FIXED POINTS OF RENORMALIZATION.

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ABSTRACT. To study the geometry of a Fibonacci map f of even degree $\ell \geq 4$, Lyubich [Ly2] defined a notion of generalized renormalization, so that f is renormalizable infinitely many times. Van Strien and Nowicki [SN] proved that the generalized renormalizations $\mathcal{R}^{\circ n}(f)$ converge to a cycle $\{f_1, f_2\}$ of order 2 depending only on ℓ. We will explicitly relate f_1 and f_2 and show the convergence in shape of Fibonacci puzzle pieces to the Julia set of an appropriate polynomial-like map.

Keywords. Holomorphic dynamics, renormalization, Fibonacci.

1. INTRODUCTION.

In this article, our goal is to study the geometry of real Fibonacci maps of degree $\ell \geq 4$. The importance of Fibonacci maps has been emphasized by Hofbauer and Keller [HK] for unimodal maps and by Branner and Hubbard [BH] for cubic polynomials. In [LM], Lyubich and Milnor studied the restriction to the real axis of a quadratic Fibonacci polynomial, and this study was enlarged to the complex plane by Lyubich in [Ly2] and [Ly3]. Existence of real Fibonacci polynomials of the form $z \mapsto z^\ell + c$ was obtained by Hofbauer and Keller for any even integer $\ell \geq 2$, and follows from a combinatorial argument due to Milnor and Thurston [MT]. However, Lyubich and Milnor [LM] observed that the geometry of Fibonacci maps was different for degree $\ell = 2$ and for degrees $\ell \geq 4$.

Fibonacci maps of degree $\ell \geq 4$ have since been studied by van Strien and Nowicki in [SN] where they obtained new results using renormalization techniques. We would like to use results by H. Epstein [E1] [E2] on fixed points of renormalization to improve the results obtained by van Strien and Nowicki.

In his survey [Ly4], Lyubich describes renormalization in the following way: the notion of renormalization of a dynamical system f consists in taking a small piece of the dynamical space, considering the *first return map* to this piece, and then rescale it to the "original" size. The new dynamical system is called the *renormalization* $\mathcal{R}(f)$ of the original one. Depending on the way one chooses the small piece, and the way one defines the first return map, one gets several definitions of renormalization.

We will show that the study of the geometry of real Fibonacci maps of degree $\ell \geq 4$ is similar to the study of the geometry of Feigenbaum maps. For this purpose, we will show that one can make a parallel approach between two notions of renormalization that have been developped during the last two decades in holomorphic dynamics.

The first notion of renormalization was introduced in 1976 by Feigenbaum [F1] [F2], and independently Coullet & Tresser [CT] for real dynamical systems and more precisely for unimodal maps. To explain a universality phenomenon, they defined a renormalization operator $\mathcal{R}$ that acts on an appropriate space of dynamical systems, and conjectured that $\mathcal{R}$ had a unique fixed point f. Lanford [La] gave a computer-assisted proof of this conjecture. Later, Epstein [E1] [E2] gave a proof of the existence of a renormalization fixed point that does not require computers. However, his proof does not give uniqueness of the fixed point. This fixed point satisfies a functional equation known as the Cvitanović-Feigenbaum equation:

$$f(z) = -\frac{1}{\alpha} f \circ f(\alpha z),$$

for some $\alpha \in]0,1[$. In 1985, the generalization to holomorphic dynamics via polynomial-like mappings, was introduced by Douady and Hubbard [DH]. The "classical renormalization theory" has been extensively studied (see Collet and Eckmann and Lanford [CE], [CEL] and [La], Cvitanović [Cv], Eckmann and Wittwer [EW], Vul, Sinai and Khanin [VSK], Epstein [E1] and [E2], Sullivan [S], de Melo and van Strien [dMvS], McMullen [McM1] and [McM2], Lyubich [Ly3], [Ly4], [Ly5] and [Ly6]). For a historical account, the reader is invited to consult [T] or [Ly5].

Lyubich generalized the notion of renormalization for polynomial-like mappings, to a wider class of maps, that we will call L-maps. This allowed him to apply the renormalization ideas to "non-renormalizable" maps as well. Lyubich and Milnor [LM] showed that this generalization could be applied to the study of Fibonacci maps. Let us define a Fibonnaci map as a branch covering $f: U^0 \cup U^1 \to V$, such that

- U^0, U^1 and V are topological open disks satisfying $\overline{U^i} \subset V$, $i = 1,2$ and $\overline{U^0} \cap \overline{U^1} = \emptyset$;
- f has a unique critical point $\omega \in U^0$:
- the orbit of the critical point satisfies some combinatorics that will be defined in section 4.

Fibonacci maps are not renormalizable in the classical sense. However, it has been the idea of Lyubich that one could define a generalized renormalization operator $\mathcal{R}$, sending the space of Fibonacci maps into itself. Hence, given a Fibonacci map f, one can define an infinite sequence of generalized renormalizations $\mathcal{R}^{\circ n}(f)$. In [SN], van Strien and Nowicki proved that if the degree ℓ of the critical point ω is larger than 2, and if the map f is real (i.e., $f(\bar{z}) = \overline{f(z)}$), then this sequence converge to a cycle $\{f_1, f_2\}$ of order 2, where f_1 and f_2 are two Fibonacci maps of degree ℓ. In [Ly4], Lyubich writes: "the combinatorial difference between f_1 and f_2 is that the restrictions of these maps on the corresponding non-critical puzzle pieces have opposite orientation". We will prove that in fact f_1 and f_2 are related in the following way.

Theorem A. *For every even integer $\ell \geq 4$, let $f_i: U_i^0 \cup U_i^1 \to V_i$, $i = 1,2$, be the two real Fibonacci maps of degree ℓ, normalized so that $\omega_i = 0$ and $f_i(\omega_i) = 1$, and satisfying $\mathcal{R}(f_1) = f_2$ and $\mathcal{R}(f_2) = f_1$. Then, there exists a neighborhood U of 0 and a neighborhood U' of 1 such that*

- $f_1|_{U \cap U_1^0} = f_2|_{U \cap U_2^0}$, *and*
- $f_1|_{U' \cap U_1^1} = -f_2|_{U' \cap U_2^1}$.

The main ingredient in our proof is a flipping operator that does not preserve the dynamics of the maps, but has the nice property of sending the space of real Fibonacci maps of degree ℓ into itself.

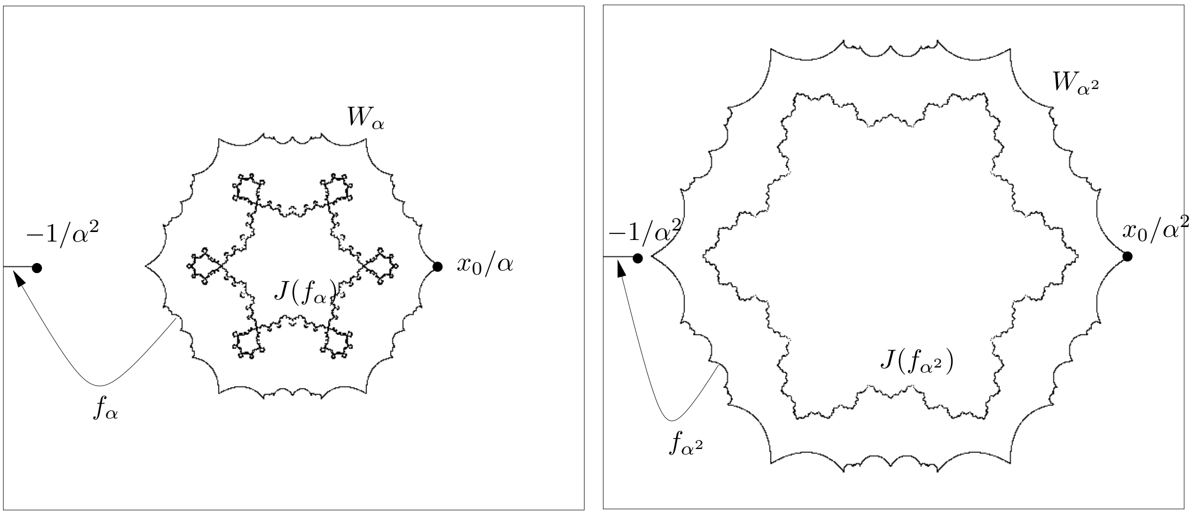


FIGURE 1. The polynomial-like maps $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ corresponding to a degree $\ell = 6$ and their Julia sets $J(f_\alpha)$ and $J(f_{\alpha^2})$.

We will then show that the restriction f of f_1 to U_1^0 satisfies a system of equations, that we will call the Cvitanović-Fibonacci equation:

$$\begin{cases} f(z) = -1/\alpha^2 f(\alpha f(\alpha z)), \quad 0 < \alpha < 1, \\ f(0) = 1 \text{ and} \\ f(z) = F(z^\ell), \text{ with } F'(0) \neq 0 \text{ and } \ell \geq 4 \text{ even.} \end{cases}$$

We will first study the geometry of the solutions of the Cvitanović-Fibonacci equation, and we will prove the following theorem.

Theorem B. *For every even integer $\ell \geq 4$, let f be the solution of the Cvitanović-Fibonacci equation in degree ℓ, and set $f_\alpha(z) = f(\alpha z)$ and $f_{\alpha^2}(z) = f(\alpha^2 z)$.*

Then, there exist domains $W_\alpha \subset \mathbb{C}$ and $W_{\alpha^2} \subset \mathbb{C}$ containing 0 such that $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ are polynomial-like mappings of degree ℓ. Besides, $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ has an attracting cycle of order 2 and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ has an attracting fixed point. In particular, the Julia set $J(f_\alpha)$ is quasi-conformally homeomorphic to the Julia set $J(z \mapsto z^\ell - 1)$ and the Julia set $J(f_{\alpha^2})$ is a quasi-circle.

Finally, the domain of analyticity of f is the quasi-disk $\widehat{W}$ bounded by the quasi-circle $\alpha J(f_{\alpha^2})$.

Figure 1 shows the two polynomial-like mappings $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ and their Julia sets.

Remark. In the context of classical renormalization, McMullen proved that the domain of analyticity of the fixed point of renormalization satisfying the Cvitanović-Feigenbaum equation is a dense open subset of $\mathbb{C}$. Our result shows that the behaviour for generalized renormalization is drastically different.

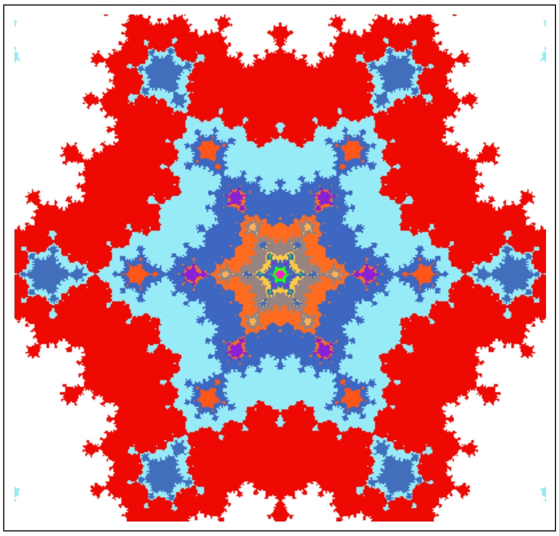


FIGURE 2. Degree six Fibonacci puzzle pieces (made by Scott Sutherland).

The next step will be to prove that any solution of the Cvitanović-Fibonacci equation gives rise to a cycle of order 2 of Fibonacci maps which is invariant under renormalization.

Theorem C. *Given any solution f of the Cvitanović-Fibonacci equation, there exists a Fibonacci map $\phi : U^0 \cup U^1 \to V$ such that ϕ and f coincide on U^0 and such that $\mathcal{R}^{\circ 2}([\phi]) = [\phi]$.*

We will then derive the following corollary.

Corollary. *For every even integer $\ell \geq 4$, there exists a unique $\alpha \in]0,1[$ such that the Cvitanović-Fibonacci equation has a solution, and this solution is itself unique.*

We will say that f is the solution of the Cvitanović-Fibonacci equation in degree ℓ.

We will then define a Yoccoz puzzle for Fibonacci maps, and study the convergence in shape of puzzle pieces. In [Ly4], Lyubich writes: “the following picture of the principal nest for degree 6 Fibonacci map show that all puzzle pieces have approximately the same shape: [see figure 2] [...] these puzzle pieces have asymptotically shapes of the Julia set of an appropriate polynomial-like map.” We will prove that this observation is true. More precisely, we will prove the following theorem.

Theorem D. *Let*

- *S_k be the Fibonacci numbers defined by $S_0 = 1$, $S_1 = 2$, and $S_{k+1} = S_k + S_{k-1}$,*
- *$\ell \geq 4$ be an even integer,*
- *$F : U^0 \cup U^1 \to V$ be a real Fibonacci map of degree ℓ normalized so that the critical point is $\omega = 0$,*

- *C_k be the connected component of $F^{-k}(V)$ that contains the critical point (it is called the critical puzzle piece of depth k),*
- *f be the solution of the Cvitanović-Fibonacci equation in degree ℓ,*
- *$\alpha \in]0,1[$ be the constant defined by the Cvitanović-Fibonacci equation, and*
- *$f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ be the polynomial-like mappings defined in Theorem B.*

Then, there exists a constant $\lambda \neq 0$ such that

- *the sequence of rescaled puzzle pieces $\dfrac{\lambda}{\alpha^{k+1}}\overline{C_{S_k-2}}$ converges for the Hausdorff topology to the filled-in Julia set $K(f_{\alpha^2})$, and*
- *the sequence of rescaled puzzle pieces $\dfrac{\lambda}{\alpha^{k-1}}\overline{C_{S_k-3}}$ converges to the filled-in Julia set $K(f_\alpha)$.*

Let us mention that a similar result has already been proved by Lyubich [Ly2] for Fibonacci maps in degree 2. He proved the convergence in shape of some puzzle pieces to the Julia set of $z \to z^2 - 1$.

2. Dynamical systems.

In this section, we will quickly recall the definition of polynomial-like mappings (see [DH]) and of generalized polynomial-like mappings (see [Ly3]). We will also define the corresponding notion of renormalization.

2.1. Polynomial-like maps.

In [DH], Douady and Hubbard introduced the concept of polynomial-like maps. A polynomial-like map is a branched covering $f : U \to V$ between two topological disks U and V, with $U \Subset V$. One defines the filled-in Julia set $K(f)$ and the Julia set $J(f)$ of a polynomial-like map $f : U \to V$ as:

$$K(f) = \{z \in U \mid (\forall n \in \mathbb{N})\ f^n(z) \in U\}, \quad \text{and} \quad J(f) = \partial K(f).$$

Definition 1. *We say that f is a DH-map if $f : U \to V$ is a polynomial-like map having a single critical point $\omega \in K(f)$.*

Remark. The Julia sets $K(f)$ and $J(f)$ are connected if and only if $K(f)$ contains all the critical points of f. Hence, the Julia set of a DH-map will always be connected.

Douady and Hubbard showed that a polynomial-like map behaves dynamically like a polynomial.

Proposition 1. (see [DH]) *For each DH-map f, there exists*

- *a unique polynomial P_c of the form $z^\ell + c$, up to conjugacy by $z \to e^{2i\pi k/(\ell-1)}z$, $k = 0, 1, \dots, \ell - 1$,*
- *topological disks U_c and V_c, and*
- *a quasi-conformal homeomorphism $\phi : V \to V_c$ satisfying $\partial\phi/\partial\overline{z} = 0$ a.e. on $K(f)$,*

such that for all $z \in U$,

$$\phi \circ f = P_c \circ \phi.$$

We will say that the two maps are in the same hybrid class.

Definition 2. *Given a DH-map $f : U \to V$, we say f is renormalizable if we can find*

- *an integer k strictly greater than 1, and*

- *topological disks* U_1 *and* V_1 *containing* ω*, such that*

$f^{\circ k}: U_1 \to V_1$ *is a DH-map.*

We will say f is k-renormalizable and that $f^{\circ k}: U_1 \to V_1$ is a renormalization of f. We will be interested in one particular polynomial, called the Feigenbaum polynomial, which is the unique polynomial, $P_{Feig} = z^2 + c_{Feig}$ with $c_{Feig} \in \mathbb{R}$, 2-renormalizable, and such that P_{Feig} and its renormalizations are in the same hybrid class. Results about the existence and uniqueness of this polynomial are discussed in [S], in [dMvS] and in [McM2].

2.2. **Generalized polynomial-like maps.** Another family of polynomial-like maps was introduced by Lyubich in [Ly3] to study what happens when the critical point escapes from U. This kind of maps appears naturally when one studies cubic polynomials with two critical points, one escaping to infinity, the other having a bounded orbit (see [BH]).

Definition 3. *A generalized polynomial-like map, which we will call an L-map, is a ramified covering map*

$$f: \bigcup_{i=0}^{k-1} U^i \to V$$

such that

- *there is a unique critical point* $\omega \in U^0$*,*
- *the orbit of* ω *is contained in the union of the* U^i*,*
- *each* U^i *contains at least one point of the orbit of* ω*, and if* $i < j$ *the orbit of* ω *visits* U^i *before* U^j*,*
- *each* U^i*,* $i = 0, \ldots k-1$ *is a topological disk compactly contained in* V*,*
- *the* U^i *are pairwised disjoint.*

There is only one way of ordering the U^i because of the third condition. We can again define

$$K_f = \{z \mid f^n(z) \in \bigcup_{i=0}^{k-1} U^i,\ \forall n \in \mathbb{N}\}, \quad \text{and} \quad J(f) = \partial K(f).$$

Figure 3 shows an L-map $f : U^0 \cup U^1 \to V$, where f is a cubic polynomial with one critical orbit escaping to infinity and one critical point having a bounded orbit.

As in [BH], [H] or [Mi], we can define the puzzles associated to an L-map.

Definition 4. *The puzzles are defined by induction:*

- *the elements of the puzzle* $\mathcal{P}_0(f)$ *of depth 0 are the open sets* U^i *(called puzzle pieces),*
- *the elements of the puzzle* $\mathcal{P}_n$ *of depth* n *are the connected components of* $f^{-n}(\mathcal{P}_0)$*.*

We can define a notion of renormalization associated to those L-maps.

Definition 5. *We will say* f *is L-renormalizable if we can find*

- *a finite collection of puzzle pieces* U_1^i*,* $i = 0, \ldots, l-1$*, a puzzle piece* V_1 *and*
- *integers* n_i*,* $i = 0, \ldots, l-1$ *with at least one* $n_i > 1$*,*

such that

- $\omega \in U_1^0$ *and*

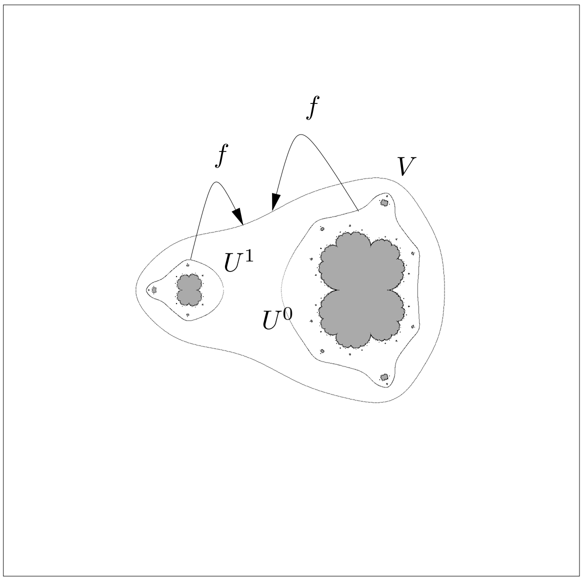


FIGURE 3. the L-map $f : U^0 \cup U^1 \to V$ and its Julia set.

- *the map* $g : \bigcup_{i=0}^{l-1} U_1^i \to V_1$ *defined by* $g|U_1^i = f^{\circ n_i}$ *is an L-map.*

We say f is $(n_0, \ldots, n_{l-1})$-renormalizable, and $g : \bigcup_{i=0}^{l-1} U_1^i \to V_1$ is an L-renormalization of f.

In this context, we will study Fibonacci maps, which are L-maps defined by some dynamical properties. Such maps were introduced in [HK] and [BH], and studied further in [LM], [Ly2] or [SN].

For the two kind of renormalizations, the maps we are interested in are infinitely renormalizable. We will assume the sequence of successive renormalizations converges to a fixed point of renormalization (cf [S], [dMvS] or [McM2] for Feigenbaum case, and [SN] for Fibonacci maps. We will then study those fixed points of renormalization, using H. Epstein's work (cf [E2]).

To do this, we must first introduce two notions. The first one is a notion of convergence, which will enable us to talk of limits, the second one is a notion of germs which will allow us to talk of fixed points of renormalization.

2.3. **Topology on the space of polynomial-like maps.** In [McM1], McMullen introduces the following topology. First of all, a pointed region is a pair (U, u), where $U \subset \mathbb{C}$ is an open set, and $u \in U$ is a point.

Definition 6. *We say that* (U_n, u_n) *converges to* (U, u) *in the Carathéodory topology if and only if*

- $u_n \to u$, *and*
- *for any Hausdorff limit* K *of the sequence* $\mathbb{P}^1 \setminus U_n$, U *is the connected component of* $\mathbb{P}^1 \setminus K$ *which contains* u.

To define a topology on the sets of polynomial-like maps, we use a theorem by McMullen.

Proposition 2. (see [McM2]) *Let*

$$g_n : (U_n, u_n) \to (V_n, v_n)$$

be a sequence of proper maps between pointed disks, with deg$(g_n) \leq d$. *Suppose* $u_n \to u$, g_n *converges uniformly to a non-constant limit on a neighborhood of* u, *and* $(V_n, v_n) \to (V, v)$. *Then* (U_n, u_n) *converges to a pointed disk* (U, u), *and* g_n *converges uniformly on compact subsets of* U *to a proper map* $g : (U, u) \to (V, v)$, *with* $1 \leq deg(g) \leq d$.

This enables us to define a topology on the sets of DH-maps or L-maps, because each branch of those maps are proper maps between disks. For DH-map and L-map, there is a natural way of choosing the basepoints. One can, for example, take the first visit of the critical orbit in the disks U^i.

2.4. **Space of germs.** In [McM1], McMullen introduces the notion of germs of polynomial-like maps. We can adapt this notion to L-maps. To do so, we just need to say two maps f_1 and f_2 are equivalent if they have the same Julia set, $K(f_1) = K(f_2) = K$, and if $f_1|K = f_2|K$.

Definition 7. *The set* $\mathcal{G}$ *of germs* $[f]$ *is the set of equivalence classes.*

McMullen gives $\mathcal{G}$ the following topology: $[f_n] \to [f]$ if and only if there are representatives f_n and f, which are DH-maps or L-maps, depending on the context, and such that $f_n \to f$ for the Carathéodory topology. Then, the space of germs is Hausdorff.

3. Feigenbaum maps

All the results we will state here have already been proved by Epstein [E1] and McMullen [McM1]. The goal is to introduce some functional equation satisfied by fixed points of renormalization, and to state some results related to it. The work we present here has been completed in [B3].

3.1. **Feigenbaum polynomial.** The Feigenbaum polynomial is the most famous example of polynomial which is infinitely renormalizable (meaning it is k-renormalizable for infinitely many k). It is the unique real quadratic polynomial which is 2^k-renormalizable for all $k \geq 1$.

Definition 8. *One can define the Feigenbaum polynomial as the unique real polynomial which is a fixed point of tuning by* -1.

Tuning is the inverse of renormalization. Given a parameter $c \in M$, such that 0 is a periodic point of period p, Douady and Hubbard have constructed a tuning map, $x \mapsto c * x$, which is a homeomorphism of M into itself, sending 0 to c, and such that if $x \neq 1/4$, then f_{c*x} is p-renormalizable, and the corresponding renormalization is in the same hybrid class as f_x. This is how they show there are small copies of the Mandelbrot set inside itself.

The Feigenbaum value, $c_{Feig} = -1.401155...$, is in the intersection of all the copies of M obtained by tuning by -1. This intersection is not known to be reduced to one point, but its intersection with the real axis is reduced to the point c_{Feig}.

3.2. **Feigenbaum renormalizations.** By construction, the Feigenbaum polynomial, P_{Feig}, is 2-renormalizable. There are several renormalizations $g : U \to V$ such that $g = f^2|U$ is a DH-map. But all those renormalizations define the same germ of DH-map. By the straightening theorem (see [DH]), there is a unique polynomial which is in the same hybrid class as g, i.e., quasi-conformally conjugate to g on a neighborhood of its Julia set, the $\overline{\partial}$ derivative of the conjugacy vanishing almost everywhere on the Julia set. This polynomial is a real polynomial, because g is real, and 2^k-renormalizable for all $k \geq 1$. Thus, it is the Feigenbaum polynomial.

We can define a renormalization operator, $\mathcal{R}_2$. Given a germ of DH-map, $[f]$, which hybrid class is the one of the Feigenbaum polynomial, let us choose a representative g corresponding to period 2 renormalization.

Definition 9. *Assume $[f]$ is a germ of a quadratic-like map which is renormalizable with period 2. There exist open sets U' and U such that the map $g : U' \to U$ defined by $g = f^{\circ 2}|U$ is a polynomial-like map with connected Julia set. The renormalization operator $\mathcal{R}_2$ is defined by*

$$\mathcal{R}_2([f]) = [\alpha^{-1} \circ g \circ \alpha],$$

with $\alpha = g(0) = f^2(0)$, and $\alpha(z) = \alpha z$.

We have normalized the germs, so that the critical value is 1.

We have seen that if $[f]$ is a germ of Feigenbaum DH-map, then $\mathcal{R}_2([f])$ is still a germ of Feigenbaum DH-map, and we can iterate this process, defining in such a way a sequence of germs : $\mathcal{R}_2^{\circ n}([P_{Feig}])$, $n \in \mathbb{N}$. The following result can be found in [S], [dMvS] or [McM2].

Proposition 3. *The sequence of germs $\mathcal{R}_2^{\circ n}([P_{Feig}])$, $n \in \mathbb{N}$, converges (for the Carathéodory topology defined in the introduction), to a point $[\phi]$. By construction this point is a fixed point of renormalization :*

$$\mathcal{R}_2([\phi]) = [\phi],$$

and is in the hybrid class of the Feigenbaum polynomial. It is the unique fixed point of $\mathcal{R}_2$.

Remark. We say that two quadratic-like germs $[f]$ and $[g]$ are in the same hybrid class if there exist representatives $f : U' \to U$ and $g : V' \to V$ which are in the same hybrid class.

Now, if $[f]$ is a fixed point of $\mathcal{R}_2$, then it satisfies the following functional equation, known as the Cvitanović-Feigenbaum equation.

Proposition 4. *Let $[f]$ be a fixed point of $\mathcal{R}_2$. Then,*

$$\begin{cases} f(z) = -\frac{1}{\alpha} f(f(\alpha z)), \quad 0 < \alpha < 1, \\ f(0) = 1, \text{ and} \\ f(z) = F(z^2), \text{ with } F'(0) \neq 0. \end{cases}$$

This equation is satisfied at least on the Julia set of f (which does not depend on the representative f of the germ $[f]$).

3.3. **Study of some functional equations.** Now the question is: what information can we obtain from this equation ? The way we can deal with it, was explained to us by H. Epstein and is developped in [E2]. There is a global theory which enables us to deal with the study of the fixed points of the three kind of renormalization at the same time. The functional equation we will study will depend on two parameters. The first one is the degree ℓ of the critical point. The second one is a parameter ν which corresponds to the case we are dealing with.

- in the case of renormalization for DH-maps, $\nu = 1$,
- in the case of renormalization for L-maps, $\nu = 1/2$, and
- in the case of renormalization for holomorphic pairs, $\nu = 2$.

Definition 10. *The universal equation is the following system of equations:*

$$\begin{cases} f(z) = -\frac{1}{\lambda} f(f(\lambda^\nu z)), \quad 0 < \lambda < 1, \\ f(0) = 1, \text{ and} \\ f(z) = F(z^\ell), \text{ with } F'(0) \neq 0. \end{cases}$$

First of all, we want to study solutions such that f and F are real analytic maps on an open interval J containing 0, and their complex extension. So let J be an open interval in $\mathbb{R}$, possibly empty, and define

$$\mathbb{C}(J) = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) \neq 0, \text{ or } z \in J\} = \mathbb{H}_+ \cup \mathbb{H}_- \cup J,$$

where

$$\mathbb{H}_+ = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\} = -\mathbb{H}_-.$$

$\mathcal{F}(J)$ is the space of holomorphic functions h in $\mathbb{C}(J)$, such that $h(\overline{z}) = \overline{h(z)}$. $\mathbb{P}^1(J) \subset \mathcal{F}(J)$ is the space of functions h such that $h(\mathbb{H}_+) \subset \mathbb{H}_+$. A function $h \in \mathbb{P}^1(J)$ is called a Herglotz function (and $-h$ is anti-Herglotz). We will study solutions of the universal equation, such that F is univalent in a neighborhood of 0, and has an anti-Herglotz inverse, F^{-1}. In fact, as the limit of renormalization can be obtained as a limit of polynomials having all their critical values in $\mathbb{R}$, this condition is satisfied by the fixed points of renormalization we will consider.

The first step is to look at the graph of f on the real axis. Figure 4 shows what this graph looks like. This graph gives the relative position of several points on the real axis.

Proposition 5. Epstein (see [E2]) *Let f be a solution of the universal equation, and $x_0 > 0$ be the first positive preimage of 0 by f. Then*

- $f(\lambda^\nu x_0) = x_0$,
- $f(1) = -\lambda$, *and*
- *the first critical point in $\mathbb{R}^+$ is x_0/λ^ν, with $f(x_0/\lambda^\nu) = -1/\lambda$.*

Besides, the universal equation can be restated in two surprising ways on the following commutative diagrams. The first diagram tells that x_0 is an attracting fixed point of the map $f(\lambda^\nu z)$. The linearizer is f.

$$\begin{array}{ccc} x_0 & \xrightarrow{f(\lambda^\nu z)} & x_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ 0 & \xrightarrow[-\lambda z]{} & 0 \end{array}$$

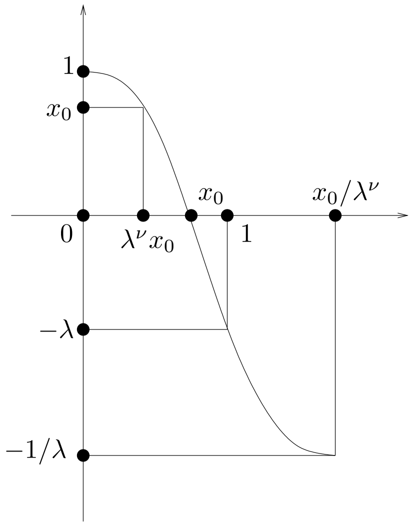
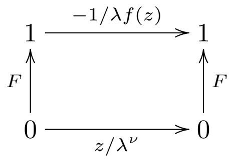


FIGURE 4. The graph of f on $\mathbb{R}^+$.

The second diagram tells that 1 is a repulsive fixed point of the map $-1/\lambda f(z)$. The parametrizer is F.



The first diagram enables Epstein to study how much one can extend F^{-1}. His results are the following.

Theorem 1. EPSTEIN (see [E2]) *Let $f(z) = F(z^\ell)$ be a solution of the universal equation, such that F^{-1} is anti-Herglotz. We then have the following results:*

- *one can extend F^{-1} such that $F^{-1} \in -\mathbb{P}^1(]-1/\lambda, 1/\lambda^2[)$,*
- *one can extend F^{-1} continuously to the boundary $\mathbb{R}$ of $\mathbb{H}_+$, and even analytically except at points $(-1/\lambda)^n$, $n \geq 1$, which are branching points of type $z^{1/\ell}$,*
- *the values of F^{-1} are never real except in $[-1/\lambda, 1/\lambda^2]$,*
- *the extension of F^{-1} to the closure of $\mathbb{H}_+$ is injective, and*
- *when z tends to infinity in $\mathbb{H}_+$, $F^{-1}(z)$ tends to a point in $\mathbb{H}_-$, which will be denoted by $F^{-1}(i\infty)$.*

By symmetry, similar statements hold in $\mathbb{H}_-$. Hence, $\mathcal{W} = F^{-1}(\mathbb{C}(]-1/\lambda, 1/\lambda^2[)$ is a bounded domain of $\mathbb{C}$. Those results are summarized in figure 5.

In the following, we will use the notations:

$$\begin{cases} \mathbb{C}_\lambda & = & \mathbb{C} \setminus \big(]-\infty, -1/\lambda] \cup [1/\lambda^2, +\infty[\big), \\ \mathcal{W} & = & F^{-1}(\mathbb{C}_\lambda), \quad \text{and} \\ W & = & \{z \in \mathbb{C} \mid z^\ell \in \mathcal{W}\}. \end{cases}$$

By construction,

$$f: \ W \to \mathbb{C}_\lambda$$

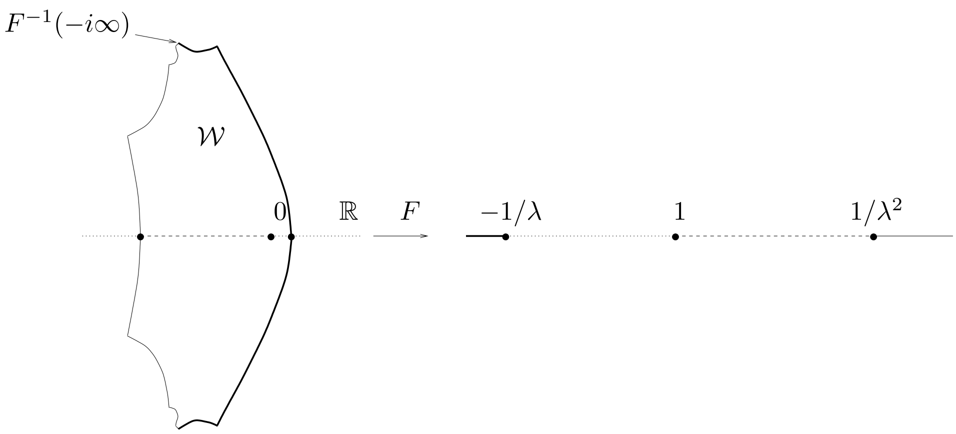


FIGURE 5. Maximal univalent extension of F.

is a ramified covering with only one critical point in 0, of degree ℓ. The set W is symmetric by rotation of angle $2i\pi/\ell$.

In [E2], Epstein uses those informations to prove the following result.

Theorem 2. (cf [E2] and figure 6) *Let f be a solution of the universal equation of parameters $\nu = 1$ and $\ell = 2$. The map $f:\ W \to \mathbb{C}_\lambda$ is a polynomial-like map. It is quasi-conformally conjugated to the Feigenbaum polynomial P_{Feig}.*

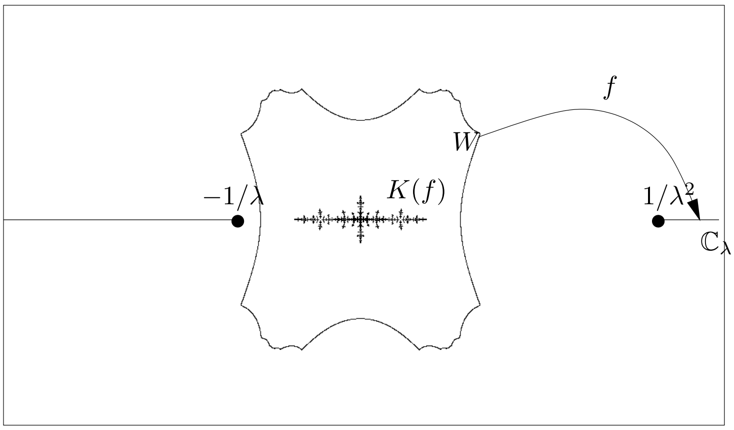


FIGURE 6. The Feigenbaum map $f:\ W \to \mathbb{C}_\lambda$.

PROOF. See [E2] or [B3]. □

To study the geometry of the Julia set of the Feigenbaum polynomial, it is sometime enough to study the geometry of the Julia set $K(f)$ of this polynomial-like map. This has been done in [B2]. Some results are easier to obtain using the fixed point of renormalization because of the invariance with respect to the scaling map $z \to \lambda z$.

4. Fibonacci maps.

In this section, we will deal with renormalization for L-maps, and more precisely, for Fibonacci maps. We will see that the dynamics of Fibonacci maps is strongly related to Fibonacci numbers, so let us first recall the definition of the Fibonacci sequence S_n.

Definition 11. *The Fibonacci sequence* S_k *is defined by* $S_0 = 1$, $S_1 = 2$, *and* $S_{k+1} = S_k + S_{k-1}$.

In particular $S_2 = 3$, $S_3 = 5$, $S_4 = 8$, $S_5 = 13$, and so on.

4.1. **Definition of Fibonacci maps.** Let us now return to the definition of Fibonacci maps. We have defined the puzzles $\mathcal{P}_n$ associated to an L-map in the introduction. Following Branner and Hubbard, we will distinguish the puzzle pieces that contain the critical point and define a notion of genealogy between those pieces.

Definition 12. *Let* $f : \bigcup U^i \to V$ *be an L-map, and for each* $z \in K(f)$, *let* $P_n(z)$ *be the puzzle piece of depth* n *which contains* z.

The critical piece C_n *of depth* n *is defined to be* $P_n(\omega)$ *if* $n \geq 0$ *and* C_{-1} *is defined to be the piece* V.

For all $n \geq -1$, *the children of* C_n *are the critical pieces* C_l *such that* $f^{\circ(l-n)}(C_l) = C_n$ *and* $f^{\circ k}(C_l)$ *does not contain the critical point* ω *for* $0 < k < l - n$.

Remark. If C_l is a child of C_n, then $f^{\circ(l-n)} : C_l \to C_n$ is a ramified covering ramified only at ω.

Let us now define what is a Fibonacci map.

Definition 13. *A Fibonacci map of degree* $\ell \geq 2$ *is an L-map* $f : U^0 \cup U^1 \to V$ *having a critical point of degree* ℓ *and satisfying the following conditions:*

- *for each* $n \geq -1$, *the critical piece* C_n *has exactly two children;*
- *if* C_l *is a child of* C_n, *then* $f^{\circ(l-n)}(0) \in C_n \setminus \overline{C}_l$.

Figure 7 shows the domain, range and Julia set of a Fibonacci map having a critical point ω of degree 6.

We would like to mention that our definition of Fibonacci maps is not the one given by Branner and Hubbard in [BH] but we will show that it is equivalent. The first condition is not sufficient to guaranty that those maps are Fibonacci maps in the sense of [BH]. The second condition says that there are no central returns in the terminology of Lyubich [Ly3].

The way Branner and Hubbard define Fibonacci maps is the following. They introduce the concept of a *tableau* in order to describe recurrence of critical orbits for cubic polynomials having one escaping critical point, and one critical point with bounded orbit. They call f a Fibonacci map if the tableau of f is the "Fibonacci tableau". The restriction of such a polynomial f to well chosen domains U^0 and U^1 gives rise to an L-map $f : U^0 \cup U^1 \to V$. It is then clear from the definition of the Fibonacci tableau that every critical piece of f has exactly two children (this is in fact the reason why the Fibonacci tableau was introduced by Branner and Hubbard), and that there are no central returns. Hence the L-map $f : U^0 \cup U^1 \to V$ is a Fibonacci map in our sense.

On the other hand, we will show that the orbit of the critical point of our Fibonacci maps returns closer to zero after each Fibonacci number of iterations in some combinatorial sense; more precisely, we will show that $f^{\circ S_n}(\omega)$ belongs

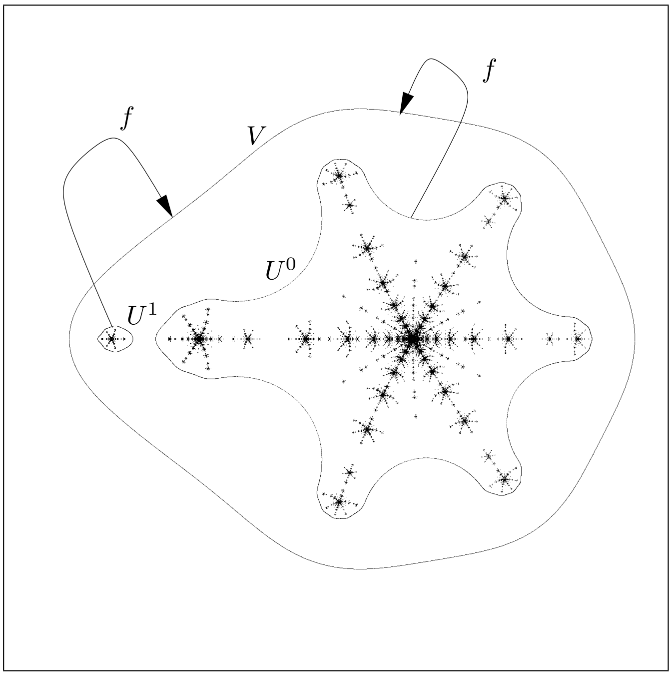


FIGURE 7. A Fibonacci map having a critical point of degree 6.

to the the critical puzzle piece of depth $S_{n+1} - 3$ but not to the critical puzzle piece of depth $S_{n+1} - 2$. This is precisely the way Branner and Hubbard define the Fibonacci tableau.

This discussion and proposition 12.8 in [BH] show that there exist Fibonacci maps having a critical point of arbitrary degree $\ell \geq 2$.

In [SN], van Strien and Nowicki mimicked an argument due to Lyubich and Milnor [LM] and prove that for every even integer $\ell \geq 2$, there exists a Fibonacci map $f : U^0 \cup U^1 \to V$ satisfying $f(\overline{z}) = \overline{f(z)}$ and $f(z) = F(z^\ell)$ with $F'(0) \neq 0$.

Definition 14. *We will say that f is a real symmetric Fibonacci map of degree ℓ if and only if $f : U^0 \cup U^1 \to V$ is a Fibonacci map satisfying $f(\overline{z}) = \overline{f(z)}$ and $f(z) = F(z^\ell)$ with $F'(0) \neq 0$.*

The proof is based on the formal machinery of kneading theory developed in [MT]. The first step consists in constructing a polynomial $P(z) = z^\ell + c$ such that the orbit of the critical point returns closer to zero after each Fibonacci number of iterations. The second step consists in renormalizing this polynomial in the sense of L-maps, so as to get an L-map $f : U^0 \cup U^1 \to V$ where $f|U^0 = P^{\circ 5}$ and $f|U^1 = P^{\circ 3}$. One can easily check that for this map f, the orbit of the critical point still returns closer to zero after each Fibonacci number of iterations so that it is a Fibonacci map.

We will now show that a Fibonacci map is infinitely renormalizable in the sense of L-maps. Afterwards, we will prove the equivalence between our definition of Fibonacci maps and the one given by Branner and Hubbard.

4.2. **Renormalization of Fibonacci maps.**

Proposition 6. (see picture 8) *Given a Fibonacci map* $f: U^0 \cup U^1 \to V$, *let* $V_1 = C_0$ *be the critical piece of depth* 0, $U_1^0 = C_2$ *be the critical piece of depth* 2, *and* $U_1^1 = P_1(f^{\circ 2}(\omega))$ *be the piece of depth* 1 *that contains* $f^{\circ 2}(\omega)$. *Then, the mapping* $g: U_1^0 \cup U_1^1 \to V_1$ *defined by* $g|_{U_1^0} = f^{\circ 2}$ *and* $g|_{U_1^1} = f$ *is a Fibonacci map.*

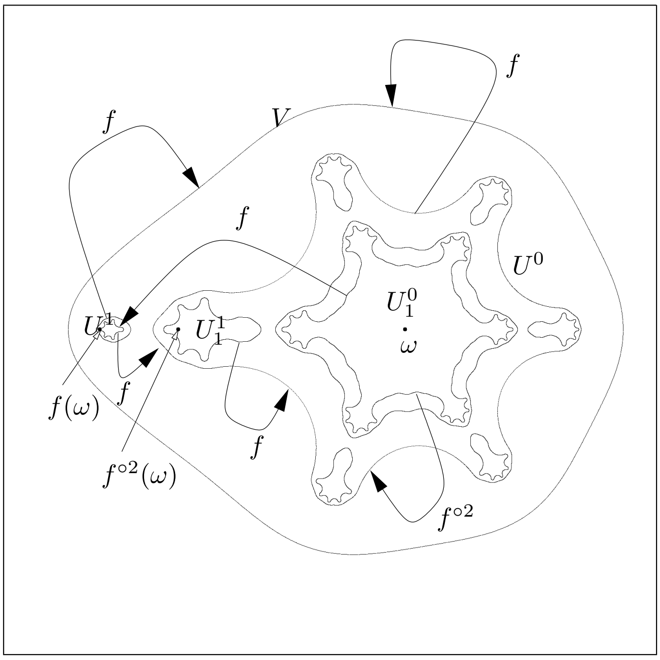


FIGURE 8. A Fibonacci map is $(2, 1)$ renormalizable.

Remark. We will call g the canonical renormalization of f.
PROOF. Let us first prove a lemma that will be useful to prove this proposition and lemma 4 below.

Lemma 1. *Let* $f: U^0 \cup U^1 \to V$ *be a Fibonacci map. Then,* $f(\omega) \in U^1$ *and if* $f^{\circ n}(\omega) \in U^1$, *we have* $f^{\circ(n+1)}(\omega) \in U^0$.

PROOF. The first statement simply follows from the fact that f has no central returns. The second statement follows from the fact that $C_{-1} = V$ has only two children. Indeed, let $C_0 = U^0$ and C_1 be the critical puzzle pieces of depth 0 and 1. Then, writing

$$C_0 \xrightarrow{f} C_{-1} \quad \text{and} \quad C_1 \xrightarrow{f} U^1 \xrightarrow{f} C_{-1},$$

we see that C_0 and C_1 are the two children of C_{-1}. In particular, there can be no extra child.

Thus, let $U = f|_{U^1}^{-1}(U^1)$ be the connected component of $f^{-1}(U^1)$ which is contained in U^1 (see picture 8). Let us prove that the critical orbit never enters U. If this were not the case, then we could define j to be the least integer such that $f^{\circ j}(\omega)$ enters U, and we could pull-back univalently the puzzle piece U along the orbit $f(\omega) \mapsto \cdots \mapsto f^{\circ j}(\omega) \in U$. Pulling-back once more by $f|_{U^0}$, we would obtain an extra child of C_{-1}.

Hence, if $f^{\circ n}(\omega) \in U^1$, then $f^{\circ(n+1)}(\omega) \in V \setminus U^1$ and since $f^{\circ(n+1)}(\omega) \in K(f) \subset U^0 \cup U^1$, we see that the proof of the lemma is completed. $\square$

Let us now prove that the map $g : U_1^0 \cup U_1^1 \to V_1$ is an L-map. Since the connected components of the domain and range of g are puzzle pieces, they are topological disks, and given two of them, we have only three possible configurations: they are equal, one is compactly contained in the other one, or their closures are disjoint. Since U_1^0 is the critical piece of depth 2 and V_1 is the critical piece of depth 0, we see that $U_1^0 \Subset V_1$. Furthermore, since $f(\omega) \in U^1$ and $f^{\circ 2}(\omega) \in U^0 = V_1$, we see that $f^{\circ 2} : U_1^0 \to V_1$ is a ramified covering with, ramified only at ω. On one hand, the "no central returns" condition implies that $f^{\circ 2}(\omega) \notin U_1^0$. On the other hand, by definition, $f^{\circ 2}(\omega) \in U_1^1$. Hence, the closures of U_1^0 and U_1^1 are disjoint. Besides, lemma 1 shows that V_1 contains $f^{\circ 2}(\omega)$. Since, V_1 is a puzzle piece of depth 0 and U_1^1 is a puzzle piece of depth 1, and since both of them contain $f^{\circ 2}(\omega)$, we see that $U_1^1 \Subset V_1$.

Let us now show that the critical orbit of g never escapes $U_1^0 \cup U_1^1$. Assume $g^{\circ n}(\omega) \in V_1 \setminus U_1^0 = C_0 \setminus C_2$. We want to show that $g^{\circ n}(\omega) \in U_1^1$. Since $g^{\circ n}(\omega) = f^{\circ k}(\omega)$ for some integer k, and since $\omega \in K(f)$, we see that $g^{\circ n}(\omega)$ belongs to a puzzle piece of f of depth 1 contained in $C_0 \setminus C_2$. It cannot be inside the critical piece C_1 since the puzzle pieces of depth 2 contained in $C_1 \setminus C_2$ are mapped by f and $f^{\circ 2}$ into U^1, so that $f^{\circ(k+1)}(\omega)$ and $f^{\circ(k+2)}(\omega)$ would both be inside U^1, contradicting lemma 1. Hence, $g^{\circ n}(\omega)$ is inside a puzzle piece of depth 1 contained in $C_0 \setminus C_1$. Assume it is a puzzle piece $U \neq U_1^1$. Then, we can use the same argument as in the proof of lemma 1: we let j be the least integer such that $f^{\circ j}(\omega)$ enters U, and we pull-back C_0 along the following orbits: $\omega \mapsto f(\omega) \mapsto f^{\circ 2}(\omega) \in C_0$, $\omega \mapsto f(\omega) \mapsto f^{\circ 2}(\omega) \in U_1^1 \mapsto\mapsto f^{\circ 3}(\omega) \in C_0$ and $\omega \mapsto f(\omega) \mapsto \cdots \mapsto f^{\circ j}(\omega) \in U \mapsto f^{\circ(j+1)} \in C_0$ showing that C_0 has at least 3 children.

Let us finally show that $g : U_1^0 \cup U_1^1 \to V_1$ is a Fibonacci map. The critical pieces of the puzzle of g are exactly the critical pieces of the puzzle of f which are children of C_0, grand-children of C_0, grand-grand-children of C_0 and so on. In particular, every critical piece of g has exactly two g-children. Besides, since f has no central returns, the same property holds for g, which concludes the proof of the proposition. □

Let us now use this renormalization result to prove that our definition of Fibonacci maps is equivalent to the one given by Branner and Hubbard.

Proposition 7. *If $f : U^0 \cup U^1 \to V$ is a Fibonacci map, then for any $n \geq 0$, $f^{\circ S_n}(\omega)$ belongs to the critical puzzle piece of depth $S_{n+1} - 3$ but not to the critical piece of depth $S_{n+1} - 2$.*

Remark. One can easily check that this correspond to the definition of the Fibonacci marked grid given in [BH], example 12.4.

Proof. Let us subdivide the proof within two lemmas that will be used again later.

Lemma 2. *Let $f : U^0 \cup U^1 \to V$ be a Fibonacci map. Then, we can define a sequence*

$$\left(f_n : U_n^0 \cup U_n^1 \to V_n\right)_{n \geq 0},$$

where $f_0 = f$ and f_{n+1} is the canonical renormalization of f_n. Then for $n \geq 1$,

- *the connected components of the range and the domain of f_n are $V_n = C_{S_{n+1}-3}$, $U_n^0 = C_{S_{n+2}-3}$ and $U_n^1 = P_{S_{n+1}+S_{n-1}-3}\left(f^{\circ S_n}(\omega)\right)$;*
- *the restrictions of f_n to U_n^0 and U_n^1 are $f_n|_{U_n^0} = f^{\circ S_n}$ and $f_n|_{U_n^1} = f^{\circ S_{n-1}}$.*

Proof. The proof is an easy induction based on the definition of the canonical renormalization. We leave the details to the reader. □

Lemma 3. *Let $f : U^0 \cup U^1 \to V$ be a Fibonacci map. Then, for any integer $n \geq 1$, $C_{S_{n+1}-2}$ is a child of $C_{S_{n-1}-2}$.*

Proof. This is again proved by induction. We first claim that the induction property holds for $n = 1$. Indeed, $C_{S_2-2} = C_1$, $C_{S_0-2} = C_{-1}$ and we have

$$C_1 \xrightarrow{f} U^1 \xrightarrow{f} C_{-1},$$

so that C_1 is a child of C_{-1}.

Next, assume that the induction property holds for some integer $n \geq 1$. Then, $f^{\circ S_{n+1}}$ restricts to a ramified covering between the critical piece of depth $S_{n+2} - 2$ and the piece of depth $S_{n+2} - 2 - S_{n+1} = S_n - 2$ which contains $f^{\circ S_{n+1}}(\omega)$. Since lemma 2 shows that $f^{\circ S_{n+1}}(\omega) \in C_{S_{n+2}-3} \subset C_{S_n-2}$, we see that $f^{\circ S_{n+1}} : C_{S_{n+2}-2} \to C_{S_n-2}$ is a ramified covering.

We still need to see that $f^{\circ S_{n+1}} : C_{S_{n+2}-2} \to C_{S_n-2}$ is ramified only at ω. To prove this result, observe that $f^{\circ S_{n+1}} = f^{\circ S_{n-1}} \circ f^{\circ S_n}$. Since $C_{S_{n+2}-2} \subset C_{S_{n+1}-2}$ and since $C_{S_{n+1}-2}$ is a child of $C_{S_{n-1}-2}$, we see that $f^{\circ S_n} : C_{S_{n+2}-2} \to f^{\circ S_n}(C_{S_{n+2}-2})$ is a ramified covering, ramified only at ω. Hence, we only need to prove that the restriction of $f^{\circ S_{n-1}}$ to $f^{\circ S_n}(C_{S_{n+2}-2})$ is univalent. We already know that this restriction is a (possibly ramified) covering onto its image. Hence, we must show that $f^{\circ S_n}(C_{S_{n+2}-2})$ does not contain a critical point of $f^{\circ S_{n-1}}$. Recall that by lemma 2, the restriction of $f^{\circ S_{n-1}}$ to $C_{S_{n+1}-3}$ has a unique critical point at ω. Hence, it is sufficient to show that $f^{\circ S_n}(C_{S_{n+2}-2}) \subset C_{S_{n+1}-3} \setminus C_{S_{n+1}-2}$ (see figure 9). By definition, $f^{\circ S_n}(C_{S_{n+2}-2})$ is the puzzle piece of depth $S_{n+2} - 2 - S_n =$

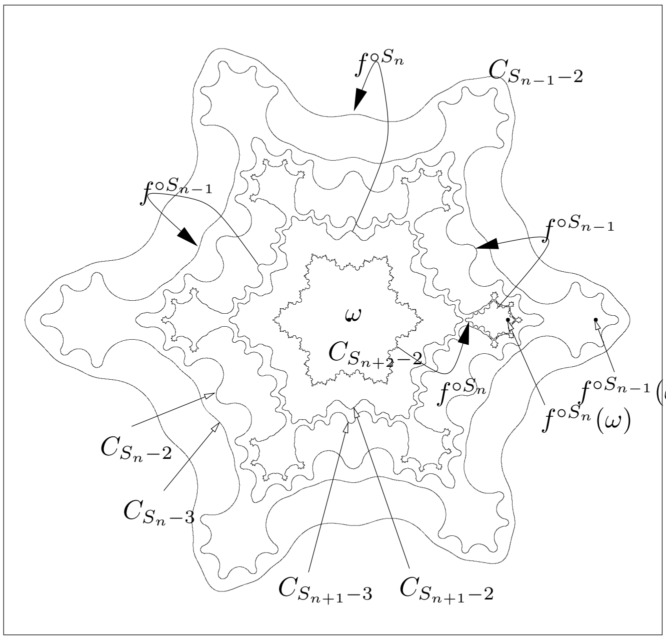


Figure 9. The position of $f^{\circ S_n}(C_{S_{n+2}-2})$.

$S_{n+1}-2$ that contains $f^{\circ S_n}(\omega)$. By lemma 2, we have $f^{\circ S_n}(\omega) \in C_{S_{n+1}-3}$, so that $f^{\circ S_n}(C_{S_{n+2}-2}) \subset C_{S_{n+1}-3}$. Besides, the induction property at level n says that $C_{S_{n+1}-2}$ is a child of $C_{S_{n-1}-2}$ and the no central return condition implies that $f^{\circ S_n}(\omega) \notin C_{S_{n+1}-2}$. In particular, $f^{\circ S_n}(C_{S_{n+2}-2}) \subset C_{S_{n+1}-3} \setminus C_{S_{n+1}-2}$. □

The proof of the proposition is contained within the proof of lemma 3, since

$$f^{\circ S_n}(0) \in f^{\circ S_n}(C_{S_{n+2}-2}) \subset C_{S_{n+1}-3} \setminus C_{S_{n+1}-2}.$$

□

From now on, all Fibonacci maps we will consider will be real symmetric Fibonacci maps. In particular, the critical point is 0. We now come back to renormalization of Fibonacci maps.

Definition 15. *We can define a renormalization operator $\mathcal{R}_{(2,1)}$, on the set of Fibonacci maps, by*

$$\mathcal{R}_{(2,1)}(f) = \alpha^{-1} \circ g \circ \alpha,$$

where g is the canonical renormalization of f, and where $\alpha(z) = g(0)\cdot z = f^{\circ 2}(0)\cdot z$.

The map $\mathcal{R}_{(2,1)}(f)$ is normalized so that its critical value is 1. This map can be projected to the space of germs of Fibonacci maps. We will keep the letter $\mathcal{R}_{(2,1)}$ to denote the projection.

The results obtained by van Strien and Nowicki in [SN] can be reformulated in the following way.

Theorem 3. (see [SN], Theorem 7.1)

- *Assume $f_1 : U_1^0 \cup U_1^1 \to V_1$ and $f_2 : U_2^0 \cup U_2^1 \to V_2$ are two real symmetric Fibonacci maps of even degree $\ell \geq 4$, such that $f_1(0)\cdot f_1^{\circ 2}(0)$ and $f_2(0)\cdot f_2^{\circ 2}(0)$ have the same sign. Then, there exists a quasi-conformal homeomorphism $\psi : V_1 \to V_2$ which conjugate the Fibonacci maps f_1 and f_2. Besides, there exists a constant $\varepsilon > 0$ such that ψ is $C^{1+\varepsilon}$ at 0.*
- *For every even integer $\ell \geq 4$, the renormalization operator $\mathcal{R}_{(2,1)}$ has a unique cycle $\{[f_1],[f_2]\}$ of order 2, where $[f_1]$ and $[f_2]$ are two germs of real symmetric Fibonacci maps of degree ℓ. If $f : U^0 \cup U^1 \to V$ is a real symmetric Fibonacci map of even degree $\ell \geq 4$, then the sequence $\mathcal{R}_{(2,1)}^{\circ n}([f])$ converges to the cycle $\{[f_1],[f_2]\}$.*

Proof. The proof given by van Strien and Nowicky consists in first obtaining real a-priori bounds which show that the closure of the post-critical set is a Cantor set with bounded geometry. If $f_1(0)\cdot f_1^{\circ 2}(0)$ and $f_2(0)\cdot f_2^{\circ 2}(0)$ have the same sign, then the ordering of the critical orbit on the real axis is the same, so that two maps are quasi-symmetrically conjugate along their critical orbit. Then, applying a pullback argument due to Sullivan and described in [dMvS], van Strien and Nowicky show that this quasi-symmetric conjugacy can be promoted to a quasi-conformal conjugacy between the two Fibonacci maps.

To prove that the conjugacy is $C^{1+\varepsilon}$ at 0, they use renormalization techniques, and the theory of towers introduced by McMullen in [McM2]. They show the convergence of renormalizations to a cycle of order 2 at the same time. □

Let us now improve this result in the following way.

Theorem A. *For every even integer $\ell \geq 4$, let $f_i : U_i^0 \cup U_i^1 \to V_i$, $i = 1,2$, be the two real symmetric Fibonacci maps of degree ℓ, normalized so that $\omega_i = 0$ and*

$f_i(\omega_i) = 1$, *and satisfying* $\mathcal{R}_{(2,1)}([f_1]) = [f_2]$ *and* $\mathcal{R}_{(2,1)}([f_2]) = [f_1]$. *Then, there exists a neighborhood* U *of* 0 *and a neighborhood* U' *of* 1 *such that*

- $f_1|_{U\cap U_1^0} = f_2|_{U\cap U_2^0}$, *and*
- $f_1|_{U'\cap U_1^1} = -f_2|_{U'\cap U_2^1}$.

PROOF. To prove this theorem, let us define a flipping operator which to a Fibonacci map f associates the new map $\tilde{f} : U^0 \cup U^1 \to V$ defined by $\tilde{f}|U^0 = f$ and $\tilde{f}|U^1 = -f$.

Lemma 4. *If* $f : U^0 \cup U^1 \to V$ *is a real symmetric Fibonacci map, then* $\tilde{f} : U^0 \cup U^1 \to V$ *is still a Fibonacci map.*

PROOF. The fact $\tilde{f}$ is still an L-map is not obvious. We must first show that the orbit of $\omega = 0$ stays in $U^0 \cup U^1$, and then show that the genealogical properties are satisfied. We define ω_n to be the n-th iterate of the critical point: $\omega_n = f^{\circ n}(\omega)$. Let us show by induction that for any $n \geq 0$, $\tilde{\omega}_n = \tilde{f}^{\circ n}(\omega) = \pm\omega_n$ if $\omega_n \in U^0$ and $\tilde{\omega}_n = \omega_n$ if $\omega_n \in U^1$. Indeed, we have already mentioned in the previous proof that the difference between two consecutive returns of the critical orbit in U^0 is at most 2. This implies that if $\omega_{n-1} \in U^1$, then $\omega_n \in U^0$. Hence, assuming the induction property holds for $n-1$, we see that

- if $\omega_{n-1} \in U^0$, then $\tilde{\omega}_n = f(\pm\omega_{n-1}) = \omega_n \in U^0 \cup U^1$, and
- if $\omega_{n-1} \in U^1$, then $\tilde{\omega}_n = -f(\omega_{n-1}) = -\omega_n \in U^0$.

This shows that the induction property is true for n. Besides, the critical pieces of the puzzle of $\tilde{f}$ are exactly the pieces C_n and $\tilde{f}^{\circ k}(C_n)$ is either $f^{\circ k}(C_n)$, or $-f^{\circ k}(C_n)$. Thus, the genealogy of f and of $\tilde{f}$ are exactly the same. Hence, $\tilde{f}$ is a Fibonacci map. □

Lemma 5. *If* $f : U^0 \cup U^1 \to V$ *is a real symmetric Fibonacci map, then the flipping operator and the renormalization operator commute:*

$$\widetilde{\mathcal{R}}_{(2,1)}([f]) = \mathcal{R}_{(2,1)}([\tilde{f}]).$$

PROOF. We have seen that the central branch of the canonical renormalization of f is $f_1 \circ f_0$, where $f_0 = f|_{U^0}$ and $f_1 = f|_{U^1}$. The other branch is f_1. Hence, $\widetilde{\mathcal{R}}_{(2,1)}([f])$ has central branch $1/\alpha f_1 \circ f_0(\alpha z)$ and outer branch $-1/\alpha f_1(\alpha z)$, where $\alpha = f^{\circ 2}(0)$.

On the other hand, the canonical renormalization of $\tilde{f}$ has central branch $-f_1 \circ f_0$ and outer branch $-f_1$, and $\tilde{f}^{\circ 2}(0) = -f^{\circ 2}(0) = -\alpha$. Hence, $\mathcal{R}_{(2,1)}([\tilde{f}])$ has central branch $-1/\alpha\big[-f_1 \circ f_0(-\alpha z)\big] = 1/\alpha f_1 \circ f_0(\alpha z)$ and outer branch $-1/\alpha\big[-f_1(-\alpha z)\big] = -\big[-1/\alpha f_1(\alpha z)\big]$. □

We now claim that if $\{[f_1],[f_2]\}$ is the cycle of order 2 of real symmetric germs of Fibonacci maps of degree ℓ which is invariant by $\mathcal{R}_{(2,1)}$, then we necessarily have $[f_2] = [\tilde{f}_1]$. Indeed, observe that $\{[\tilde{f}_1],[\tilde{f}_2]\}$ is a cycle of order 2 of real symmetric Fibonacci maps of degree ℓ which is invariant by $\mathcal{R}_{(2,1)}$. By uniqueness of such a cycle (see theorem 3), we have $\{[f_1],[f_2]\} = \{[\tilde{f}_1],[\tilde{f}_2]\}$, and since $[f_1] \neq [\tilde{f}_1]$, we have $[f_2] = [\tilde{f}_1]$.

This shows that f_1 and f_2 coincide in a neighborhood of $K(f_1)\cap U_1^0 = K(f_2)\cap U_2^0$, and f_1 and $-f_2$ coincide in a neighborhood of $K(f_1)\cap U_1^1 = K(f_2)\cap U_2^0$, which concludes the proof of theorem A. □

We will now show that if $f : U^0 \cup U^1 \to V$ is a real symmetric Fibonacci map such that $\mathcal{R}_{(2,1)}([f]) = [\tilde{f}]$, then the restriction $f_0 = f|_{U^0}$ of f to U^0 satisfies the following system of equations, that we will call the Cvitanović-Fibonacci equation:

$$\begin{cases} f(z) = -\dfrac{1}{\alpha^2} f(\alpha f(\alpha z)), \quad 0 < \alpha < 1, \\ f(0) = 1 \text{ and} \\ f(z) = F(z^\ell), \text{ with } F'(0) \neq 0 \text{ and } \ell \geq 4 \text{ even.} \end{cases}$$

Indeed, let $f_1 = f|U^1$. Then, writing down $\mathcal{R}_{(2,1)}([f]) = [\tilde{f}]$ gives:

$$\begin{cases} f_0(z) &= -1/\alpha^2 f_0(\alpha f_0(\alpha z)), \text{ and} \\ f_1(z) &= -1/\alpha f_0(\alpha z), \end{cases}$$

Indeed, we have seen that the central branch of $\mathcal{R}_{(2,1)}([f])$ is $1/\alpha f_1 \circ f_0(\alpha z)$ and the outer branch is $1/\alpha f_1(\alpha z)$. As $\mathcal{R}_{(2,1)}([f]) = [\tilde{f}]$, we get

$$f_0(z) = \frac{1}{\alpha} f_1 \circ f_0(\alpha z)$$

and

$$f_1(z) = -\frac{1}{\alpha} f_0(\alpha z)$$

for all z in the Julia set $K(f)$, which enables us to conclude, replacing f_1 in the first equation.

4.3. **Solutions of the Cvitanović-Fibonacci equation.** We will now study the geometry of the solutions of the Cvitanović-Fibonacci equation. In particular, we will study the domain of analyticity of such solutions.

Definition 16. *Let f and g be two holomorphic functions defined on open connected domains of $\mathbb{C}$: U_f and U_g. We say g is an analytic extension of f if g is equal to f on some non-empty open set. Moreover, if all such analytic extension are restriction of a single map*

$$\hat{f} : \widehat{W} \to \mathbb{C},$$

we will say that $\hat{f}$ is the maximal analytic extension of f .

Theorem B. *For every even integer $\ell \geq 4$, let f be the solution of the Cvitanović-Fibonacci equation in degree ℓ, and set $f_\alpha(z) = f(\alpha z)$ and $f_{\alpha^2}(z) = f(\alpha^2 z)$.*

Then, there exist domains $W_\alpha \subset \mathbb{C}$ and $W_{\alpha^2} \subset \mathbb{C}$ containing 0 such that $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ are polynomial-like mappings of degree ℓ. Besides, $f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ has an attracting cycle of order 2 and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ has an attracting fixed point. In particular, the Julia set $J(f_\alpha)$ is quasi-conformally homeomorphic to the Julia set $J(z \mapsto z^\ell - 1)$ and the Julia set $J(f_{\alpha^2})$ is a quasi-circle.

Finally, the domain of analyticity of f is the quasi-disk $\widehat{W}$ bounded by the quasi-circle $\alpha J(f_{\alpha^2})$.

Remark. The functions f_α and f_{α^2} are not conjugated to f. As we will see, their dynamical behavior is really different.

Proof. The map f_α is a solution of the universal equation we introduced in the preceding chapter for $\lambda = \alpha^2$ and $\nu = 1/2$:

$$f_\alpha(z) = -\frac{1}{\lambda} f_\alpha(f_\alpha(\lambda^2 z)).$$

We will use the notations of the preceding chapter, indexing all the sets with the letter α. We have seen that there exists a bounded domain $W_\alpha \subset \mathbb{C}$ such that $f_\alpha : W_\alpha \to \mathbb{C}_{\alpha^2} = \mathbb{C}(]-1/\alpha^2, 1/\alpha^4[)$ is a ramified covering.

Let us now show that the interpretation of the universal equation in terms of a linearization equation enables us to prove the following lemma.

Lemma 6. (see figure 1) *The maps*

$$f_\alpha : W_\alpha \to \mathbb{C}_{\alpha^2} \quad \text{and} \quad f_{\alpha^2} : W_{\alpha^2} = W_\alpha/\alpha \to \mathbb{C}_{\alpha^2}$$

are DH-maps with attracting cycles.

Proof. By definition of W_α, the mapping f_α is a ramified covering map from W_α to $\mathbb{C}_{\alpha^2}$ with only one critical point of degree ℓ in 0. Hence, the map f_{α^2} is a ramified covering from W_{α^2} to $\mathbb{C}_{\alpha^2}$. Moreover, as the degree is even,

$$\overline{W}_\alpha \cap \mathbb{R} = [-x_0/\alpha, x_0/\alpha] \subset]-1/\alpha, 1/\alpha[,$$

with $f_\alpha(x_0) = 0$. The inclusion is given by the relative position of points obtained from figure 4. Hence, f_α and f_{α^2} are both polynomial-like maps. We will show they are DH-maps, i.e., that the critical point does not escape. To do this, it is enough to show that f_α has an attracting cycle of period 2, and f_{α^2} has an attracting fixed point. Those cycles must attract the critical point.

We can rewrite the Cvitanović-Fibonacci equation, using the functions f_α and f_{α^2} in two different ways:

$$\begin{cases} f_\alpha(z) = -\dfrac{1}{\alpha^2} f_\alpha(f_{\alpha^2}(z)) \\ f_{\alpha^2}(z) = -\dfrac{1}{\alpha^2} f_\alpha(f_\alpha(-\alpha^2 z)) \end{cases}$$

The first equation tells us that f_α linearizes f_{α^2} in a neighborhood of x_0:

$$\begin{array}{ccc} x_0 & \xrightarrow{f_{\alpha^2}} & x_0 \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ 0 & \xrightarrow[-\alpha^2 z]{} & 0. \end{array}$$

Hence f_{α^2} has an attracting fixed point, x_0, of multiplier α^2.

The second equation tells us $f_\alpha \circ f_\alpha$ is conjugated by $z \to -\alpha^2 z$ to f_{α^2}. Hence, $f_\alpha \circ f_\alpha$ has an attracting fixed point: $-\alpha^2 x_0 < 0$. Since $f_\alpha(-\alpha^2 x_0) > 0$, f_α has a cycle of period 2: $\{-\alpha^2 x_0, f_\alpha(-\alpha^2 x_0)\}$. □

We have just shown that f_{α^2} is a DH-map with an attracting fixed point. It follows immediately that the attracting basin $\widehat{W}_\alpha$ of f_{α^2} is a quasi-disk. Moreover, as f_α is the linearizer of f_{α^2}, it has a maximal analytic extension $\hat{f}_\alpha : \widehat{W}_\alpha \to \mathbb{C}$. To conclude the proof, just remind that $f(z) = f_\alpha(z/\alpha)$, and define $\widehat{W} = \alpha \widehat{W}_\alpha$. □

Remark. The basin of attraction of the DH-map $f_{\alpha^2} : W_{\alpha^2} \to \mathbb{C}_{\alpha^2}$ being $\widehat{W}/\alpha$, and f_{α^2} being conjugate to $f_\alpha^{\circ 2}$ by the scaling map $z \mapsto -\alpha^2 z$, we see that the immediate basin of the DH-map $f_\alpha : W_\alpha \to \mathbb{C}_{\alpha^2}$ has two connected components, the one containing 0 being $\alpha\widehat{W}$, the other one being $f_\alpha(\alpha\widehat{W})$.

Before going further, let us observe some consequences of this theorem. The following lemma will be useful in the construction of a particular Fibonacci map (see theorem C).

Lemma 7. *The mapping* $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ *is a DH-map representing the same germ as* $f_\alpha : \widehat{W}_\alpha \to \mathbb{C}_{\alpha^2}$.

Proof. Since $\widehat{W}_\alpha$ is by definition the basin of attraction of the DH-map $f_{\alpha^2} : W_{\alpha^2} \to \mathbb{C}_{\alpha^2}$, we see that the map $f_{\alpha^2} : \widehat{W}_\alpha \to \widehat{W}_\alpha$ is a ramified covering, ramified only at 0. Hence, the same property holds for the map $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$.

Let us now show that $\widehat{W}$ is relatively compact in $\widehat{W}_\alpha$. This is an immediate consequence of the following inclusion of sets (see figure 10): $W \subset \widehat{W} \Subset W_\alpha = W/\alpha$. The first inclusion is obvious because f is analytic on W, which has to be inside

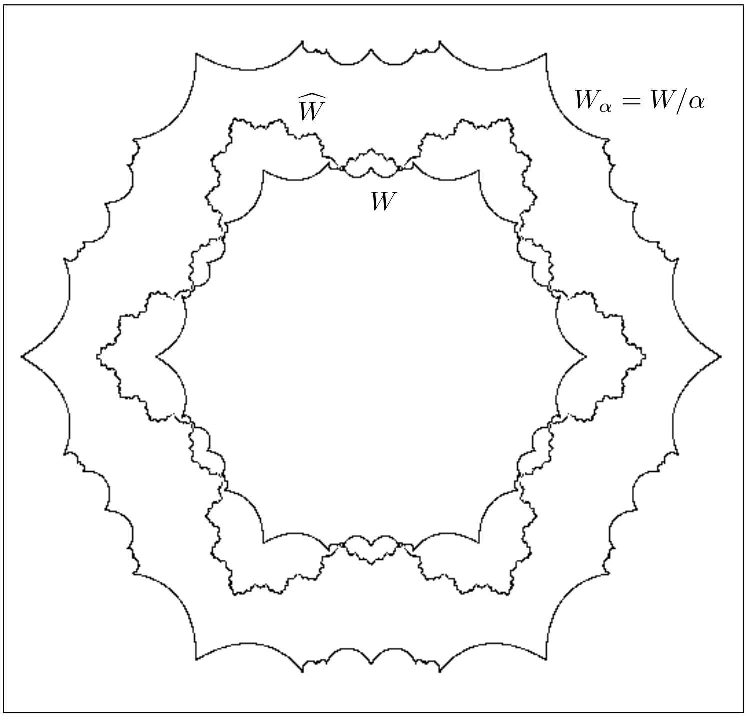


Figure 10. We have $W \subset \widehat{W} \Subset W_\alpha = W/\alpha$.

the domain of analyticity $\widehat{W}$ of f. To show the second inclusion, note that the map $f_{\alpha^2} : W_{\alpha^2} \to \mathbb{C}_{\alpha^2}$ is a DH-map. Hence its filled-in Julia set, i.e., the closure of $\widehat{W}_\alpha$, is contained in W_{α^2}.

This concludes the proof of the lemma, since any polynomial-like restriction of a DH-map represent the same germ. □

Definition 17. *For* $k \geq -1$*, we define* $\mathcal{D}_k$ *and* $\mathcal{D}'_k$ *to be the sets*

$$\mathcal{D}_k = \left\{ z \in \widehat{W}_\alpha \mid f_{\alpha^2}^{\circ(k+1)}(z) \in \widehat{W} \right\} \quad \text{and} \quad \mathcal{D}'_k = \left\{ z \in \widehat{W} \mid f_{\alpha}^{\circ(k+1)}(z) \in \widehat{W} \right\}.$$

We can now prove the following geometric result, which will be used in the proofs of theorems C and D.

Lemma 8. *For any $k \geq -1$, the sets $\mathcal{D}_k$ and $\mathcal{D}'_k$ are quasi-disks. We have the inclusions*

$$\mathcal{D}_k \Subset \mathcal{D}_{k+1} \Subset \widehat{W}_\alpha \quad \text{and} \quad K(f_\alpha) \subset \mathcal{D}'_{k+1} \Subset \mathcal{D}'_k.$$

Besides, the mappings $f_{\alpha^2} : \mathcal{D}_{k+1} \to \mathcal{D}_k$ and $f_\alpha : \mathcal{D}'_{k+1} \to \mathcal{D}'_k$ are ramified coverings, ramified only at 0.

Furthermore, the closure of the set $\mathcal{D}_k$ – resp. $\mathcal{D}'_k$ – converges, for the Hausdorff topology on compact subsets of $\mathbb{P}^1$, to the filled-in Julia set $K(f_{\alpha^2})$ – resp. $K(f_\alpha)$ – as k tends to infinity.

Proof. The statement for the sets $\mathcal{D}'_k$ is an immediate consequence of the fact that $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is a DH-map whose Julia set is $K(f_\alpha)$, combined with the fact that $\mathcal{D}_k = [f_\alpha|_{\widehat{W}}]^{-(k+1)}(\widehat{W})$.

Proving the statement for the sets $\mathcal{D}_k$ is of the same order of difficulty. Indeed, the set $\widehat{W} = \mathcal{D}_{-1}$ is contained in the basin of attraction of f_{α^2}, i.e. $\widehat{W}_\alpha$. Besides,

$$\mathcal{D}_k = [f_{\alpha^2}|_{\widehat{W}_\alpha}]^{-(k+1)}(\widehat{W}).$$

Finally, $f_{\alpha^2}(\widehat{W}) = f_\alpha(\alpha\widehat{W})$, and since the immediate basin of the DH-map $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is $\alpha\widehat{W} \cup f_\alpha(\alpha\widehat{W})$, we see that $f_{\alpha^2}(\widehat{W}) \subset \widehat{W}$. □

4.4. **Construction of a Fibonacci map.** We will now prove that any solution of the Cvitanović-Fibonacci equation gives rise to a cycle of order 2 of Fibonacci maps which is invariant under renormalization.

Theorem C. (see figure 11) *Given any solution f of the Cvitanović-Fibonacci equation, there exists a Fibonacci map $\phi : \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ such that ϕ and f coincide on U^0 and such that $\mathcal{R}^{\circ 2}_{(2,1)}([\phi]) = [\phi]$.*

Proof. We first need to define the map ϕ. We define $\mathcal{V}$ to be the domain $\widehat{W}$. The domain $\mathcal{U}_0$ is defined to be equal to $\alpha^2\mathcal{D}_0$ (defined in definition 17). Combining lemmas 1 and 7, we see that the mapping $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is a DH-map having a cycle of period 2: $\{-\alpha^2 x_0, f_\alpha(-\alpha^2 x_0)\}$. The immediate basin of this cycle has two connected components. The one containing 0 is $\alpha\widehat{W}$. We define $\mathcal{U}^1$ to be the other connected component. It is clear that $\mathcal{U}^0$, $\mathcal{U}^1$ and $\mathcal{V}$ are quasi-disks.

We claim that the map $\phi : \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$, defined by

$$\begin{cases} \phi|_{\mathcal{U}^0}(z) = f(z) \\ \phi|_{\mathcal{U}^1}(z) = \dfrac{1}{\alpha} f(\alpha z), \end{cases}$$

is a Fibonacci map, and that the germ $[\phi]$ is a fixed point of $\widetilde{\mathcal{R}}_{(2,1)}$.

Step 1. Let us first show that $\overline{\mathcal{U}^0}$ and $\overline{\mathcal{U}^1}$ are disjoint and contained in $\mathcal{V}$. Lemma 8 says that $\mathcal{D}_0 = \mathcal{U}^0/\alpha^2$ is compactly contained in $\widehat{W}_\alpha = \widehat{W}/\alpha$. Hence, the closure of $\mathcal{U}^0$ is contained in $\alpha\widehat{W}$. Since the immediate basin of the DH-map $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is $\alpha\widehat{W} \sqcup \mathcal{U}^1$, we see that $\overline{\mathcal{U}^0}$ and $\overline{\mathcal{U}^1}$ are disjoint and contained in $\mathcal{V} = \widehat{W}$ (see figure 12).

Step 2. The map $\phi : \mathcal{U}^0 \to \mathcal{V}$ is a ramified covering, ramified only at 0 and the map $\phi : \mathcal{U}^1 \to \mathcal{V}$ is an isomorphism. Indeed, lemma 8 states that $f_{\alpha^2} : \mathcal{D}_0 \to \mathcal{D}_{-1}$ is a ramified covering ramified only at 0. Using $f_{\alpha^2}(z) = f(\alpha^2 z)$, $\mathcal{D}_0 = \mathcal{U}_0/\alpha^2$ and $\mathcal{D}_{-1} = \mathcal{V}$, the first statement is proved. Using again that the immediate basin of the DH-map $f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is $\alpha\widehat{W} \sqcup \mathcal{U}^1$, and that the critical point of this

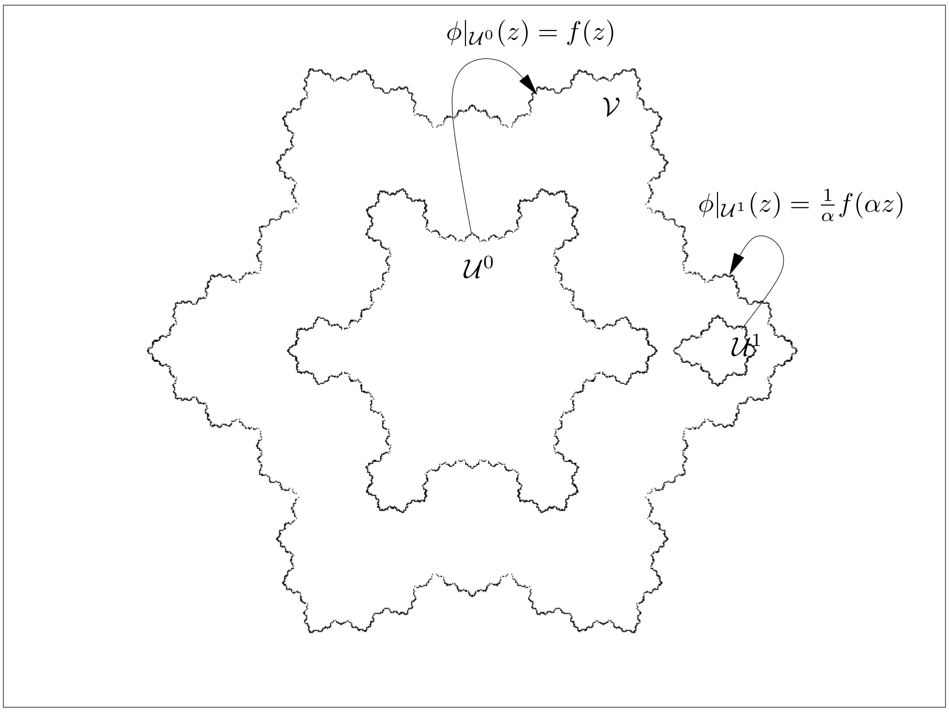


FIGURE 11. The Fibonacci map ϕ.

DH-map is contained in $\alpha\widehat{W}$, we see that $f_\alpha : \mathcal{U}^1 \to \alpha\widehat{W}$ is an isomorphism, which immediately implies the second statement, since $\phi|_{\mathcal{U}^1} = f_\alpha/\alpha$.

Step 3. We now need to show that the critical orbit does not escape from $\mathcal{U}^0 \cup \mathcal{U}^1$. For this purpose, we will prove a result that will be used again later, in the study of the shape of puzzle pieces. In the following lemma, $[x]$ denotes the integer part of x.

Lemma 9. *For any $k \geq 0$, $\phi^{\circ S_{2k}}$ is well defined on $\alpha^{2k+2}\mathcal{D}_k$ and $\phi^{\circ S_{2k+1}}$ is well defined on $\alpha^{2k+2}\widehat{W}$. Besides,*

$$\phi^{\circ S_{2k}} : \alpha^{2k+2}\mathcal{D}_k \to \alpha^{2k}\mathcal{D}_{k-1} \quad \text{and} \quad \phi^{\circ S_{2k+1}} : \alpha^{2k+2}\widehat{W} \to \alpha^{2k}\widehat{W}$$

are ramified coverings ramified only at 0. In both cases, the iterate $\phi^{\circ S_n}$ coincides with the map

$$z \mapsto (-1)^{[(n+1)/2]}\alpha^n f\left(\frac{z}{\alpha^n}\right).$$

PROOF. We first claim that this property holds for $k = 0$. Indeed, it says that

- ϕ is well defined on $\alpha^2\mathcal{D}_0 = \mathcal{U}^0$, $\phi : \alpha^2\mathcal{D}_0 = \mathcal{U}^0 \to \mathcal{D}_{-1} = \mathcal{V}$ is a ramified covering ramified only at 0 which coincides with f;
- $\phi^{\circ 2}$ is well defined on $\alpha^2\widehat{W}$, and $\phi^{\circ 2} : \alpha^2\widehat{W} \to \widehat{W}$ is a ramified covering ramified only at 0 which coincides with $-\alpha f(z/\alpha)$.

The first point is obvious (by definition of ϕ). The second point requires some argumentation. To prove this, remember that the immediate basin of the DH-map

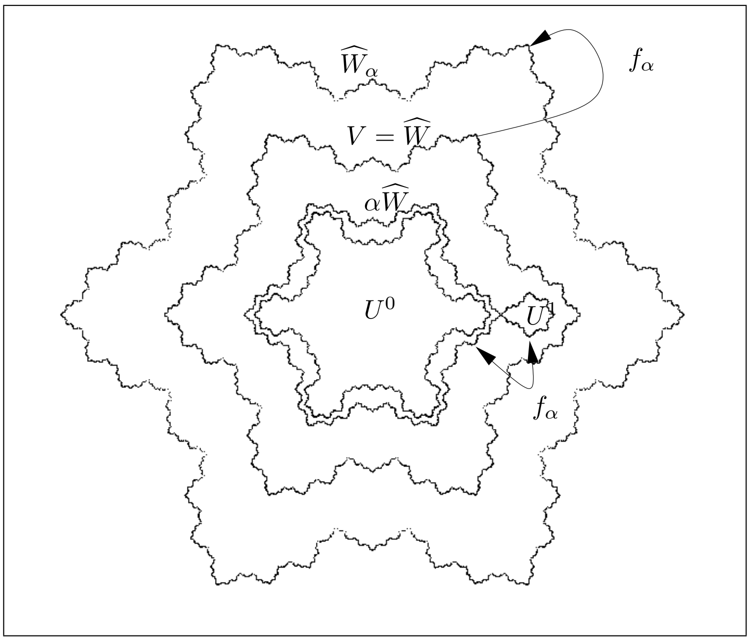


FIGURE 12. $\mathcal{U}^0$ and $\mathcal{U}^1$ are quasi-disks relatively compact in $\mathcal{V}$ and their closures are disjoint.

$f_\alpha : \widehat{W} \to \widehat{W}_\alpha$ is $\alpha\widehat{W} \sqcup \mathcal{U}^1$. Hence, $f_\alpha : \alpha\widehat{W} \to \mathcal{U}^1$ is a ramified covering, ramified only at 0, so that

$$\phi(\alpha^2\widehat{W}) = f(\alpha^2\widehat{W}) = f_\alpha(\alpha\widehat{W}) = \mathcal{U}^1,$$

and $\phi : \alpha^2\widehat{W} \to \mathcal{U}^1$ is a ramified covering, ramified only at 0. Post-composing with the isomorphism $\phi : \mathcal{U}^1 \to \mathcal{V} = \widehat{W}$, we are done. Indeed, we see that $\phi^{\circ 2} : \alpha^2\widehat{W} \to \widehat{W}$ is a ramified covering ramified only at 0 which coincides with

$$\frac{1}{\alpha} f(\alpha f(z)) = -\alpha\Big[-\frac{1}{\alpha^2} f\Big(\alpha f\big(\alpha \frac{z}{\alpha}\big)\Big)\Big] = -\alpha f(z/\alpha).$$

Let us now assume that the property holds for some integer $k-1 \geq 0$. We need to show that it holds for k. Remember that $\mathcal{D}_k \subset \widehat{W}/\alpha$. Hence, using the induction property at level $k-1$, we see that $\phi^{\circ S_{2k-1}}$ is well defined on $\alpha^{2k+2}\mathcal{D}_k \subset \alpha^{2k-1}\widehat{W} \subset \alpha^{2k}\widehat{W}$. Besides,

$$\phi^{\circ S_{2k-1}}(\alpha^{2k+2}\mathcal{D}_k) = \alpha^{2k-1} f\left(\frac{\alpha^{2k+2}\mathcal{D}_k}{\alpha^{2k-1}}\right) = \alpha^{2k-1} f(\alpha^3\mathcal{D}_k).$$

Since $\alpha^3\mathcal{D}_k \subset \alpha\widehat{W}$, we have

$$\phi^{\circ S_{2k-1}}(\alpha^{2k+2}\mathcal{D}_k) \subset \alpha^{2k-1} f_\alpha(\alpha\widehat{W}) = \alpha^{2k-1}\mathcal{U}^1 \subset \alpha^{2k-1}\widehat{W}.$$

Hence, the induction property at level $k-1$ shows that $\phi^{\circ S_{2k-2}}$ is well defined on $\phi^{\circ S_{2k-1}}(\alpha^{2k+2}\mathcal{D}_k)$ and coincides with $(-1)^{[(2k-1)/2]}\alpha^{2k-1} f(z/\alpha^{2k-1})$. This shows that $\phi^{\circ S_{2k}} = \phi^{\circ S_{2k-2}} \circ \phi^{\circ S_{2k-1}}$ is well defined on $\alpha^{2k+2}\mathcal{D}_k$ and coincides with

$$(-1)^{[(2k-1)/2]}\alpha^{2k-2} f\Big[\frac{1}{\alpha^{2k-2}}\Big((-1)^{[(2k)/2]}\alpha^{2k-1} f\big(\frac{z}{\alpha^{2k-1}}\big)\Big)\Big]$$

$$= (-1)^{[(2k-1)/2]}\alpha^{2k}\Big[\frac{1}{\alpha^2}f\Big(\alpha f\big(\alpha\frac{z}{\alpha^{2k}}\big)\Big)\Big] = (-1)^{[(2k+1)/2]}\alpha^{2k} f\left(\frac{z}{\alpha^{2k}}\right).$$

In particular, we can write the following diagram:

$$\begin{array}{ccc} \alpha^{2k+2}\mathcal{D}_k & \xrightarrow{\phi^{S_{2k}}} & \alpha^{2k}\mathcal{D}_{k-1} \\ {\scriptstyle z\mapsto z/\alpha^{2k+2}}\downarrow & & \uparrow{\scriptstyle z\mapsto(-1)^{[(2k+1)/2]}\alpha^{2k}z} \\ \mathcal{D}_k & \xrightarrow{f_{\alpha^2}} & \mathcal{D}_{k-1}. \end{array}$$

Since $f_{\alpha^2} : \mathcal{D}_k \to \mathcal{D}_{k-1}$ is a ramified covering, ramified only at 0, we see that $\phi^{S_{2k}} : \alpha^{2k+2}\mathcal{D}_k \to \alpha^{2k}\mathcal{D}_{k-1}$ is a ramified covering, ramified only at 0.

The same analysis can be performed for $\phi^{S_{2k+1}} = \phi^{S_{2k-1}} \circ \phi^{S_{2k}}$ and yields the diagram:

$$\begin{array}{ccc} \alpha^{2k+2}\widehat{W} & \xrightarrow{\phi^{S_{2k}}} & \alpha^{2k}\widehat{W} \\ {\scriptstyle z\mapsto z/\alpha^{2k+3}}\downarrow & & \uparrow{\scriptstyle z\mapsto(-1)^{[(2k+2)/2]}\alpha^{2k+1}z} \\ \widehat{W}_\alpha & \xrightarrow{f_{\alpha^2}} & \widehat{W}_\alpha \end{array}$$

We leave this analysis to the reader. □

Since for any $k \geq 0$, ϕ^{S_n} is well defined in a neighborhood of 0, we have proved that the critical orbit never escapes from $\mathcal{U}^0 \cup \mathcal{U}^1$. In particular, we have proved that the mapping $\phi : \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ is an L-map.

Step 4. Let us now prove that this L-map is a Fibonacci map. We will show that $\phi^{\circ S_n}(0)$ belongs to the critical piece of depth $S_{n+1} - 3$ but not to the critical piece of depth $S_{n+1} - 2$ (this corresponds to Branner Hubbard definition of Fibonacci maps).

Lemma 10. *For any $k \geq 0$, the critical piece of depth $S_{2k} - 2$ is equal to $\alpha^{2k}\widehat{W}$, and the critical piece of depth $S_{2k+1} - 2$ is equal to $\alpha^{2k+2}\mathcal{D}_k$. For any $n \geq 0$, the critical piece of depth $S_{n+1} - 3$ is equal to $\alpha^n\mathcal{D}'_{n-1}$.*

Proof. Once more, this is proved by induction. Let us first do it for pieces of depth $S_{2k} - 2$ and $S_{2k+1} - 2$. For $k = 0$ observe that the critical piece of depth $S_0 - 2 = -1$ is $\widehat{W}$ and the critical piece of depth $S_1 - 2 = 0$ is $\alpha^2\mathcal{D}_0$. This is clear by definition. Then, lemma 9 shows that we have the following ramified coverings:

$$\alpha^{2k+2}\widehat{W} \xrightarrow{\phi^{\circ S_{2k+1}}} \alpha^{2k}\widehat{W}$$

$$\alpha^{2k+4}\mathcal{D}_{k+1} \xrightarrow{\phi^{\circ S_{2k+2}}} \alpha^{2k+2}\mathcal{D}_k$$

which shows that if the critical piece of depth $S_{2k} - 2$ (resp. $S_{2k+1} - 2$) is $\alpha^{2k}\widehat{W}$ (resp. $\alpha^{2k+2}\mathcal{D}_k$), then $\alpha^{2k+2}\widehat{W}$ (resp. $\alpha^{2(k+1)+2}\mathcal{D}_{k+1}$) is the critical piece of depth $S_{2k} - 2 + S_{2k+1} = S_{2k+2} - 2$ (resp. $S_{2k+1} - 2 + S_{2k+2} = S_{2k+3} - 2$).

To prove the result for the pieces of depth $S_n - 3$, observe that for $n = 0$ the critical piece of depth $S_1 - 3 = -1$ is $\mathcal{D}'_1 = \widehat{W}$. Then, assume that the critical piece of depth $S_{n+1} - 3$ is equal to $\alpha^n\mathcal{D}'_{n-1}$. If n is odd, i.e., $n = 2k + 1$, then we can argue that $\alpha^{n+1}\mathcal{D}'_n \subset \alpha^{n+1}\widehat{W} = \alpha^{2k+2}\widehat{W}$. Hence, lemma 9 shows that ϕ^{S_n} is well defined on $\alpha^{n+1}\mathcal{D}'_n$ and we have:

$$\begin{array}{ccc}
\alpha^{n+1}\mathcal{D}'_n \subset \alpha^{n+1}\widehat{W} & \xrightarrow{\phi^{\circ S_n}} & \alpha^n\mathcal{D}'_{n-1} \subset \alpha^{n-1}\widehat{W} \\
{\scriptstyle z\mapsto z/\alpha^{n+1}}\downarrow & & \uparrow{\scriptstyle z\mapsto(-1)^{[(n+1)/2]}\alpha^n z} \\
\mathcal{D}'_n \subset \widehat{W} & \xrightarrow[f_\alpha]{} & \mathcal{D}'_{n-1} \subset \widehat{W}_\alpha.
\end{array}$$

This shows that if n is odd, then $\phi^{\circ S_n}: \alpha^{n+1}\mathcal{D}'_n \to \alpha^n\mathcal{D}'_{n-1}$ is a ramified covering, so that $\alpha^{n+1}\mathcal{D}'_n$ is the critical piece of depth $S_{n+1}-3+S_n = S_{n+2}-3$. To treat the case when n is even, we will use the same argument. However, this requires first proving that $\alpha^{2k+1}\mathcal{D}'_{2k} \subset \alpha^{2k+2}\mathcal{D}_k$, i.e., we need to prove the following sub-lemma.

Sub-lemma. *For any $k \geq 0$, $\mathcal{D}'_{2k} \subset \alpha\mathcal{D}_k$.*

Proof of the Sub-lemma. This property holds for $k=0$. Indeed, we already mentioned that $\mathcal{D}'_0 = \alpha\mathcal{D}_0$. Let us now assume that it holds for some integer $k \geq 0$. Since, by definition, $\mathcal{D}'_{2k+2} \subset \widehat{W}$, we have $\mathcal{D}'_{2k+2}/\alpha \subset \widehat{W}_\alpha$ and

$$f_{\alpha^2}(\mathcal{D}'_{2k+2}/\alpha) = f_\alpha(\mathcal{D}'_{2k+2}) = \mathcal{D}'_{2k+1}.$$

Besides, lemma 8 shows that $\mathcal{D}'_{2k+1} \Subset \mathcal{D}'_{2k}$. Hence

$$f_{\alpha^2}(\mathcal{D}'_{2k+2}/\alpha) \subset \mathcal{D}_k,$$

and by definition of $\mathcal{D}_{k+1}$, we see that $\mathcal{D}'_{2k+2}/\alpha \subset \mathcal{D}_{k+1}$. □

We now return to the proof of our lemma. When $n = 2k$ is even, lemma 9 shows that ϕ^{S_n} is well defined on $\alpha^{n+1}\mathcal{D}'_n$ and we have

$$\begin{array}{ccc}
\alpha^{n+1}\mathcal{D}'_n \subset \alpha^{2k+2}\mathcal{D}_k & \xrightarrow{\phi^{\circ S_n}} & \alpha^n\mathcal{D}'_{n-1} \subset \alpha^{2k}\mathcal{D}_{k-1} \\
{\scriptstyle z\mapsto z/\alpha^{n+1}}\downarrow & & \uparrow{\scriptstyle z\mapsto(-1)^{[(n+1)/2]}\alpha^n z} \\
\mathcal{D}'_n \subset \alpha\mathcal{D}_k & \xrightarrow[f_\alpha]{} & \mathcal{D}'_{n-1} \subset \mathcal{D}_{k-1}.
\end{array}$$

This concludes the proof of the lemma □

To conclude the proof of step 4, observe that lemma 9 implies that

$$\phi^{\circ S_n}(0) = (-1)^{[(n+1)/2]}\alpha^n f(0) = (-1)^{[(n+1)/2]}\alpha^n.$$

Hence, to prove that $\phi^{\circ S_n}(0)$ belongs to the critical piece of depth $S_{n+1}-3$ but not to the critical piece of depth $S_{n+1}-2$, we need to show that for all $n \geq 0$, $\alpha^n \in \alpha^n\mathcal{D}'_{n-1}$, for all even integer $n \geq 0$, $\alpha^n \notin \alpha^{n+2}\mathcal{D}_{n/2}$ and for all odd integer $n \geq 0$, $\alpha^n \notin \alpha^{n+1}\widehat{W}$. Since for all even integer $n \geq 0$, we have $\alpha^{n+2}\mathcal{D}_{n/2} \subset \alpha^{n+1}\widehat{W}$, it is sufficient to prove that $1 \in \mathcal{D}'_{n-1} \setminus \alpha\widehat{W}$. This is clear since 1 is contained $\mathcal{U}^1$. Indeed, $\mathcal{U}^1 \subset K(f_\alpha) \subset \mathcal{D}'_{n-1}$, and $\mathcal{U}^1 \cap \alpha\widehat{W} = \emptyset$ since they are the two connected components of the immediate basin of attraction of the DH-map $f_\alpha: \widehat{W} \to \widehat{W}_\alpha$.

Step 5. We finally show that $\widetilde{\mathcal{R}}_{(2,1)}([\phi]) = [\phi]$. The canonical renormalization of $\phi: \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ is $\psi: \mathcal{U}^0_1 \cup \mathcal{U}^1_1 \to \mathcal{V}_1$, where

- $\mathcal{V}_1$ is equal to $\mathcal{U}^0$,
- $\mathcal{U}^0_1$ is the critical piece of depth $2 = S_3 - 3$, i.e., $\alpha^2\mathcal{D}'_1$,
- $\mathcal{U}^1_1$ is the piece of depth 1 that contains $\phi^{\circ 2}(\omega) = -\alpha$,
- $\psi|_{\mathcal{U}^0_1}(z) = \phi^{\circ 2}(z) = -\alpha f(z/\alpha)$, and

- $\psi|_{\mathcal{U}_1^1}(z) = \phi(z) = f(z)$.

Hence the central branch of $\widetilde{\mathcal{R}}_{(2,1)}(\phi)$ coïncides with $-1/\alpha\big[-\alpha f(-\alpha z/\alpha)\big] = f$ and the outer branch of $\widetilde{\mathcal{R}}_{(2,1)}([\phi])$ coincides with $1/\alpha\big[f(-\alpha z)\big] = 1/\alpha f(\alpha z)$. Since the two branches of the Fibonacci maps $\widetilde{\mathcal{R}}_{(2,1)}(\phi)$ and ϕ coincide, and since the range of $\widetilde{\mathcal{R}}_{(2,1)}([\phi])$ is contained in the range of ϕ – indeed, $\mathcal{V}_1/\alpha = \alpha\mathcal{D}_0 \subset \widehat{W} = \mathcal{V}$ – we see that $\widetilde{\mathcal{R}}_{(2,1)}(\phi)$ is a restriction of ϕ. Thus, we only need to prove that there are no points of $K(\phi)$ in $\mathcal{V}\setminus\mathcal{V}_1$. This immediately follows from the following inclusion of sets:

$$K(\phi) \subset \mathcal{U}^0 \cup \mathcal{U}^1 \subset K(f_\alpha) \subset \mathcal{D}'_0 = \alpha\mathcal{D}_0 = \mathcal{V}_1.$$

□

Corollary. *For every even integer $\ell \geq 4$, there exists a unique $\alpha \in]0,1[$ such that the Cvitanović-Fibonacci equation has a solution, and this solution is itself unique.*

Proof. For every even integer $\ell \geq 4$, we have seen that there exists a real number $\alpha \in]0,1[$ such that the Cvitanović-Fibonacci equation has a solution. This solution was obtained as a limit of renormalizations. Now, if there was another possible value of α or another solution, then the renormalization operator $\mathcal{R}_{(2,1)}$ would have at least two cycles of order 2 of real Fibonacci maps of degree ℓ. But this contradicts theorem 3. □

4.5. **Shape of the Fibonacci puzzle pieces.** To conclude our study of Fibonacci maps, we will show the following geometric result describing the shape of some Fibonacci critical puzzle pieces.

Theorem D. *Let*

- *$\ell \geq 4$ be an even integer,*
- *$F: U^0 \cup U^1 \to V$ be a real symmetric Fibonacci map of degree ℓ normalized so that the critical point is $\omega = 0$,*
- *C_k be the critical puzzle piece of depth k,*
- *f be the solution of the Cvitanović-Fibonacci equation in degree ℓ,*
- *$\alpha \in]0,1[$ be the constant defined by the Cvitanović-Fibonacci equation, and*
- *$f_\alpha : W_\alpha \to f_\alpha(W_\alpha)$ and $f_{\alpha^2} : W_{\alpha^2} \to f_{\alpha^2}(W_{\alpha^2})$ be the polynomial-like mappings defined in Theorem B.*

Then, there exists a constant $\lambda \neq 0$ such that

- *the sequence of rescaled puzzle pieces $\dfrac{\lambda}{\alpha^{k+1}}\overline{C_{S_k-2}}$ converges for the Hausdorff topology to the filled-in Julia set $K(f_{\alpha^2})$, and*
- *the sequence of rescaled puzzle pieces $\dfrac{\lambda}{\alpha^{k-1}}\overline{C_{S_k-3}}$ converges to the filled-in Julia set $K(f_\alpha)$.*

Let us mention that a similar result has already been proved by Lyubich [Ly2] for Fibonacci maps in degree 2. He proved the convergence in shape of some puzzle pieces to the Julia set of $z \to z^2 - 1$.

Proof. We will first show that the theorem holds for the Fibonacci map $\phi : \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ constructed in theorem C. Since the critical pieces of $\tilde{\phi} : \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ are the same as the critical pieces of ϕ, the statement also holds for $\tilde{\phi}$. We will then use theorem 3 which says that there exists a quasiconformal homeomorphism $\psi: V \to \mathcal{V}$ which conjugates F to either ϕ or $\tilde{\phi}$. The fact that this conjugacy is $C^{1+\varepsilon}$ at 0 will enable us conclude the proof of theorem D.

The statement for the Fibonacci map $\phi: \mathcal{U}^0 \cup \mathcal{U}^1 \to \mathcal{V}$ constructed in theorem C is an immediate consequence of lemmas 8 and 10. Indeed, let us call $\mathcal{C}_n$ the critical puzzle piece of depth n for this Fibonacci map. Then, lemma 10 says that

$$\mathcal{C}_{S_{2k}-2} = \alpha^{2k+1}\widehat{W}_\alpha, \quad \mathcal{C}_{S_{2k+1}-2} = \alpha^{2k+2}\mathcal{D}_k \quad \text{and} \quad \mathcal{C}_{S_n-3} = \alpha^{n-1}\mathcal{D}'_{n-2}$$

Then, $\overline{\mathcal{C}_{S_{2k}-2}}/\alpha^{2k+1}$ is constantly equal to the closure of $\widehat{W}_\alpha$ which is precisely $K(f_{\alpha^2})$, and lemma 8 shows that $\overline{\mathcal{C}_{S_{2k+1}-2}}/\alpha^{2k+2} = \overline{\mathcal{D}_k}$ converges to $K(f_{\alpha^2})$ whereas $\overline{\mathcal{C}_{S_n-3}}/\alpha^{n-1} = \overline{\mathcal{D}'_{n-2}}$ converges to $K(f_\alpha)$.

Theorem 3 says that there exists a quasiconformal homeomorphism $\psi: V \to \mathcal{V}$ which conjugates F to either ϕ or $\tilde{\phi}$. Without loss of generality, we may assume that F is conjugate to ϕ. Since ψ is $C^{1+\varepsilon}$ at 0, we have

$$\psi(z) = \lambda z + \mathcal{O}\left(|z|^{1+\varepsilon}\right),$$

for some real number $\lambda \neq 0$. Observe that

$$\frac{1}{\alpha^k}\psi(\alpha^k z) = \lambda z + \frac{\mathcal{O}\left(|\alpha^k z|^{1+\varepsilon}\right)}{\alpha^k}$$

converges uniformly on every compact subset of $\mathbb{C}$, as k tends to infinity, to the scaling map $z \mapsto \lambda z$. Besides, it sends C_{S_k-2}/α^{k+1} – resp. C_{S_k-3}/α^{k-1} – to $\mathcal{C}_{S_k-2}/\alpha^{k+1}$ – resp. $\mathcal{C}_{S_k-3}/\alpha^{k-1}$. In particular, the compact sets $\lambda\overline{C_{S_k-2}}/\alpha^{k+1}$ and $\overline{\mathcal{C}_{S_k-2}}/\alpha^{k+1}$ have the same limit. This is also true for the compact sets $\lambda\overline{C_{S_k-3}}/\alpha^{k-1}$ and $\overline{\mathcal{C}_{S_k-3}}/\alpha^{k-1}$. Hence, the theorem is proved. □

Aknowledgements.

I would like to thank B. Branner, A. Douady, H. Epstein, M. Flexor, J.H. Hubbard, C.T. McMullen and S. van Strien for their disponibility in answering any of my questions and in solving many of my problems.

References

[BH] B. Branner & J.H. Hubbard, *The Iteration of Cubic Polynomials, Part II: Patterns and Parapatterns*, Acta Mathematatica, Vol 169, (1992), 229-325.

[BKNS] H. Bruin, G. Keller, T. Nowicki & S. van Strien - *Absorbing Cantor sets in dynamical systems : Fibonacci maps*, Stonybrook IMS preprint 1994/2. To appear in Annals of Mathematics.

[B1] X. Buff, *Points fixes de renormalisation*, Thesis, Orsay (1996).

[B2] X. Buff, *Ensembles de Julia de mesure strictement positive*, Séminaire Bourbaki n° 820, 1996.

[B3] X. Buff, *Geometry of the Feigenbaum fixed point of renormalization*, Conformal Geometry and Dynamics, (1999).

[CE] P. Collet & J.P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, Boston, (1980).

[CEL] P. Collet, J.P. Eckmann & O.E. Lanford III, *Universal properties of maps on an interval*, Comm. Math. Physics **76**, 211-254, (1980).

[CT] P. Coullet & C. Tresser, *Itération d'endomorphismes et groupe de renormalisation*, J. Phys. Colloque C 539, C5-25 (1978).

[Cv] P. Cvitanović, *Universality in Chaos*, Adam Hilger Ltd., Bristol, (1984).

[DH] A. Douady & J.H. Hubbard, *On the Dynamics of Polynomial-like Mappings*, Ann. Scient., Ec. Norm. Sup. 4^e series, 18 (1985) 287-343.

[EW] J.P. Eckmann & P. Wittwer, *A complete proof of the Feigenbaum conjectures*, J. Stat. Phys., vol 46 (1987), 455-477.

[E1] H. Epstein, *Fixed points of composition operators II*, Nonlinearity, vol. 2, (1989), 305-310.

[E2] H. Epstein, *Fixed points of the period-doubling operator*, Lecture notes, Lausanne (1992).

[F1] M.J. Feigenbaum, *Quantitative universality for a class of non-linear transformations*, J. Stat. Phys., vol. 19, (1978), 25-52.

[F2] M.J. Feigenbaum, *The universal metric properties of non-linear transformations*, J. Stat. Phys., vol. 21, (1979), 669-706.

[H] J.H. Hubbard, *Local Connectivity of Julia Sets and Bifurcation Loci : Three Theorems of J.C. Yoccoz, Topological methods in Modern Mathematics*, Publish or Perrish, (1993), 467-511.

[HK] F. Hofbauer & G. Keller - *Some remarks on recent results about S-unimodal maps*, Ann. Institut Henri Poincaré **Vol. 53** (1990), 413-425.

[La] O.E. Lanford III *A computer assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math.Soc., vol. 6, (1982), 427-434.

[Ly1] M. Lyubich, *On the Lebesgue measure of the Julia set of a quadratic polynomial*, Preprint, Institute for Mathematical Sciences, SUNY at Stony Brook, (1991).

[Ly2] M. Lyubich, *Teichmüller Space of Fibonacci Maps*, Preprint, Institute for Mathematical Sciences, SUNY at Stony Brook, (1993).

[Ly3] M. Lyubich, *Dynamics of quadratic polynomials, I-II*, Acta Mathematica, **vol** 178 (1997), 185-297.

[Ly4] M. Lyubich, *Renormalization Ideas in Conformal Dynamics*, Cambridge Seminar "Current Developments in Math.", May 1995. International Press, 1995. Cambridge, MA, 155-184.

[Ly5] M. Lyubich, *Feigenbaum-Coullet-Tresser universality and Milnor's Hairiness Conjecture*, Annals of Math, **149** (1999), 319-420.

[Ly6] M. Lyubich, *Regular and stochastic dynamics in the real quadratic family*, Proc. Natl. Acad. Sci. USA **95**, 14025-14027, (1998).

[LM] M. Lyubich & J. Milnor, The unimodal Fibonacci map, Journal of the A.M.S., **vol** 6, 425-457, (1993).

[McM1] C.T. McMullen, *Complex Dynamics and Renormalization*, Annals of Math Studies, vol. 135, Princeton University Press, (1994).

[McM2] C.T. McMullen, *Renormalization and 3-manifolds which fibers over the circle*, Annals of Math Studies vol. 142, Princeton University Press, (1996).

[dMvS] W. de Melo & S. van Strien, *One dimensional dynamics*, Springer Verlag, (1993).

[Mi] J. Milnor, *Local Connectivity of Julia Sets : Expository Lectures*, Stony Brook IMS Preprint 1992/11, to appear in "The Mandelbrot set, Theme and Variations" edit: Tan Lei, LMS Lecture Note Series, Cambr. U. Press.

[MT] J. Milnor & W. Thurston *On iterated maps of the interval*, Dynamical Systems, Proc. U. Md., 1986-87, (J. Alexander, ed.), Lecture Notes in Math., vol. 1342, Springer, New York, 465-563, (1988).

[SN] S. van Strien & T. Nowicki, *Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps*, Preprint, Institute for Mathematical Sciences, SUNY at Stony Brook, (1994).

[S] D. Sullivan, *Bounds, quadratic differentials and renormalization conjectures*, In F. Browder, editor, Mathematics into Twenty-first Century: 1988 Centennial Symposium, August 8-12, pages 417-466. Amer. Math. Soc., 1992.

[T] C. Tresser, *Aspects of renormalization in dynamical systems theory*, in Chaos and Complexity, 11-19, Editions Frontiers, 1995.

[VSK] E.B. Vul, Ya. G. Sinai & K.M. Khanin, *Feigenbaum universality and the thermodynamical formalism*, Russian Math. Surveys **39**, 1-40, (1984).

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