_k}(\mathcal{P})$. Note that the weight of any piece $\mathcal{P}'_j$ is at most $1/2$, since otherwise, $\mathcal{P}'_j$ would map isomorphically to $\mathcal{S}^{\pm}$ whose boundary intersects $\mathbb{R}$ in a single point $\mp\alpha$, whereas the boundary of $\mathcal{P}'_j \subset \mathcal{C}_0$ intersects $\mathbb{R}$ in two points. It follows that

$$\begin{cases} \text{weight}(\mathcal{P}_n) = \frac{1}{2}\text{weight}(\mathcal{P}'_{n-m_k}) \geq \text{weight}(\mathcal{P}'_{n-m_k}) - \frac{1}{4} & \text{for } n \in [m_k+1, m_\star - 1], \\ \text{weight}(\mathcal{P}_{m_\star}) \geq 0 \geq \text{weight}(\mathcal{P}'_{m_\star - m_k}) - \frac{1}{2} & \text{for } n = m_\star \\ \text{weight}(\mathcal{P}_n) = \text{weight}(\mathcal{P}'_{n-m_k}) & \text{for } n \in [m_\star+1, m]. \end{cases}$$

Thus,

$$\begin{aligned} \text{height}(\mathcal{P}) &\geq \text{height}(\mathcal{C}_{m_k}) + \text{height}(\mathcal{P}') - \frac{m_\star - m_k - 1}{4} - \frac{1}{2} \\ &= \text{height}(\mathcal{C}_{m_k}) + \text{height}(\mathcal{P}') - \frac{m_\star - m_k + 1}{4}. \end{aligned}$$

Since f is strongly regular, $n_k - n_0 \le k/4 \le m_k/8$. So,

$$m_\star - m_k + 1 \le m_{k+1} - m_k + 3 = n_k + 3 \le n_0 + 3 + \frac{m_k}{8}.$$

According to Lemma 10.4 and the induction hypothesis applied to $\mathcal{P}'$ whose depth is $m_\star - m_k$,

$$\mathrm{height}(\mathcal{P}) \ge \frac{n_0+3}{4} + \frac{3m_k}{32} + \frac{m-m_k-1}{16} - \frac{n_0+3}{4} - \frac{m_k}{32} = \frac{m-1}{16}.$$

Case 3.b. $m_\star \ge m_{k+1}+3$. Since $\mathcal{P}$ is not contained in $\mathcal{C}_{m_{k+1}}$, for $n \in [m_{k+1}, m_\star]$, the pieces $\mathcal{P}_n$ are either unmarked or semi-critical. According to Lemma 7.2, the piece $\mathcal{P}_{m_{k+1}}$ cannot be semi-critical, thus it is unmarked. It follows from the rule for marked positions (Lemma 5.4) that for $n \in [m_{k+1}+1, m_\star]$ with $m - m_{k+1} - 1$ even, the piece $\mathcal{P}_n$ is semi-critical. According to Lemma 7.4, if in addition $n \ge m_{k+1}+3$, the piece $\mathcal{P}_n$ is a semi-critical child of $\mathcal{C}_0$, thus has weight $1/4$ except possibly for $\mathcal{P}_{m_\star}$ which may have weight 0. There are $(m_\star - m_{k+1} - 3)/2$ such pieces. If $n \in [m_\star+1, m]$, the piece $\mathcal{P}_n$ is either unmarked or off-critical. In both cases $\mathrm{weight}(\mathcal{P}_n) = \mathrm{weight}(\mathcal{P}'_{n-m_\star})$, where $\mathcal{P}'_{n-m_\star} := f^{\circ m_\star}(\mathcal{P}_n)$ is the piece of depth $n - m_\star$ containing $\mathcal{P}' := f^{\circ m_\star}(\mathcal{P})$. Thus,

$$\begin{aligned}\mathrm{height}(\mathcal{P}) &\ge \mathrm{height}(\mathcal{C}_{m_k}) + \frac{m_\star - m_{k+1} - 3}{8} + \mathrm{height}(\mathcal{P}') \\ &\ge \frac{n_0+3}{4} + \frac{3m_k}{32} + \frac{m_\star - m_k - n_k - 3}{8} + \frac{m - m_\star - 1}{16} \\ &= \frac{n_0}{4} + \frac{3}{8} - \frac{m_k}{32} + \frac{m_\star}{16} - \frac{n_k}{8} + \frac{m-1}{16}.\end{aligned}$$

Since f is strongly regular, $n_k - n_0 \le k/4 \le m_k/8$ and

$$\mathrm{height}(\mathcal{P}) \ge \underbrace{\frac{n_0}{8} + \frac{3}{8}}_{>0} + \underbrace{\frac{m_\star}{16} - \frac{3m_k}{64}}_{>0} + \frac{m-1}{16} \ge \frac{m-1}{16}. \qquad \square$$

11. Plowing in the dynamical space

We are now ready to prove Proposition 2.6. We must prove that there exist $C > 0$, $\rho < 1$ and $N \ge 1$ such that if f is strongly regular with $n_0 \ge N$, then

$$\forall n \ge 1, \qquad \frac{\mathrm{Leb}\big\{x \in I \mid \boldsymbol{n}(x) \ge n\big\}}{\mathrm{Leb}(I)} \le C\rho^n.$$

Proof of Proposition 2.6. Assume f is strongly regular with $n_0 \ge 7$. According to Proposition 10.3, if $\mathcal{P} \in \mathfrak{S}_m$ with $m \ge 0$, then

$$\mathrm{height}(\mathcal{P}) \ge \frac{m-1}{16},$$

so that, according to equation (1),

$$\frac{\mathrm{Leb}(\mathcal{P}\cap\mathbb{R})}{\mathrm{Leb}(\mathcal{C}_0\cap\mathbb{R})} \le \mathrm{e}^\pi \cdot \lambda^{(m-1)/16} \quad \text{with} \quad \lambda := \mathrm{e}^{-2\pi\mathrm{m}} < 1.$$

According to Proposition 8.2, there are at most $3\cdot(2n_0)^{m/n_0}$ singular pieces of depth $m \ge 0$. Choose N sufficiently large so that $\rho := \lambda^{1/16}\cdot(2N)^{1/N} < 1$.

Then, if $n_0 \ge N$ and $n \ge 1$,

$$\frac{\mathrm{Leb}\big\{x \in I \mid \boldsymbol{n}(x) \ge n\big\}}{\mathrm{Leb}(I)} = \sum_{\mathcal{P}\in\mathfrak{S}_{n-1}} \frac{\mathrm{Leb}(\mathcal{P}\cap\mathbb{R})}{\mathrm{Leb}(\mathcal{C}_0\cap\mathbb{R})} \le C\rho^n$$

with

$$C := \frac{3\,\mathrm{e}^\pi}{(2N)^{1/N}\lambda^{1/8}}. \qquad \square$$

12. Parapuzzle

To transfer the dynamical estimates to the parameter space, we shall use the Yoccoz parapuzzle. The definitions given in the introduction for $c \in [-2,-1]$ generalize to $c \in \mathbb{C}$ as follows.

12.1. The parapuzzle in the 1/2-wake. For $c \in \mathbb{C}$, denote by $\mathcal{K}_c$ the filled-in Julia set of f_c, i.e. the set of points $z \in \mathbb{C}$ with bounded orbit under iteration of f_c. The Green function $\mathfrak{g}_c : \mathbb{C} \to [0,+\infty)$ is defined by

$$\mathfrak{g}_c(z) = \lim_{n\to+\infty} \frac{1}{2^n} \log(1 + \left|f_c^{\circ n}(z)\right|).$$

It vanishes precisely on $\mathcal{K}_c$ and satisfies $\mathfrak{g}_c \circ f_c(z) = 2\mathfrak{g}_c(z)$. The Mandelbrot set is the set

$$\mathscr{M} := \left\{c \in \mathbb{C} \mid \mathfrak{g}_c(c) = 0\right\}.$$

The Böttcher coordinate ϕ_c which conjugates f_c to f_0 in a neighborhood of ∞ extends as a conformal representation

$$\phi_c : \left\{z \in \mathbb{C} \mid \mathfrak{g}_c(z) > \mathfrak{g}_c(0)\right\} \to \left\{z \in \mathbb{C} \mid \log|z| > \mathfrak{g}_c(0)\right\}.$$

In particular, if $c \in \mathbb{C}\smallsetminus\mathscr{M}$, $\phi_c(c)$ is well defined since $\mathfrak{g}_c(c) = 2\mathfrak{g}_c(0) > 0$. It is well known that the map

$$\Phi : \mathbb{C}\smallsetminus\mathscr{M} \ni c \mapsto \phi_c(c) \in \mathbb{C}\smallsetminus\overline{\mathbb{D}}$$

is a conformal representation. For $t \in \mathbb{R}/\mathbb{Z}$, the parameter ray of angle t is

$$\mathscr{R}(t) := \Phi^{-1}\left\{r\,\mathrm{e}^{\mathrm{i}2\pi t} \mid r > 1\right\}.$$

The parameter rays $\mathscr{R}(1/3)$ and $\mathscr{R}(2/3)$ land at $c = -3/4$. The 1/2-wake $\mathscr{W}$ is the connected component of $\mathbb{C}\smallsetminus\left(\mathscr{R}(1/3) \cup \mathscr{R}(2/3) \cup \{-3/4\}\right)$ which contains $[-2,-1]$.

For $c \in \mathscr{W}$, the dynamical rays $\mathcal{R}_c(1/3)$ and $\mathcal{R}_c(2/3)$ land at a common repelling fixed point α_c – those rays are defined as the gradient lines of $\mathfrak{g}_c$ stemming from infinity with angle 1/3 and 2/3 (measured via the Böttcher coordinate). Set

$$\mathcal{U}_c = \left\{z \in \mathbb{C} \mid \mathfrak{g}_c(z) < 1\right\}\smallsetminus\left(\mathcal{R}_c(1/3) \cup \mathcal{R}_c(2/3) \cup \{\alpha_c\}\right).$$

For $m \geq -2$, set

$$\mathscr{U}_m = \left\{c \in \mathscr{W} \mid f_c^{\circ(m+2)}(c) \in \mathcal{U}_c\right\}.$$

The parapuzzle pieces of depth m are the connected components of $\mathscr{U}_m$. If $c \in \mathscr{U}_{-2}$ and $m \geq -2$ are such that $f_c^{\circ(m+2)}(c) \in \mathcal{U}_c$, there is a parapuzzle piece of depth m which contains the parameter c. We denote it by $\mathscr{P}_m(c)$. Since $f_c^{-1}(\mathcal{U}_c) \subset \mathcal{U}_c$, we see that $f_c^{\circ(m+2)}(c) \in \mathcal{U}_c$ as soon as $f_c^{\circ(m+3)}(c) \in \mathcal{U}_c$. It follows that parapuzzle pieces are either disjoint or nested: $\mathscr{P}_{m+1}(c) \subset \mathscr{P}_m(c)$ for all $m \geq -2$. Figure 24 shows the parapuzzle pieces of depth -2 through 3.

12.2. The dynamical puzzle. In the introduction, we only defined the dynamical puzzle for $c \in [-2,-1]$. The definition generalizes for parameters $c \in \mathscr{U}_{-2}$: the puzzle pieces of depth m are the connected components of $f_c^{-(m+2)}(\mathcal{U}_c)$. We denote by $\mathrm{Puzzle}_m(c)$ the set of puzzle pieces of depth m for f_c.

If $x \in f_c^{-(m+2)}(\mathcal{U}_c)$, we denote by $\mathcal{P}_{m,c}(x) \in \mathrm{Puzzle}_m(c)$ the puzzle piece of depth m which contains x. We set $\mathcal{C}_{m,c} := \mathcal{P}_{m,c}(0)$. Again, two puzzle pieces are either nested or disjoint. Figure 25 shows the puzzle pieces of depth -2 through 3 for a polynomial f_c with disconnected Julia set.

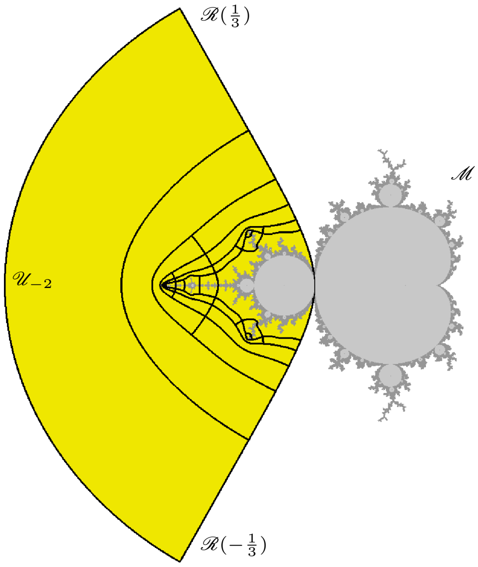


Figure 24: The Mandelbrot set and the parapuzzle pieces of depth -2 through 3 in the 1/2-wake. The set $\mathscr{U}_{-2}$ is colored yellow.

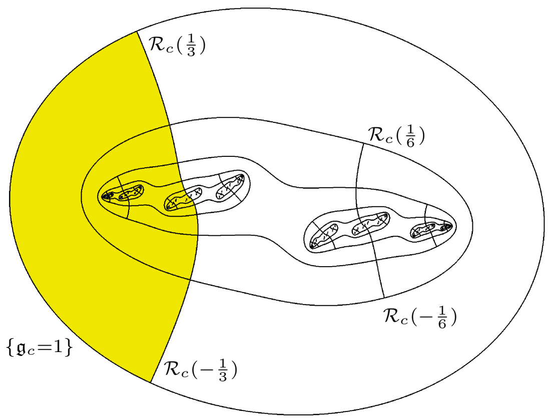


Figure 25: The puzzle of depths -2 through 3 for $f_c(z) = z^2 - 2 + \mathrm{i}/2$. The Julia set is not connected. The puzzle piece of depth 0 containing the critical value c is colored yellow.

12.3. **Boundaries of puzzle pieces.** We are interested in showing that when c varies in a parapuzzle piece of depth $m \geq -2$, the boundaries of the dynamical puzzle pieces of depth $n \in [-2, m+1]$ vary holomorphically. Those boundaries are composed of equipotentials, external rays and their landing points. According to the λ-Lemma of Mañe-Sad-Sullivan, it is enough to prove that the equipotentials and external rays vary holomorphically. This is the case if and only if they do not bifurcate on an iterated preimage of the critical point.

For $c \in \mathscr{W} \cup \{0\}$, let $\mathcal{G}_c$ be the union of the external rays of angles in $\pm 1/3$ and the equipotentials of levels 2^k, $k \geq 0$. For $m \geq -2$, set

$$\mathcal{G}_c^m := f_c^{-(m+2)}(\mathcal{G}_c) \quad \text{and} \quad \mathscr{G}^m := \Phi^{-1}(\mathcal{G}_0^m).$$

Then, $f_c(\mathcal{G}_c) \subset \mathcal{G}_c$, and for $m \geq 0$, $f_c(\mathcal{G}_c^m) \subset \mathcal{G}_c^m = f_c(\mathcal{G}_c^{m+1})$. Moreover, $c \in \mathscr{G}^m$ if and only if $c \in \mathcal{G}_c^m$. In addition, for $c \in \mathscr{U}_{-2}$, the closure of $\mathcal{G}_c^m$ contains the boundaries of puzzle pieces of depth $m \geq -2$; and the closure of $\mathscr{G}^m$ contains the boundaries of the parapuzzle pieces of depth $m \geq -2$.

We say that the graph $\mathcal{G}_c^m$ bifurcates if it contains an iterated preimage of 0.

Lemma 12.1. *As c varies in a parapuzzle piece of depth $m \geq -2$, the graph $\mathcal{G}_c^{m+1}$ does not bifurcate.*

Proof. Assume $\mathcal{G}_c^{m+1}$ bifurcates. Since $f_c(\mathcal{G}_c^{m+1}) \subset \mathcal{G}_c^{m+1}$, $0 \in \mathcal{G}_c^{m+1}$. It follows that $f_c^{\circ(m+2)}(c) = f_c^{\circ(m+3)}(0) \in \mathcal{G}_c$. This is not possible if c belongs to a parapuzzle piece of depth m since in this case, $f_c^{\circ(m+2)}(c) \in \mathcal{U}_c \subset \mathbb{C} \smallsetminus \mathcal{G}_c$. □

Corollary 12.2. *Let $\mathscr{P}$ be a parapuzzle piece of depth $m \geq -2$. Then the boundaries of the dynamical puzzle pieces of depth $n \in [-2, m+1]$ vary holomorphically with respect to $c \in \mathscr{P}$.*

12.4. **Enlarged and thickened pieces.** As in the introduction, we now define enlarged and thickened dynamical pieces. Given $c \in \mathscr{U}_{-2}$, set

$$\mathcal{S}_c^{\pm} := \mathcal{P}_{-1,c}(\pm\beta_c) \quad \text{and} \quad \widehat{\mathcal{S}}_c^{\pm} := \pm\mathcal{C}_{-2,c},$$

where β_c is the landing point of the ray of angle 0, i.e., the fixed point of f_c different from α_c.

Definition 12.3 (Good and enlarged pieces)**.** *A puzzle piece $\mathcal{P} \in \mathrm{Puzzle}_m(c)$ of depth $m \geq -1$ is a* good piece *if $f_c^{\circ(m+1)}(\mathcal{P}) = \mathcal{S}_c^{\pm}$. In that case, the* enlarged piece *$\widehat{\mathcal{P}}$ is the component of $f_c^{-(m+1)}(\widehat{\mathcal{S}}_c^{\pm})$ containing $\mathcal{P}$.*

Lemma 12.4. *Assume $\mathscr{P}$ is a parapuzzle piece of depth $m \geq -2$. The boundaries of enlarged pieces of depth $n \in [-1, m+1]$ vary holomorphically with respect to $c \in \mathscr{P}$.*

Proof. The enlarged pieces $\widehat{\mathcal{S}}_c^{\pm}$ have depth -1. Their boundaries are contained in $\mathcal{G}_c^{-1}$. It follows that the boundaries of enlarged pieces of depth $n \in [-1, m+1]$ are contained in $\mathcal{G}_c^{m+1}$. The result follows from Lemma 12.1. □

Let us now define the thickened pieces. For $c \in \mathscr{W}$, the rays $\mathcal{R}_c(\pm 5/12)$ do not bifurcate and land at a common iterated preimage of α_c. Let $\widetilde{\mathcal{C}}_{-2,c}$ be the component of

$$\{z \in \mathbb{C} \mid \mathfrak{g}_c(z) < 1\} \smallsetminus \left(\overline{\mathcal{R}_c}(5/12) \cup \overline{\mathcal{R}_c}(-5/12)\right)$$

which contains $\mathcal{C}_{-2,c}$.

Definition 12.5 (Thickened pieces)**.** *If $\mathcal{P} \in \mathrm{Puzzle}_m(c)$ is a good piece of depth $m \geq -1$, the* thickened piece *$\widetilde{\mathcal{P}}$ is the component of $f_c^{-(m+2)}(\widetilde{\mathcal{C}}_{-2,c})$ containing $\mathcal{P}$.*

Lemma 12.6. *Assume $\mathscr{P}$ is a parapuzzle piece of depth $m \geq 0$. The boundaries of thickened pieces of depth $n \in [-1, m-1]$ vary holomorphically with respect to $c \in \mathscr{P}$.*

Proof. The thickened piece $\widetilde{\mathcal{C}}_{-2,c}$ have depth -2. Its boundary is contained in $\mathcal{G}_c^0$. It follows that the boundaries of thickened puzzle pieces of depth $n \in [-1, m-1]$ are contained in $\mathcal{G}_c^{m+1}$. The result follows from Lemma 12.1. □

It will be convenient to restrict our study to the parapuzzle pieces $\mathscr{V} \Subset \mathscr{V}'$ containing -2 of respective depths 1 and 0:

$$\mathscr{V} := \mathscr{P}_1(-2) \quad \text{and} \quad \mathscr{V}' := \mathscr{P}_0(-2).$$

The parapuzzle piece $\mathscr{V}'$ is bounded by an arc of equipotential of level $1/4$ and arcs of the parameter rays $\mathscr{R}(\pm 5/12)$ which land at a common parameter $c \in \mathscr{M}$ (see Figure 26). As c varies in $\mathscr{V}$, the thickened parapuzzle pieces $\widetilde{\mathcal{S}}_c^{\pm}$ move holomorphically. The parapuzzle piece $\mathscr{V}$ is relatively compact in $\mathscr{V}'$.

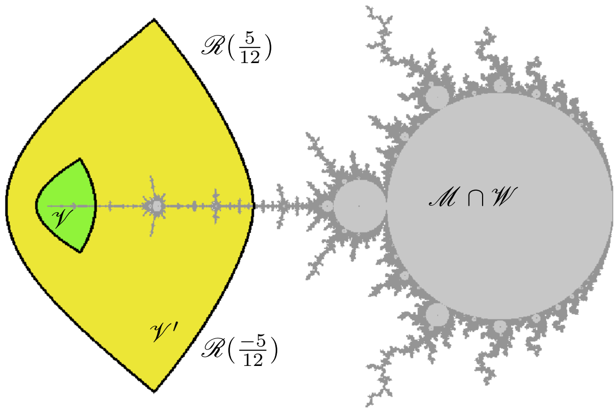


Figure 26: The 1/2-limb $\mathscr{M} \cap \mathscr{W}$ and the parapuzzle piece $\mathscr{V} := \mathscr{P}_1(-2)$ (green) which is relatively compact in the parapuzzle piece $\mathscr{V}' := \mathscr{P}_0(-2)$ (yellow).

12.5. **Regularity.** As in §2.2, a puzzle piece $\mathcal{P} \in \text{Puzzle}_m(c)$ of depth $m \geq 1$ is regular if

$$f_c^{\circ m}(\mathcal{P}) = \mathcal{C}_{0,c} \quad \text{and} \quad f_c^{\circ m} : \widehat{\mathcal{P}} \to \widehat{\mathcal{C}}_{0,c} \text{ is an isomorphism;}$$

and $\boldsymbol{n}_c : \mathbb{C} \to \mathbb{N} \cup \{+\infty\}$ is defined by

$$\boldsymbol{n}_c(x) := \inf\bigl\{n \geq 1 \mid x \text{ is contained in a regular piece of depth } n\bigr\}$$

with the convention that $\inf \varnothing := +\infty$.

Assume $\mathscr{P} \subseteq \mathscr{V}$ is a parapuzzle piece of depth $m \geq 1$. According to Lemma 12.4, as c varies in $\mathscr{P}$, the boundaries of enlarged pieces of depth $n \in [-1, m+1]$ vary holomorphically. Let $c \mapsto \widehat{\mathcal{P}}_c$ be such a holomorphically varying puzzle piece. Note that if $\mathcal{P}_{c_0}$ is regular for some $c_0 \in \mathscr{P}$, then $\mathcal{P}_c$ is regular for all $c \in \mathscr{P}$. Indeed, no critical point of $f_c^{\circ n}$ can enter $\widehat{\mathcal{P}}_c$ as it moves holomorphically, so that the degree of $f_c^{\circ n} : \widehat{\mathcal{P}}_c \to \widehat{\mathcal{C}}_{0,c}$ remains constant (equal to 1).

In particular, if $\mathcal{P}_{m,c_0}(c_0)$ is regular for some $c_0 \in \mathscr{P}$, then $\mathcal{P}_{m,c}(c)$ is regular for all $c \in \mathscr{P}$. In that case, we say that the parapuzzle piece $\mathscr{P}$ is regular and we define $\boldsymbol{N}_{\mathscr{P}} : \mathscr{P} \to \mathbb{N} \cup \{+\infty\}$ by

$$\boldsymbol{N}_{\mathscr{P}}(c) := \boldsymbol{n}_c\bigl(f_c^{\circ m}(c)\bigr).$$

Then, $\boldsymbol{N}_{\mathscr{P}}(c)$ is finite if and only if c belongs to a regular piece $\mathscr{Q} \subsetneq \mathscr{P}$; and for each $n \geq 1$, $\boldsymbol{N}_{\mathscr{P}}^{-1}(n)$ is a union of regular parapuzzle pieces of depth $m+n$.

13. Harvesting in parameter space

We now state the main result which will enable us to transfer results from the dynamical space to the parameter space. This transfer technique was first used by Shishikura to simplify the proof by Yoccoz that the Mandelbrot set is locally connected at non renormalizable parameters (see [R]).

Proposition 13.1. *Assume $\mathscr{P} \subset \mathscr{V}$ is a regular parapuzzle piece of depth $m \geq 1$ and $c_0 \in \mathscr{P}$. Then there exists a homeomorphism $\Psi : \mathscr{P} \to \mathcal{C}_{0,c_0}$ such that if $\mathscr{Q} \subset \mathscr{P}$ is a parapuzzle piece of depth $m+n$ with $n \in [1, m-1]$, then*

- *$\Psi(\mathscr{Q})$ is a puzzle piece of depth n for f_{c_0};*
- *$\mathscr{Q}$ is regular if and only if $\Psi(\mathscr{Q})$ is regular;*
- *$\operatorname{modulus}(\mathscr{P} \smallsetminus \overline{\mathscr{Q}}) \geq \kappa \cdot \operatorname{height}\big(\Psi(\mathscr{Q})\big)$ for some constant universal κ which depends neither on c_0, nor on $\mathscr{P}$.*

Proof. According to Lemma 12.1, as c varies in $\mathscr{P}$, the graph $\mathcal{G}_c^{m+1}$ does not bifurcate, thus varies holomorphically. Let $c \mapsto (\psi_c : \mathcal{G}_{c_0}^{m+1} \to \mathcal{G}_c^{m+1})$ be a holomorphic motion parameterized by $c \in \mathscr{P}$, respecting potentials and external arguments. According to Slodkowski's Theorem, this holomorphic motion extends to a holomorphic motion $c \mapsto (\psi_c : \mathbb{C} \to \mathbb{C})$. Define $\Psi : \mathscr{P} \to \mathcal{C}_{0,c_0}$ by

$$\Psi(c) := \psi_c^{-1} \circ f_c^{\circ m}(c).$$

Then, $\Psi : \mathscr{P} \to \mathcal{C}_{0,c_0}$ is a (locally quasiconformal) homeomorphism which maps parapuzzle pieces of depth $m+n$ with $n \in [0, m+1]$ to puzzle pieces of f_{c_0} of depth n.

Next, $\mathscr{Q}$ is regular if and only if $\mathcal{P} := \mathcal{P}_{n,c}\big(f_c^{\circ m}(c)\big)$ is regular. And this is the case if and only if $\Psi(\mathscr{Q}) = \psi_c^{-1}(\mathcal{P})$ is regular.

To control the modulus of $\mathscr{P} \smallsetminus \overline{\mathscr{Q}}$, we use the following result.

Lemma 13.2. *Assume $\mathcal{P}' \in \operatorname{Puzzle}_n(c_0)$ is a good piece of depth $n \in [1, m-1]$ containing $\Psi(\mathscr{Q})$. Set $\mathscr{A}' = \Psi^{-1}\big(\mathcal{A}(\mathcal{P}')\big)$. Then,*

$$\operatorname{modulus}(\mathscr{A}') \geq \frac{1}{K} \cdot \operatorname{modulus}\big(\mathcal{A}(\mathcal{P}')\big)$$

for some constant K which only depends on the hyperbolic diameter of $\mathscr{V}$ in $\mathscr{V}'$.

Proof. Let $c \mapsto \big(\chi_c : \mathcal{A}(\mathcal{S}_{c_0}^{+}) \to \mathcal{A}(\mathcal{S}_c^{\pm})\big)$ be a holomorphic motion parameterized by $c \in \mathscr{V}'$. Since $\mathscr{V}$ is relatively compact in $\mathscr{V}'$, there is a constant K such that χ_c is K-quasiconformal for all $c \in \mathscr{V}$. This constant K only depends on the hyperbolic diameter of $\mathscr{V}$ in $\mathscr{V}'$. For $c \in \mathscr{P}$, set $\mathcal{P}'_c := \psi_c(\mathcal{P}')$. According to Lemmas 12.4 and 12.6, the enlarged and thickened pieces $\widehat{\mathcal{P}}'_c$ and $\widetilde{\mathcal{P}}'c$ vary holomorphically with respect to $c \in \mathscr{P}$. Thus, $f_c^{\circ n} : \mathcal{A}(\mathcal{P}'_c) \to \mathcal{A}(\mathcal{S}_c^{\pm})$ is a covering map for all $c \in \mathscr{P}$, and we may lift the holomorphic motion $c \mapsto \chi_c$ to a holomorphic motion $c \mapsto \big(\phi_c : \mathcal{A}(\mathcal{P}') \to \mathcal{A}(\mathcal{P}'_c)\big)$ such that $f_c^{\circ n} \circ \phi_c = \chi_c \circ f_{c_0}^{\circ n}$ on $\mathcal{A}(\mathcal{P}')$. Then, the homeomorphism $\Phi : \mathscr{A}' \to \mathcal{A}(\mathcal{P}')$ defined by

$$\Phi(c) := \phi_c^{-1} \circ f_c^{\circ n}(c)$$

is K-quasiconformal. □

The proposition follows with

$$\kappa := \frac{1}{K} \cdot \inf_{c \in \mathscr{V}} \operatorname{modulus}\big(\mathcal{A}(\mathcal{S}_c^{\pm})\big). \qquad \square$$

A regular parapuzzle piece $\mathscr{P} \subset \mathscr{V}$ is regular of order ℓ if it is contained in exactly ℓ regular parapuzzle pieces

$$\mathscr{P}_1 \supset \mathscr{P}_2 \supset \cdots \supset \mathscr{P}_\ell = \mathscr{P}.$$

Set

$$n_0 := \operatorname{depth}(\mathscr{P}_1) \quad \text{and} \quad n_k := \operatorname{depth}(\mathscr{P}_{k+1}) - \operatorname{depth}(\mathscr{P}_k) \quad \text{for } k \in [1, \ell].$$

The parapuzzle piece $\mathscr{P}$ is strongly regular of order ℓ if for all $k \in [1, \ell]$,

$$\frac{1}{k}\sum_{j=1}^{k} \max(0, n_j - n_0) \leq \frac{1}{4}.$$

Proposition 2.8 may be reformulated as follows.

Proposition 13.3. *There exist $K > 0$, $\sigma < 1$ and $N \geq 1$ such that the following holds. If $\mathscr{P} \subset \mathscr{P}_N(-2)$ is strongly regular of order ℓ, then*

$$\forall n \in \big[1, \mathrm{depth}(\mathscr{P})\big], \quad \frac{\mathrm{Leb}\{c \in \mathscr{P} \cap \mathbb{R} \mid \boldsymbol{N}_{\mathcal{P}}(c) \geq n\}}{\mathrm{Leb}(\mathscr{P} \cap \mathbb{R})} \leq K\sigma^n.$$

Proof. Assume $N \geq 7$, and fix $c_0 \in \mathscr{P}$. Let $\Psi : \mathscr{P} \to \mathcal{C}_{0,c_0}$ be a homeomorphism provided by Proposition 13.1.

Let us say that a parapuzzle piece $\mathscr{Q} \subset \mathscr{P}$ is singular if the parapuzzle pieces $\mathscr{P}'$ satisfying $\mathscr{Q} \subseteq \mathscr{P}' \subsetneq \mathscr{P}$ are not regular.

Assume $n \in [2, m]$ with $m := \mathrm{depth}(\mathscr{P})$. Then, $c \in \mathscr{P} \cap \mathbb{R}$ and $\boldsymbol{N}_{\mathscr{P}}(c) \geq n$ if and only if

- either c is an iterated preimage of α_c for f_c
- or the parapuzzle piece $\mathscr{Q} := \mathscr{P}_{m+n-1}(c)$ is singular.

The first case occurs for countably many c, thus has Lebesgue measure 0. The second case occurs if and only if $\Psi(\mathscr{Q}) \in \mathrm{Puzzle}_{n-1}(c_0)$ is a singular piece. It follows from Propositions 10.3 and 13.1 that

$$\mathrm{modulus}(\mathscr{P} \smallsetminus \overline{\mathscr{Q}}) \geq \kappa \cdot \frac{n-2}{16}$$

for some constant κ which depends neither on c_0, nor on $\mathscr{P}$; and from Proposition 8.2 that $\mathscr{P}$ contains at most $3 \cdot (2N)^{n/N}$ singular pieces of depth $m + n - 1$. As a consequence

$$\frac{\mathrm{Leb}\{c \in \mathscr{P} \cap \mathbb{R} \mid \boldsymbol{N}_{\mathcal{P}}(c) \geq n\}}{\mathrm{Leb}(\mathscr{P} \cap \mathbb{R})} \leq 3 \cdot (2N)^{n/N} \cdot \mathrm{e}^{\pi} \cdot \exp\left(-2\pi\kappa\frac{n-2}{16}\right).$$

The result now follows with $K = 3\,\mathrm{e}^{\pi}\,\mathrm{e}^{\pi\kappa/4}$, any $\sigma \in (\mathrm{e}^{-\pi\kappa/8}, 1)$ and N sufficiently large so that $(2N)^{1/N} \cdot \mathrm{e}^{-\pi\kappa/8} < \sigma$. □

14. The probabilistic argument

In this section, we finally prove Lemma 2.9. Recall that (Y, p) is a probability space and $(M_k : Y \to \mathbb{N})_{k \geq 1}$ are random variables. Consider the associated sum

$$S_k := M_1 + \cdots + M_k.$$

Given $(m_1, \ldots, m_k) \in \mathbb{N}^k$, the set $Y(m_1, \ldots, m_k)$, the conditional expectations $E(m_1, \ldots, m_k)$ and the conditional variances $V(m_1, \ldots, m_k)$ are defined in §2.5. We assume that the quantities

$$E := \sup_{(m_1,\ldots,m_k)} E(m_1, \ldots, m_k) \quad \text{and} \quad V := \sup_{(m_1,\ldots,m_k)} V(m_1, \ldots, m_k).$$

are finite. We will actually need that they are small.

Proposition 14.1. *For any $\gamma > 1/2$ there is a constant C_γ such that for all $\delta > 0$,*

$$p\big\{y \in Y \mid \forall k \geq 1, \quad S_k(y) < kE + \delta k^\gamma \log_2 k\big\} > 1 - \frac{C_\gamma V}{\delta^2}.$$

Proof. For $k \geq 1$, let $N_k : Y \to \mathbb{R}$ and $T_k : Y \to \mathbb{R}$ be the random variables defined by

$$N_k(y) := M_k(y) - E\big(M_1(y), \ldots, M_{k-1}(y)\big) \quad \text{and} \quad T_k := \sum_{1 \leq j \leq k} N_j.$$

First, observe that

$$\text{for } i \neq j, \quad \int_Y N_i \cdot N_j \, \mathrm{d}p = 0.$$

Indeed, without loss of generality, assume that $i < j$. Then, N_i is constant on each set $Y(m_1, \ldots, m_j)$ and the average of N_j on each such set is 0 by construction.

Next, let $\varphi : \mathbb{N}^* \to \mathbb{N}$ be the map that forgets the rightmost 1 in the binary expansion of numbers:

$$\varphi\big(2^r \cdot (2s+1)\big) = 2^{r+1} s.$$

For $k \geq 1$, set

$$U_k := T_k - T_{\varphi(k)} = \sum_{j=\varphi(k)+1}^{k} N_j.$$

Lemma 14.2. *Given $\gamma > 1/2$, there exists a constant C_γ such that for all $\delta \geq 0$*

$$p\big\{y \in Y \mid \exists k \textit{ with } U_k \geq \delta k^\gamma\big\} \leq C_\gamma \cdot \frac{V}{\delta^2}.$$

Proof. For $k = 2^r \cdot (2s+1)$, we have $k - \varphi(k) = 2^r$ and so,

$$\delta^2 k^{2\gamma} \cdot p(U_k \geq \delta k^\gamma) \leq \int_Y U_k^2 \, \mathrm{d}p = \sum_{\varphi(k) < j \leq k} \int_Y {N_j}^2 \, \mathrm{d}p \leq 2^r V.$$

Thus,

$$p\{y \in Y \mid \exists k \text{ with } U_k \geq \delta k^\gamma\} \leq \sum_{r \geq 0, s \geq 0} \frac{2^r V}{\delta^2 \cdot (2^r)^{2\gamma} \cdot (2s+1)^{2\gamma}} = C_\gamma \cdot \frac{V}{\delta^2}$$

with

$$C_\gamma := \left(\sum_{r \geq 0} \frac{1}{(2^{2\gamma-1})^r}\right) \cdot \left(\sum_{s \geq 0} \frac{1}{(2s+1)^{2\gamma}}\right). \qquad \square$$

Outside a set of measure $C_\gamma V/\delta^2$, we have $U_k < \delta k^\gamma$ for all $k \geq 1$. Forgetting all the 1s in the binary expansion of k requires at most $\log_2 k$ steps. Thus, outside a set of measure $C_\gamma V/\delta^2$, we have

$$\text{for all } k \geq 1 \quad T_k < \delta k^\gamma \log_2 k \quad \text{so that} \quad S_k < kE + \delta k^\gamma \log_2 k.$$

This completes the proof of Proposition 14.1. $\square$

Lemma 2.9 is a corollary of this proposition.

Proof of Lemma 2.9. We need to show that for all $\varepsilon > 0$ and $\eta > 0$, if E and V are sufficiently small, then

$$p\big\{y \in Y \mid \forall k \geq 1,\ S_k(y) \leq k\varepsilon\big\} \geq 1 - \eta.$$

Given $\varepsilon > 0$ and $\eta > 0$, fix $\gamma \in (1/2, 1)$. Let $\delta > 0$ be sufficiently small so that $\delta k^\gamma \log_2 k < \varepsilon k/2$ for all $k \geq 1$. Assume $E < \varepsilon/2$ and $V < \delta^2 \eta / C_\gamma$. The result now follows from Proposition 14.1. $\square$

References

[BDH] G. Blé, C. Henriksen & A. Douady, *Round Annuli*, Contemporary Mathematics. 355 (2004), 71–76.

[BH] B. Branner & J.H. Hubbard, *The iteration of cubic polynomials. II. Patterns and para-patterns*, Acta Math. 169 (1992), no. 3-4, 229–325.

[H] J.H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, in Topological methods in modern mathematics (Stony Brook, NY, 1991), 467–511, Publish or Perish, Houston, TX, 1993.

[M] J. Milnor, *Local connectivity of Julia sets: expository lectures*, in The Mandelbrot set, theme and variations, 67–116,

[R] P. Roesch, *Holomorphic motions and puzzles (following M. Shishikura)* in The Mandelbrot set, theme and variations, 117–131, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000.

[Y] J.C. Yoccoz, A proof of Jakobson's theorem, preprint.