Dynamics of quadratic polynomials

Journée à la mémoire de Jean-Christophe Yoccoz
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Polynomial dynamics

- $P : \mathbb{C} \to \mathbb{C}$ is a complex polynomial.
- $(z_n)_{n\geq 0}$ is defined by $z_0 \in \mathbb{C}$ and $z_{n+1} = P(z_n)$.

Question

What is the long term behavior of the sequence (z_n) ?

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Question

What is the long term behavior of the sequence (z_n) ?

- $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is compact.
- $\omega(z_0)$ is the set of limit values of the sequence (z_n).

Question

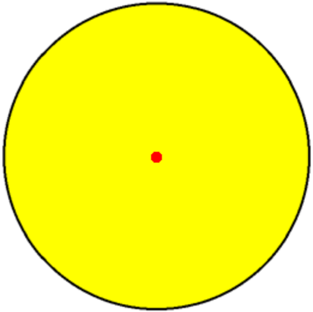
How does $\omega(z_0)$ depend on z_0 ?

The filled-in Julia set

- Study initiated by Fatou (1919) and Julia (1920).
- Contributions by Cremer (1932-36), Siegel (1942), Brolin, Guckenheimer, Jakobson. . . .
- Very active in the last 20 years of the last century with the help of computer pictures.

Definition

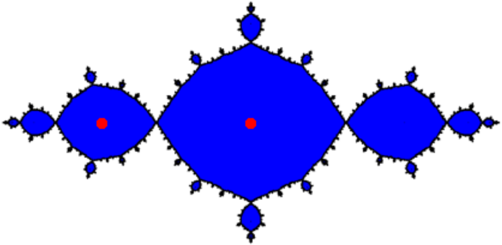
The filled-in Julia set of P is the set $K(P)$ of points $z_0 \in \mathbb{C}$ for which the sequence (z_n) is bounded.



$$P(z) = z^2$$

- If $|z_0| < 1$, $\omega(z_0) = \{0\}$ and if $|z_0| > 1$, $\omega(z_0) = \{\infty\}$.

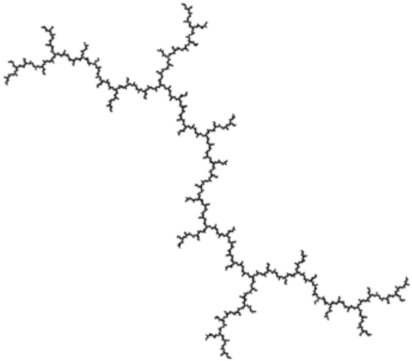
Examples



$$P(z) = z^2 - 1$$

- If z_0 is blue, $\omega(z_0) = \{0, -1\}$.

Examples



$$P(z) = z^2 + i$$

- $K(z^2 + \mathrm{i})$ is a dendrite. For almost every z_0, $\omega(z_0) = \{\infty\}$.

Examples

$$P(z) = z^2 + 1/2$$

- $K(z^2 + 1/2)$ is a Cantor set.

Fatou and Julia sets

Definition

The Julia set $J(P)$ is the topological boundary of $K(P)$.

Definition

The Fatou set is the complement of $J(P)$.

- Stability : the map $z \mapsto \omega(z)$ is continuous on the Fatou set.

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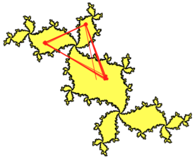
The Fatou set is the complement of $J(P)$.

- Stability : the map $z \mapsto \omega(z)$ is continuous on the Fatou set.
- Chaos :
  - $J(P)$ is the closure of the set of repelling periodic points ;
  - the map $z \mapsto \omega(z)$ is discontinuous on the Julia set.

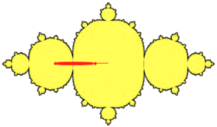
Theorem (Fatou)

The Fatou set of a polynomial consists of attracting basins, parabolic basins and/or rotation domains.



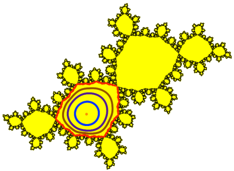
attracting basin

$\omega(z)$ is a cycle in the Fatou set



parabolic basin

$\omega(z)$ is a cycle in the Julia set



rotation domain

$\omega(z)$ is a topological circle

Existence of Siegel disks

- $Q_\theta(z) = \mathrm{e}^{2\pi i\theta} z + z^2$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- Q_θ fixes $z = 0$ with multiplier $\mathrm{e}^{2\pi i\theta}$.

Theorem (Siegel 1942)

If θ is Diophantine, then Q_θ has a Siegel disk.

- p_k/q_k are the approximants of θ given by the continued fraction algorithm.
- θ is a Brjuno number if $\displaystyle\sum_{k\geq 0} \frac{\log q_{k+1}}{q_k} < +\infty$.

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Theorem (Yoccoz 1988)

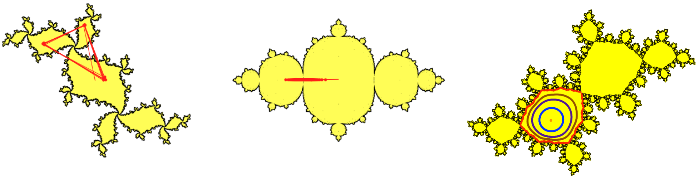
If Q_θ has a Siegel disk, then θ is a Brjuno number.

The critical orbit

- $P_c(z) = z^2 + c$.
- P_c has a critical point at $z = 0$.

Theorem (Fatou)

An attracting basin or a parabolic basin contains a critical point. The boundary of a Siegel disk is accumulated by a critical orbit.



Theorem (Fatou-Shishikura)

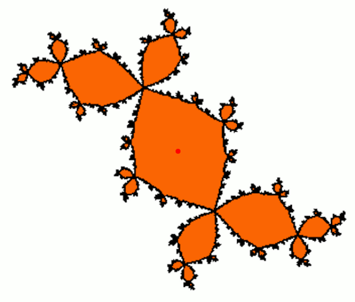
A quadratic polynomial has at most one non repelling cycle.

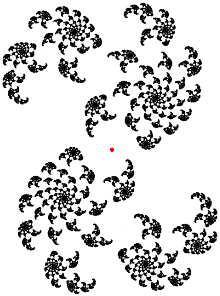
The critical orbit

- $P_c(z) = z^2 + c$.
- P_c has a critical point at $z = 0$.

Theorem (Fatou)

The filled-in Julia set $K(P_c)$ is connected if $0 \in K(P_c)$. Otherwise, it is a Cantor set.

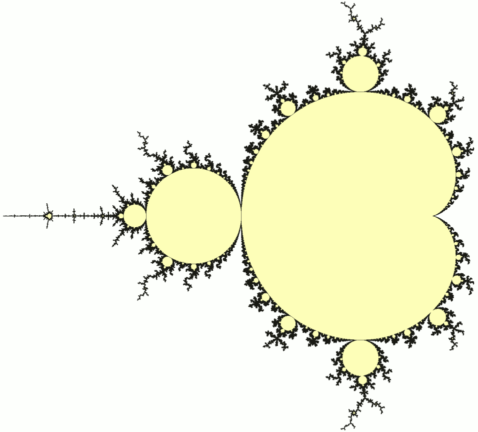




The Mandelbrot set

Definition

The Mandelbrot set is the set M of parameters $c \in \mathbb{C}$ for which $K(P_c)$ is connected.



The boundary of M

Conjecture

$\mathit{Area}(\partial M) = 0.$

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Theorem (Jakobson 1981)

$Length(\mathbb{R} \cap \partial M) > 0.$

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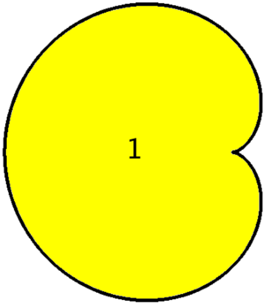
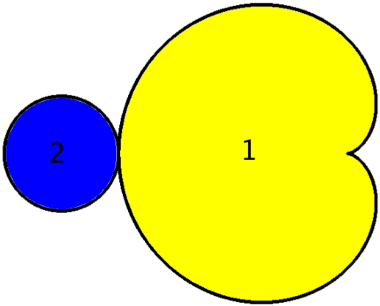
Theorem (Shishikura 1991)

$Hdim(\partial M) = 2$.

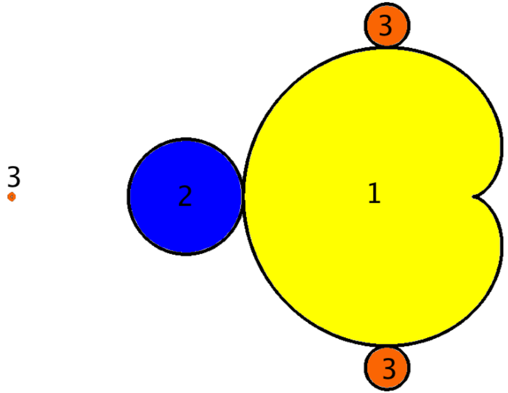
The interior of M : hyperbolic component of period 1



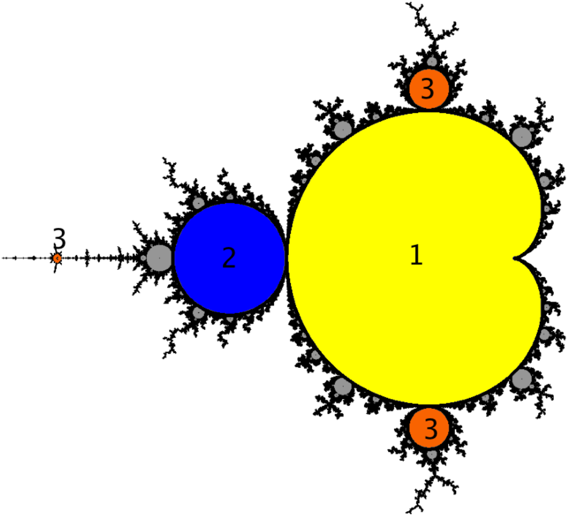
The interior of M : hyperbolic component of period 2



The interior of M : hyperbolic components of period 3



The interior of M : hyperbolic components



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Theorem (Douady-Hubbard 1982)

If the Mandelbrot set is locally connected, the conjecture is true.

Conjecture (MLC)

The Mandelbrot set is Locally Connected.

- Yoccoz has a major contribution towards the proof of MLC.

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Theorem (Graczyk-Swiatek, Lyubich 1997)

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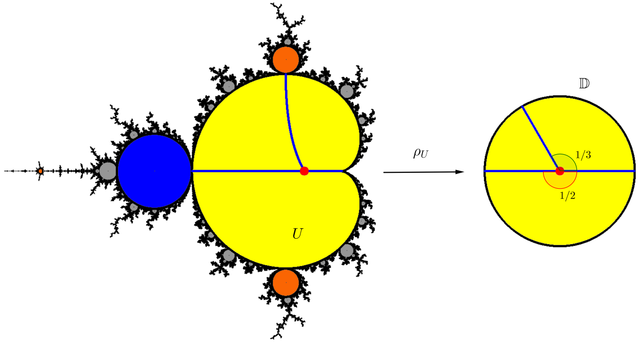
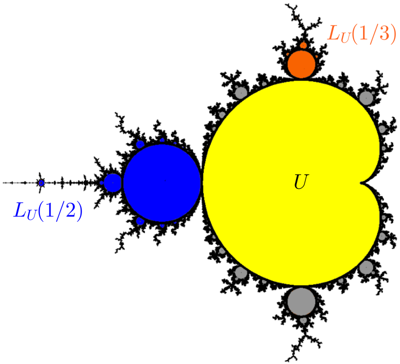


- U is a hyperbolic component.
- $\rho_U(c)$ is the multiplier of the attracting cycle of P_c
- $\rho_U : U \to \mathbb{D}$ is an isomorphism and extends as a homeomorphism $\rho_U : \overline{U} \to \overline{\mathbb{D}}$.
- $\gamma_U(\theta) := \rho_U^{-1}(\mathrm{e}^{2\pi\mathrm{i}\theta})$.

Limbs of the Mandelbrot set

Theorem (Yoccoz)

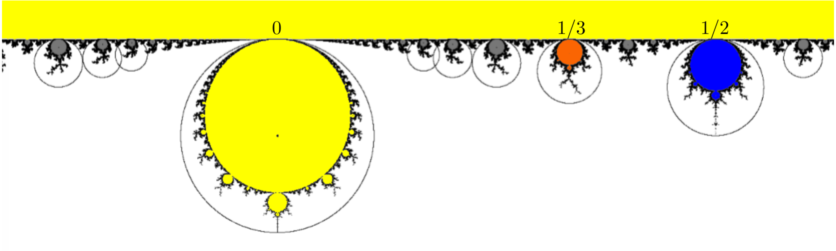
If U a hyperbolic component, then $M \setminus \overline{U} = \bigsqcup_{\theta \in \mathbb{Q}/\mathbb{Z}} L_U(\theta)$ *with* $L_U(\theta)$ *connected,* $\overline{L_U(\theta)} \cap \overline{U} = \gamma_U(\theta)$ *and* $\mathrm{diam}\big(L_U(\frac{p}{q})\big) \underset{q\to\infty}{\longrightarrow} 0$.



The Yoccoz Inequality

Theorem (Yoccoz)

If U has period $n \geq 1$ and $c \in L_U(p/q)$, then P_c has a cycle of period n with multiplier $\rho_U(c) = \mathrm{e}^{2\pi\mathrm{i}\tau}$ where τ belongs to the disk contained in the lower half-plane, tangent to $\mathbb{R}$ at p/q, with radius $n \log 2/(2\pi q)$.



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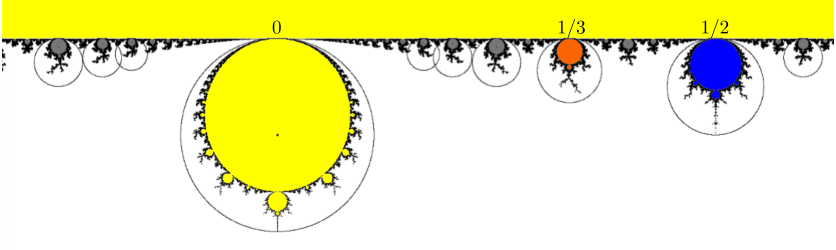
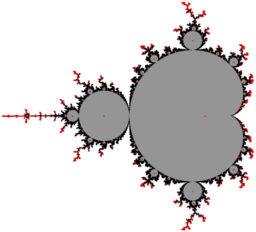


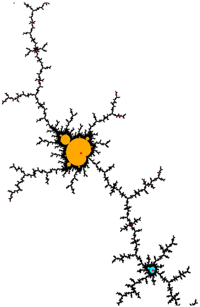
Corollary (Yoccoz)

If P_c has an indifferent cycle, then M is locally connected at c.

Theorem (Douady-Hubbard)

The Mandelbrot set contains copies of itself.





Theorem (Douady-Hubbard)

Every parameter c such that $P_c^{\circ n}(0) = 0$ *is the center of a copy of the Mandelbrot set.*

Renormalization

Definition

The polynomial P_c is renormalizable if c belongs to a copy of M.

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The polynomial P_c is infinitely renormalizable if c belongs to an infinite sequence of nested copies of M.

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Theorem (Yoccoz)

If P_c has no indifferent cycle and is not infinitely renormalizable,

- *$K(P_c)$ is locally connected and*
- *M is locally connected at c.*

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If P_c has no indifferent cycle and is not infinitely renormalizable,

- *$K(P_c)$ is locally connected and*
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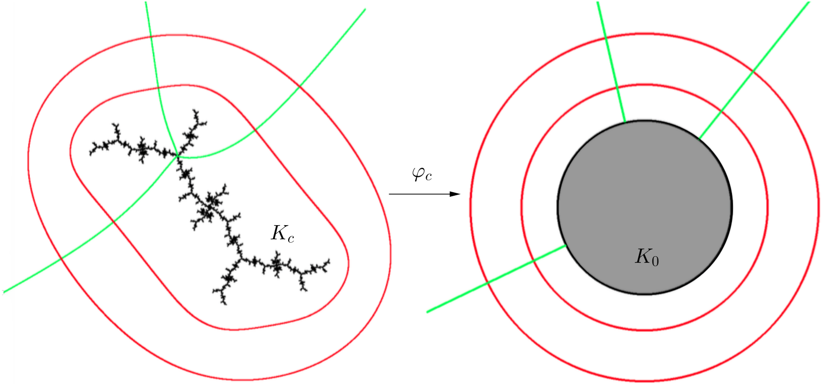
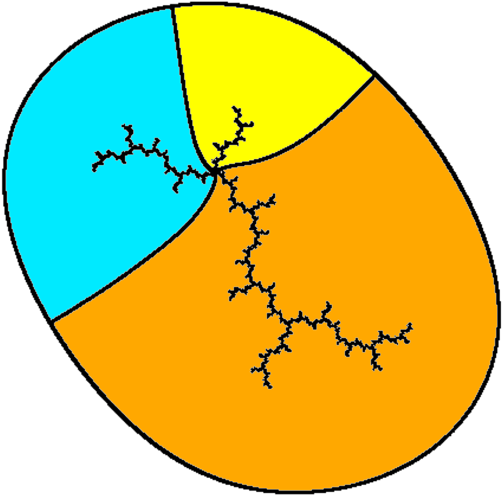
Corollary

If c is in the interior of M but not in an infinite sequence of nested copies of M, then P_c has an attracting cycle.

External rays and equipotentials



The Yoccoz Puzzles



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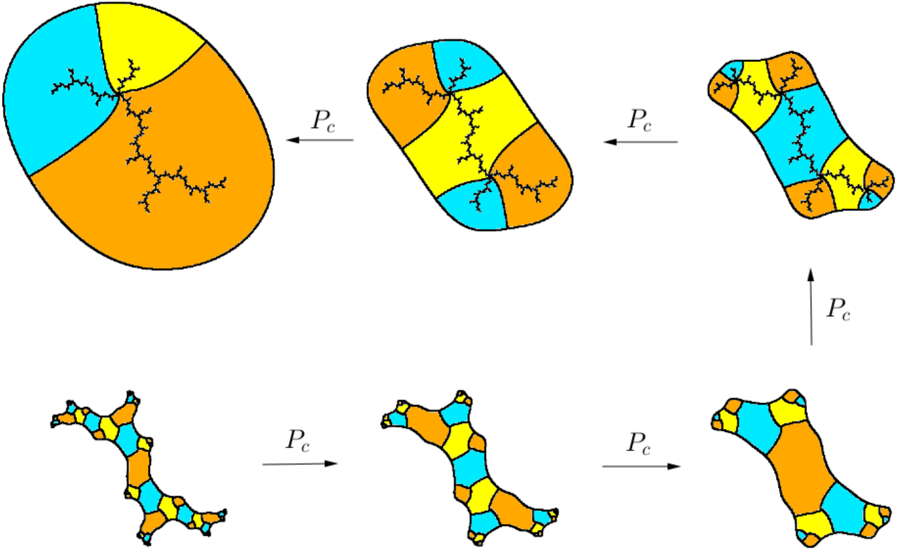

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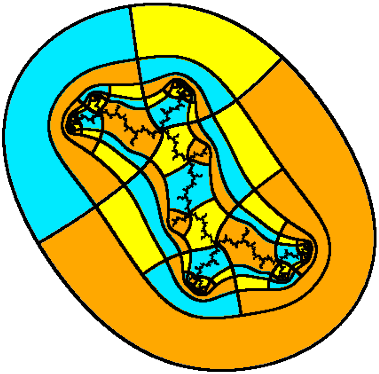
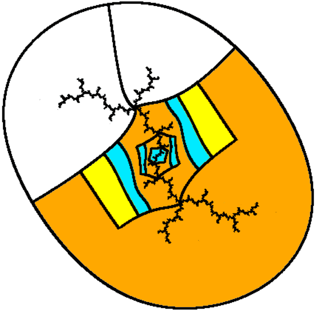
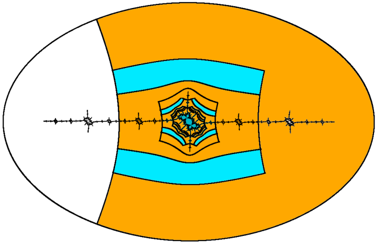


- The puzzle pieces are either disjoint or nested.
- The intersection of a puzzle piece with K_c is connected.

Proposition

- *If P_c is not renormalizable, any sequence of nested puzzle pieces shrinks down to a point.*
- *If P_c is renormalizable, the sequence of puzzle pieces containing* 0 *shrinks down to a copy of a filled-in Julia set.*



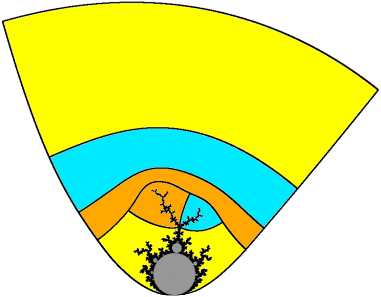


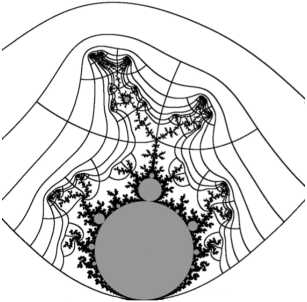
The Yoccoz Parapuzzles

- Yoccoz constructs parapuzzles covering the limbs of M.

Proposition

- *If P_c is not renormalizable, the sequence of parapuzzle pieces containing c shrinks down to $\{c\}$.*
- *If P_c is renormalizable, the sequence of puzzle pieces containing c shrinks down to the copy of M containing c.*





The Yoccoz Parapuzzles for the 1/2-limb