Sequences, differential equations and billiards

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Sequences

- X is a manifold.
- $F : X \to X$ is a map.
- $x_0 \in X$ and $x_{n+1} := F(x_n)$.

Question

What is the behavior of the sequence (x_n)?

- X is a compact manifold.
- $F : X \to X$ is a continuous map.
- $x_0 \in X$ and $x_{n+1} := F(x_n)$.

Question

What is the behavior of the sequence (x_n)?

Example

- $X = \mathbb{R}/\mathbb{Z}$ and $F : x \mapsto x + \theta$ is the rotation of angle θ.
- If θ is irrational, orbits are dense.
- If θ is rational, orbits are finite sets.

The ω-limit set

- The set $\omega(x_0)$ of accumulation points of the sequence (x_n) is an invariant compact set.

Question

Does the compact set $\omega(x)$ depend continuously on x?

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Does the compact set $\omega(x)$ depend continuously on x?

No in general.

Example

- $X = \mathbb{R} \cup \{\infty\}$ and $F : x \mapsto x^2$.
- If $|x| < 1$, $\omega(x) = \{0\}$.
- If $|x| > 1$, $\omega(x) = \{\infty\}$.
- If $|x| = 1$, $\omega(x) = \{1\}$.

Holomorphic dynamical systems

- X is a compact complex manifold.
- $F : X \to X$ is a holomorphic map.

Example

- $X = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- $F : X \to X$ is a polynomial of degree ≥ 2.

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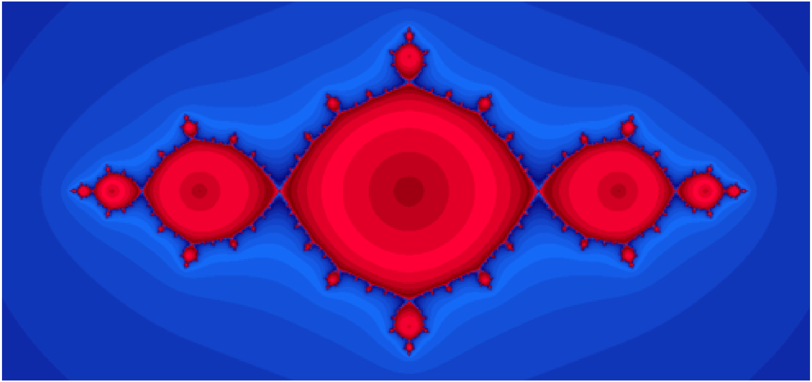
Definition

- The Fatou set $\mathcal{F}_F$ is the largest open set on which the family of iterates $(F^{\circ n})$ is equicontinuous.
- The Julia set $\mathcal{J}_F \subseteq X$ is the complement of $\mathcal{F}_F$.

Proposition

The map $x \mapsto \omega(x)$ is continuous on $\mathcal{F}_F$.

The Fatou set of $F : z \mapsto z^2 - 1$



Differential equations

- X is a manifold.
- $\vec{v}$ is a vector field on X.
- We study solutions of $\dot{\gamma} = \vec{v} \circ \gamma$ with $\dot{\gamma}(t) := \dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}$.
- $\gamma_x : I_x \to X$ is the maximal solution satisfying $\gamma_x(0) = x$.

Example

- $X = \mathbb{R}^n$.
- $\vec{v}(x) = x$ is the radial vector field.
- $\gamma_x(t) = \mathrm{e}^t \cdot x$.

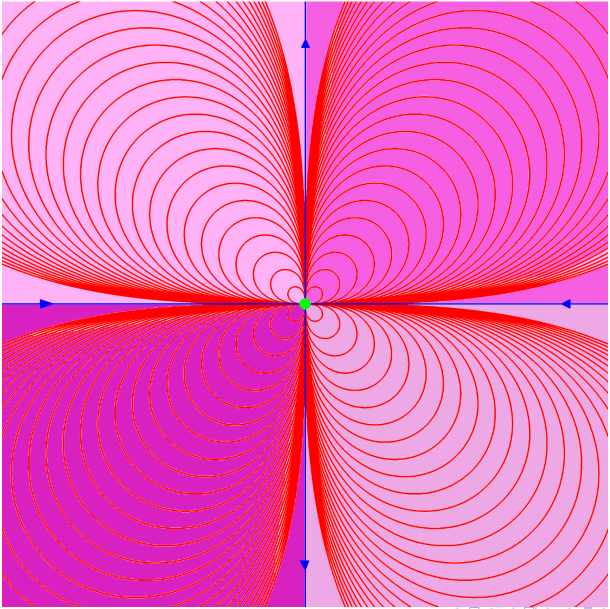
Fluid dynamics

- Consider the stationary flow of a fluid.
- In a Lagrangian approach, the solution $\gamma_x(t)$ is the time-parametrization of the trajectory of a particle which passes at the point x at time $t = 0$.
- In an Eulerian approach, the vector field $\vec{v}$ is the velocity field, i.e., the vector $\vec{v}(x)$ is the speed of a particle as it passes at the point x:

$$\vec{v}(x) = \dot{\gamma}_x(0).$$

- The fluid may be incompressible, irrotational, ...

Trajectories for $\vec{v}(z) = -z^3$ in $\mathbb{C}$



In order to draw trajectories of a differential equation, one may use an iterative scheme.

Definition

If $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$, the Euler method with step $\varepsilon \in (0, +\infty)$ for the vector field $\vec{v}$ is the map $F_\varepsilon : X \to X$ defined by

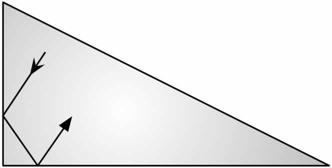
$$F_\varepsilon(x) := x + \varepsilon \vec{v}(x).$$

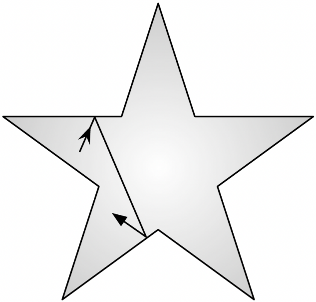
Proposition

If $n = \mathcal{O}(1/\varepsilon)$, then $F_\varepsilon^{\circ j}(x)$ remains close to $\gamma_x(j)$ for $j \in [0, n]$.

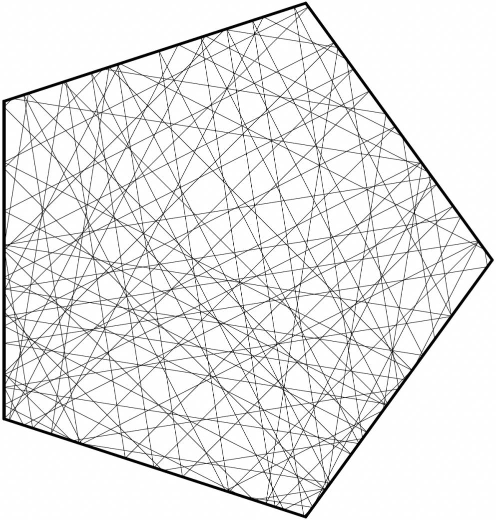
Billiards

- A *billiard table* is a connected polygon in $\mathbb{R}^2$.
- A billiard trajectory is a straight-line path which begins at some point in the interior of the table, and bounces off the edges with angle of reflection equal to the angle of incidence.





A billiard trajectory on the regular pentagon



Triangular billiards

Proposition

In a triangular billiard, all orbits are dense or periodic.

Proposition

In an acute triangle there is always a periodic trajectory.

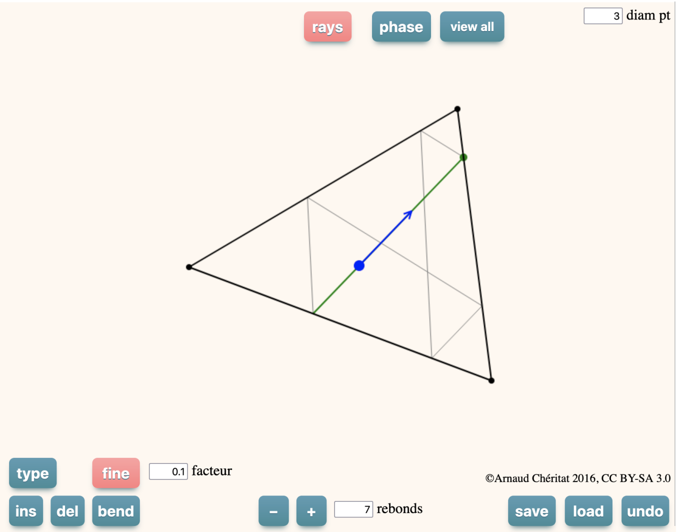
Question

In a triangular billiard, does there always exist a periodic trajectory?

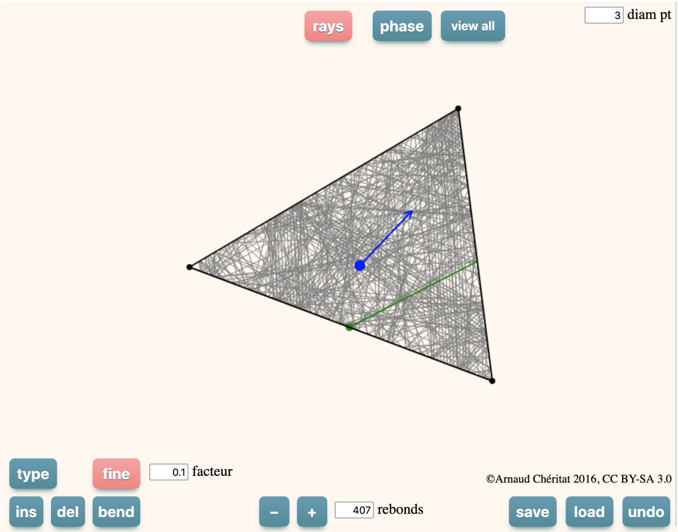
Answer

- Yes in an acute triangle.
- Yes if the angle are rational multiples of π.

A periodic trajectory in an acute triangle



A dense trajectory in an acute triangle



Equilateral triangle

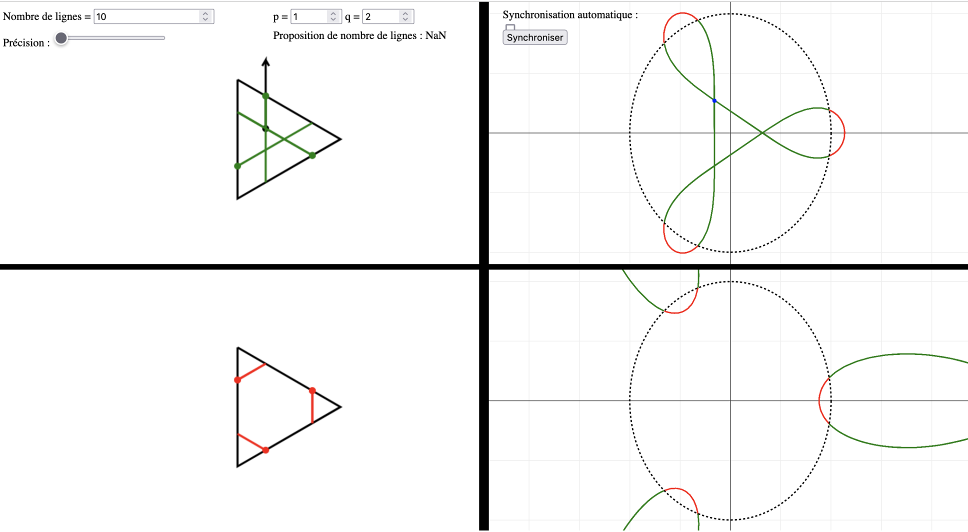
Proposition

In an equilateral triangle, a trajectory is periodic if and only if its slope (with respect to the sides of the triangle) is rational.

Theorem (Valdez)

The trajectories for the vector field $\vec{v}(x, y) = (y^2, x^2)$ *in* $\mathbb{C}^2$ *are understandable in terms of the trajectories in an equilateral triangle.*

Equilateral triangle and $\vec{v}(x, y) = (y^2, x^2)$ on $\mathbb{C}^2$



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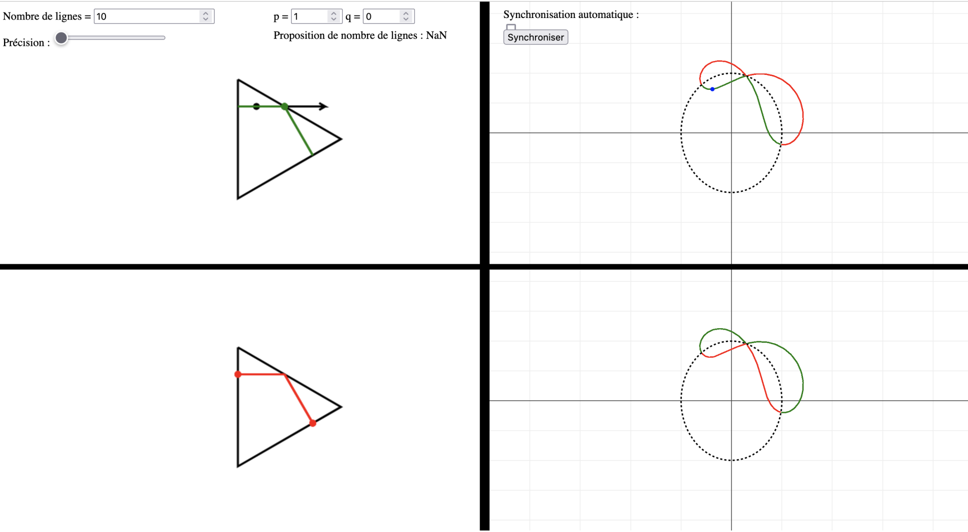

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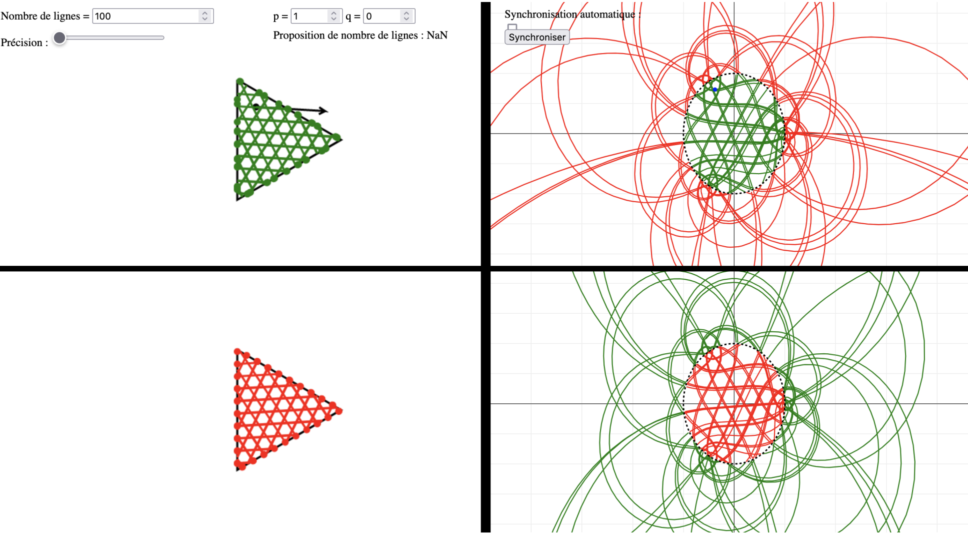


Work in progress with Jasmin Raissy

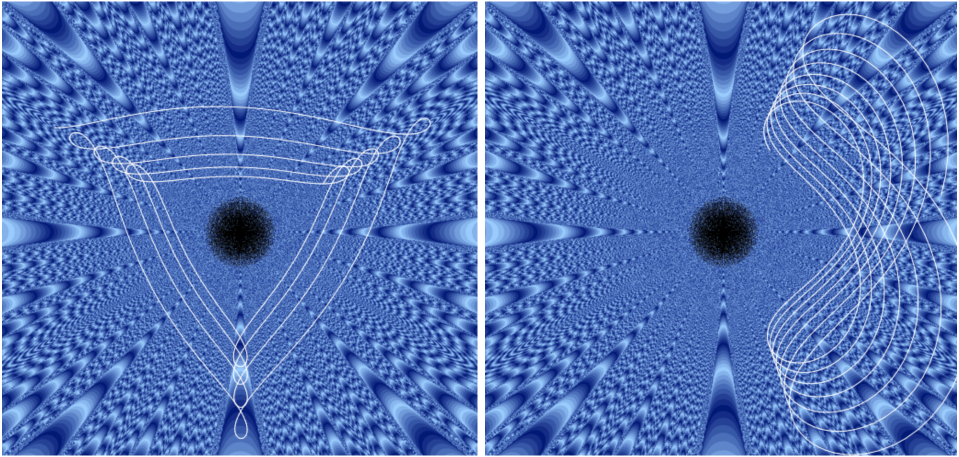
Theorem

For $\lambda \in (1, +\infty)$, the polynomial endomorphism $F_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

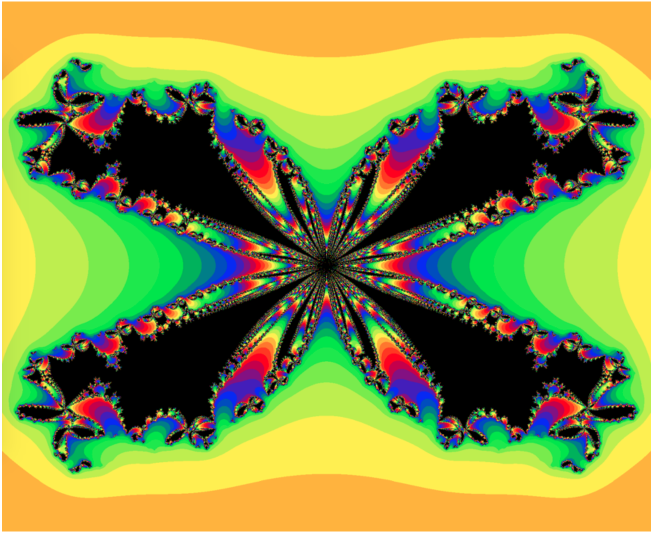
$$F_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + \lambda xy \begin{pmatrix} x \\ y \end{pmatrix}$$

has infinitely many spiralling domains contained in distinct Fatou components.

The dynamics of $F_0(x, y) = (x + y^2, y + x^2)$



The dynamics of $F_1(x, y) = (x + y^2 + x^2y, y + x^2 + xy^2)$



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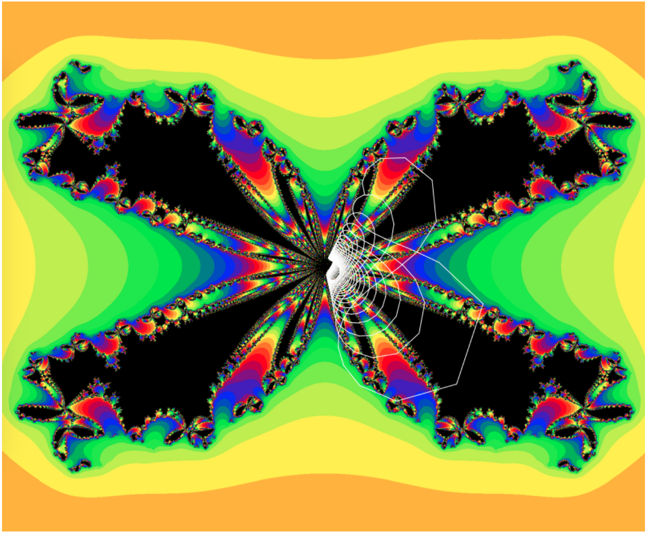

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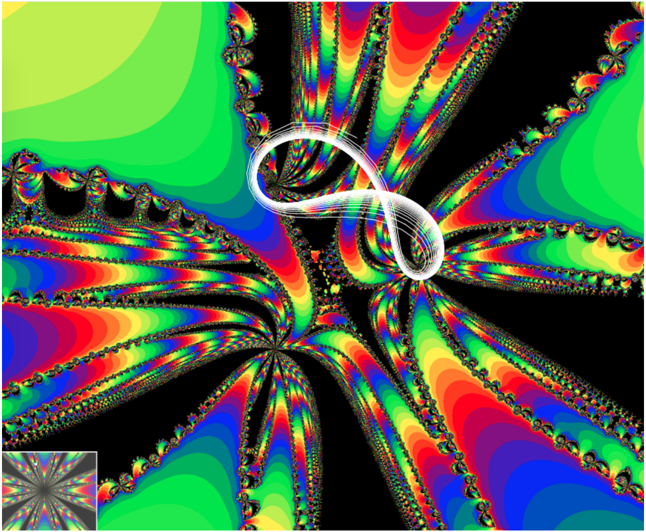

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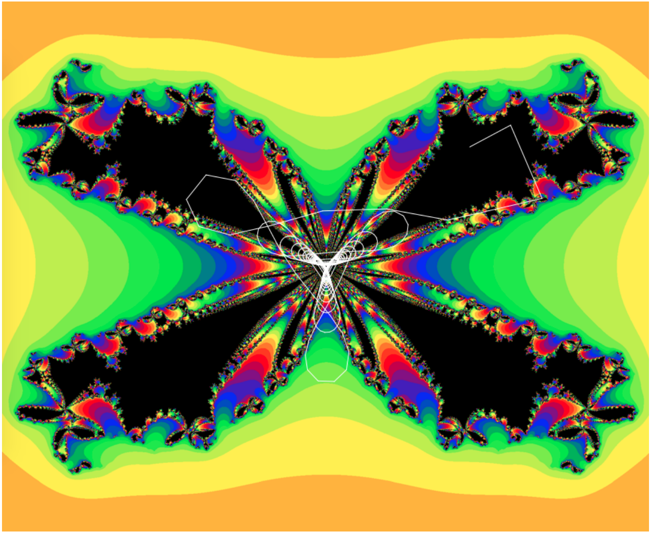

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