Stability of eigenvalues and observable diameter in $\mathrm{RCD}(1, \infty)$ spaces

Jerôme Bertrand and Max Fathi

November 24, 2021

Abstract

We study stability of the spectral gap and observable diameter for metric measure spaces satisfying the $\mathrm{RCD}(1, \infty)$ condition. We show that if such a space has an almost maximal spectral gap, then it almost contains a Gaussian component, and the Laplacian has eigenvalues that are close to any integers, with dimension-free quantitative bounds. Under the additional assumption that the space admits a needle disintegration, we show that the spectral gap is almost maximal iff the observable diameter is almost maximal, again with quantitative dimension-free bounds.

1 Introduction

A classical topic in geometry is to understand the structure of spaces that maximize a given geometric quantity, within a suitable class of spaces. An emblematic example that motivates some of our work here is the Cheeger-Gromoll splitting theorem, which states that manifolds with nonnegative Ricci curvature that contain an infinite-length minimizing geodesic must split off a line. There are many examples of results with a similar flavor, involving quantities such as the diameter, the spectral gap, isoperimetric-type properties, etc... We shall discuss relevant examples later in this introduction.

Once such a result is known, it is natural to investigate the structure of spaces that *almost* maximize the geometric quantity, and determine whether they are close in some sense to having the same structure as an extremal space. When investigating classes of compact spaces, a non-explicit stability estimate can often be derived by a compactness argument, via Gromov's precompactness theorem, and showing that a sequence of spaces that saturate in the limit the geometric constraint must converge to an extremal space. See [14, Theorem 1.5] for an example where this strategy is used. Our work here will deal with a class of non-compact spaces, for which this strategy is unavailable. Moreover, our focus will be on deriving estimates that have a fully explicit dependence on the various parameters (and in particular, making sure the estimates are dimension-free).

In this work, we shall consider metric measures spaces (M, d, m) which belong to the class of $\mathrm{RCD}(1, \infty)$ spaces, normalized so that $m(M) = 1$. These spaces are

1

a natural extension of Riemannian manifolds satisfying a curvature condition. It is stronger than the Lott-Sturm-Villani curvature dimension condition $CD(1,\infty)$, but more appropriate for the type of questions we investigate here, since it is for example the right setting for the non-smooth generalization of the Cheeger-Gromoll splitting theorem [21]. For some of the results we obtain here, we shall further assume the existence of a $CD(1,\infty)$ disintegration of the metric measure space. Such spaces are studied -for general K and N- in [13] and denoted by $CD^1(K,N)$. These notions will be fully defined in Section 2.

Examples of metric measures spaces (m.m.s. for short) admitting a $CD(1,\infty)$ disintegration are weighted Riemannian manifolds whose Bakry-Émery curvature is greater or equal to one, as proved by Klartag [29], and $RCD(1,N)$ m.m.s. (for any $1 < N < +\infty$) as proved by Cavalletti and Mondino in [14].

In an $RCD(1,\infty)$ space, the spectral gap is bounded from below by 1, that is

Theorem 1.1. *Let (M,d,m) be an $RCD(1,\infty)$ space, with λ_1 the smallest positive eigenvalue of $-\Delta$. Then $\lambda_1 \geq 1$.*

Note that $\lambda_1 = 1$ holds true for the Gauss space. This result goes back to Lichnerowicz [30] for smooth manifolds. In [23, Theorem 1.1], Gigli, Ketterer, Kuwada, and Ohta proved the following rigidity result:

Theorem 1.2. *Let (M,d,m) be an $RCD(1,\infty)$ space, and assume that $\lambda_1 = 1$. Then there exists an $RCD(1,\infty)$ space (Y,d_Y,m_Y) such that:*

- *The metric space (M,d) is isometric to the product space $(\mathbb{R},|\cdot|) \times (Y,d_Y)$ with the product metric.*

- *Through the isometry above, the measure m coincides with the product measure $e^{-x^2/2}dx \otimes m_Y$.*

Other rigidity results for $RCD(1,\infty)$ spaces were obtained in [35, 26]. For smooth Riemannian manifolds endowed with their volume measure, the analogous inequality is due to Lichnerowicz and the case of equality -which characterizes the unit sphere- is due to Obata. Obata's theorem has been generalized to $RCD(N-1,N)$ spaces by Ketterer (with $N < \infty$) [28].

Recall that $RCD(N-1,N)$ spaces have diameter bounded from above by π and thus are uniformly doubling. Moreover, the $RCD(N-1,N)$ condition is stable by measured Gromov-Hausdorff convergence. Therefore, according to Gromov's precompactness theorem, $RCD(N-1,N)$ spaces form a compact class of metric measure spaces for the measured Gromov Hausdorff distance. Therefore Ketterer's result also leads to stability results for $RCD(N-1,N)$ spaces with almost minimal first eigenvalue (or equivalently, almost maximal diameter). In the setting of Riemannian manifolds, the equivalence follows from work of Cheng [15] and Croke [18]. No such compactness property is available for the $RCD(1,\infty)$ spaces, which leaves the properties of such spaces with almost minimal first eigenvalue unknown.

Part of our work here will be to study RCD spaces with almost minimal spectral gap, that is $RCD(1,\infty)$ spaces with $\lambda_1 \leq 1 + \epsilon$. Let us recall that an $RCD(1,\infty)$

space has a discrete spectrum [24, Proposition 6.7]. One of our main results is the following about higher eigenvalues:

Theorem 1.3. *Consider an* $RCD(1,\infty)$ *space, and assume its spectral gap* λ_1 *is smaller than* $1 + \epsilon$ *with* $\epsilon > 0$ *less than some fixed* $\epsilon_0 > 0$. *Then for any* $n \geq 1$, $\theta < 1/2$, *there is an eigenvalue* $-\lambda_n$ *of the Laplacian* Δ *such that* $|\lambda_n - n\lambda_1| \leq C(n, \theta, \epsilon_0)\epsilon^{1/2-\theta}$.

Since by assumption λ_1 is close to one, we hence prove that positive integers are close to being eigenvalues of the Laplacian when the spectral gap is almost minimal. The integers appear here as the eigenvalues of the Ornstein-Uhlenbeck operator, which is the Laplacian on the Gaussian space.

In the class of $RCD(1,\infty)$ spaces, higher eigenvalues are stable under pointed measured Gromov-Hausdorff convergence [24, Theorem 7.2]. The argument in that work uses a compactness argument (see [24, Theorem 7.8]), so it is not explicitly quantitative. Moreover, since the eigenfunctions are not necessarily globally Lipschitz, it does not seem like this type of statement can be immediately deduced from a statement on closeness to a Gaussian factor, such as Theorem 1.4 below.

In the Euclidean setting, it is known that the sequence of ordered eigenvalues is bounded from below by the ordered eigenvalues of the Ornstein-Uhlenbeck operator on the same space [34, Theorem 1.1], as a consequence of Caffarelli's contraction theorem. To our knowledge, an analogous result in the $RCD(1,\infty)$ setting is an open problem at the time of writing.

In the smooth finite-dimensional setting, Petersen [37] proved that an n-dimensional manifold with Ricci curvature bounded from below by $n-1$, whose $(n+1)$-th eigenvalue is close to n, is close in the Gromov-Hausdorff sense to a sphere. Aubry [4] later showed that the statement still holds for the n-th eigenvalue, with quantitative bounds, and that a control on the $(n-1)$-th eigenvalue is not enough. More recently, Takatsu [43] proved convergence of the spectral structure of low-dimensional projections of spheres of high-dimension to Gaussian spaces, which is a particular example to which all the results presented here apply.

We shall also show that the pushforward of the reference measure by a normalized eigenfunction is close to Gaussian, with dimension-free quantitative bounds, partially extending results of [19, 17] from the Euclidean setting to the full setting of RCD spaces. Similar questions for manifolds with small deficit in the Bakry-Ledoux isoperimetric inequality were raised in [31, Remark 7.6], and can be positively answered with the tools we use here.

Theorem 1.4. *Let* (M, d, μ) *be an* $RCD(1,\infty)$ *space such that* $\mu(M) = 1$, *and* f *be a eigenfunction of* $-\Delta$ *associated to the eigenvalue* λ, *and normalized by* $\int_M f^2 \, d\mu = 1$. *Let* $\nu = f_\sharp \mu$, *and* γ *be the standard one-dimensional Gaussian measure. Then,*
$$W_1(\nu, \gamma) \leq 4 \times 2^{\lambda/2}\sqrt{\lambda - 1}.$$

Here W_1 stands for the L^1 Wasserstein distance. Asymptotic normality for eigenfunctions on manifolds was studied by E. Meckes [32], and (a variant of)

Theorem 1.4 can be directly deduced from her abstract theorem via our main Lemma 3.3 below. When the eigenspace associated with the spectral gap is of dimension higher than one, or if there are several eigenvalues close to one, then a multivariate version of Theorem 1.4 holds, and can be proved with the arguments we develop here, as in the Euclidean setting [17].

A problem left open by this work is a complete stability estimate for Theorem 1.2. We would expect a space with almost minimal spectral gap to be close in some sense to a product space with Gaussian factor, but we have not been able to prove a satisfactory statement showing an approximate splitting at the level of the metric.

A second part of our work will deal with stability of the observable diameter, motivated by Ketterer's work on Obata's theorem in RCD spaces [28, 27], as well as [31]. While seeking for upper bound on the diameter of an $RCD(1,\infty)$ space is meaningless, Gromov's observable diameter is a good candidate to replace standard diameter. Let us recall its definition.

Definition 1.5 (Observable diameter). *The observable diameter D_{obs} of a metric measure space (M, d, μ) is a function of $\kappa \in (0, 1)$ defined by*

$$D_{obs}((M, d, \mu); \kappa) := \sup_{f:M \to \mathbb{R}} \left\{ \inf_{E \, \text{Borel}} \left\{ \text{diam}(E); f_\sharp \mu(E) \geq 1 - \kappa \right\}; f \, 1\text{-Lipschitz} \right\}.$$

For ease of notation, the observable diameter is also denoted by $D_{obs}(\mu; \kappa)$.

As proved in Theorem 2.10 below, the observable diameter of an $RCD(1, \infty)$ space is pointwise bounded from above by that of the Gauss space.

We shall prove some results connecting $RCD(1, \infty)$ spaces with almost minimal spectral gap to those with almost maximal observable diameter. More precisely, we prove

Theorem 1.6. *Let (M, d, μ) be an $RCD(1, \infty)$ space admitting a $CD(1, \infty)$ disintegration in the sense of Assumption 2.1, and whose spectral gap satisfies $\lambda_1 \leq 1 + \epsilon$ for $\epsilon > 0$. Then for any $\theta > 0$ small enough, there exists a constant $C(\theta)$ that only depends on θ such that for any $1 > \kappa \geq C(\theta)\epsilon^{1/20-\theta} > 0$, the following inequalities hold*

$$D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa - C(\theta)\epsilon^{1/20-\theta}) \geq$$
$$D_{obs}((M, d, \mu); \kappa - C(\theta)\epsilon^{1/20-\theta}) \geq D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa).$$

The assumption on the $CD(1, \infty)$ disintegration in this result could be relaxed, in that we only need a disintegration with respect to one particular test function, namely an eigenfunction associated with the spectral gap.

This theorem can be compared with [12, Theorem 4.4], which proved that under a $CD(N - 1, N)$ condition if the spectral gap is almost minimal then the diameter is almost maximal, with dimension-dependent exponent and constants.

Finally, we also obtain a converse result, deducing closeness of the spectral gap from closeness of the observable diameter:

Theorem 1.7. *Let (M, d, μ) be an $RCD(1, \infty)$ space admitting a $CD(1, \infty)$ disintegration in the sense of Assumption 2.1, and assume that*

$$D_{obs}((M, d, \mu); \kappa) \geq D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa) - \epsilon$$

for a given $0 < \kappa < 1$ and $\epsilon > 0$ small enough. Then there exists $C = C(\kappa)$ such that the spectral gap λ_1 satisfies

$$\lambda_1 \leq 1 + C\epsilon^{1/22}.$$

As a corollary, this implies rigidity of the observable diameter, under the extra Assumption 2.1, via [23, Theorem 1.1]. This statement seems to be new.

The remainder of the article is as follows: in Section 2, we present some necessary preliminaries on RCD spaces and needle decompositions. In Section 3, we prove Theorems 1.3 and 1.4, and Section 4 contains the proofs of Theorems 1.6 and 1.7. We summarize some of the notations used at the end of Section 2.

2 Preliminaries

2.1 RCD spaces

2.1.1 Definitions

In this work, we shall always work with separable, complete metric spaces, which we shall denote by (M, d). We shall endow it with a Borel probability measure μ. Additionally, we shall assume the metric space is *geodesic*: every pair of points x, y are connected by a *minimal geodesic*, namely a map $\gamma : [0, 1] \longrightarrow M$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $d(\gamma(s), \gamma(t)) = |t - s| d(x, y)$ for all $s, t \in [0, 1]$.

We can endow the space of Borel probability measures on M with finite second order moment, denoted by $\mathcal{P}_2(M)$, with the L^2 Wasserstein distance from optimal transport. We refer to the monograph [44] for the definition of L^p Wasserstein distances.

The relative entropy functional with respect to μ on $\mathcal{P}_2(M)$ is defined as

$$\mathrm{Ent}_\mu(\nu) := \int_M \rho \log \rho \, d\mu$$

if $\nu = \rho\mu$ is an absolutely continuous probability measure, and $+\infty$ otherwise.

With these objects in mind, we can define the curvature-dimension condition $\mathrm{CD}(K, \infty)$:

Definition 2.1. *A metric measure space (M, d, μ) is said to satisfy the curvature-dimension condition $CD(K, \infty)$ for $K \in \mathbb{R}$ if the relative entropy is K-convex along W_2-geodesics on $\mathcal{P}_2(M)$, that is for any pair of probability measures ν_1, ν_2 with finite relative entropy with respect to μ, there is a W_2 minimal geodesic $(\nu_t)_{t \in [0,1]}$ connecting them, and such that for all $t \in [0, 1]$*

$$\mathrm{Ent}_\mu(\nu_t) \leq (1 - t)\mathrm{Ent}_\mu(\nu_0) + t\,\mathrm{Ent}_\mu(\nu_1) - \frac{Kt(1-t)}{2}W_2(\nu_0, \nu_1)^2.$$

The parameter K plays the role of a Ricci curvature lower bound. Indeed, when the space is a smooth Riemannian manifold endowed with its (normalized) volume measure, it satisfies this condition iff the Ricci curvature tensor (viewed as a quadratic form acting on a single vector field) is bounded from below by K times the metric tensor. This definition is a particular instance of the more general curvature dimension $\mathrm{CD}(K, N)$, that takes into account the dimension. Here we shall mostly deal with the infinite-dimensional setting, so we omit the full definition, and refer to [45] for the more general setting.

Given the metric measure structure, we can also define the Cheeger energy

$$\mathrm{Ch}(f) := \frac{1}{2} \inf_{(f_i)} \liminf \int_M |\operatorname{Lip} f_i|^2 d\mu$$

where the infimum runs over the set of all sequences of locally-Lipschitz functions that converge to f in $L^2(\mu)$, and where $\operatorname{Lip}(f)(x)$ is the local Lipschitz constant of f around x.

Given an L^2 function with finite Cheeger energy, there exists a minimal weak upper gradient $|\nabla f| \in L^2(\mu)$ such that

$$\mathrm{Ch}(f) = \frac{1}{2} \int_M |\nabla f|^2 d\mu.$$

The Sobolev space $W^{1,2}$ is the space of L^2 functions with finite Cheeger energy. We refer to [1] for more details about these notions.

Definition 2.2. *A metric measure space (M, d, μ) that satisfies the $\mathrm{CD}(K, \infty)$ condition is said to satisfy the Riemannian curvature-dimension condition $\mathrm{RCD}(K, \infty)$ if the Cheeger energy is a quadratic form, that is*

$$\mathrm{Ch}(f + g) + \mathrm{Ch}(f - g) = 2\,\mathrm{Ch}(f) + 2\,\mathrm{Ch}(g)$$

for all $f, g \in W^{1,2}$.

The most basic example of an $\mathrm{RCD}(K, \infty)$ space with positive K is a Gaussian measure on $\mathbb{R}$ with variance K^{-1}. More generally, a probability measure on $\mathbb{R}^d$ with density e^{-V} w.r.t. the Lebesgue measure satisfies the $\mathrm{RCD}(K, \infty)$ condition iff $\operatorname{Hess} V \geq K$ in the weak sense.

On a smooth Riemannian manifold $(M, \tilde{g})$ (where $\tilde{g}$ is the Riemannian metric of the manifold) endowed with its volume measure, the $\mathrm{RCD}(K, \infty)$ condition is equivalent to requiring the Ricci curvature tensor to satisfy $\operatorname{Ric}(U, U) \geq K\tilde{g}(U, U)$ for vector fields U. More generally, if the measure has smooth density e^{-V} w.r.t. the volume measure, the $\mathrm{RCD}(K, \infty)$ condition is equivalent to the Bakry-Émery condition

$$\operatorname{Ric} + \operatorname{Hess} V \geq K\tilde{g}.$$

However, there are also non-smooth spaces that satisfy the $\mathrm{RCD}(K, \infty)$ condition, including Alexandrov spaces [38] and some stratified spaces [8].

On an RCD(K, ∞) space, we can then define by polarization the scalar product of two elements of $W^{1,2}$ and the Dirichlet form

$$\langle \nabla f, \nabla g \rangle := \frac{1}{4}(|\nabla(f+g)|^2 - |\nabla(f-g)|^2) \in L^1(\mu); \ \ \mathcal{E}(f,g) := \int_M \langle \nabla f, \nabla g \rangle d\mu.$$

The natural analog of the Laplacian on (M, d, g) is the operator $\Delta : D(\Delta) \longrightarrow L^2$ such that

$$\mathcal{E}(f,g) = -\int_M g(\Delta f) d\mu \ \ \forall g \in W^{1,2}.$$

The domain $D(\Delta)$ is dense in $W^{1,2}$. We refer to [2] for details of the construction. Note that even when the space is a smooth Riemannian manifold, Δ is not simply the Laplace-Beltrami operator, since it takes into account the reference measure μ, that is not necessarily the volume measure. For example, for the Euclidean space endowed with a measure with density e^{-V}, the natural Laplace operator is $\Delta_{eucl} - \nabla V \cdot \nabla$, where Δ_{eucl} is the usual Laplacian on $\mathbb{R}^d$. Finally, note that since the Cheeger energy is used as a Dirichlet form, the induced Laplacian is a local operator, which will allow us to use the diffusion property.

2.1.2 Properties of RCD(K, ∞) spaces

An important property of RCD(K, ∞) is the tensorization property, which states that this class of spaces is stable for the product [2, Section 6]:

Proposition 2.3 (Tensorization property). *If two spaces (M_i, d_i, μ_i) satisfy the RCD(K, ∞) condition, then the product space $(M_1 \times M_2, d_1 \oplus d_2, \mu_1 \otimes \mu_2)$ also does. Here $d_1 \oplus d_2((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$.*

In particular this property explains why rigidity statements such as Theorem 1.2 involve a splitting: the product of an RCD$(1, \infty)$ space with minimal spectral gap and any other RCD$(1, \infty)$ space is still an RCD$(1, \infty)$ space, and still has minimal spectral gap.

The spectral gap λ_1 can be reformulated as the sharp constant in the Poincaré inequality for μ, that is the largest constant C_P such that

$$\mathrm{Var}_\mu(f) \leq \frac{1}{C_P} \int_M |\nabla f|^2 d\mu \ \ \forall f \in W^{1,2}.$$

As we have mentioned in Theorem 1.1 (with $K = 1$), an RCD(K, ∞) space with positive K satisfies a Poincaré inequality with constant K^{-1}, that is $\lambda_1 \geq K$. Moreover, as proved in [23] (with results in the smooth setting obtained in [16]), this bound is rigid: if the sharp spectral gap of an RCD(K, ∞) space is equal to K, then the space splits off a Gaussian factor that has variance K^{-1}.

The sharp constant of the Poincaré inequality is indeed the inverse of the smallest positive eigenvalue λ_1 of $-\Delta$, that is there exists an eigenfunction f such that

$$\Delta f = -C_P^{-1} f.$$

Existence is ensured by the validity of the logarithmic Sobolev inequality, which is strictly stronger than the Poincaré inequality, and provides enough compactness to ensure there is a function that achieves equality in the sharp Poincaré inequality, see [24, Proposition 6.7].

2.2 Localization of the curvature condition

Part of our work requires an additional assumption on the metric measure space, namely the existence of a suitable needle decomposition. A brief reminder on that decomposition is provided below. This notion originates in Klartag's work [29] in the smooth setting, and was extended to metric measure spaces by Cavalletti and Mondino [14]. Recently, [11] showed that the needle decomposition provides an equivalent way of defining curvature-dimension conditions with finite dimension (which unfortunately is not our setting here).

Theorem 2.4 (Needle decomposition). *Let (M, d, μ) be an $RCD(1, N)$ space ($N < +\infty$) and f be a centered, L^1 function such that*

$$\int_M |f(x)| d(x, x_0) d\mu < \infty.$$

Then the space M can be written as the disjoint union of two measurable sets Z and T, where T admits a partition in the following sense. There exists a 1-Lipschitz function $g : M \longrightarrow \mathbb{R}$, a partition $(X_q)_{q \in Q}$ of T, a probability measure $\mathfrak{q}$ on Q and a family of probability measures $(m_q)_{q \in Q}$, each supported on X_q, such that

1. *For any measurable A, $\mu(A \cap T) = \int_Q m_q(A) d\mathfrak{q}(q)$;*

2. *For $\mathfrak{q}$ a.e. q, for any x, y in X_q, $d(x, y) = |g(x) - g(y)|$, and for any $z \notin X_q$, there exists $x \in X_q$ such that $|g(z) - g(x)| < d(x, z)$;*

3. *For $\mathfrak{q}$ a.e. q, X_q is a minimal geodesic of (M, d) not reduced to a single point, $m_q \ll \mathcal{H}^1$, and (X_q, d, m_q) is $RCD(1, N)$;*

4. *For $\mathfrak{q}$ a.e. q, $\int_{X_q} f dm_q = 0$ and $f = 0$ μ-a.e. in Z.*

Note that the properties of the needle decomposition require that $\mathfrak{q}$-almost every needle is not reduced to a singleton, and that $\text{Lip}(g) = 1$ almost everywhere.

The 1-Lipschitz function g is called a guiding function, and is obtained as an optimizer in the variational problem

$$\sup_{u \; 1-\text{Lip}} \int_M f u d\mu.$$

It can be interpreted as the potential in the Kantorovitch-Rubinstein dual formulation of the L^1 optimal transport problem between the positive measures $f_+ \mu$ and $f_- \mu$, which have same mass since f is centered.

Since we consider infinite-dimensional spaces, the current state of knowledge on needle decompositions does not guarantee existence of the decomposition, as the

proof uses compactness of finite-dimensional positively curved spaces. However, it may nonetheless exist (indeed, its study in the Euclidean context goes back to work of Payne and Weinberger [36]). So we make the following assumption for Theorems 1.6 and 1.7.

Assumption 2.1. *The space (M, d, μ) satisfies the conclusion of the above theorem with $N = \infty$.*

Note that since the second moment of μ is finite and the eigenfunction is L^2, the integrability condition $\int_M |f(x)| d(x, x_0) d\mu < \infty$ is automatically satisfied.

Proposition 2.5 (Structure of needles)**.** *Without loss of generality, for $\mathfrak{q}$ a.e. q we can identify (X_q, d, m_q) with a space of the form $(\mathbb{R}, |\cdot|, e^{-\psi_q} dx)$ where $\psi_q'' \geq 1$ (in the weak sense) and ψ_q is finite in some interval I, and such that, on X_q, $g(t) = t + c_q$.*

Proof. Let $I = \{g(x), x \in X_q\}$. According to the property 3 of Theorem 2.4 and the continuity of g, for $\mathfrak{q}$ a.e q, the set I is an interval of $\mathbb{R}$. By the property 2 of Theorem 2.4, we can parameterize X_q by I, and the measure m_q is of the form $e^{-\psi_q} dx$ on I. The fact that (non-degenerate) log-concave measures have a density is due to Prékopa [39]. We can then extend ψ_q to take the value $+\infty$ outside I, which allows us to identify $(I, |\cdot|, e^{-\psi_q} dx)$ with $(\mathbb{R}, |\cdot|, e^{-\psi_q} dx)$. $\square$

2.3 Observable diameter

We now define the notion of observable diameter, which plays the role of diameter for RCD$(1, \infty)$ spaces, by taking advantage of concentration inequalities. This notion was introduced by Gromov in [25]. Informally, a control on the observable diameter states that, even if the diameter itself is not finite, most of the mass is concentrated inside a set whose diameter satisfies an a priori bound. Below, we recall the definition of the observable diameter for the reader convenience.

Definition 2.6. *The observable diameter D_{obs} of the metric measure space (M, d, μ) is a function of $\kappa \in (0, 1)$ defined by*

$$D_{obs}((M, d, \mu); \kappa) := \sup_{f: M \to \mathbb{R}} \left\{ \inf_{E \, \mathrm{Borel}} \left\{ \mathrm{diam}(E); f_\sharp \mu(E) \geq 1 - \kappa \right\}; f \, 1\text{-Lipschitz} \right\}.$$

The observable diameter can be estimated by means of the separation distance whose definition (in the case of two sets with the same lower bound on their mass) is recalled below.

Definition 2.7. *The separation distance is a function of $\kappa \in (0, 1)$ defined by*

$$Sep((M, d, \mu); \kappa) := \sup_{A_1, A_2 \subset M \, \mathrm{Borel}} \left\{ d(A_1, A_2); \mu(A_i) \geq \kappa, i \in \{1, 2\} \right\}.$$

The following inequality holds (see [42, Proposition 2.26] for a proof):

Proposition 2.8. *Let (M, d, μ) be a metric measure space such that $\mu(M) = 1$. Then, for any $\kappa \in (0, 1)$*

$$D_{obs}((M, d, \mu); \kappa) \leq Sep((M, d, \mu); \kappa/2). \tag{1}$$

Remark 2.1. *In the case of the Gaussian space $(\mathbb{R}, |\cdot|, \gamma)$, the symmetry of the density with respect to the origin and the fact that half-spaces saturate the Gaussian isoperimetric inequality [9] yield for any $\kappa \in (0, 1)$*

$$D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa) = Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2).$$

If we denote these quantities by s, they satisfy

$$\gamma\left(\left(-\infty, \frac{s}{2}\right]\right) = \frac{\kappa}{2}.$$

On RCD$(1, \infty)$ spaces, the Lévy-Gromov type isoperimetric inequality has been proved by Ambrosio and Mondino [3, Theorem 4.2]:

Theorem 2.9. *Let (M, d, μ) be an RCD$(1, \infty)$ space such that $\mu(M) = 1$. Then, for any Borel subset $E \subset X$ it holds*

$$\mathcal{P}(E) \geq \frac{1}{\sqrt{2\pi}} e^{-a_E^2/2},$$

where $\mathcal{P}$ stands for perimeter, and $\mu(E) = \gamma((-\infty, a_E])$.

In the nonsmooth setting, the perimeter of a Borel subset E is defined as

$$\mathcal{P}(E) := \inf\left\{\liminf_n \int |\nabla f_n| d\mu; \ f_n \text{ globally Lipschitz and } \lim_n \int |f_n - \mathbb{1}_E| d\mu = 0\right\}.$$

As a consequence, one gets the following comparison inequality for the observable diameter.

Theorem 2.10. *Let (M, d, μ) be an RCD$(1, \infty)$ probability space. Then for any $\kappa \in (0, 1)$, the following inequalities hold*

$$D_{obs}((M, d, \mu); \kappa) \leq Sep((M, d, \mu); \kappa/2)$$
$$\leq Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2) = D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa).$$

Proof. According to (1), the only property that is yet to prove is the inequality involving the separation distances of $(\mathbb{R}, |\cdot|, \gamma)$ and (M, d, μ). The proof is the same as the one of [42, Lemma 2.32], we report it since we need the argument later in this work. Without loss of generality, we can assume that $Sep((M, d, \mu); \kappa/2) > 0$ since otherwise there is nothing to prove.

Let $A_1, A_2 \subset M$ be two mutually disjoint sets such that $\mu(A_i) \geq \kappa/2$ for $i \in \{1, 2\}$. Set $r := d(A_1, A_2)$ and notice that $A_1 \cap V_r(A_2) = \emptyset$, where $V_r(A_2)$ stands for the open r-neighborhood of A_2. Let us also define $\sigma(s) := \gamma((-\infty, s])$

and note that σ is an increasing function of s. Consequently, making use of the isoperimetric inequality for RCD$(1, \infty)$ spaces in integrated form, we infer for any $s > 0$

$$\mu(V_s(A_2)) \geq \gamma((-\infty, -a + \delta + s]) \geq \gamma((-\infty, -a + s]),$$

where $-a + \delta := \sigma^{-1}(\mu(A_2)) \geq -a := \sigma^{-1}(\kappa/2)$. With this notation,

$$Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2) = 2|\sigma^{-1}(\kappa/2)| = 2a.$$

Now, observe that

$$
\begin{aligned}
\kappa/2 \leq \mu(A_1) & \leq & 1 - \mu(V_r(A_2)) \\
& \leq & 1 - \gamma((-\infty, -a + r]).
\end{aligned}
$$

Therefore, by symmetry of γ,

$$\gamma((-\infty, -a]) + \gamma(([a - r, +\infty)) = \kappa/2 + \gamma((-\infty, -a + r]) \leq 1.$$

Thus, $a - r \geq -a$ hence $2|\sigma^{-1}(\kappa/2)| \geq r$ and the result is proved. $\qquad\square$

2.4 Summary of notations

We shall use the following notations throughout the manuscript:

- Given a metric-measure space (M, d, μ), $L^p(\mu)$ stands for the space of measurable functions with finite L^p moment with respect to μ. Whenever there is no risk of ambiguity on the reference measure, we shall simply denote this space by L^p. We shall denote by Δ the Laplace operator on (M, d, μ), and P_t the associated semigroup.

- We shall denote by $\mathcal{P}_p(M)$ the space of Borel probability measures with finite p-moment. W_p shall stand for the L^p Wasserstein distance on $\mathcal{P}_p(M)$.

- The operator Γ_2 is the *carré du champ operator* of Bakry-Emery, that is

$$\Gamma_2(f) = \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle.$$

- We shall occasionally abbreviate *metric measure space* as *m.m.s.*

- We denote by $\mathrm{Var}_\mu(f)$ the variance of an L^2 function with respect to μ, that is

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2.$$

3 Properties of RCD$(1, \infty)$ spaces with almost minimal spectral gap

In this section, we shall investigate properties of RCD$(1, \infty)$ spaces when the spectral gap is close to one, and prove Theorems 1.3 and 1.4.

In the sequel, L^p norms will always be with respect to the reference measure μ, that is

$$||f||_p := \left(\int_M |f|^p d\mu \right)^{1/p}.$$

3.1 Main lemma

We shall first prove the main technical lemma that underlies the proofs of our results, Lemma 3.3, which states that if the spectral gap is close to one, then the gradient of an associated eigenfunction is almost constant.

We shall use the following L^p version of Poincaré inequalities:

Proposition 3.1. *Under the curvature condition RCD$(1, \infty)$, for any $p \geq 1$ and any funciton f with $|\nabla f| \in L^p$ we have*

$$\left\| f - \int_M f d\mu \right\|_p \leq 2p \, |||\nabla f|||_p.$$

This inequality is strongest for $p = 1$, where it corresponds to the classical Cheeger inequality. See for example [33, Proposition 2.5] for a proof (using $p' = q' = p$ with the value of p we consider, and taking value 1 for the other two parameters involved). The constant in this formulation is not sharp, since for $p = 2$ this is not the best estimate on the spectral gap.

When considering eigenfunctions in positive curvature, we have nice integrability estimates in all L^p spaces, which are summarized in the following proposition:

Proposition 3.2 (Integrability of eigenvectors)**.** *Assume that curvature is bounded from below by 1. Let f be an eigenfunction of $-\Delta$ with eigenvalue λ and normalized so that $||f||_2 = 1$. Then*

1. *$||f||_p \leq (p-1)^{\lambda/2}$ for any $p \geq 2$;*

2. *$|||\nabla f|^2||_p \leq 2^\lambda ||f||_{2p}^2 \leq (4p-2)^\lambda$ for any $p \geq 1$.*

Proof. The proof of 1 can be found in [7, Section 5.3]. It is a conseuqance of the logarithmic Sobolev inequality, which holds for RCD$(1, \infty)$ spaces.

To prove 2, we use the gradient bound $|\nabla P_t f|^2 \leq (e^{2t} - 1)^{-1} P_t(f^2)$ (see [7, Theorem 4.7.2]). We apply it to the eigenfunction f with $t = (\ln 2)/2$ to get

$$|\nabla f|^{2p} \leq \left(2^\lambda P_t(f^2) \right)^p \leq 2^{\lambda p} P_t(f^{2p})$$

and then integrate with respect to the reference measure μ (which is rpeserved by the semigroup) to conclude. □

The key lemma in our proofs is the following result, stating that the gradient of the eigenfunction is almost constant when λ is close to one, with quantitative bounds in all L^p norms. A version of this result in the Euclidean setting was derived in [19], and proved using optimal transport and Caffarelli's contraction theorem. The proof we present here, in addition to being more general, is simpler even in the Euclidean setting.

Lemma 3.3 (Key Lemma). *Let f be an eigenfunction of $-\Delta$ associated with (non-zero) eigenvalue $\lambda \leq 1 + \epsilon$, with $\int_M f^2 d\mu = 1$. Then for any $p \in [1, 2[$, we have*

$$|||\nabla f|^2 - \lambda||_p \leq 4p(\lambda - 1)^{1/2}\lambda^{1/2}\left(\frac{6p-4}{2-p}\right)^{\lambda/2}.$$

Assume now $\lambda \leq 1 + \epsilon_0$ for some fixed threshold ϵ_0. Then for any $\theta \in (0, 1/2)$ and $p \geq 2$ there exists $C = C(p, \theta, \epsilon_0)$ such that

$$|||\nabla f|^2 - \lambda||_p \leq C(\lambda - 1)^{1/2 - \theta}.$$

Proof. In the non-smooth setting, under the curvature condition we have the following pointwise estimate of [41, Theorem 3.4] (generalizing a result of [5, 6]) that holds almost everywhere:

$$|\nabla|\nabla f|^2| \leq 2|\nabla f|\sqrt{\Gamma_2(f) - |\nabla f|^2} \qquad (2)$$

This is only proved for globally Lipschitz f in [41], but we shall only use its integrated version, so we can proceed by approximation, by taking $h_{n,t} = P_t \max(-n, \min(f, n))$ for $t, n > 0$, which is globally Lipschitz (see for example [23, Proposition 2.5]), and then letting n go to infinity, then using the fact that f is an eigenfunction so $P_t f = e^{-\lambda t} f$ to let t go to zero. In the smooth setting, $|\nabla|\nabla f|^2|^2 \leq 4||\operatorname{Hes} f||_2^2|\nabla f|^2$. A version of this estimate involving Hessians in the nonsmooth context was investigated in [22].

For an eigenfunction f with eigenvalue λ, we have $\int \Gamma_2(f) d\mu = \lambda \int |\nabla f|^2 d\mu$. Therefore, using Holder's inequality, we have for $1 \leq p < 2$

$$\begin{aligned}
|||\nabla|\nabla f|^2|||_p^p &\leq 2^p \left(\int_M (\Gamma_2(f) - |\nabla f|^2) d\mu\right)^{p/2} \left(\int_M |\nabla f|^{2p/(2-p)} d\mu\right)^{(2-p)/2} \\
&\leq 2^p(\lambda - 1)^{p/2}\left(\int_M |\nabla f|^2 d\mu\right)^{p/2}\left(\int_M |\nabla f|^{2p/(2-p)} d\mu\right)^{(2-p)/2} \\
&\leq 2^p(\lambda - 1)^{p/2}\lambda^{p/2}\left(\frac{6p-4}{2-p}\right)^{\lambda p/2}
\end{aligned}$$

To conclude, we just apply the L^p Poincaré inequality to $|\nabla f|^2$.

To obtain the inequality for $p \geq 2$, using Holder's inequality we have

$$|||\nabla f|^2 - \lambda||_p \leq |||\nabla f|^2 - \lambda||_1^{1-\theta}|||\nabla f|^2 - \lambda||_{p'}^{\theta} \leq |||\nabla f|^2 - \lambda||_1^{1-\theta}(\lambda + |||\nabla f|^2||_{p'})^{\theta}$$

with p' suitably large enough, and we then apply Proposition 3.2. $\qquad\square$

3.2 Spectral properties

Let us start with the following general lemma:

Lemma 3.4. *Given $\mathcal{L}$ an unbounded operator on $L^2(\mu)$ with discrete spectrum. Assume that there is a function f such that $\mathcal{L}f = \alpha f + g$ with $||f||_2 = 1$ and $||g||_2 \leq \epsilon$. Then there is an eigenvalue of $\mathcal{L}$ at distance at most ϵ of α.*

Proof. Let α_i be the collection of eigenvalues of $\mathcal{L}$, with associated orthonormal basis of eigenfunctions f_i. We can write $f = \sum \beta_i f_i$ and $g = \sum \delta_i f_i$ with $\sum \beta_i^2 = 1$ and $\sum \delta_i^2 \leq \epsilon^2$. Then

$$\sum (\alpha_i - \alpha)\beta_i f_i = \sum \delta_i f_i$$

and hence for any i, $(\alpha_i - \alpha)\beta_i = \delta_i$, so that $\sum (\alpha_i - \alpha)^2 \beta_i^2 \leq \epsilon^2$. But since $\sum \beta_i^2 = 1$, there is at least one index i such that $|\alpha_i - \alpha| \leq \epsilon$. $\square$

With all this in hand, we can now prove Theorem 1.3:

Proof of Theorem 1.3. Consider H_n the n-th Hermitte polynomial, defined via $H_0 = 1$, $H_1(x) = x$ and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. Recall that $H_n''(x) - xH_n'(x) = -nH_n(x)$, and that they form an orthogonal basis of eigenfunctions for the one-dimensional Ornstein-Uhlenbeck generator.

Let f be a normalized eigenfunction of $-\Delta$ with eigenvalue λ. From the diffusion property [41, Section 2.2],

$$\Delta H_n(f) = H_n'(f)Lf + H_n''(f)|\nabla f|^2 = -\lambda n H_n(f) + H_n''(f)(|\nabla f|^2 - \lambda). \qquad (3)$$

According to Lemma 3.3, we have a suitable L^p bound on $|\nabla f|^2 - \lambda$ for all p, and $H_n''(f)$ has controlled L^p norm via Proposition 3.2, we control the L^2 norm of $H_n''(f)(|\nabla f|^2 - \lambda)$ by some $C(n, \theta, \epsilon_0)(\lambda - 1)^{1/2 - \theta}$, and can then apply Lemma 3.4 to conclude. $\square$

3.3 Proof of Theorem 1.4

We shall now prove that the pushforward of the measure by the eigenfunction is approximately Gaussian, generalizing part of the main result of [17] to the RCD setting. The proof will be based on (a version of) Stein's lemma, a classical result in probability and statistics, which allows to control distances to a Gaussian target measure via deficits in an integration-by parts formula. This was already the strategy used in [32, 17]. We refer to [40] for the proof, and an introduction to Stein's method. We could also proceed by directly combining the main abstract theorem of [32] with Lemma 3.3.

Lemma 3.5. *Let γ be the standard Gaussian measure on $\mathbb{R}$. Then for any $\mu \in \mathcal{P}(\mathbb{R})$, we have*

$$\max\left(d_{TV}(\mu, \gamma), W_1(\mu, \gamma)\right) \leq \sup_{||f||_\infty, ||f'||_\infty \leq 4} \int f' - xf d\mu.$$

Here W_1 stands for the L^1 optimal transport distance, and d_{TV} for the total variation distance on $\mathcal{P}(M)$.

To keep the statement compact, we have not given the sharp required bounds on f and f' in the integration by parts formula. If we only wish to control W_1, we can additionally assume the second derivative to be bounded. Higher-dimensional versions of these estimates are available, although typically getting good estimates on the total variation distance in higher dimension is much harder than estimates on W_1, due to regularity issues for solutions to certain Poisson equations.

Let $g : \mathbb{R} \to \mathbb{R}$ be a 1-Lipschitz function. According to Stein's lemma,

$$W_1(\nu, \gamma) \leq 4 \sup_{g \, \text{1-Lipschitz}} \int (g' - xg) d\nu.$$

Actually, the factor 4 is not necessary, if one uses a version of Stein's lemma with sharper constants. We have

$$\int xg(x)d\nu = \int f(x)g(f(x))d\mu = -\lambda^{-1} \int Lf(x)g(f(x))d\mu$$
$$= \lambda^{-1} \int \nabla f \cdot \nabla(g \circ f)d\mu = \lambda^{-1} \int g'(f(x))|\nabla f|^2(x)d\mu$$

Hence, using Lemma 3.3, we get

$$\sup_{g \, \text{1-Lipschitz}} \int (g' - xg)d\nu \leq \lambda^{-1} \int ||\nabla f|^2 - \lambda|d\mu$$
$$\leq 4 \times 2^{\lambda/2}\sqrt{\lambda - 1}.$$

4 Spectral gap and observable diameter

This section aims at proving that for an RCD$(1, \infty)$ space that admits CD$(1, \infty)$ disintegration, the spectral gap is close to that of the Gaussian space if and only if the observable diameter is close to that of the Gauss space.

4.1 Proof of Theorem 1.6

Let f be a normalized eigenfunction of $-\Delta$, with eigenvalue λ_1, as in the previous section. We consider the guiding function in the needle decomposition

$$g := \underset{u \, \text{1-Lipschitz}}{\operatorname{argmax}} \int_M uf d\mu. \tag{4}$$

Without loss of generality, we may assume that g is centered. Note that since f is an eigenfunction, g also maximizes $\int_M \nabla u \cdot \nabla f d\mu$.

The approach follows in the spirit the same one as in [31]. In a first lemma, we estimate the L^2 norm of f and $|\nabla f|$ along many needles.

Lemma 4.1. *Assume $\lambda_1 \leq 1 + \epsilon$ with $\epsilon \leq 1$. There is a set of needles Q_δ such that $\mathfrak{q}(Q_\delta) \geq 1 - \delta$ and for any needle $q \in Q_\delta$ we have*

$$1 - \frac{48\sqrt{\epsilon} + 2\epsilon}{\delta} \leq \int f^2 dm_q \leq 1 + \frac{48\sqrt{\epsilon}}{\delta}$$

and

$$\int f^2 dm_q \leq \int |\nabla f|^2 dm_q \leq \int f^2 dm_q + \frac{2\epsilon}{\delta}.$$

Proof. On $\mathfrak{q}$ a.e. needle q, we have by the Poincaré inequality on the needle

$$\int f^2 dm_q \leq \int |f'|^2 dm_q \leq \int |\nabla f|^2 dm_q.$$

Therefore, by Markov's inequality, we have

$$\mathfrak{q}\left(\left\{q; \int |\nabla f|^2 dm_q \geq \int f^2 dm_q + 2\epsilon/\delta\right\}\right) \leq \frac{\delta\left(\iint |\nabla f|^2 dm_q d\mathfrak{q}(q) - \iint f^2 dm_q d\mathfrak{q}(q)\right)}{2\epsilon}$$

$$\leq \delta/2.$$

So for an arbitrarily large proportion of needles, we almost have equality in Poincaré's inequality for f, up to a deficit of order ϵ.

Then from Lemma 3.3 with $p = 1$,

$$\int_M \left|\int_M (|\nabla f|^2 - 1) dm_q\right| d\mathfrak{q}(q) \leq \int_M ||\nabla f|^2 - 1| d\mu \leq 24\sqrt{\epsilon}.$$

Hence

$$\mathfrak{q}\left(\left\{q; \left|\int |\nabla f|^2 dm_q - 1\right| \geq 48\sqrt{\epsilon}/\delta\right\}\right) \leq \delta/2.$$

Taking the intersection, we get a set of needles of mass at least $1 - \delta$ on which

$$1 - \frac{48\sqrt{\epsilon}}{\delta} \leq \int |\nabla f|^2 dm_q \leq 1 + \frac{48\sqrt{\epsilon}}{\delta}$$

and

$$\int f^2 dm_q \leq \int |\nabla f|^2 dm_q \leq \int f^2 dm_q + \frac{2\epsilon}{\delta}.$$

Reordering these bounds leads to the desired result. $\qquad\square$

Next, we prove the L^2 closeness of f and g for a large set of needles.

Lemma 4.2. *For any $q \in Q_\delta$, there exists a constant c_q such that*

$$\int (f - (g - c_q))^2 dm_q \leq \frac{C\sqrt{\epsilon}}{\sqrt{\delta}}.$$

Remark 4.1. *Recall that for $\mathfrak{q}$-a.e. q, $\int_{X_q} f \, dm_q = 0$. Since the average is an L^2 projection, we can therefore replace c_q by $\int_{X_q} g \, dm_q$ in the above estimate. Moreover, we can infer from the lemma above that*

$$\left| \int_{X_q} g \, dm_q - c_q \right| \leq \frac{C\epsilon^{1/4}}{\delta^{1/4}}. \tag{5}$$

Proof. Let $\Theta = \left(\int_{X_q} f^2 dm_q \right)^{-1/2}$. Recall that we can identify $\mathfrak{q}$ a.e. needle X_q with the space $(\mathbb{R}, |\cdot|, e^{-\psi_q} dt)$, by Proposition 2.5. Using the results of [17] on stability of the Bakry-Émery theorem on an Euclidean space, and since $\int_{X_q} f dm_q = 0$, we have

$$\int_{\mathbb{R}} (\Theta f - t)^2 e^{-\psi_q(t)} dt \leq \frac{\sqrt{18\epsilon}}{\sqrt{\delta}}$$

and therefore, from the parameterization of g on the needle,

$$\int_{\mathbb{R}} (\Theta f - (g - c_q))^2 e^{-\psi_q(t)} dt \leq \frac{\sqrt{18\epsilon}}{\sqrt{\delta}}.$$

From Lemma 4.1, and provided $\sqrt{\epsilon}/\delta \leq C_0$, we have $|\Theta^2 - 1| \leq C\sqrt{\epsilon}/\delta$; therefore

$$|\Theta - 1|^2 \leq C\epsilon/\delta^2.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}} (f - (g - c_q))^2 e^{-\psi_q(t)} dt \quad &\leq \quad 2(\Theta - 1)^2 \int_{\mathbb{R}} f^2 e^{-\psi_q(t)} dt \\
&\qquad\qquad + 2 \int_{\mathbb{R}} (\Theta f - (g - c_q))^2 e^{-\psi_q(t)} dt, \\
&\leq \quad C\sqrt{\epsilon}/\sqrt{\delta}.
\end{aligned}$$

$\square$

In the last technical lemma of this section, we show that g is almost centered on a large set of needles.

Lemma 4.3. *There exists a set of needles $\overline{Q}_\alpha$ with $\mathfrak{q}(\overline{Q}_\alpha) \geq 1 - \alpha$ such that for any $q \in \overline{Q}_\alpha$ we have*

$$\left| \int_{X_q} g \, dm_q \right|^2 \leq C\epsilon^{1/5}/\alpha$$

Proof. Recall that we assumed g is centered with respect to μ. Therefore, since g is 1-Lipschitz, the L^2 version of the Poincaré inequality yields

$$1 \geq \mathrm{Var}_\mu(g) = \int_Q \mathrm{Var}_{m_q}(g) d\mathfrak{q}(q) + \int_Q \left(\int_{X_q} g dm_q \right)^2 d\mathfrak{q}(q).$$

If $q \in Q_\delta$ (which is still the set provided by Lemma 4.1) then

$$\mathrm{Var}_{m_q}(g) \geq \left(\mathrm{Var}_{m_q}(f)^{1/2} - \mathrm{Var}_{m_q}(f-g)^{1/2}\right)^2 \geq 1 - C\epsilon^{1/4}/\delta^{1/4},$$

where we use Lemma 4.2 and (5) to get the inequality. Consequently, we obtain

$$\int \mathrm{Var}_{m_q}(g)d\mathfrak{q}(q) \geq \mathfrak{q}(Q_\delta)(1 - C\epsilon^{1/4}/\delta^{1/4}) \geq (1-\delta)(1 - C\epsilon^{1/4}/\delta^{1/4}),$$

so that

$$\int \left(\int g \, dm_q\right)^2 d\mathfrak{q}(q) \leq 1 - (1-\delta)(1 - C\epsilon^{1/4}/\delta^{1/4}) \leq \delta + C\epsilon^{1/4}/\delta^{1/4}.$$

Taking δ of order $\epsilon^{1/5}$ and using Markov's inequality yields the result. $\qquad\square$

We finally prove the H^1-closeness of f and g provided that the spectral gap is close to 1.

Theorem 4.4. *Let (M, d, μ) be an $RCD(1, \infty)$ space admitting $CD(1, \infty)$ disintegration. Assume $\lambda_1 \leq 1 + \epsilon$ with $\epsilon < 1$. For any θ small enough, there exists $C(\theta) > 0$ such that*

$$\int_M (f-g)^2 d\mu \leq C(\theta)\epsilon^{1/10-\theta}$$

and

$$\int_M |\nabla f - \nabla g|^2 d\mu \leq C(\theta)\epsilon^{1/20-\theta}.$$

Proof. Let us decompose $\|f - g\|_{L^2}^2$ as follows:

$$\int_M (f-g)^2 d\mu = \int_{Q_\delta \cap \overline{Q}_\alpha} \int (f-g)^2 dm_q d\mathfrak{q}(q) + \int_{(Q_\delta \cap \overline{Q}_\alpha)^c} \int (f-g)^2 dm_q d\mathfrak{q}(q).$$

To estimate the first term on the right hand side, we build on Lemmas 4.2 and 4.3:

$$\int_{Q_\delta \cap \overline{Q}_\alpha} \int (f-g)^2 dm_q d\mathfrak{q}(q)$$

$$= \int_{Q_\delta \cap \overline{Q}_\alpha} Var_{m_q}(f-g)d\mathfrak{q}(q) + \int_{Q_\delta \cap \overline{Q}_\alpha} \left(\int f - g \, dm_q\right)^2 d\mathfrak{q}(q)$$

$$\leq \int_{Q_\delta \cap \overline{Q}_\alpha} \int (f - (g - c_q))^2 dm_q d\mathfrak{q}(q) + \int_{Q_\delta \cap \overline{Q}_\alpha} \left(\int g \, dm_q\right)^2 d\mathfrak{q}(q)$$

$$\leq C\sqrt{\epsilon}/\sqrt{\delta} + C\epsilon^{1/5}/\alpha.$$

In order to estimate the second term, we set p and q two positive numbers with $p^{-1} + q^{-1} = 1$.

$$\int_{(Q_\delta \cap \overline{Q}_\alpha)^c} \int (f - g)^2 dm_q d\mathfrak{q}(q) \leq \mathfrak{q}((Q_\delta \cap \overline{Q}_\alpha)^c)^{1/p} \left(\int (f - g)^{2q} d\mu \right)^{1/q}$$

$$\leq 2(\alpha + \delta)^{1/p} \left(\left(\int f^{2q} d\mu \right)^{1/q} + \left(\int g^{2q} d\mu \right)^{1/q} \right)$$

$$\leq 2(\alpha + \delta)^{1/p}((2q - 1)^{1+\epsilon} + 16q^2)$$

$$\leq Cq^2(\alpha + \delta)^{1/p}.$$

Here we used Proposition 3.2 to control the L^{2q} norm of f and Proposition 3.1 to control the L^{2q} norm of g.

Combining the two estimates yields

$$\int_M (f - g)^2 d\mu \leq C(q^2(\alpha + \delta)^{(q-1)/q} + \sqrt{\epsilon}/\sqrt{\delta} + C\epsilon^{1/5}/\alpha).$$

We then take α of order $\epsilon^{1/10}$, q large enough and δ of order say $\epsilon^{3/10}$ (the influence of δ is on lower order terms anyway).

For the second estimate, we expand the norm of the gradients as follows:

$$\int_M |\nabla f - \nabla g|^2 d\mu = \int_M |\nabla f|^2 + |\nabla g|^2 - 2\nabla f \cdot \nabla g d\mu$$

$$\leq 2 + \epsilon + 2 \int_M (Lf)g d\mu$$

$$= 2 + \epsilon - 2\lambda_1 \int_M fg d\mu$$

$$= 2 + \epsilon - 2\lambda_1 \int_M f^2 d\mu - 2\lambda_1 \int_M f(g - f) d\mu$$

$$\leq \epsilon + 2\lambda_1 ||f||_{L^2} ||f - g||_{L^2}$$

$$\leq C(\theta)\epsilon^{1/20-\theta}.$$

$\square$

Proceeding as in the proof of Theorem 1.4, we also obtain

Corollary 4.5. *Let (M, d, μ) be an $RCD(1, \infty)$ space admitting $CD(1, \infty)$ disintegration, and whose spectral gap satisfies $\lambda_1 \leq 1 + \epsilon$. Let g be the guiding function as in (4). Then,*

$$W_1(g_\sharp \mu, \gamma) \leq C(\theta)\epsilon^{1/20-\theta}$$

$$d_{TV}(g_\sharp \mu, \gamma) \leq 2C(\theta)\epsilon^{1/20-\theta}$$

We shall use the estimate on the total variation distance to get an estimate on the observable diameter via the following lemma:

Lemma 4.6. *Assume that μ and ν are two probability measures on a metric space (M, d) such that $d_{TV}(\mu, \nu) \leq \epsilon$, where $\epsilon > 0$. Then for any $1 > \kappa > \epsilon$, the following inequality holds*

$$D_{obs}((M, d, \mu); \kappa - \epsilon) \geq D_{obs}((M, d, \nu); \kappa).$$

Proof. We shall argue by contradiction. Fix κ, and assume that $D_{obs}((M, d, \mu); \kappa - \epsilon) < D_{obs}((M, d, \nu); \kappa) - t$ for some $t > 0$. Then for any f 1-Lipschitz there exists $E \subset \mathbb{R}$ such that $\mu(E) \geq 1 - \kappa + \epsilon$ and $\text{diam}(f(E)) \leq D_{obs}(\nu; \kappa) - t$. But then $\nu(E) \geq 1 - \kappa + \epsilon - d_{TV}(\mu, \nu) \geq 1 - \kappa$, and therefore, taking the sup over all f, we get $D_{obs}((M, d, \nu); \kappa) \leq D_{obs}((M, d, \nu); \kappa) - t$ hence a contradiction. $\square$

Since the function g in Corollary 4.5 is 1-Lipschitz, by definition of observable diameter, we infer

$$D_{obs}((M, d, \mu); \cdot) \geq D_{obs}((\mathbb{R}, |\cdot|, g_\sharp \mu); \cdot)$$

on $(0, 1)$ (see [42] for a proof). Therefore, by combining Theorem 1, Corollary 4.5, and Lemma 4.6, we get the proof of Theorem 1.6

4.2 Proof of Theorem 1.7

We assume throughout this part that

$$D_{obs}((M, d, \mu); \kappa) \geq D_{obs}((\mathbb{R}, |\cdot|, \gamma); \kappa) - \epsilon \tag{6}$$

for a given $0 < \kappa < 1$ and $\epsilon > 0$ small enough. Our aim is to show Theorem 1.7, namely that such an RCD space has almost minimal spectral gap.

Let us denote by $V_s(A)$ the s-open neighborhood of a subset A of a given metric space $((\mathbb{R}, |\cdot|)$ in most cases).

As a consequence of Theorem 2.10, we get

$$Sep((M, d, \mu); \kappa/2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2) - \epsilon. \tag{7}$$

Instead of the observable diameter, we shall mainly deal with the separation distance through (7). The proof mostly relies on a careful study of the 1D case. We will then infer the bound on the spectral gap by means of the disintegration property.

In what follows, $(\mathbb{R}, |\cdot|, m)$ stands for an RCD$(1, \infty)$ space, namely $m = e^{-\phi(x)} dx$ is a probability measure such that $\phi(x) - x^2/2$ is convex.

4.2.1 The $1D$ case

In this part we assume that $(\mathbb{R}, |\cdot|, m)$ is an RCD$(1, \infty)$ space such that for some $1 > \theta > 0$ and for $\eta > 0$ sufficiently small

$$D_{obs}((\mathbb{R}, |\cdot|, m); \theta) \geq D_{obs}((\mathbb{R}, |\cdot|, \gamma); \theta) - \eta. \tag{8}$$

Let us start with some basic properties regarding the Gauss measure. Recall that $\sigma(x) = \gamma(] - \infty, x])$.

Lemma 4.7. *For $0 < \kappa_1 \leq a < b < 1/2$ where κ_1 is fixed and sufficiently small, the following estimates hold*

$$\sqrt{2\pi}(b-a) \leq \sigma^{-1}(b) - \sigma^{-1}(a) \leq \frac{\sqrt{2\pi}}{\kappa_1^2}(b-a). \tag{9}$$

Proof. We denote by $e^{-\phi_\gamma}$ the density of the standard Gauss measure on $\mathbb{R}$. By definition of σ^{-1}, we have

$$\int_{\sigma^{-1}(a)}^{\sigma^{-1}(b)} e^{-\phi_\gamma(s)} \, ds = b - a.$$

The first inequality follows readily from this equality. For the second one, we need to estimate $|\sigma^{-1}(\kappa_1)|$ from above. We write for $\kappa_1 < 1/2$

$$\begin{aligned}
\kappa_1 &= \int_{-\infty}^{\sigma^{-1}(\kappa_1)} e^{-\phi_\gamma(s)} \, ds \\
&\leq \frac{1}{\sqrt{2\pi}|\sigma^{-1}(\kappa_1)|} e^{-(\sigma^{-1}(\kappa_1))^2/2}.
\end{aligned}$$

From the inequality above, we infer $|\sigma^{-1}(\kappa_1)| \leq 2\sqrt{-\ln \kappa_1}$ as soon as $\sigma^{-1}(\kappa_1) \leq -1$, which in turn implies the needed lower bound on $e^{-\phi_\gamma(s)}$ on $[\sigma^{-1}(a), \sigma^{-1}(b)]$ for $\kappa_1 < \gamma(]-\infty, -1])$.

$\square$

Remark 4.1. *Recall that $Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) = 2|\sigma^{-1}(\theta/2)|$. Thus, the preceding lemma also gives bounds on the separation distance on the Gauss space .*

Easy computations assert that the interval $[\sigma^{-1}(\theta/2), -\sigma^{-1}(\theta/2)]$ is the only closed interval of given length $R := 2|\sigma^{-1}(\theta/2)|$ whose γ-measure is maximal, namely $1-\theta$. More precisely, there exists a constant $C(R)$, depending continuously on $R > 0$, such that

$$|\gamma([s - R/2, s + R/2]) - (1-\theta)| \geq C(R)\, s^2, \tag{10}$$

for any s sufficiently small (say, such that $\gamma([s - R/2, s + R/2]) \geq 1/2$).

Lemma 4.8. *Let $(\mathbb{R}, |\cdot|, m)$ be a $CD(1, \infty)$ space. Let $A_1, A_2 \subset \mathbb{R}$ be two Borel subsets. Then, the following inequality holds*

$$d(A_1, A_2) \leq -\sigma^{-1}(m(A_1)) - \sigma^{-1}(m(A_2)).$$

In particular, given $0 < \theta < 1$, and assuming $\min\{m(A_1), m(A_2)\} \geq \theta/2 > 0$ and

$$d(A_1, A_2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta > 0,$$

the following inequality holds

$$\max\{m(A_1), m(A_2)\} \leq \theta/2 + \sqrt{2/\pi}\, \eta.$$

Proof. The proof is a simple consequence of an integrated version of the isoperimetric inequality: given a Borel subset A of $\mathbb{R}$ and $t \geq 0$,

$$m(V_t(A)) \geq \gamma((-\infty, \sigma^{-1}(m(A)) + t]). \tag{11}$$

In particular, for $r = d(A_1, A_2)$,

$$
\begin{aligned}
\gamma((-\infty, \sigma^{-1}(m(A_1)) + r]) &\leq m(V_r(A_1)) \\
&\leq 1 - m(A_2), \\
&\leq \gamma((-\infty, -\sigma^{-1}(m(A_2))]),
\end{aligned}
$$

where the second inequality follows from $A_2 \cap V_r(A_1) = \emptyset$. The inequality relating the expressions at the ends implies

$$\sigma^{-1}(m(A_1)) + r \leq -\sigma^{-1}(m(A_2))$$

and the first inequality is proved.

To prove the second statement let us rewrite the first inequality as

$$d(A_1, A_2) \leq 2|\sigma^{-1}(\theta/2)| + (\sigma^{-1}(\theta/2) - \sigma^{-1}(m(A_1))) + (\sigma^{-1}(\theta/2) - \sigma^{-1}(m(A_2))). \tag{12}$$

We are left with estimating $(\sigma^{-1}(\theta/2) - \sigma^{-1}(m(A_i)))$ from above. According to (9), we have

$$\sigma^{-1}(m(A_i)) - \sigma^{-1}(\theta/2) \geq \sqrt{2\pi}(m(A_i) - \theta/2).$$

By combining (12) together with the above inequality, we get the second claim. $\qquad \square$

Remark 4.2. *The proof above applies verbatim to general $RCD(1, \infty)$ spaces since it is mainly based on the isoperimetric inequality stated in Theorem 2.9. It gives the same results.*

In the same vein, we shall also need the following estimate:

Lemma 4.9. *Let $(\mathbb{R}, |\cdot|, m)$ be a $CD(1, \infty)$ space and $0 < \theta < 1$. Consider $A_1, A_2 \subset \mathbb{R}$ two Borel subsets such that $\min\{m(A_1), m(A_2)\} \geq \theta/2 > 0$ and*

$$r := d(A_1, A_2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta > 0.$$

Then, for any $s \in [0, r]$ and $i \in \{1, 2\}$

$$m((V_s(A_i)) \leq \gamma((-\infty, \sigma^{-1}(m(A_i)) + s]) + \sqrt{2/\pi}\, \eta.$$

Proof. We argue as in the proof of the previous lemma. Set $i = 1$.

$$
\begin{aligned}
m(V_s(A_1)) &\leq 1 - m(V_{r-s}(A_2)), \\
&\leq 1 - \gamma([-\sigma^{-1}(m(A_2)) - r + s, +\infty)) \\
&\leq \gamma((-\infty, -\sigma^{-1}(m(A_2)) + 2\sigma^{-1}(\theta/2) + 2\eta + s]), \\
&\leq \gamma((-\infty, \sigma^{-1}(\theta/2)) + s + 2\eta]),
\end{aligned}
$$

where we use the symmetry of γ, the inequality on $r = d(A_1, A_2)$ to infer the inequality on the third line, and $m(A_1) \geq \theta/2$ to get the estimate in the last one. The proof then follows from $\gamma((c, c + \delta)) \leq \frac{1}{\sqrt{2\pi}}\delta$, where c is arbitrary.

$\square$

Let us add a few remarks regarding the separation distance on $(\mathbb{R}, |\cdot|, m)$ and fix some notation. First notice that if A_1, A_2 are two admissible Borel sets for $Sep((\mathbb{R}, |\cdot|, m); \theta/2)$, then so are the closures $\overline{A_1}$ and $\overline{A_2}$. Without loss of generality, one can assume that the interval $[a_-, a_+]$ is such that $(a_-, a_+,) \cap (A_1 \cup A_2) = \emptyset$ and $a_+ - a_- = d(A_1, A_2)$. Up to reversing A_1 and A_2, we can further assume that $a_- \in A_1$ and $a_+ \in A_2$. If we further assume that $d(A_1, A_2) > Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - \eta > 0$ for $\eta \leq \eta(\theta)$ sufficiently small, we claim that A_1 and A_2 satisfy $A_2 \cap]-\infty, a_-] = \emptyset$ and $A_1 \cap [a_+, +\infty[= \emptyset$. Indeed, if it were otherwise the positive distance between the intertwined sets A_1 and A_2 would give us a set $S \subset (\mathbb{R} \setminus A_1 \cup \mathbb{R} \setminus A_2)$ such that $\text{diam}(S) \geq 2d(A_1, A_2)$; this would imply

$$D_{obs}((\mathbb{R}, |\cdot|, \gamma); \theta/2) \geq D_{obs}((\mathbb{R}, |\cdot|, m); \theta/2) \geq \text{diam}(S) \geq 2D_{\text{obs}}((\mathbb{R}, |\cdot|, \gamma); \theta) - 2\eta$$

according to (8) and Theorem 2.10. Finally, by combining this together with the approximation

$$D_{obs}((\mathbb{R}, |\cdot|, \gamma); \theta) = Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) = 2|\sigma^{-1}(\sqrt{\theta/2})| = 2\sqrt{2\ln(2/\theta)}(1 + \varepsilon(\theta))$$

(with $\lim_{\theta \to 0} \varepsilon(\theta) = 0$) which prevents the observable diameter of γ of almost doubling when θ is replaced by $\theta/2$, we get a contradiction. In the rest of this part, we thus assume that $A_1 = (-\infty, a_-]$ and $A_2 = [a_+, +\infty)$.

Our next goal is to show that ϕ is uniformly close to ϕ_γ on $[a_-, a_+]$. To this aim, we shall use properties of the optimal map T such that $T_\sharp \gamma = m$. Let us recall that T is non-decreasing and 1-Lipschitz, as a consequence of Caffarelli's contraction theorem [10, 20].

We set

$$\alpha_- := \sigma^{-1}(m(A_1)) \text{ and } \alpha_+ := \sigma^{-1}(1 - m(A_2)) = -\sigma^{-1}(m(A_2)), \qquad (13)$$

in other terms $T(\alpha_\pm) = a_\pm$.

Proposition 4.10. *Let* $(\mathbb{R}, |\cdot|, m = e^{-\phi(s)}ds)$ *be a* $CD(1, \infty)$ *space such that for* $1 > \theta > 0$ *and* $0 < \eta < \eta_0(\theta)$,

$$d(A_1, A_2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta > 0.$$

Then, up to translating m *so that the median is at the origin, there exists* $C = C(\theta) > 0$ *such that for any* $s \in [a_-, a_+]$,

$$|\phi(s) - \phi_\gamma(s)| \leq C\eta^{1/2}.$$

Proof. In this proof, $C(\theta)$ will be a constant that only depends on θ, but which may change from line to line. Set $\delta_+ := a_+ - \alpha_+$ and $\delta_- := a_- - \alpha_-$. We first claim that

$$
\begin{aligned}
\phi(s) &\leq \phi_\gamma(s - \delta_+) \quad \text{on } [s_l, a_+] \\
\phi(s) &\leq \phi_\gamma(s - \delta_-) \quad \text{on } [a_-, s_l],
\end{aligned}
$$

where $s_l = \arg\min \phi$ (recall ϕ is 1-convex). We shall only prove the first inequality, the second one is proved in the same way.

Setting the median of m at the origin is equivalent to enforcing $T(0) = 0$, since the median of the Gaussian measure is also at the origin. Thanks to the monotonicity of T and the Lipschitz bound recalled above, we notice that $T(s) \geq s + \delta_+$ on $(-\infty, \alpha_+]$. By combining this together with the fact that $e^{-\phi}$ is decreasing on $[s_l, +\infty)$, we get on $[s_l - \delta_+, \alpha_+]$

$$
e^{-\phi(T(s))} \leq e^{-\phi(s + \delta_+)}.
$$

By definition of T, we have for almost every s,

$$
e^{-\phi_\gamma(s)} = e^{-\phi(T(s))} T'(s) \leq e^{-\phi(T(s))},
$$

and the proof of the inequality is complete by continuity of ϕ and ϕ_γ.

Next, we prove that $\delta_+ \simeq \delta_- \simeq T(0) = 0$. Using that $s \mapsto T(s) - s$ is non-increasing on $[\alpha_-, \alpha_+]$, we get that

$$
0 \leq (T(\alpha_-) - \alpha_-) - (T(\alpha_+)) - \alpha_+) = \delta_- - \delta_+ = (\alpha_+ - \alpha_-) - (a_+ - a_-)
$$

By assumption on $d(A_1, A_2) = a_+ - a_-$ and using that $-\sigma^{-1}$ is non-increasing, we obtain

$$
\begin{aligned}
a_+ - a_- &\geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta \\
&\geq -2\sigma^{-1}(\theta/2) - 2\eta \\
&\geq -\sigma^{-1}(m(A_1)) - \sigma^{-1}(m(A_2)) - 2\eta \\
&\geq \alpha_+ - \alpha_- - 2\eta,
\end{aligned}
$$

where we use (13) to get the last inequality. Therefore, we obtain

$$
\begin{cases}
\quad 0 \leq \delta_- - \delta_+ \leq 2\eta \\
\text{and} \quad \delta_+ \leq T(0) = 0 \leq \delta_-
\end{cases} \tag{14}
$$

by monotonicity. As a consequence, $|\delta_+| + |\delta_-| \leq 2\eta$.

Now recall that by definition of $\alpha_\pm$, $\min(\gamma((-\infty, \alpha_-]), \gamma([\alpha_+, +\infty))) \geq \theta/2$. Moreover, Lemma 4.8 implies

$$
\max(\gamma((-\infty, \alpha_-]), \gamma([\alpha_+, +\infty))) \leq \theta/2 + \sqrt{2/\pi}\,\eta. \tag{15}
$$

Hence, (10) implies the existence of $C(\theta) > 0$ such that

$$
|(\alpha_+ + \alpha_-)|/2 \leq C(\theta)\sqrt{\eta}, \tag{16}
$$

By combining this together with (14), we infer

$$\left| \frac{a_+ + a_-}{2} \right| \leq C\sqrt{\eta}. \tag{17}$$

As a consequence,

$$\begin{aligned}
\phi(s) &\leq \phi_\gamma(s - \delta_+) \\
&= \phi_\gamma(s) - \delta_+ s + \delta_+^2/2 \\
&\leq \phi_\gamma(s) + C(\theta)\eta
\end{aligned} \tag{18}$$

on $[a_-, a_+] \subset [-C(\theta), C(\theta)]$.

The last step of the proof relies on Markov's inequality and the fact that $\phi - \phi_\gamma$ is a convex function. More precisely, given $\nu > 0$, we set

$$S_\nu := \{s \in [a_-, a_+] \, | \, \phi(s) \geq \phi_\gamma(s) - \nu\}.$$

Observe that $(a_-, a_+) \cap (A_1 \cup A_2) = \emptyset$ implies

$$\int_{[a_-, a_+]} e^{-\phi(s)} \, ds \leq 1 - m(A_1) - m(A_2) = \int_{[\alpha_-, \alpha_+]} e^{-\phi_\gamma(s)} \, ds. \tag{19}$$

Besides, (15) implies the existence of $C > 0$ such that

$$\int_{[a_-, a_+]} e^{-\phi_\gamma(s)} \, ds \geq \int_{[\alpha_-, \alpha_+]} e^{-\phi_\gamma(s)} \, ds - C\eta.$$

Using (18), we have

$$\begin{aligned}
\int_{[a_-, a_+]} e^{-\phi(s)} \, ds &\geq e^\nu \int_{S_\nu^c} e^{-\phi_\gamma(s)} \, ds + e^{-C\eta} \int_{S_\nu} e^{-\phi_\gamma(s)} \, ds \\
&\geq (e^\nu - 1) \int_{S_\nu^c} e^{-\phi_\gamma(s)} \, ds + e^{-C\eta} \left(\int_{[\alpha_-, \alpha_+]} e^{-\phi_\gamma(s)} \, ds - C\,\eta \right).
\end{aligned}$$

This estimate and (19) yield

$$\gamma(S_\nu^c) \leq C \, \frac{\eta}{e^\nu - 1} \leq C \, \frac{\eta}{\nu},$$

which suggests to choose $\nu = \eta^{1/2}$.

Now, the convexity of $h := \phi - \phi_\gamma$ forces S_ν^c to be an interval, and the Lebesgue measure of $S_{\eta^{1/2}}^c$ is bounded from above by $C(\theta)\eta^{1/2}$. Thales' theorem and the convexity of h then imply $\min h \geq -C(\theta)\eta^{1/2}$ and the proof is complete.

$\square$

Let us set $a_\eta := |\sigma^{-1}(\sqrt{\eta})|$. Our goal is to extend the estimate on $|\phi - \phi_\gamma|$ from $[a_-, a_+]$ to a much larger interval (depending on η). This is the content of the next lemma.

Lemma 4.11. *Let* $(\mathbb{R}, |\cdot|, m = e^{-\phi(s)}ds)$ *be a* $CD(1, \infty)$ *space such that for* $1 > \theta > 0$ *and* $0 < \eta < \eta_0(\theta)$

$$Sep((\mathbb{R}, |\cdot|, m); \theta/2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta > 0.$$

Then, up to translating m *so that the median is at the origin, there exists* $C = C(\theta) > 0$ *and* $b_\eta > 0$ *such that, for any* $s \in [-b_\eta, b_\eta]$,

$$|\phi(s) - \phi_\gamma(s)| \leq C\eta^{1/10},$$

and $m((-\infty, -b_\eta]) \leq \gamma((-\infty, -b_\eta]) \leq C\eta^{1/10}.$

Proof. First of all, by definition of a_η, $\gamma((-\infty, -a_\eta]) = \sqrt{\eta}$, and since the median of m is at the origin, the Gaussian concentration inequality (which is the integrated form of the isoperimetric inequality for RCD$(1, \infty)$ spaces states that $m((-\infty, -a_\eta]) \leq \gamma((-\infty, -a_\eta])$.

Set $h := \phi - \phi_\gamma$. Recall that $|h| \leq C\eta^{1/2}$ on $[a_-, a_+]$ and that h is convex. Markov's inequality yields the existence of $\hat{a}_+ \in [a_-, a_+]$ such that

$$|\phi'(\hat{a}_+) - \phi'_\gamma(\hat{a}_+)| \leq C\eta^{1/4} \text{ and } |a_+ - \hat{a}_+| \leq \eta^{1/4}.$$

For $s \geq \hat{a}_+$, the 1-convexity of ϕ yields

$$\phi(s) \geq \phi(\hat{a}_+) + \phi'(\hat{a}_+)(s - \hat{a}_+) + 1/2(s - \hat{a}_+)^2.$$

Noticing that $\phi_\gamma(\hat{a}_+) + \phi'_\gamma(\hat{a}_+)(s - \hat{a}_+) + 1/2(s - \hat{a}+)^2 = \phi_\gamma()$, we infer from the previous inequality:

$$\phi(s) \geq \phi_\gamma(s) + E(s),$$

where $E(s) = -C\eta^{1/4}(1 + (s - \hat{a}_+))$.

Recall that $a_\eta = |\sigma^{-1}(\sqrt{\eta})| \sim \sqrt{\ln(1/\eta)}$ for $\eta \sim 0$ thus, for η small enough, there exists a constant C such that

$$a_\eta \leq C\sqrt{\ln(1/\eta)}, \tag{20}$$

and there exists a constant C such that

$$E(s) \geq -C\eta^{1/4}(1 + a_\eta) \geq -C\eta^{1/5}$$

on $[\hat{a}_+, a_\eta]$. Consequently,

$$\phi \geq \phi_\gamma - C\eta^{1/5} \tag{21}$$

on $[a_+, a_\eta] \subset [\hat{a}_+, a_\eta]$.

The same method applies on $[-a_\eta, \hat{a}_-]$ (where the definition of $\hat{a}_-$ is clear from the context) and leads to the same estimate $\phi \geq \phi_\gamma - C\eta^{1/5}$.

We then have, using the estimates on the interval $[a_-, a_+]$ and that $|a_+ - \hat{a}_+| \leq \eta^{1/4}$,

$$\left| \int_{[-a_\eta, a_\eta] \setminus [a_-, a_+]} e^{-\phi(s)} - e^{-\phi_\gamma(s)} \, ds \right| = |\gamma([a_-, a_+]) - m([a_-, a_+])$$
$$+ \gamma([-a_\eta, a_\eta]^c) - m([-a_\eta, a_\eta]^c)|$$
$$\leq 2\gamma([a_\eta, +\infty)) + m((-\infty, -a_\eta])$$
$$+ m([a_\eta, +\infty)) + C\sqrt{\eta}$$
$$\leq C\sqrt{\eta}.$$

As in the proof of Proposition 4.10, Markov's inequality and the estimates above imply that

$$S_\eta := \{s \in [-a_\eta, a_\eta] \setminus [a_-, a_+] | \, (\phi - \phi_\gamma)(s) \leq \eta^{1/10}\}$$

satisfies

$$\gamma(S_\eta^c) \leq C\eta^{1/10}.$$

Finally, the convexity of $\phi - \phi_\gamma$ implies that S_η is the union of two intervals; besides, thanks to Proposition 4.10, S_η contains a neighborhood of a_- and a_+. Therefore S_η^c is the union of two intervals $[-a_\eta, -b_\eta] \cup [b_\eta, a_\eta]$. We also have

$$m((-\infty, -b_\eta]) \leq \gamma((-\infty, -b_\eta]) \leq \gamma((-\infty, -a_\eta]) + \gamma((-a_\eta, -b_\eta]) \leq C\eta^{1/10}.$$

$\square$

We are left with estimating the variance of the identity map relative to the measure m.

Lemma 4.12. *Let* $(\mathbb{R}, |\cdot|, m = e^{-\phi(s)}ds)$ *be a* $CD(1, \infty)$ *space such that for* $1 > \theta > 0$ *and* $0 < \eta < \eta_0(\theta)$,

$$Sep((\mathbb{R}, |\cdot|, m); \theta/2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \theta/2) - 2\eta > 0.$$

Then, up to translating m so that its median is at the origin, the following estimates hold

$$\left| \int_\mathbb{R} s \, dm(s) \right| \leq C\eta^{1/11},$$

and

$$\int_\mathbb{R} s^2 \, dm(s) \geq 1 - C\eta^{1/11}.$$

In particular,

$$Var_m(s) \geq 1 - C\eta^{1/11}.$$

Proof. Let us start with some estimates relative to the Gaussian measure. Using

$$\frac{e^{-\phi_\gamma}(\mathrm{b}_\eta)}{\mathrm{b}_\eta} \sim \int_{\mathrm{b}_\eta}^{+\infty} e^{-\phi_\gamma(s)}\,ds \le C\eta^{1/10},$$

we infer, for η small enough,

$$\int_{\mathrm{b}_\eta}^{+\infty} s\,e^{-\phi_\gamma(s)}\,ds = e^{-\phi_\gamma(\mathrm{b}_\eta)} \le C\,\mathrm{b}_\eta\,\eta^{1/10} \le C\eta^{1/10}|\sigma^{-1}(\eta^{1/10})| \le C\eta^{1/11}.$$

Moreover, thanks to $\int_{\mathrm{b}_\eta}^{+\infty} s^2\,e^{-\phi_\gamma(s)}\,ds = \mathrm{b}_\eta\,e^{-\phi_\gamma(\mathrm{b}_\eta)} + \int_{\mathrm{b}_\eta}^{+\infty} e^{-\phi_\gamma(s)}\,ds$, we get

$$\int_{\mathrm{b}_\eta}^{+\infty} s^2\,e^{-\phi_\gamma(s)}\,ds \le C\eta^{1/11}.$$

Now, we use Lemma 4.11 to infer similar bounds for m. More precisely,

$$
\begin{aligned}
\left|\int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} s\,e^{-\phi(s)}\,ds - \int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} s\,e^{-\phi_\gamma(s)}\,ds\right| &\le \int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} |s|\big(e^{|\phi(s)-\phi_\gamma(s)|}-1\big)e^{-\phi_\gamma(s)}\,ds \\
&\le \int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} |s|\big(e^{C\eta^{1/10}}-1\big)e^{-\phi_\gamma(s)}\,ds \\
&\le 2(1/\sqrt{2\pi}-e^{-\phi_\gamma(\mathrm{b}_\eta)})C\eta^{1/10} \le C\eta^{1/10}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\left|\int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} s^2 e^{-\phi(s)}\,ds - \int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} s^2 e^{-\phi_\gamma(s)}\,ds\right| &\le \int_{[-\mathrm{b}_\eta,\mathrm{b}_\eta]} s^2\big(e^{C\eta^{1/10}}-1\big)e^{-\phi_\gamma(s)}\,ds \\
&\le C\eta^{1/10}.
\end{aligned}
$$

Last, we claim that $0 \le T(s) \le s$ (resp. $0 \ge T(s) \ge s$) on $[\mathrm{b}_\eta,+\infty) \subset [\alpha_+,+\infty)$ (resp. on $(-\infty,-\mathrm{b}_\eta]$), since T is monotone and 1-Lipschitz and $T(0) = 0$.

With this property, we can estimate the remaining part of the expectation of m:

$$
\begin{aligned}
\int_{[\mathrm{b}_\eta,+\infty]} s\,e^{-\phi(s)}\,ds &\le \int_{[T^{-1}(\mathrm{b}_\eta),+\infty)} T(u)\,e^{-\phi_\gamma(u)}\,du \\
&\le \int_{[\mathrm{b}_\eta,+\infty)} u e^{-\phi_\gamma(u)}\,du \\
&\le C\eta^{1/11},
\end{aligned}
$$

where we use the upper tail bound for the Gaussian measure. The same method applies on $(-\infty,-\mathrm{b}_\eta]$, and the proof is complete. $\qquad\square$

4.2.2 The general case

According to (7), there exists sets A_1 and A_2 such that

$$d(A_1, A_2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2) - 2\epsilon,$$

and $\min(\mu(A_1), \mu(A_2)) \geq \kappa/2$.

We now introduce a needle decomposition relative to a function f defined in terms of the sets A_1 and A_2 above. Precisely, we set

$$f = \chi_{A_1} - \chi_{A_2} - \mu(A_1) + \mu(A_2). \tag{22}$$

This function satisfies

$$\int_M |f(x)| d(x, x_0) \, d\mu < \infty \text{ and } \int_M f \, d\mu = 0.$$

Applying the needle decomposition associated to f defined on (M, d, μ), we get that for $\mathfrak{q}$ a.e q, $\int_{X_q} f \, dm_q = 0$. This yields for a.e q

$$m_q(A_1 \cap X_q) - m_q(A_2 \cap X_q) = \mu(A_1) - \mu(A_2).$$

Therefore, according to Lemma 4.8 (see Remark 4.2), we get

$$|m_q(A_1 \cap X_q) - m_q(A_2 \cap X_q)| \leq \sqrt{2/\pi} \, \epsilon. \tag{23}$$

From this estimate, we infer the following result

Lemma 4.13. *Let (M, d, μ) be an $RCD(1, \infty)$ space admitting $CD(1, \infty)$ disintegration and such that (6) holds. Then, relatively to the needle decomposition associated to f as in (22), there exists a set of needles Q_δ such that $\mathfrak{q}(Q_\delta) \geq 1 - \delta$, and for any needle $q \in Q_\delta$, we have*

$$\min\{m_q(A_1), m_q(A_2)\} \geq \kappa/2 - \frac{\sqrt{8/\pi} \, \epsilon}{\delta} - \sqrt{2/\pi} \, \epsilon. \tag{24}$$

Proof. Since we assume there is a needle decomposition, there exists a partition of M: $M = \mathrm{T} \sqcup Z$ such that $\mu|_\mathrm{T}$ admits a needle decomposition and $f = 0$ μ-a.e. on Z. Now, by the very definition of f, $Z \subset (A_1 \cup A_2)^c$ (actually $Z = \emptyset$ whenever $\mu(A_1) \neq \mu(A_2)$). Therefore, we obtain

$$\mu(A_i) = \int_\mathrm{T} \chi_{A_i} \, d\mu = \int_Q m_q(A_i) \, d\mathfrak{q}. \tag{25}$$

Recall that $\mathfrak{q}$ a.e. needle is in particular a geodesic, thus the distance between $A_1 \cap X_q$ and $A_2 \cap X_q$ along X_q is greater or equal to $d(A_1, A_2)$, where

$$d(A_1, A_2) \geq Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2) - 2\epsilon > Sep((\mathbb{R}, |\cdot|, \gamma); \kappa/2 + \sqrt{2/\pi}\epsilon)$$

according to (9) and Remark 4.1. Using Theorem 2.10, we infer

$$d(A_1, A_2) > Sep((X_q, |\cdot|, m_q); \kappa/2 + \sqrt{2/\pi}\epsilon).$$

This inequality yields

$$\min\{m_q(A_1), m_q(A_2)\} \leq \kappa/2 + \sqrt{2/\pi}\epsilon,$$

and we can replace min by max in the above equation thanks to (23), provided that $\sqrt{2/\pi}\epsilon$ is replaced by $2\sqrt{2/\pi}\epsilon$.

The end of the proof consists in applying Markov's inequality in (25) combined with $\max\{m_q(A_1), m_q(A_2)\} \leq \kappa/2 + \sqrt{8/\pi}\epsilon$.

Defining $Q_\delta := \{q \in Q \mid m_q(A_1) \geq \kappa/2 - \frac{\sqrt{8/\pi}\,\epsilon}{\delta}\}$, we get

$$
\begin{aligned}
\mu(A_1) &\leq \int_{Q_\delta^c} m_q(A_1)\, d\mathfrak{q} + \int_{Q_\delta} m_q(A_1)\, d\mathfrak{q} \\
&< (\kappa/2 - \frac{\sqrt{8/\pi}\epsilon}{\delta})\mathfrak{q}(Q_\delta^c) + (\kappa/2 + \sqrt{8/\pi}\epsilon)(1 - \mathfrak{q}(Q_\delta^c)),
\end{aligned}
$$

from which we infer (since $\mu(A_1) \geq \kappa/2$)

$$\mathfrak{q}(Q_\delta^c) \leq \delta.$$

Applying (23) yields the result.

$\square$

For $\mathfrak{q}$ a.e. $q \in Q_{\sqrt{\epsilon}}$, there exists a constant $\widehat{C} > 0$ such that

$$\min\{m_q(A_1), m_q(A_2)\} \geq \kappa/2 - \widehat{C}\sqrt{\epsilon},$$

while, as explained in the proof above, for almost every q,

$$d(A_1 \cap X_q, A_2 \cap X_q) \geq d(A_1, A_2) \geq Sep((X_q, |\cdot|, m_q); \kappa/2) - 2\epsilon.$$

Besides, using (9) and Remark 4.1, we obtain

$$Sep((X_q, |\cdot|, m_q); \kappa/2) - 2\epsilon \geq Sep((X_q, |\cdot|, m_q); \kappa/2 - \widehat{C}\sqrt{\epsilon}) - C\sqrt{\epsilon}.$$

According to Lemma 4.12, this implies for $\mathfrak{q}$ a.e $q \in Q_{\sqrt{\epsilon}}$ the existence of a constant $C = C(\kappa) > 0$ such that

$$
\begin{aligned}
Var_{m_q}(s) &\geq 1 - C\epsilon^{1/22}, \\
\text{and} \qquad \mathfrak{q}(Q_{\sqrt{\epsilon}}) &\geq 1 - \sqrt{\epsilon}.
\end{aligned}
\tag{26}
$$

4.2.3 Proof of Theorem 1.7

The proof builds upon the decomposition relative to f as in (22). According to Theorem 2.4, there exists a guiding function u which is 1-Lipschitz, and such that for $\mathfrak{q}$ a.e. $q \in Q$, $u(s) = s + c_q$ on X_q (with $c_q \in \mathbb{R}$). As a consequence, we have

$$\int_M |\nabla u|^2\, d\mu \leq 1,$$

and

$$
\begin{aligned}
Var_\mu(u) &= \int_M u^2 - \left(\int_M u\,d\mu\right)^2 d\mu \\
&\geq \int_Q \int_{X_q} u^2\,dm_q\,d\mathfrak{q}(q) - \int_Q \left(\int_{X_q} u\,dm_q\right)^2 d\mathfrak{q}(q) \\
&= \int_Q Var_{m_q}(u)\,d\mathfrak{q}(q),
\end{aligned}
$$

where the inequality follows from Cauchy-Schwarz. Using (26), we then get

$$
Var_\mu(u) \geq \mathfrak{q}(Q_{\sqrt{\epsilon}})\left(1 - C\epsilon^{1/22}\right) \geq 1 - C'\epsilon^{1/22} \geq (1 - C'\epsilon^{1/22})\int_M |\nabla u|^2 d\mu,
$$

which gives the upper bound on the spectral gap.

References

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, 195(2):289–391, 2014.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.*, 163(7):1405–1490, 2014.

[3] Luigi Ambrosio and Andrea Mondino. Gaussian-type isoperimetric inequalities in $\mathsf{RCD}(K, \infty)$ probability spaces for positive K. *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.*, 27(4):497–514, 2016.

[4] Erwann Aubry. Pincement sur le spectre et le volume en courbure de Ricci positive. *Ann. Sci. École Norm. Sup. (4)*, 38(3):387–405, 2005.

[5] D. Bakry. Transformations de Riesz pour les semi-groupes symétriques. II: Étude sous la condition $\Gamma_2 \geq 0$. Sémin. de probabilités XIX, Univ. Strasbourg 1983/84, Proc., Lect. Notes Math. 1123, 145-174 (1985)., 1985.

[6] Dominique Bakry. L'hypercontractivité et son utilisation en théorie des semi-groupes. In *Lectures on probability theory. Ecole d'Eté de Probabilités de Saint- Flour XXII-1992. Summer School, 9th-25th July, 1992, Saint-Flour, France*, pages 1–114. Berlin: Springer, 1994.

[7] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.

[8] Jérôme Bertrand, Christian Ketterer, Ilaria Mondello, and Thomas Richard. Stratified spaces and synthetic Ricci curvature bounds. *To appear in Ann. Inst. Fourier*, 2018.

[9] Christer Borell, The Brunn-Minkowski inequality in Gauss space. *Invent. Math.* 30 (1975), no. 2, 207–216.

[10] Luis A. Caffarelli. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Commun. Math. Phys.*, 214(3):547–563, 2000.

[11] F. Cavalletti, N. Gigi, and F. Santarcangelo. Displacement convexity of entropy and the distance cost optimal transportation. *to appear in Annales de la Faculté Scientifique de Toulouse*, 2020.

[12] F. Cavalletti, A. Mondino, and D. Semola. Quantitative obata's theorem. *Arxiv preprint*, 2019.

[13] Fabio Cavalletti and Emanuel Milman. The globalization theorem for the curvature-dimension condition. *to appear in Invent. Math.*, 2021.

[14] Fabio Cavalletti and Andrea Mondino. Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *Invent. Math.*, 208(3):803–849, 2017.

[15] Shiu Yuen Cheng. Eigenvalue comparison theorems and its geometric applications. *Math Z.*, 143(3):1289–297, 1975.

[16] Xu Cheng and Detang Zhou. Eigenvalues of the drifted Laplacian on complete metric measure spaces. *Commun. Contemp. Math.*, 19(1):1650001, 17, 2017.

[17] Thomas A. Courtade and Max Fathi. Stability of the Bakry-Émery theorem on $\mathbb{R}^n$. *J. Funct. Anal.*, 279(2):108523, 28, 2020.

[18] Christopher B. Croke. An eigenvalue pinching theorem. *Invent. Math.*, 68(2):253–256, 1982.

[19] Guido De Philippis and Alessio Figalli. Rigidity and stability of Caffarelli's log-concave perturbation theorem. *Nonlinear Anal.*, 154:59–70, 2017.

[20] Max Fathi, Nathael Gozlan, and Maxime Prod'homme. A proof of the Caffarelli contraction theorem via entropic regularization. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 96, 18, 2020.

[21] Nicola Gigli. An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature. *Anal. Geom. Metr. Spaces*, 2:169–213, 2014.

[22] Nicola Gigli. *Nonsmooth differential geometry – an approach tailored for spaces with Ricci curvature bounded from below*, volume 1196. Providence, RI: American Mathematical Society (AMS), 2018.

[23] Nicola Gigli, Christian Ketterer, Kazumasa Kuwada, and Shin-ichi Ohta. Rigidity for the spectral gap on $\text{Rcd}(K, \infty)$-spaces. *Amer. J. Math.*, 142(5):1559–1594, 2020.

[24] Nicola Gigli, Andrea Mondino, and Giuseppe Savaré. Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. *Proc. Lond. Math. Soc. (3)*, 111(5):1071–1129, 2015.

[25] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces. Transl. from the French by Sean Michael Bates. With appendices by M. Katz, P. Pansu, and S. Semmes. Edited by J. LaFontaine and P. Pansu*, volume 152. Boston, MA: Birkhäuser, 1999.

[26] Bang-Xian Han. Rigidity of some functional inequalities on RCD spaces. *J. Math. Pures Appl. (9)*, 145:163–203, 2021.

[27] Christian Ketterer. Cones over metric measure spaces and the maximal diameter theorem. *J. Math. Pures Appl. (9)*, 103(5):1228–1275, 2015.

[28] Christian Ketterer. Obata's rigidity theorem for metric measure spaces. *Anal. Geom. Metr. Spaces*, 3:278–295, 2015.

[29] Bo'az Klartag. Needle decompositions in Riemannian geometry. *Mem. Amer. Math. Soc.*, 249(1180):v+77, 2017.

[30] André Lichnerowicz, Géométrie des groupes de transformations. (French) Travaux et Recherches Mathématiques, III. Dunod, Paris 1958 ix+193 pp.

[31] C.H. Mai and S. Ohta. Quantitative estimates for the bakry-ledoux isoperimetric inequality. *to appear in Comment. Math. Helv.*, 2021.

[32] Elizabeth Meckes. On the approximate normality of eigenfunctions of the Laplacian. *Trans. Am. Math. Soc.*, 361(10):5377–5399, 2009.

[33] Emanuel Milman. On the role of convexity in isoperimetry, spectral gap and concentration. *Invent. Math.*, 177(1):1–43, 2009.

[34] Emanuel Milman. Spectral estimates, contractions and hypercontractivity. *J. Spectr. Theory*, 8(2):669–714, 2018.

[35] Shin-ichi Ohta and Asuka Takatsu. Equality in the logarithmic Sobolev inequality. *Manuscripta Math.*, 162(1-2):271–282, 2020.

[36] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Ration. Mech. Anal.*, 5:286–292, 1960.

[37] Peter Petersen. On eigenvalue pinching in positive Ricci curvature. *Invent. Math.*, 138(1):1–21, 1999.

[38] Anton Petrunin. Alexandrov meets Lott-Villani-Sturm. *Münster J. Math.*, 4:53–64, 2011.

[39] Prékopa, András On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)* 34 (1973), 335–343.

[40] Nathan Ross. Fundamentals of Stein's method. *Probab. Surv.*, 8:210–293, 2011.

[41] Giuseppe Savaré. Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $\mathrm{RCD}(K, \infty)$ metric measure spaces. *Discrete Contin. Dyn. Syst.*, 34(4):1641–1661, 2014.

[42] Takashi Shioya. *Metric measure geometry*, volume 25 of *IRMA Lectures in Mathematics and Theoretical Physics*. EMS Publishing House, Zürich, 2016. Gromov's theory of convergence and concentration of metrics and measures.

[43] Asuka Takatsu. Spectral convergence of high-dimensional spheres to gaussian spaces. *preprint*, 2021.

[44] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[45] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.