ADAMS INEQUALITY ON PINCHED HADAMARD MANIFOLDS

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Abstract. In this article we prove the Adams type inequality for $W^{m,p}(M)$ functions, where m is an even integer and (M, g) is a Hadamard manifold with Ricci curvature bounded from below and sectional curvature bounded from above by a negative constant.

1. Introduction

In this article we focus on the Adams inequality on Hadamard manifolds. Recall, a Hadamard manifold is a complete simply connected manifold of nonpositive sectional curvature and Adams inequalities are the optimal Sobolev embedding of the Sobolev space $W^{k,p}$ when $kp = n$ where n is the dimension of the space.

There are many works on Sobolev embeddings on Riemannian manifolds and we know in particular that the Sobolev embedding holds when the manifold is compact. To be precise, let (M, g) be a compact Riemannian manifold then the Sobolev embedding states that the Sobolev space $W^{k,p}(M)$ is continuously embedded in to $L^q(M)$ where $q = \frac{np}{n-kp}$ provided $1 \leq p < \frac{n}{k}$. The precise inequalities with precise constants describing these embeddings are of importance in both partial differential equations and geometric analysis and has been a hot topic of research for the past many decades (see [14] and the references therein). However when M is a complete noncompact manifold then the Sobolev embedding is a nontrivial issue. In fact there exists a complete noncompact Riemannian manifold M for which the Sobolev embedding $W^{k,p}(M) \hookrightarrow L^q(M)$ does not hold for any p satisfying $kp < n$ where $q = \frac{np}{n-kp}$.

When M is compact and $p = \frac{n}{k}$ one can easily see that $W^{k,p}(M)$ is continuously embedded in to $L^q(M)$ for all $q < \infty$ but not for $q = \infty$ and hence none of the above embeddings $W^{k,p}(M) \hookrightarrow L^q(M)$, for $q < \infty$ are optimal. When $M = \Omega$, a bounded domain in $\mathbb{R}^n$ with smooth boundary and $k = 1$, an embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into an Orlicz space establishing the exponential integrability of these functions was obtained by Pohožaev ([26]) and Trudinger([30]). In 1971 J.Moser ([23]) while trying to study the question of prescribing the Gaussian curvature on sphere understood the need for establishing a sharp form of the embedding obtained by Pohožaev and Trudinger. He showed that there exists a positive constant C_0

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depending only on N such that

$$\sup_{u\in C_c^\infty(\Omega),\int_\Omega |\nabla u|^n\leq 1} \int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}\,dx \leq C_0|\Omega|. \tag{1.1}$$

holds for all $\alpha \leq \alpha_n = n\,[\omega_{n-1}]^{\frac{1}{n-1}}$ where Ω is a bounded domain in $\mathbb{R}^n$, and $|\Omega|$ denotes the volume of Ω and ω_{n-1} denotes the $n-1$ dimentional area of the sphere S^{n-1}. Moreover when $\alpha > \alpha_n$, the above supremum is infinite. Moser, in the same paper, established the appropriate version of this sharp inequality on the sphere S^2 and later Cherrier ([7]) proved it for the case of any compact Riemannian manifold. These optimal inequalities of the Sobolev space $W^{1,n}(M)$ where n is the dimension of M are called the Moser-Trudinger inequalities.

Even though one expects a similar type inequality to hold for higher order Sobolev spaces, it is not at all obvious how to modify the proofs of the case $k=1$ to $k>1$ due to the failure of Polya-Szego type inequalities for higher order gradients ∇^k. In a significant work, D.R. Adams ([1]) established the sharp embedding in the case of higher order Sobolev spaces $W_0^{k,p}(\Omega)$ when $kp=n$. He found the sharp constant β_0 for the higher order Trudinger-Moser type inequality. More precisely he proved that if k is a positive integer less than n, then there exists a constant $c_0 = c_0(k,n)$ such that

$$\sup_{u\in C_c^k(\Omega),\int_\Omega |\nabla^k u|^p\leq 1} \int_\Omega e^{\beta|u(x)|^{p'}}\,dx \leq c_0|\Omega|, \tag{1.2}$$

for all $\beta \leq \beta_0(k,n)$ and for all bounded domains Ω in $\mathbb{R}^n$ where $p=\frac{n}{k}$, $p'=\frac{p}{p-1}$,

$$\beta_0(k,n) = \begin{cases} \dfrac{n}{\omega_n}\left[\dfrac{\pi^{\frac{n}{2}}2^k\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)}\right]^{p'}, & \text{if } k \text{ is odd},\\ \dfrac{n}{\omega_n}\left[\dfrac{\pi^{\frac{n}{2}}2^k\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}\right]^{p'}, & \text{if } k \text{ is even}, \end{cases} \tag{1.3}$$

and ∇^k is defined by

$$\nabla^k := \begin{cases} \Delta^{\frac{k}{2}}, & \text{if } k \text{ is even},\\ \nabla\Delta^{\frac{k-1}{2}}, & \text{if } k \text{ is odd}. \end{cases} \tag{1.4}$$

Furthermore, if $\beta > \beta_0$, then the supremum in (1.2) is infinite.

Subsequently Fontana in [10] obtained the following sharp version of (1.2) on compact Riemannian manifolds :
Let (M,g) be an n dimensional compact Riemannian manifold without boundary, and k be a positive integer less than n, then there exists a constant $c_0 = c_0(k,M)$ such that

$$\sup_{u\in C^k(M),\int_M u=0,\int_\Omega |\nabla^k u|^p\leq 1} \int_M e^{\beta|u(x)|^{p'}}\,dx \leq c_0 \tag{1.5}$$

if $\beta \leq \beta_0(k,n)$, where p, p', ∇_g^k as above where ∇_g and Δ_g are the gradient and Laplace Beltrami operators with respect to the metric g. Furthermore, if $\beta > \beta_0$, then the supremum in (1.5) is infinite. These type of sharp inequalities satisfied by the $W^{k,p}(M)$ functions when $kp = n$ are called the Adams inequalities.

In this article our focus will be on Adams inequalities on Hadamard manifolds. First observe that Hadamard manifolds have infinite volume and hence $\int_M e^{\beta|u(x)|^{p'}}\, dx$ is infinite even for the trivial function $u = 0$. To tackle these issues we modify the exponential function and look for inequalities of the form

$$\sup_{u\in C_c^m(M), \int_M |\nabla_g^m u|^p \leq 1} \int_M E_k(\beta|u(x)|^{p'})\, d\mu_g(x) < \infty \tag{1.6}$$

holds for $\beta \leq \beta_0(m,n)$, where $\beta_0(m,n)$ is as defined in (1.3) and $E_k(x) = e^x - \sum_{i=0}^{k-1} \frac{x^i}{i!}$ for some $k \in \mathbb{N}$.

First observe that if (1.6) holds for some $k \in \mathbb{N}$, then as a consequence we will have the inequality

$$\left[\int_M |u(x)|^{kp'}\, d\mu_g(x)\right]^{\frac{p}{kp'}} \leq C \int_M |\nabla_g^m u|^p\, d\mu_g(x), \quad \forall u \in C_c^m(M) \tag{1.7}$$

When M is the Euclidean space $\mathbb{R}^n$ using standard scaling arguments we can see that such inequalities and hence (1.6) are impossible as $mp = n$. However in this case one can prove embeddings if one replaces the constraint $\int_M |\nabla^m u|^p \leq 1$ by $\int_M |\nabla^m u|^p + \lambda \int_M |u|^p \leq 1$ for some positive constant λ, see Cao ([5]), Panda ([24]), J.M. do Ó ([9]), Ruf ([27]), Li-Ruf ([16]) and the references therein.

When the sectional curvature is bounded from above by a negative constant we do have inequalities like (1.7). For example we have the Poincare inequality which follows from Theorem 2.5. Therefore one type of spaces where we expect Adams inequality of the form (1.6) is this set of strictly negatively curved spaces. In the case of constant negative curvature, namely the hyperbolic space, Trudinger-Moser and Adams inequalities have been investigated in detail. For $k = 1, n = 2$, Mancini-Sandeep ([21]) proved the Trudinger-Moser inequality in the hyperbolic space or in other words $W^{1,2}(\mathbb{H}^2)$ is embedded into the Zygmund space Z_ϕ determined by the function $\phi = (e^{4\pi u^2} - 1)$. Another proof of this inequality was given by Adimurthi-Tinterev ([2]). In fact in [21], they obtained the following general theorem:
Let $\mathbb{D}$ be the unit open disc in $\mathbb{R}^2$, endowed with a conformal metric $h = \rho g_e$, where g_e denotes the Euclidean metric and $\rho \in C^2(\mathbb{D}), \rho > 0$, then

$$\sup_{u\in C_c^\infty(\mathbb{D}), \int_{\mathbb{D}} |\nabla_h u|^2 \leq 1} \int_{\mathbb{D}} \left(e^{4\pi u^2} - 1\right)\, dv_h < \infty \tag{1.8}$$

holds true if and only if $h \leq c\, g_{\mathbb{H}^2}$ for some positive constant c. Here ∇_h, dv_h denotes respectively the gradient and volume element for the metric h and $g_{\mathbb{H}^2} = \sum_{i=1}^{2} \left(\frac{2}{1-|x|^2}\right)^2 dx_i^2$ is the Poincare metric in the disc.
Extensions of this inequality to $n > 2$ were obtained in Lu-Tang ([19]) and Battaglia-Mancini ([4]). See also [22] for another proof and related issues.

Various forms of Adams inequality in the Hyperbolic space were proved by Karmakar and Sandeep [15] and Fontana and Morpurgo [12]. In [12] it was shown that (1.6) holds when M is the hyperbolic space and $k = [p-1]$ where $[x]$ denotes the smallest integer greater than or equal to x. In [15] another approach was taken from the point of view of prescribing the Q-curvature and proved the following inequality with $p = 2$:

$$\sup_{u\in C_c^\infty(M),\ \int_M (P_{\frac{n}{2}}u)u\ d\mu_g\ \le 1} \int_M \left(e^{\beta u^2}-1\right)\ d\mu_g < +\infty \tag{1.9}$$

iff $\beta \le \beta_0(\frac{n}{2}, n)$, where β_0 is as before and M is the n-dimensional hyperbolic space and $P_{\frac{n}{2}}$ is the critical GJMS operator in the Hyperbolic space. Related inequalities with Hardy type potentials were obtained in [20].

Moser-Trudinger inequality has been proved for general Hadamard manifolds in [31]. i.e, they showed that when M is a Hadamard manifold then for any $\lambda > 0$ the inequality

$$\sup_{u\in C_c^1(M),\int_M(|\nabla u|^n+\lambda|u|^n)\ d\mu_g\le 1} \int_M E_{n-1}(\beta|u(x)|^{\frac{n}{n-1}})\ d\mu_g(x) < \infty \tag{1.10}$$

holds with the optimal choice of β as $n\left[\omega_{n-1}\right]^{\frac{1}{n-1}}$.

In this article we investigate the validity of Adams inequality of the form (1.6) in general pinched Hadamard manifolds. The main difficulty one faces in this task is to handle the case of infinite volume. Also unlike in the constant curvature space estimates on balls of fixed radius will depend on the center of the ball. To handle these situations we make some assumptions on the curvature. Following is the main result in this article.

Theorem 1.1. *Let (M, g) be an n-dimensional pinched Hadamard manifold satisfying $K_g \le -a^2$ and $Ric_g \ge -(n-1)b^2$ for some $a, b > 0$[1]. Let m be an even integer satisfying $2 \le m < n$ and $p = \frac{n}{m}$. Then for any $\lambda > 0$,*

$$\sup_{u\in C_c^m(M),\int_M[|\nabla^m u|^p+\lambda|u|^p]\ \le 1} \int_M E_{[p-1]}(\beta|u(x)|^{p'})\ d\mu_g(x) < \infty \tag{1.11}$$

iff $\beta \le \beta_0(m,n)$, where $\beta_0(m,n)$ is as defined in (1.3). If $n \le 2m$ then the theorem holds with $\lambda = 0$.

We will prove the result by converting it in to an estimate on operators given by kernels, an idea initiated in this case by Adams [1] and developed further in [10], [11] and [12]. We will implement this scheme by writing the functions as integral operators given by kernels. The properties of these kernels leading to Adams type inequalities has been given in [12]. The real issue in our case is to establish these conditions on kernels. In the constant curvature case explicit formulas makes this job easy, but in our case we lack these explicit formulas for kernels. One of the main tool we use to overcome this difficulty is comparison theorems.

[1]Consequently, K_g is bounded from below as well.

We divide this article into four sections. Section 2 will be devoted to preliminary materials, Section 3 will develop the details required on Green's function and the proof of theorem will be given in Section 4.

2. Notations and Preliminaries

In this section we will introduce our notations and recall some results from Riemannian geometry which we will be using in this article. For more details and proofs of theorems we refer to any standard book on Riemannian Geometry like [6], [13], [25].

2.1. **Notations.** We will denote by (M, g) a Riemannian manifold with inner product $g(\cdot,\cdot)$. The Ricci and Sectional curvatures will be denoted by Ric_g and K_g respectively.

A Hadamard manifold is a complete simply connected Riemannian manifold (M, g) with $K_g(x) \leq 0$ for all $x \in M$. We will denote the n dimensional hyperbolic space of constant curvature $\lambda < 0$ by $\mathbb{H}^n_\lambda$.

The Riemannian distance between x and y will be denoted by $d_g(x, y)$ and the Riemannian measure will be denoted by μ_g. The surface measure of the Euclidean unit sphere S^{n-1} will be denoted by ω_{n-1}.

Let us denote by ∇_g and $\Delta_g = +\mathrm{Tr}\,\mathrm{Hess}$ the gradient and the Laplace Beltrami operator associated with the metric g. Moreover for a positive integer k, let Δ_g^k be the k-th iterated Laplacian, we define the k th order gradient ∇_g^k by,

$$\nabla_g^k := \begin{cases} \Delta_g^{\frac{k}{2}}, & \text{if } k \text{ is even,} \\ \nabla_g \Delta_g^{\frac{k-1}{2}}, & \text{if } k \text{ is odd,} \end{cases} \tag{2.1}$$

For $u \in C^k(M)$ we define $|\nabla_g^k u(x)|,\ x \in M$ as the Euclidean modulus of $\Delta_g^{\frac{k}{2}} u(x)$ when k is even and $\sqrt{g\left(\nabla_g \Delta_g^{\frac{k-1}{2}} u(x), \nabla_g \Delta_g^{\frac{k-1}{2}} u(x)\right)}$ when k is odd.

2.2. **Some results from Riemannian Geometry.** One of the main difficulties we will face in proving our result comes from the infinite measure of these manifolds. First we will recall some results on the volume.

Let $V_\lambda^n(r)$ denotes the volume of a ball with radius $r > 0$ in the n-dimensional simply connected space form of constant curvature $\lambda \in \mathbb{R}$ then we have

$$V_\lambda^n(r) = \begin{cases} \frac{\omega_{n-1}}{n} r^n, & \text{if } \ \lambda = 0, \\ \frac{\omega_{n-1}}{a^n} \int\limits_0^{ar} \sinh^{n-1} s \, ds, & \text{if } \ \lambda = -a^2 < 0 \end{cases} \tag{2.2}$$

In the general case we have the following volume comparison theorem:

Theorem 2.1. *Let (M,g) be an n-dimensional complete Riemannian manifold with $Ric \geq (n-1)\lambda$ for some $\lambda \in \mathbb{R}$ then for any $x \in M$ the volume ratio*

$$\frac{\mu_g(B(x,r))}{V_\lambda^n(r)}$$

is a nonincreasing function of r. In particular

$$\frac{\mu_g(B(x,r))}{V_\lambda^n(r)} \leq \lim_{R\to 0} \frac{\mu_g(B(x,R))}{V_\lambda^n(R)} = 1$$

and hence $\mu_g(B(x,r)) \leq V_\lambda^n(r)$.

More precisely for the volume element we will need the following:

Theorem 2.2. *Let (M,g) be an n dimensional complete Riemannian manifold with $Ric \geq -(n-1)b^2$ for some $b > 0$. For $x \in M$ let $r^{n-1}A_x(r,\theta)\, d\theta dr$ denotes the Riemannian measure in normal coordinates centered at x, then*

$$r^{n-1}A_x(r,\theta) \leq \left(\frac{\sinh(br)}{b}\right)^{n-1}.$$

Next we recall the Hessian comparison theorem:

Theorem 2.3. *Let (M,g) be a Riemannian manifold such that $K_g \leq -a^2$ with $a > 0$. Let $y \in M$, then at any point $x \neq y$, it holds*

$$D_g^2(d_g(y,\cdot))(x) \geq a \coth(a\, d_g(y,x))\bar{g},$$

where D_g^2 denotes the Hessian of the distance function and $\bar{g}$ the restriction of the metric g to $\{\nabla_g d_g(y,\cdot)(x)\}^\perp \subset T_xM$. Taking the trace, we get

$$\Delta_g(d_g(y,\cdot))(x) \geq (n-1)a \coth(a\, d_g(y,x)).$$

If $a = 0$ then

$$D_g^2(d_g(y,\cdot))(x) \geq \frac{1}{d_g(y,x)}\bar{g} \text{ and } \Delta_g(d_g(y,\cdot))(x) \geq \frac{(n-1)}{d_g(y,x)}.$$

Next we recall the Laplacian comparison theorem:

Theorem 2.4. *Let (M,g) be a Riemannian manifold such that $Ric_g \geq -(n-1)b^2$. Let $y \in M$, then at any point $x \neq y$, it holds*

$$\Delta_g(d_g(y,\cdot))(x) \leq (n-1)b \coth(b\, d_g(x,y)).$$

2.3. **Poincaré type Inequalities.** In this final subsection we recall some inequalities in Sobolev space and deduce some corollaries.
The following theorem is due to McKean for $p = 2$ (see [6]) and generalized further by Strichartz ([29], Theorem 5.4).

Theorem 2.5. *Let (M,g) be a Hadamard manifold with $K_g \leq -a^2 < 0$ then for $1 \leq p < \infty$ the inequality*

$$\left(\frac{(n-1)a}{p}\right)^p \int_M |u|^p \, d\mu_g \leq \int_M |\nabla_g u|^p \, d\mu_g \tag{2.3}$$

holds for all $u \in C_c^\infty(M)$.

Let us also recall the following multiplicative inequality (see [8, Theorem 4.1] for a proof).

Theorem 2.6. *Let (M, g) be a complete Riemannian manifold and $1 < p \leq 2$, then there exists a constant $C_p > 0$ such that*

$$|||\nabla_g u|||_p \;\leq\; C_p \,(||u||_p)^{\frac{1}{2}}\,(||\Delta_g u||_p)^{\frac{1}{2}} \tag{2.4}$$

holds for all $u \in C_c^\infty(M)$.

Combining the above two theorems and a recursive application will give the following inequality:

Theorem 2.7. *Let (M, g) be a Hadamard manifold with $K_g \leq -a^2 < 0$ then for $1 < p \leq 2$ and $k \in \mathbb{N}$, there exists $C_{k,p} > 0$ such that the*

$$\int\limits_M |u|^p \, d\mu_g \;\leq\; C_{k,p} \int\limits_M |\nabla_g^k u|^p \, d\mu_g \tag{2.5}$$

holds for all $u \in C_c^\infty(M)$.

3. Green's function

One of the crucial tool which we will be using to prove our results is the information on the Green's function of the Laplace operator. In this section, following the approach due to Li and Tam [17], we will construct a Green's function on a Hadamard manifold and show that it can be bounded by terms depending only on the curvature bounds; we will also establish integral estimates for this Green's function and its gradient. First let us recall the definition of entire Green's function.

Definition 3.1. Let (M, g) be a Riemannian manifold, then an entire Green's function of the Laplace Beltrami operator $-\Delta_g$ is a function $G : M \times M \setminus \{(x, x) : x \in M\} \to [0, \infty)$ satisfying

- (i) For each fixed $x \in M$, $\Delta_g G^x(y) = 0$ for all $y \in M \setminus \{x\}$, where G^x is the function $y \to G(x, y)$.
- (ii) $G(x, y) = G(y, x)$ for all $x \neq y$.
- (iii) For each fixed $x \in M$,

$$G^x(y) \;=\; \begin{cases} \frac{d_g(x,y)^{2-n}}{(n-2)\omega_{n-1}}\,[1 + o(1)] & if \quad n \geq 3 \\ \frac{-\log d_g(x,y)}{2\pi}\,[1 + o(1)] & if \quad n = 2. \end{cases}$$

We need the entire Green's function for the following representation formula:

Remark. Let (M, g) be a Riemannian manifold and G be an entire Green's function then for $u \in C_c^2(M)$

$$u(x) \;=\; \int\limits_M G(x, y)(-\Delta_g u(y))\, d\mu_g(y) \tag{3.1}$$

and

$$u(x) \;=\; -\Delta_g \left(\int_M G(x,y)u(y)\; d\mu_g(y) \right) \tag{3.2}$$

Let $\Phi : (0,\infty) \to (0,\infty)$ be defined by

$$\Phi(r) = \frac{r^{2-n}}{(n-2)\omega_{n-1}} \tag{3.3}$$

then we know that an entire Green's function of $-\Delta$ in the Euclidean space $\mathbb{R}^n$, $n \geq 3$ is given by $G(x,y) = \Phi(|x-y|)$. Similarly for $a > 0$ if $\Psi_a : (0,\infty) \to (0,\infty)$ is defined by

$$\Psi_a(r) = \frac{a^{n-2}}{\omega_{n-1}} \int_{ar}^{\infty} (\sinh t)^{1-n}\; dt \tag{3.4}$$

then one can easily see that an entire Green's function of the hyperbolic space $\mathbb{H}^n_{-a^2}$ is given by $G(x,y) \;=\; \Psi_a(d_{g_a}(x,y))$ where d_{g_a} is the distance in $\mathbb{H}^n_{-a^2}$.

In general we have the following theorem regarding the entire Green's function of a Hadamard manifold:

Theorem 3.1. *Let (M,g) be a Hadamard manifold of dimension $n \geq 3$, then (M,g) admits an entire Green's function G satisfying the estimate*

$$0 \;<\; G(x,y) \leq \Phi(d_g(x,y)) \tag{3.5}$$

where Φ is as in (3.3). Moreover if (M,g) satisfies :
(i) $K_g \leq -a^2 < 0$ then

$$0 \;<\; G(x,y) \leq \Psi_a(d_g(x,y)). \tag{3.6}$$

(ii) $Ric_g \geq -(n-1)b^2$, $b > 0$ then

$$0 < \Psi_b(d_g(x,y)) \leq G(x,y) \tag{3.7}$$

where Ψ_a and Ψ_b are as in (3.4).

We also need the following estimates on the L^2 and L^1 norms of G and its gradient:

Theorem 3.2. *Let (M,g) be a Hadamard manifold satisfying $K_g \leq -a^2$ and $Ric_g \geq -(n-1)b^2$ for some $a > 0, b > 0$. Let G be the entire Green's function established in Theorem 3.1, then for every $R > 0$ there exists $A_R > 0$ such that*

$$\Phi(d_g(x,y))\,[1 - A_R\, d_g(x,y)] \leq G(x,y) \leq \Phi(d_g(x,y))\;, \quad \text{whenever} \;\; d_g(x,y) < R. \tag{3.8}$$

Moreover there exists a $C > 0$ such that for every $x \in M$

$$\int_{B(x,R)} G(x,y)\; d\mu_g(y) \leq C(1+R), \tag{3.9}$$

$$\int_{M \setminus B(x,R)} G^2(x,y)\; d\mu_g(y) \leq C\Psi_a(R), \tag{3.10}$$

and

$$\int_{M\setminus B(x,R)} |\nabla_g G(x,\cdot)|^2 \, d\mu_g(y) \le \frac{C}{R^2}\Psi_a(R). \tag{3.11}$$

We need a few lemmas before going to the proofs of Theorem 3.1 and Theorem 3.2. First let us recall the theorem concerning the existence of Green's function for the Laplace operator with Dirichlet boundary condition in bounded domains. For details we refer to [3].

Lemma 3.3. *Let (M, g) be a Riemannian manifold of dimension $n \ge 3$ and Ω be a bounded open subset of Ω with smooth boundary, then there exists a $G : \overline{\Omega}\times\overline{\Omega}\setminus\{(x, x) : x \in \overline{\Omega}\} \to (0, \infty)$ such that*

(i) $G(x, y) = G(y, x)$, $\forall x \ne y$
(ii) $G(x, y) = 0$, if $x \in \partial\Omega$ *or* $y \in \partial\Omega$.
(iii) $-\Delta_g G(x, \cdot) = \delta_x,\ -\Delta_g G(\cdot, y) = \delta_y$
(iv) *For each x fixed,* $G(x, y) = \frac{[d_g(x,y)]^{2-n}}{(n-2)\omega_{n-1}}\,[1 + o(1)]$ *as $y \to x$.*

We are going to get our Green's function as the limit of Dirichlet Green's functions in bounded domains. The following lemma plays a crucial role in getting the bounds on the Green's function.

Lemma 3.4. *Let (M, g) be a Hadamard manifold of dimension $n \ge 3$ and Φ, Ψ_a be as in (3.3) and (3.4). For $x \in M$ define $\Phi^x, \Psi_a^x : M \setminus \{x\} \to (0, \infty)$ by $\Phi^x(y) = \Phi(d_g(x, y))$ and $\Psi_a^x(y) = \Psi_a(d_g(x, y))$ for all $y \in M \setminus \{x\}$. Then:*

(i) $-\Delta_g \Phi^x(y) \ge 0$ for all $y \in M \setminus \{x\}$.
(ii) $-\Delta_g \Psi_a^x(y) \ge 0$ for all $y \in M \setminus \{x\}$ *if* $K_g \le -a^2 < 0$.
(iii) $-\Delta_g \Psi_b^x(y) \le 0$ for all $y \in M \setminus \{x\}$ *if* $Ric_g \ge -(n-1)b^2$.

Proof. Let us recall that, given a C^2 function $f : (0, +\infty) \to (0, +\infty)$,

$$\Delta_g(f(d_g(x, \cdot)) = f'(d_g(x, \cdot))\Delta_g d_g(x, \cdot) + f''(d_g(x, \cdot))$$

(we use $|\nabla_g d_g(x, \cdot)| = 1$ on $M \setminus \{x\}$). Note that $\Phi''(r) = \frac{n-1}{r}\Phi'(r)$; a similar formula holds for Ψ_a. The conclusions (i) and (ii) then follows from Theorem 2.3 while (iii) follows from Theorem 2.4 □

Proof of Theorem 3.1. Fix a point $O \in M$ and define for $R > 0$,

$$B_R := \{x \in M : d_g(O, x) < R\}.$$

Let G_R denotes the unique Dirichlet Green's function of B_R given by Lemma 3.3, we will show that the limit of G_R as $R \to \infty$ exists and is the required Green's function. We will present the arguments in several steps.

Step 1: Let $0 < R_1 < R_2 < \infty$ and $x, y \in B_{R_1},\ x \ne y$ then $G_{R_1}(x, y) \le G_{R_2}(x, y)$.
Proof of Step 1. Fix $x \in B_{R_1}$ and consider the function $g_\epsilon : B_{R_1} \setminus \{x\} \to \mathbb{R}$ defined by

$$g_\epsilon(y) = (1 + \epsilon)G_{R_2}(x, y) - G_{R_1}(x, y)$$

Then for any $\delta > 0$, g_ϵ is harmonic in $B_{R_1} \setminus \overline{B(x, \delta)}$ and $g_\epsilon \ge 0$ on $\partial(B_{R_1} \setminus \overline{B(x, \delta)})$ for δ small enough thanks to (iv) of Lemma 3.3. Thus by maximum principle $g_\epsilon \ge 0$ in $B_{R_1} \setminus \overline{B_\delta}$ for δ small enough and hence in $B_{R_1} \setminus \{x\}$. Now Step 1 follows by taking

$\epsilon \to 0$.

Step 2 : For every $R > 0, G_R(x,y) \le \Phi^x(y)$ for all $x, y \in B_R$, where Φ^x is defined as in Corollary 3.4.

Proof of Step 2. Fix $x \in B_R$ and $\delta > 0$ small enough and consider the function $g^{x,\delta} : B_R \setminus B(x,\delta) \to \mathbb{R}$ defined by

$$g^{x,\delta}(y) = \Phi^x(y) - m_\delta G_R(x,y)$$

where $m_\delta = \frac{\Phi(\delta)}{\max\{G_R(x,y): d_g(x,y)=\delta\}}$. Then it follows from maximum principle that $g^{x,\delta}(y) \ge 0$ in $B_R \setminus B(x,\delta)$. Note that $m_\delta \to 1$ as $\delta \to 0$. Thus Step 2 follows by taking $\delta \to 0$ in $g^{x,\delta}(y) \ge 0$ for $y \in B_R \setminus B(x,\delta)$.

Step 3 : Define for $x, y \in M,\ x \ne y, G(x,y) = \lim\limits_{R\to\infty} G_R(x,y)$, then G is the required Green's function.

Proof of Step 3. First observe that G is well defined thanks to Step 1 and Step 2. The estimate (3.5) on G follows from Step 2 by taking the limit $R \to \infty$. Also $G(x,y) = G(y,x)$ as it holds for each G_R. For any $x \in M$ the function $\Phi_x \in L^1_{loc}(M)$ and $G_R \le \Phi_x$. Thus $\Delta_g G_R(x,.) \to \Delta_g G(x,.)$ in the sense of distributions which implies $-\Delta_g G(x,.) = \delta_x$ for all $x \in M$, in particular $\Delta_g G^x = 0$ in $M \setminus \{x\}$.
It remains to show that G satisfies the last condition of the definition of entire Green's function. Fix $x \in M$ and $R > 0$ such that $x \in B_{\frac{R}{2}}$, then as $y \to x$, we have

$$\frac{[d_g(x,y)]^{2-n}}{(n-2)\omega_{n-1}}[1 + o(1)] = G_R(x,y) \le G(x,y) \le \frac{[d_g(x,y)]^{2-n}}{(n-2)\omega_{n-1}}$$

and hence G satisfies (iii) of the definition.

When (M,g) satisfies *Sect.* $\le -a^2 < 0$ we can repeat steps 2 and 3 with Ψ_a instead of Φ to establish (3.6).

To prove (3.7) fix $x \in M$. For $\delta > 0$ define $h^{x,\delta}$ by

$$h^{x,\delta}(y) \ = \ m_\delta G^x(y) - \Psi_b(d_g(x,y)) \ , \ y \in M \setminus B(x,\delta)$$

where $m_\delta = \frac{\Psi_b(\delta)}{\min\limits_{\{y: d_g(x,y)=\delta\}} G^x(y)}$. Then using (iii) of Lemma 3.4 we get $-\Delta_g h^{x,\delta} \ge 0$ and hence using maximum principle $h^{x,\delta} \ge 0$ in $M \setminus B(x,\delta)$. Taking the limit as $\delta \to 0$ and observing that $m_\delta \to 1$ we get $G^x(y) - \Psi_b(d_g(x,y)) \ge 0$ for $y \in M \setminus \{x\}$. This completes the proof of the theorem. □

Proof of Theorem 3.2. The upper and lower bound of G namely (3.8) follows from (3.5) and (3.7).
Fix $x \in M$ and define $G^x : M \setminus \{x\} \to (0,\infty)$ by $G^x(y) = G(x,y)$. Then it follows from Theorem 3.1 that

$$B(x, \Psi_b^{-1}(t)) \subset \{y : G^x(y) > t\} \subset B(x, \Psi_a^{-1}(t)). \tag{3.12}$$

Let B_R and G_R be as in the proof of Theorem 3.1 and define $G^x_R(y) = G_R(x,y), y \ne x$, then we know that G^x_R monotonically converges to G^x. For $t > 0, R > 0$ define the compactly supported function

$$H_R^t(y) \;=\; \min\{t, G_R^x(y)\}.$$

Then using Theorem 2.5 with $p = 2$ we get

$$\left[\frac{(n-1)a}{2}\right]^2 \int\limits_{B_R} (H_R^t)^2 \, d\mu_g \;\leq\; \int\limits_{B_R} |\nabla_g H_R^t|^2 d\mu_g. \tag{3.13}$$

Now,

$$\begin{aligned}
\int\limits_{B_R} |\nabla_g H_R^t|^2 d\mu_g &= \int\limits_{B_R \cap \{G_R^x < t\}} |\nabla_g G_R^x|^2 d\mu_g \\
&= -\int\limits_{B_R \cap \{G_R^x < t\}} (\Delta_g G_R^x) G_R^x d\mu_g - \int\limits_{\{G_R^x = t\}} (\frac{\partial G_R^x}{\partial \nu}) G_R^x \\
&= -t \int\limits_{\{G_R^x = t\}} (\frac{\partial G_R^x}{\partial \nu})
\end{aligned} \tag{3.14}$$

where ν is the outward unit normal of $\{G_R^x > t\}$ and we have used that $\Delta_g G_R^x = 0$ in $M \setminus \{x\}$. For small enough $\epsilon > 0$, we get by applying Green's formula on $\{G_R^x > t\} \setminus B(x, \epsilon)$:

$$\int\limits_{\{G_R^x = t\}} (\frac{\partial G_R^x}{\partial \nu}) + \int\limits_{\partial B(x,\epsilon)} (\frac{\partial G_R^x}{\partial \nu}) = \int\limits_{\{G_R^x > t\} \setminus B(x,\epsilon)} \Delta_g G_R^x \, d\mu_g = 0.$$

where ν on $\partial B(x, \epsilon)$ is the unit inward normal of $B(x, \epsilon)$. Inserting this relation into (3.14), we get, by definition of G_R^x,

$$\int\limits_{B_R} |\nabla_g H_R^t|^2 d\mu_g = t \lim_{\epsilon \to 0} \int\limits_{\partial B(x,\epsilon)} (\frac{\partial G_R^x}{\partial \nu}) = t$$

Using this estimate in (3.13) and taking the limit $R \to \infty$ we get

$$\left[\frac{(n-1)a}{2}\right]^2 \left[t^2 \mu_g(\{G^x \geq t\}) + \int\limits_{\{G^x < t\}} (G^x)^2 \, d\mu_g \right] \;\leq\; t \tag{3.15}$$

Hence $\int\limits_{\{G^x < t\}} (G^x)^2 \, d\mu_g \;\leq Ct$ and (3.10) follows from (3.12).

To prove (3.9), first observe from (3.15) that

$$\mu_g(\{G^x > t\}) \leq \left[\frac{2}{(n-1)a}\right]^2 \frac{1}{t}, \quad \forall \, t > 0. \tag{3.16}$$

Also from (3.12), Theorem 2.1 and (2.2) we have ,

$$\mu_g(\{G^x > t\}) \leq \mu_g(B(x, \Phi^{-1}(t))) \leq V_{-b^2}^n(\Phi^{-1}(t)) \leq C \left(\frac{1}{t}\right)^{\frac{n}{n-2}}, \; t \geq 1 \tag{3.17}$$

Thus using (3.12) and (3.16), we get

$$\begin{aligned}
\int\limits_{B(x,R)} G(x,y)\, d\mu_g(y) &\leq \int\limits_{G^x > \Psi_b(R)} G^x(y)\, d\mu_g(y) \\
&= \int\limits_0^\infty \mu_g\left(\{G^x > t\} \cap \{G^x > \Psi_b(R)\}\right) dt \\
&= \int\limits_0^{\Psi_b(R)} \mu_g\left(\{G^x > \Psi_b(R)\}\right) dt + \int\limits_{\Psi_b(R)}^\infty \mu_g\left(\{G^x > t\}\right) dt \\
&= \left[\frac{2}{(n-1)a}\right]^2 + \int\limits_{\Psi_b(R)}^\infty \mu_g\left(\{G^x > t\}\right) dt
\end{aligned}$$

If $\Psi_b(R) \geq 1$ then (3.17) implies that $\int\limits_{\Psi_b(R)}^\infty \mu_g\left(\{G^x > t\}\right) dt \leq C$ where C is independent of x. If $\Psi_b(R) < 1$ then (3.16) and (3.17) gives

$$\int\limits_{\Psi_b(R)}^\infty \mu_g\left(\{G^x > t\}\right) dt = \int\limits_{\Psi_b(R)}^1 \mu_g\left(\{G^x > t\}\right) dt + \int\limits_1^\infty \mu_g\left(\{G^x > t\}\right) dt \leq C(1+R)$$

This proves (3.9).
To prove (3.11) choose a smooth function $f : \mathbb{R} \to [0,1]$ such that $f(r) = 0$ if $r \leq 1$ and $f(r) = 1$ if $r \geq 2$ and define $f_R : M \to [0,1]$ by $f_R(y) = f(\frac{d_g(x,y)}{R})$.
Since $\Delta_g G^x_{\tilde{R}} = 0$ in $B_{\tilde{R}} \setminus \{x\}$ we get

$$\int\limits_{B_{\tilde{R}}} \Delta_g G^x_{\tilde{R}}(y)(f_R(y))^2 G^x_{\tilde{R}}(y)\, d\mu_g(y) = 0$$

This implies

$$\int\limits_{B_{\tilde{R}}} |\nabla_g G^x_{\tilde{R}}(y)|^2 (f_R(y))^2\, d\mu_g(y) \leq 2 \int\limits_{B_{\tilde{R}}} |\nabla_g f_R(y)||\nabla_g G^x_{\tilde{R}}(y)| f_R(y) G^x_{\tilde{R}}(y)\, d\mu_g(y)$$

$$\leq \frac{C}{R}\left(\int\limits_{B_{\tilde{R}}} |\nabla_g G^x_{\tilde{R}}(y)|^2 (f_R(y))^2\, d\mu_g(y)\right)^{\frac{1}{2}} \left(\int\limits_{\{y: R \leq d_g(x,y) \leq 2R\}} (G^x_{\tilde{R}})^2\right)^{\frac{1}{2}}$$

Now (3.11) follows by taking $\tilde{R} \to \infty$ and using (3.10).

Remark. The gradient estimate (3.11) also follows from (3.10) once we use the pointwise estimate on positive harmonic functions defined on balls in terms of the lower bound on the Ricci curvature proved by Yau ([32]) and the subsequent improvement obtained in [18]. Using these results we get

$$|\nabla_g(\log G(x,\cdot))| \;\leq\; \frac{C}{d_g(x,\cdot)} \tag{3.18}$$

□

4. Proof of Theorem

In this section we will prove our main theorem. We follow the idea of converting the problem into a convolution type estimate problem introduced by Adams ([1]) and further developed by Fontana ([10]) and Fontana-Morpurgo ([11], [12]). The main part of the proof is to represent functions in terms of kernels. First we will introduce these kernels and prove the necessary estimates on these kernels.

4.1. **Estimates on the Kernel.** Define for $m = 2j,\ j = 1, 2, ...,$ the kernel $K^m : M \times M \setminus \{(x, x) : x \in M\} \to (0, \infty)$ by

$$K^m(x,y) = \begin{cases} G(x,y) & \text{if} \quad m = 2 \\ \int\limits_M K^{m-2}(x,z)G(z,y)\, d\mu_g(z) & \text{if } m \geq 4 \end{cases} \tag{4.1}$$

Lemma 4.1. *Let (M, g) be an n dimensional Hadamard manifold satisfying $K_g \leq -a^2$ and $Ric_g \geq -(n-1)b^2$ for some positive numbers a, b then for $m < n$, K^m is well defined and satisfies the estimate*

$$K^m(x,y) \leq \begin{cases} \alpha_{n,m} \left[d_g(x,y)\right]^{m-n} \left(1 + C\left[d_g(x,y)\right]^{\frac{1}{2}}\right) & \textit{if} \quad d_g(x,y) < 1 \\ Ce^{-\beta_m d_g(x,y)} \quad \textit{if} \qquad d_g(x,y) \geq 1 \end{cases} \tag{4.2}$$

for some $\beta_m > 0$ and $\alpha_{n,m} = \frac{\Gamma(\frac{n-m}{2})}{\omega_{n-1} 2^{m-1} (\frac{m-2}{2})! \Gamma(\frac{n}{2})}$. Moreover there exists $\alpha_m > 0$ such that

$$\int\limits_{M \setminus B(x,R)} \left(K^m(x,y)\right)^2\, d\mu_g(y) \leq Ce^{-\alpha_m R}, \quad \text{for all } R \geq 1 \tag{4.3}$$

for some $C > 0$.

Proof. First observe that when $m = 2$ the Lemma 4.1 follows from (3.6) and the estimate (3.10). Next we show that if the lemma is true for an even m then it holds for $m + 2$ if $m + 2 < n$ and hence it will follow for all even $m < n$. Also observe that if (4.2) holds with $R = 1$ as threshold then, up to modifying the constants C, it also holds for any $R > 0$.
Let us consider the cases $d_g(x, y) < 2$ and $d_g(x, y) \geq 2$ separately.

Case 1 : Let $x, y \in M$ be such that $d_g(x, y) < 2$.

$$\int\limits_M K^m(x,z)G(z,y)\, d\mu_g(z) = \int\limits_{B(x,2)} K^m(x,z)G(z,y)\, d\mu_g(z)$$

$$+ \int\limits_{M \setminus B(x,2)} K^m(x,z)G(z,y)\, d\mu_g(z)$$

The second integral on the right is uniformly bounded independent of x as it is bounded from above by

$$\left(\int_{M\setminus B(x,2)} (K^m(x,z))^2 \, d\mu_g(z)\right)^{\frac{1}{2}} \left(\int_{M\setminus B(y,1)} (G(z,y))^2 \, d\mu_g(z)\right)^{\frac{1}{2}}$$

and the estimates (3.10) and (4.3).
Next we will estimate the first term. From (3.8) and the fact that K^m satisfies (4.2) we get

$$\int_{B(x,2)} K^m(x,z)G(z,y) \, d\mu_g(z) \leq$$

$$\int_{B(x,2)} \alpha_{n,m} \left[d_g(x,z)\right]^{m-n} \left(1 + C \left[d_g(x,z)\right]^{\frac{1}{2}}\right) \Phi(d_g(z,y)) d\mu_g(z).$$

We will estimate the right-hand side by writing it in the normal coordinates centered at x. Let us identify isometrically the tangent spaces of M at x with the Euclidean space $\mathbb{R}^n$ by fixing a g-orthonormal basis. Let $Exp_x : \mathbb{R}^n \to M$ be the exponential map. Since $K_g \leq 0$, by Rauch comparison theorem we get for any two points $z_i \in \mathbb{R}^n, i = 1, 2,$

$$d_g(Exp_x(z_1), Exp_x(z_2)) \geq |z_1 - z_2|.$$

Since Φ is decreasing we get $\Phi(d_g(Exp_x(z_1), Exp_x(z_2))) \leq \Phi(|z_1 - z_2|)$. We also set $Exp_x^{-1}(z) = \tilde{z}$ for an arbitrary point $z \in M$.

Using the Ricci curvature lower bound, we can estimate from above the volume element; precisely, if we set $d\tilde{z}$ the Lebesgue measure, Theorem 2.2 can be rephrased as

$$d\mu_g(z) \leq \left(\frac{\sinh(b|\tilde{z}|)}{b|\tilde{z}|}\right)^{n-1} d\tilde{z}.$$

Combining these facts together, we get

$$\mathcal{I} := \int_{B(x,2)} \left[d_g(x,z)\right]^{m-n} \left(1 + C \left[d_g(x,z)\right]^{\frac{1}{2}}\right) \Phi(d_g(z,y)) d\mu_g(z)$$

$$\leq \int_{B(0,2)} |\tilde{z}|^{m-n} \left(1 + C|\tilde{z}|^{\frac{1}{2}}\right) \frac{|\tilde{z} - \tilde{y}|^{2-n}}{(n-2)\omega_{n-1}} \left(\frac{\sinh(b|\tilde{z}|)}{b|\tilde{z}|}\right)^{n-1} d\tilde{z}.$$

For $\tilde{z} \neq 0$, we decompose the integrand as follows

$$|\tilde{z}|^{m-n} \left(1 + C|\tilde{z}|^{\frac{1}{2}}\right) \frac{|\tilde{z} - \tilde{y}|^{2-n}}{(n-2)\omega_{n-1}} \left(\frac{\sinh(b|\tilde{z}|)}{b|\tilde{z}|}\right)^{n-1} =$$

$$|\tilde{z}|^{m-n} \left(1 + C|\tilde{z}|^{\frac{1}{2}}\right) \frac{|\tilde{z} - \tilde{y}|^{2-n}}{(n-2)\omega_{n-1}} \left(\left(\frac{\sinh(b|\tilde{z}|)}{b|\tilde{z}|}\right)^{n-1} - 1\right) +$$

$$|\tilde{z}|^{m-n} \left(1 + C|\tilde{z}|^{\frac{1}{2}}\right) \frac{|\tilde{z} - \tilde{y}|^{2-n}}{(n-2)\omega_{n-1}}.$$

Note that each term above is nonnegative and, for $|\tilde{z}| \leq 2$,

$$0 \leq \left(\frac{\sinh(b|\tilde{z}|)}{b|\tilde{z}|}\right)^{n-1} - 1 \leq \tilde{C}|\tilde{z}|^2.$$

Combining these facts together, we obtain

$$\mathcal{I} \leq \int\limits_{B(0,2)} |\tilde{z}|^{m-n} \frac{|\tilde{z}-\tilde{y}|^{2-n}}{(n-2)\omega_{n-1}}\, d\tilde{z} + \int\limits_{B(0,2)} |\tilde{z}|^{m-n+1/2} \frac{|\tilde{z}-\tilde{y}|^{2-n}}{(n-2)\omega_{n-1}}\, d\tilde{z}$$
$$+ \int\limits_{B(0,2)} |\tilde{z}|^{m-n+2+1/2} \frac{|\tilde{z}-\tilde{y}|^{2-n}}{(n-2)\omega_{n-1}}\, d\tilde{z}$$

Bounding each term by integrating over $\mathbb{R}^n$ instead of $B(0,2)$ and using, for $0 < \alpha, \beta < n$ such that $\alpha + \beta < n$,

$$\int\limits_{\mathbb{R}^n} |x|^{\alpha-n}|x-y|^{\beta-n}\, dx = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)}|y|^{\alpha+\beta-n}$$

where

$$\gamma(x) = 2^x \pi^{\frac{n}{2}} \frac{\Gamma(\frac{x}{2})}{\Gamma(\frac{n-x}{2})};$$

(see [28], Chapter 5) we get the estimate in this case.

Case 2 : Let $x, y \in M$ be such that $d_g(x,y) \geq 2$.

Let us denote $d := d_g(x,y)$, then

$$\int\limits_M K^m(x,z)G(z,y)\, d\mu_g(z) = \int\limits_{B(y,\frac{d}{2})} K^m(x,z)G(z,y)\, d\mu_g(z)$$
$$+ \int\limits_{M\setminus B(y,\frac{d}{2})} K^m(x,z)G(z,y)\, d\mu_g(z)$$

Since K^m satisfies the lemma we have using (3.9)

$$\int\limits_{B(y,\frac{d}{2})} K^m(x,z)G(z,y)\, d\mu_g(z) \leq Ce^{-\alpha_m \frac{d}{2}} \int\limits_{B(y,\frac{d}{2})} G(z,y)\, d\mu_g(z) \leq Ce^{-\alpha_m \frac{d}{4}}$$

where C, β, α_m are independent of x and y. Now

$$\int\limits_{M\setminus B(y,\frac{d}{2})} K^m(x,z)G(z,y)\, d\mu_g(z) = \int\limits_{B(x,\frac{1}{2})} K^m(x,z)G(z,y)\, d\mu_g(z)$$
$$+ \int\limits_{M\setminus\left[B(y,\frac{d}{2})\cup B(x,\frac{1}{2})\right]} K^m(x,z)G(z,y)\, d\mu_g(z) \leq C\Psi_a(\frac{d}{2}) \int\limits_{B(x,\frac{1}{2})} K^m(x,z)\, d\mu_g(z)$$

$$+\left(\int\limits_{M\setminus B(x,\frac{1}{2})} (K^m(x,z))^2\, d\mu_g(z)\right)^{\frac{1}{2}} \left(\int\limits_{M\setminus B(y,\frac{d}{2})} (G(z,y))^2\, d\mu_g(z)\right)^{\frac{1}{2}}$$

Using (3.10) and (4.3) we get a bound of the form $Ce^{-\beta d}$ for the last term in the above inequality for some positive constants β, C independent of x, y. Next we show that $\int\limits_{B(x,\frac{1}{2})} K^m(x,z)\, d\mu_g(z)$ is bounded independent of x. Since K^m satisfies the lemma, writing in the normal coordinates centered at x and using Theorem 2.2 we get

$$\int\limits_{B(x,\frac{1}{2})} K^m(x,z)\, d\mu_g(z) \leq C \int\limits_0^{\frac{1}{2}} \int\limits_{S^{n-1}} r^{m-n} r^{n-1} A(r,\theta) dr\, d\theta$$

$$\leq C \int\limits_0^{\frac{1}{2}} r^{m-n} (\sinh(br))^{n-1} dr \leq C.$$

Combining all the above estimates we see that (4.2) holds for K^{m+2}. It remains to show that (4.3) holds for K^{m+2}.
For this purpose let us define $K^m_R(x,y)$ for $x, y \in B_R,\ x \neq y$ as in (4.1) with G_R instead of G where G_R is as in the Proof of Theorem 3.1. Then using monotone convergence theorem we see that for all $x \neq y$, $K^m_R(x,y) \to K^m(x,y)$ as $R \to \infty$ and for any fixed $x \in M$, $K^m_R(x,\cdot)$ solves

$$-\Delta_g K^{m+2}_R(x,\cdot) = K^m_R(x,\cdot)\ ,\quad K^{m+2}_R(x,y) = 0 \quad y \in \partial B_R.$$

Let $f \in C^1(M)$ be such that $0 \leq f \leq 1$,and $f = 0$ in a neighbourhood of x.
Multiplying the above equation by $f^2 K^{m+2}_R$ we get

$$\int\limits_{B_R} -\Delta_g K^{m+2}_R(x,y)(f(y))^2 K^{m+2}_R(x,y) d\mu_g(y) = \int\limits_{B_R} K^m_R(x,y) K^{m+2}_R(x,y)(f(y))^2\, d\mu_g(y) \tag{4.4}$$

The term on the left-hand side can be rewritten as

$$\int\limits_{B_R} -\Delta_g K^{m+2}_R(x,y)(f(y))^2 K^{m+2}_R(x,y) d\mu_g(y) =$$

$$\int\limits_{B_R} |\nabla_g (f(y) K^{m+2}_R(x,y))|^2 d\mu_g(y) - \int\limits_{B_R} |\nabla_g f|^2 (K^{m+2}_R(x,y))^2 d\mu_g(y)$$

Inserting this into (4.4),we obtain

$$\int\limits_{B_R} |\nabla_g (f(y) K^{m+2}_R(x,y))|^2 d\mu_g(y) - \int\limits_{B_R} |\nabla_g f|^2 (K^{m+2}_R(x,y))^2 d\mu_g(y)$$

$$\leq \left(\int\limits_{B_R} (K^m_R(x,y) f(y))^2\, d\mu_g(y)\right)^{\frac{1}{2}} \left(\int\limits_{B_R} (K^{m+2}_R(x,y) f(y))^2\, d\mu_g(y)\right)^{\frac{1}{2}}$$

$$\leq \ C \int_{B_R} (K_R^m(x,y)f(y))^2 \, d\mu_g(y) + \frac{1}{2}\left(\frac{(n-1)a}{2}\right)^2 \int_{B_R} (K_R^{m+2}(x,y)f(y))^2 \, d\mu_g(y)$$

Using Theorem 2.5 and taking the limit $R \to \infty$ we get

$$\begin{aligned} \frac{1}{2}\left(\tfrac{(n-1)a}{2}\right)^2 \int_M (f(y)K^{m+2}(x,y))^2 d\mu_g(y) - & \int_M |\nabla_g f|^2 (K^{m+2}(x,y))^2 d\mu_g(y) \\ & \leq C \int_M (K^m(x,y)f(y))^2 \, d\mu_g(y) \end{aligned} \tag{4.5}$$

Taking f such that $f = 0$ in $B(x, \frac{1}{2})$ and $f = 1$ in $M \setminus B(x,1)$ we get

$$\int_{M\setminus B(x,1)} (K^{m+2}(x,y))^2 d\mu_g(y) \leq C \int_{B(x,1)\setminus B(x,\frac{1}{2})} (K^{m+2}(x,y))^2 d\mu_g(y)$$

$$+ C \int_{M\setminus B(x,\frac{1}{2})} (K^m(x,y))^2 d\mu_g(y)$$

The first term on the right hand side of the above inequality is bounded independent of x as $K^{m+2}(x,y) \leq C(d_g(x,y))^{m+2-n}$ and the measure of the annulus is bounded independent of x thanks to the lower bound on Ric_g. The second term is bounded by assumption. Thus there exists a $C > 0$ such that for all $x \in M$

$$\int_{M\setminus B(x,1)} (K^{m+2}(x,y))^2 d\mu_g(y) \leq C \tag{4.6}$$

Let $R > 0$ and choose $f_R \in C^1(M)$ such that

$$f_R = 0 \text{ in } B(x,R), \; f_R = 1 \text{ in } M \setminus B(x, R+1), \; |\nabla_g f_R| \leq 1, \; 0 \leq f_R \leq 1.$$

Then by taking $f = f_R$ in (4.5) and using the fact that $B(x,R+1) \setminus B(x,R) = (M \setminus B(x,R)) \setminus (M \setminus B(x,R+1))$ the equation (4.5) simplifies to

$$\left[\frac{1}{2}\left(\frac{(n-1)a}{2}\right)^2 + 1\right] \int_{M\setminus B(x,R+1)} (K^{m+2}(x,y))^2 d\mu_g(y)$$

$$\leq \int_{M\setminus B(x,R)} (K^{m+2}(x,y))^2 d\mu_g(y) + C \int_{M\setminus B(x,R)} (K^m(x,y))^2 \, d\mu_g(y)$$

Thus if we denote $\alpha = \left[\frac{1}{2}\left(\frac{(n-1)a}{2}\right)^2 + 1\right]^{-1}$ then $0 < \alpha < 1$ and satisfies for all $R > 0$

$$\int_{M\setminus B(x,R+1)} (K^{m+2}(x,y))^2 d\mu_g(y) \leq \alpha \int_{M\setminus B(x,R)} (K^{m+2}(x,y))^2 d\mu_g(y)$$

$$+ C\alpha \int_{M\setminus B(x,R)} (K^m(x,y))^2 \, d\mu_g(y).$$

Let $R > 1$, then $k \le R < k+1$ for some $k \in \mathbb{N}$ and hence a repeated use of the above identity gives

$$\int_{M\setminus B(x,R)} (K^{m+2}(x,y))^2 d\mu_g(y) \le \int_{M\setminus B(x,k)} (K^{m+2}(x,y))^2 d\mu_g(y)$$

$$\le \alpha^{k-1} \int_{M\setminus B(x,1)} (K^{m+2}(x,y))^2 d\mu_g(y) \;+\; \sum_{i=1}^{k-1} C\alpha^i \int_{M\setminus B(x,k-i)} (K^m(x,y))^2 \, d\mu_g(y)$$

$$\le \alpha^{R-2} \int_{M\setminus B(x,1)} (K^{m+2}(x,y))^2 d\mu_g(y) \;+\; \sum_{i<\frac{k}{2}} C\alpha^i \int_{M\setminus B(x,k-i)} (K^m(x,y))^2 \, d\mu_g(y)$$

$$+ \sum_{i\ge\frac{k}{2}}^{k-1} C\alpha^i \int_{M\setminus B(x,k-i)} (K^m(x,y))^2 \, d\mu_g(y)$$

$$\le \alpha^{R-2} \int_{M\setminus B(x,1)} (K^{m+2}(x,y))^2 d\mu_g(y) + Ck \int_{M\setminus B(x,\frac{k}{2})} (K^m(x,y))^2 \, d\mu_g(y)$$

$$+ Ck\alpha^{\frac{k}{2}} \int_{M\setminus B(x,1)} (K^m(x,y))^2 \, d\mu_g(y)$$

$$\le Ce^{(R-2)log\alpha} + Cke^{-\alpha_m \frac{k}{2}} + Ck\alpha^{\frac{k}{2}} \le Ce^{-\alpha_{m+2}R}$$

for some $\alpha_{m+2} > 0$ thanks to (4.6). □

4.2. **Symmetrization of the kernel.** Recall, for a function $f : M \to [-\infty, \infty]$ the distribution function of f is given by

$$\lambda_f(t) \;=\; \mu_g(\{x \in M : |f(x)| > t\}) \;,\; t \in \mathbb{R}$$

and its nonincreasing rearrangement $f^* : (0, \infty) \to (0, \infty)$ is defined by

$$f^*(t) \;=\; \inf\{s : \lambda_f(s) \le t\} \;,\; t > 0.$$

For $K : M \times M \to [-\infty, \infty]$, denote by K^x the function $y \to K(x, y)$. Denote by K^* and K^{**} the functions

$$K^*(t) \;=\; \sup_{x\in M}(K^x)^*(t) \;\;,\;\; K^{**}(t) \;=\; \frac{1}{t}\int_0^t K^*(s)\, ds \;,\; t > 0.$$

We have the following estimate on the kernel K^m introduced in (4.1) .

Theorem 4.2. *Let (M, g), K^m be as in Lemma 4.1 then*

(i) *there exist constants $A, \beta > 0$ such that*

$$(K^m)^*(t) \;\le\; [\beta_0(m,n)t]^{\frac{m-n}{n}} \left[1 + At^{\beta}\right] \quad \textit{for } 0 < t \le 1 \tag{4.7}$$

(ii) *For any $\sigma \in (0, 1)$, there exists $B_\sigma > 0$ such that*

$$(K^m)^*(t) \;\le\; \frac{B_\sigma}{t^\sigma} \quad \textit{for } t > 1. \tag{4.8}$$

Proof. First note that if $f(t) = At^{-\alpha}\left[1+Bt^{\beta}\right]$, $t>0$, for positive constants A, B, α, β such that $\beta<\alpha$ then there exists a $C>0$ such that

$$f^{-1}(t) \leq \left[At^{-1}\right]^{\frac{1}{\alpha}}\left[1+Ct^{-\frac{\beta}{\alpha}}\right] \quad \text{for} \quad t>1$$

Using this together with (4.2) and Theorem 2.1 we get for $t>1$,

$$\mu_g(\{y\in M : K^m(x,y)>t\}) \leq \mu_g(B(x,f^{-1}(t))) \leq V^n_{-b^2}(f^{-1}(t))$$

where f is as above with $A=\alpha_{n,m}$ and $\alpha=n-m$, $B=C$ and $\beta=\frac{1}{2}$. Now substituting $V^n_{-b^2}(f^{-1}(t))$ using (2.2) we get for any $x\in M$,

$$\mu_g(\{y\in M : K^m(x,y)>t\}) \;\leq\; \frac{\omega_{n-1}}{n}\left(\frac{\alpha_{n,m}}{t}\right)^{\frac{n}{n-m}}\left[1+Ct^{\frac{-1}{2(n-m)}}\right] \quad \text{for} \quad t>1$$

Again if $g(t)=At^{-\alpha}\left[1+Bt^{-\beta}\right]$, $t>0$, for positive constants A,B,α,β then there exists a $C>0$ such that

$$g^{-1}(t) \leq \left[At^{-1}\right]^{\frac{1}{\alpha}}\left[1+Ct^{\frac{\beta}{\alpha}}\right] \quad \text{for} \quad 0<t\leq 1$$

Using this fact together with the above estimate proves (4.7).
To prove (4.8), first recall from (3.16) and (3.17) we have for any $x\in M$,

$$\mu_g(\{y\in M : G^x(y)>t\}) \;\leq \begin{cases} \frac{C}{t} & \text{for} \quad 0<t\leq 1 \\ \frac{C}{t^{\frac{n}{n-2}}} & \text{for} \quad t>1 \end{cases}$$

where C is independent of x. Hence

$$G^*(t) \;\leq \begin{cases} \frac{C}{t^{\frac{n-2}{n}}} & \text{for} \quad 0<t\leq 1 \\ \frac{C}{t} & \text{for} \quad t>1 \end{cases} \tag{4.9}$$

This immediately proves (4.8) when $m=2$. Now assume the result is true for m. We claim that it will be true for $m+2$ if $m+2<n$.
Fix $x\in M$, then

$$K^{m+2}(x,y) = \int_M K^m(x,z)G(z,y)d\mu_g(z) = \int_M G(y,z)(K^m)^x(z)d\mu_g(z)$$

i.e., for $x\in M$, $(K^{m+2})^x$ is obtained by integrating $(K^m)^x$ against the kernel G. Thus it follows from the improved version of O'Neil's lemma (see [11], Lemma 2) that

$$[(K^{m+2})^x]^*(t) \leq [(K^{m+2})^x]^{**}(t) \leq tG^{**}(t)[(K^m)^x]^{**}(t) + \int_t^{\infty} G^*(s)[(K^m)^x]^*(s)\,ds$$

Now the estimate (4.8) on K^{m+2} follows from the induction assumption and (4.9). □

4.3. **Proof of theorem.** As stated before we will prove our theorem by writing the functions as integrals of the corresponding derivatives agaist kernel an idea initiated in [1] and developed further by Fontana and collaborators. Let us recall the following theorem which is essentially Theorem 3 of [12].

Theorem 4.3. *Let (M, g) be a Hadamard manifold and $K : M \times M \to [-\infty, \infty]$ be a measurable function satisfying $K(x, y) = K(y, x)$ for all x, y and for some $1 < q < \infty$,*

$$K^*(t) \leq \begin{cases} [At]^{\frac{-1}{q'}} \left[1 + Ct^{\beta}\right] & \text{for} \quad 0 < t \leq 1 \\ Bt^{\frac{-1}{q'}} & \text{for} \quad t > 1 \end{cases} \tag{4.10}$$

and

$$\int_1^{\infty} (K^*(t))^{q'} \, dt < \infty \tag{4.11}$$

where $q' = \frac{q}{q-1}$ and A, B, C are fixed constants. For a measurable functions $f : M \to \mathbb{R}$ define

$$Tf(x) \; = \int_M K(x, y) f(y) \, d\mu_g(y) \; , \; x \in M \tag{4.12}$$

whenever the integral exists. Then $Tf(x)$ is defined for a.e. $x \in M$ when $f \in L^q(M)$ and there exists a constant $\tilde{C} > 0$ such that

$$\int_M E_{[q-1]} \left(A|Tf(x)|^{q'}\right) \, d\mu_g(x) \; \leq \tilde{C} \left(1 + \int_M |Tf(x)|^q \, d\mu_g(x)\right) \tag{4.13}$$

holds for all $f \in L^q(M)$ with $||f||_q \leq 1$.

Proof of Theorem 1.1 : First note that a repeated use of (3.1) gives

$$u(x) = T(\nabla_g^k u)$$

where T is defines as in (4.12) with $K(x, y) = K^m(x, y)$ when $x \neq y$ and $K(x, x) = 0$. Moreover from Theorem 4.2 we see that K^m satisfies the assumptions of the above theorem with $q = p = \frac{n}{m}$ and $A = \beta_0(m, n)$. Thus Theorem 1.1 applies and we get for $u \in C_c^m(M)$ with $\int_M \left[|\nabla_g^m u|^P \; + \lambda \; |u|^p\right] \; d\mu_g \; \leq 1$,

$$\int_M E_{[p-1]} \left(\beta_0(m, n)|u(x)|^{p'}\right) \, d\mu_g(x) \; \leq \tilde{C} \left(1 + \int_M |u|^p \, d\mu_g(x)\right) \leq \tilde{C} \left(1 + \lambda^{-1}\right).$$

When $n \leq 2m$, we have $p \leq 2$ and hence from Theorem 2.7 we see that if $u \in C_c^m(M)$ with $\int_M |\nabla_g^m u|^P \; d\mu_g \; \leq 1$, then $\int_M |u|^P \; d\mu_g \; \leq C_{m,p}$ and hence the conclusion of the theorem follows if $\lambda = 0$. The optimality of the constant $\beta_0(m, n)$ follows using standard test functions (see [1] for the proof in the Euclidean case and [10] Proposition 3.6 for the Riemannian case).This completes the proof. ☐

References

[1] D.R. Adams,*A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. (2), 128 (2) (1988), pp. 385-398

[2] Adimurthi and K. Tinterev, *On a version of Trudinger-Moser inequality with Möbius shift invariance*, Calc. Var. Partial Differential Equations 39 (2010), no. 1-2, 203-212

[3] Aubin, Thierry *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[4] Battaglia, Luca; Mancini, *Gabriele Remarks on the Moser-Trudinger inequality*, Adv. Nonlinear Anal. 2 (2013), no. 4, 389-425
[5] D. Cao *Nontrivial solution of semilinear elliptic equations with critical exponent in* $\mathbb{R}^2$, Communications in Partial Differential Equations, vol. 17, 407-435, 1992.
[6] Chavel, Isaac, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.
[7] Cherrier, Pascal *Une inégalité de Sobolev sur les variétés riemanniennes* Bull. Sci. Math. (2) 103 (1979), no. 4, 353-374.
[8] Coulhon, Thierry; Duong, Xuan Thinh *Riesz transform and related inequalities on noncompact Riemannian manifolds*, Comm. Pure Appl. Math. 56 (2003), no. 12, 1728-1751.
[9] J.M. do Ó,*N-Laplacian equations in* $\mathbb{R}^N$ *with critical growth,* Abstract and Applied Analysis, vol. 2, pp. 301-315 (1997).
[10] Fontana, Luigi *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. 68 (1993), no. 3, 415-454.
[11] Fontana, Luigi; Morpurgo, Carlo *Adams inequalities on measure spaces*, Adv. Math. 226 (2011), no. 6, 5066-5119.
[12] L. Fontana, C. Morpurgo ,*Sharp Adams and Moser-Trudinger inequalities on* $\mathbb{R}^n$ *and other spaces of infinite measure*,Preprint, arXiv:1504.04678 [math.AP]
[13] Gallot, Sylvestre; Hulin, Dominique; Lafontaine, Jacques *Riemannian geometry*, Third edition. Universitext. Springer-Verlag, Berlin, 2004.
[14] Hebey, Emmanuel *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics, 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
[15] Karmakar, Debabrata; Sandeep, Kunnath *Adams inequality on the hyperbolic space*, J. Funct. Anal. 270 (2016), no. 5, 1792-1817.
[16] Y. Li, B. Ruf *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^{\mathbb{N}}$, Indiana University Mathematics Journal, vol. 57, no. 1, 451-480, 2008.
[17] Peter Li , Luen-Fai Tam *Symmetric Green's functions on complete manifolds*, Amer. J. Math., 109(6):1129 -1154, 1987.
[18] Li, Peter; Wang, Jiaping *Complete manifolds with positive spectrum*, II. J. Differential Geom. 62 (2002), no. 1, 143-162
[19] G. Lu, H. Tang *Best constants for Moser-Trudinger inequalities on high dimensional hyperbolic spaces*, Adv. Nonlinear Stud. 13 (2013), no. 4, 1035-1052.
[20] Li, Jungang; Lu, Guozhen; Yang, Qiaohua *Fourier analysis and optimal Hardy-Adams inequalities on hyperbolic spaces of any even dimension*, Adv. Math. 333 (2018), 350-385.
[21] G. Mancini, K. Sandeep *Moser-Trudinger inequality on conformal discs*, Communications in Contemporary Mathematics, vol. 12, no. 6, 1055-1068, 2010.
[22] G. Mancini, K. Sandeep, C. Tintarev *Trudinger-Moser inequality in the hyperbolic space* $\mathbb{H}^N$, Adv. Nonlinear Anal. 2 (2013), no. 3, 309-324.
[23] Moser, J. *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1970/71), 1077-1092.
[24] R. Panda *Nontrivial solution of a quasilinear elliptic equation with critical growth in* $\mathbb{R}^{\mathbb{N}}$, Proceedings of the Indian Academy of Science, vol. 105, pp. 425-444, 1995.
[25] Petersen, Peter *Riemannian geometry*, Second edition. Graduate Texts in Mathematics, 171. Springer, New York, 2006.
[26] S.I. Pohozhaev,*The Sobolev imbedding in the case* $pl = n$, Proc.Tech.Sci. Conf. on Adv. Sci. Research 1964-1965 , Mathematics Section, Moskov. Energet. Inst., Moscow (1965), 158-170
[27] B. Ruf *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$, Journal of Functional Analysis, vol. 219, no. 2, pp. 340-367, 2005.
[28] Stein, Elias M *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J.
[29] Strichartz, Robert S.*Analysis of the Laplacian on the complete Riemannian manifold* J. Funct. Anal. 52 (1983), no. 1, 48-79.

[30] Trudinger, Neil S. *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17 1967 473-483.

[31] Yang, Qiaohua; Su, Dan; Kong, Yinying *Sharp Moser-Trudinger inequalities on Riemannian manifolds with negative curvature* Ann. Mat. Pura Appl. (4) 195 (2016), no. 2, 459-471.

[32] Yau, Shing Tung *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201-228.