TOWARDS A GEOMETRIC CORRESPONDENCE FOR σ-DERIVATIONS

by Marc Reversat

Abstract. — This paper is a review of a tentative to give a common formalism for an analog of the geometric Satake correspondance, when the usual derivation is replaced by a σ-derivation, for essentially two kinds of morphisms σ: on one hand σ is a Frobenius morphism and then we work with schemes, bundles... over a base field of positive characteristic, on the other hand σ is a q-difference morphism and we are over $\mathbb{C}$. First we explain the motivation of the problem, second we give the main ideas of our construction and sketches of proofs.

Résumé (*Vers une correspondance géométrique pour les σ-dérivations*)

Nous expliquons ici une tentative de construction d'une sorte de correspondance géométrique de Satake, quand la dérivation usuelle est remplacée par des σ-dérivations, ceci dans deux "cas extrêmes" : lorsque σ est un morphisme de Frobenius, alors nous travaillons avec des schémas, fibrés... sur un corps de base de caractéristique positive, ou bien lorque σ est un morphisme aux q-différences, le corps de base étant cette fois $\mathbb{C}$. D'abord nous expliquons les motivations d' un tel problŔme, ensuite nous donnons les principaux arguments de notre construction ainsi qu'une idée des démonstrations.

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1. Introduction

During the seventies has appeared clearly analogies between Drinfeld modules, a notion coming from the arithmetic in positive characteristics, and a class of differential equations (of course living in spaces of zero characteristic), the Krichever modules.

Let $\mathcal{C}$ be a smooth projective and geometrically irreducible curve over a field $\mathbb{F}_q$ with q elements (resp. over the field $\mathbb{C}$ of complex numbers), let ∞ be a close point of $\mathcal{C}$ and $A = \mathcal{O}_\mathcal{C}(\mathcal{C} - \{\infty\})$ be the ring of regular functions outside ∞. Let k be an algebra over $\mathbb{F}_q$ (resp. be a differential algebra over $\mathbb{C}$). When the base field is $\mathbb{F}_q$, we denote by $L = k\{\tau\}$ the ring of Ore polynomials (cf [34]), where τ is the Frobenius morphism such that for all λ in k, $\tau\lambda = \lambda^q\tau$. When $\mathbb{C}$ is the base field, we set $L = k[D]$, where D is a differential operator such that for all λ in k, $D\lambda - \lambda D$ is the derivative of λ. Let

$$u = \sum_{0\le i\le d} u_i\tau^i \text{ , resp. } u = \sum_{0\le i\le d} u_i D^i$$

be an element of L, the order of u is d if $u_d \neq 0$ and u is called elliptic if u_d is invertible in k. A morphism of algebras

$$\phi : A \to L \tag{1}$$

is called a Drinfeld module (resp. a Krichever module) if $\phi(a)$ is elliptic for all non zero element of A and if there exists an $a \in A$ such that $\phi(a)$ has non zero order.

Drinfeld modules has been introduced, under the name of elliptic modules, by V.G.Drinfeld ([11]), to construct arithmetic objects of positive characteristic analog to those coming from the elliptic curves over the field of rationals $\mathbb{Q}$, as (non commutative) rings of isogenies and their orders, torsion points and Tate modules, modular schemes..., which permit to Drinfeld to give a proof of the Langlands correspondence in positive characteristic for GL_2 and for Galois representations that "look like the special representation at infinity"(op. cit. and [13]).

Commutative subrings of $L = k[D]$, especially when $k = \mathbb{C}[[t]]$ and $D = d/dt$, has been introduced by I.M. Krichever to study differential equations, he proved that for any commutative subring B of $L = \mathbb{C}[[t]][d/dt]$ containing two operators with coprime orders, there exists a curve $\mathcal{C}$ over $\mathbb{C}$ and a morphism as in (1), with image equal to B ([19], see also [20]); this means that B is what is called today a Krichever module.

V.G. Drinfeld has given the common framework for this two kind of modules: in both case (positive characteristic case and differential case), the commutative subrings of L are of the form (1), this is proved by introducing a common formalism ([13]). At almost the same time, D. Mumford ([33]) has given dictionaries relating in three different cases, on one hand an algebraic curve, one or many of its points and a vector bundle on this curve, on the other hand commutative subrings of rings of operators. Firstly D. Mumford considers the case of a difference operator (for instance, with the previous notation $L = k(t)\{\sigma\}$, where σ is defined by $\sigma(f(t)) = f(t+1)$), in the second case D. Mumford recovers Krichever results recalled above, the third ditionary extends (at least partially) the Drinfeld framework (see above) to the case where $L = k\{\sigma\}$, k beeing any field and $\sigma \in \mathrm{Aut}(k)$ beeing of infinite order. Note that the three situations studied here can be decribed by derivations, ordinary derivation in the Krichever case, σ-derivation[(1)] in the other cases, the derivation over k beeing $\delta = \sigma - \mathrm{Id}_k$.

Modular schemes exist for Drinfeld modules ([11], indeed they have been generalized, see [29] and [22]) and for Krichever modules ([24], unfortunately unpublished). In a forthcoming paper ([36]) we will explain how to construct modular schemes for σ-difference equations, at least in the q-difference case (this construction uses arguments closed to those of [11] and [24]). But the aim of this paper is slitly different, it is in the spirit of works [12] of Drinfeld and [33] of Mumford: to find a common formalism for phenomenons occuring for σ-derivations, when σ is the identity map (the ordinary derivation case), or when σ is a Frobenius

(1) Definition is given in § 4.1.

morphism (a situation closed to the Drinfeld one) and in other cases, especially when σ is a q-difference morphism (let $q \in k$, $q \neq 0, 1$, this is for instance the case where $L = k(t)\{\sigma\}$ with σ defined by $\sigma(f(t)) = f(qt)$). Because our problem comes from the Langlands program, we recall now some of its aspects (indeed a few of them!).

Let $\mathcal{C}/\mathbb{F}_q$ be a projective, smooth and geometrically irreducible curve over the finite field $\mathbb{F}_q$ and let F be its field of rational functions (i.e. F is a global field of positive characteristic). V.G. Drinfeld has proved for $\mathrm{GL}_2(F)$ the so-called Langlands correspondence (see [13]), which has been completely generalized to $\mathrm{GL}_n(F)$ by L. Lafforgue ([23]), but we are here rather interested by an other aspect of the Langlands philosophy, which appear clearly in [14]. Let l be a prime number, $l \neq 0$ in $\mathbb{F}_q$. In [14], Drinfeld has proved that there exists a natural correspondence between 2-dimensional continuous l- adic representations of $\pi_1(\mathcal{C})$ and the set of unramified cuspidal Hecke eigenforms. We give some details for this statement, because it is important for the sequel of this introduction.

Let F^{alg} be an algebraic closure of F (the field of rational functions over $\mathcal{C}$) and $\mathbb{A}_F$ de the adèle ring of F. For any closed point ν of $\mathcal{C}$ we denote by F_ν the completion of F at the place ν, by O_ν the valuation ring of F_ν, by q_ν the cardinality of its residue field and by $\sigma_\nu \in \pi_1(\mathcal{C}) = \mathrm{Gal}(F^{\mathrm{alg}}/F)$ a lift of the Frobenius morphism at ν. An unramified cusp form is a function $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{Q}_l^{\mathrm{alg}}$ (an algebraic closure of $\mathbb{Q}_l$) which is right invariant under $\mathrm{GL}_2(F)$ and left invariant under $\mathrm{GL}_2(\mathbb{O})$ (where $\mathbb{O}$ is the product of all O_ν), satifying the cusp condition: for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$

$$\int_{h\in\mathbb{A}_F/F} f(g\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix})dh \;=\; 0.$$

For every closed point ν of $\mathcal{C}$ let T_ν and U_ν be the two Hecke operators defined on the space of cusp forms by

$$T_\nu(f)(g) = \int_{M_\nu} f(h^{-1}g)dh \quad \text{and} \quad U_\nu(f)(g) = f(\begin{pmatrix}\varpi_\nu^{-1} & 0\\ 0 & \varpi_\nu^{-1}\end{pmatrix} g),$$

where ϖ_ν is a uniformizing parameter in O_ν, where M_ν is the set of 2×2 matrices with entries in O_ν and determinant in $\varpi_\nu O_\nu^*$, and where dh is here the Haar measure on $\mathrm{GL}_2(F_\nu)$ normalized by $mes(\mathrm{GL}_2(O_\nu)) = 1$. Drinfeld has proved (see [14])

Theorem 1.1. — *Let E be a finite extension of $\mathbb{Q}_l$ and $\rho : \pi_1(\mathcal{C}) \to \mathrm{GL}_2(E)$ be a geometracally irreducible continuous l-adic representation.*

Then there exists a (unique up to multiplicative constant) nonzero unramified cusp form f such that, for any closed point ν of $\mathcal{C}$, $T_\nu(f) = \mathrm{Tr}\rho(\sigma_\nu)f$ and $U_\nu(f) = q_\nu^{-1}\det\rho(\sigma_\nu)f$.

This statement admits a geometric counterpart ([25]), first conjecturally extends to GL_n by G. Laumon ([25] and [26]) and proved by E. Frenkel-D. Gaitsgory-D. Kazhdan-K. Vilonen

Theorem 1.2. — *([25] for $n = 2$ and [15]) Each rank $n \geq 2$ and geometrically irreducible l-adic representation of $\pi^1(\mathcal{C})$ corresponds to a (bounded) complex of $\mathbb{Q}_l$-sheaves K_E (a perverse sheaf) on the stack $\mathrm{Bun}_{\mathcal{C},n}$ of rank n vector bundles on $\mathcal{C}$, which is in a certain sense cuspidal and an eigenvector for Hecke operators.*

One recovers the Drinfeld theorem and its extensions by using the Grothendieck dictionary "fonctions-faisceaux" and the adelic interpretation of the vector bundles on $\mathcal{C}$ (given by A. Weil), the automorphic form f (unramified, cuspidal, eigenvector for the Hecke operators) being given by the following formula: for all $g \in \mathrm{GL}_n(\mathbb{A}_F)$

$$f(g) = \chi_{K_E}(g) = \sum_{i\geq 0}(-1)^i\mathrm{Tr}(\sigma_g, \mathcal{H}^i_g(K_E)),$$

where σ_g is the Frobenius morphism acting on the stalks at g of the cohomology sheaves $\mathcal{H}^i(K_E)$.

When one considers the theorem 1.2, it is natural to note that the field of definition of the curve $\mathcal{C}$ does not matter (this is very well explained in [18]). Then one can settle a problem in zero characteristic, closed to 1.2, which is what we called the geometric correspondence [(2)]. Now let $\mathcal{C}$ be a smooth projective curve over the field of complex numbers $\mathbb{C}$ and let G be a reductive goup also defined over $\mathbb{C}$, one denotes by ${}^LG$ the Langlands dual of G (when $G = \mathrm{GL}_n$ one has ${}^LG = \mathrm{GL}_n(\mathbb{C})$) and by $\mathrm{Bun}_{\mathcal{C},G}$ the stack of G-bundles on $\mathcal{C}$.

Conjecture 1.3. — *Geometric correspondence. There a natural correspondence between ${}^LG$-local $\mathbb{Q}_l$-systems on $\mathcal{C}$ and perverse Hecke eigen-$\mathbb{Q}_l$-sheaves on $\mathrm{Bun}_{\mathcal{C},G}$.*

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(2) Sometimes named geometric Langlands correspondence, Drinfeld-Langlands correspondence, or Beilinson correspondence; indeed it is clear that it comes from main works by Langlands, Drinfeld, Laumon and Beilinson.

(This conjecture admits a precise statement, which needs a too long preparation to be settled here.) For ${}^LG$-local systems which are "opers", this is a theorem of Beilinson and Drinfeld ([6], (5.2.6)). It has been recently solved when $G = \mathrm{GL}_n$ by E. Frenkel-D. Gaitsgory-K. Vilonen ([16]) and D. Gaitsgory ([17], see also the review [27]). Problems related to 1.3 are sometimes studied with $\mathcal{D}$-modules in place of perverse sheaves, this is the same up to the Riemann-Hilbert correspondence.

We are now interested by the conjecture 1.3 when the curve $\mathcal{C}$ is reduced to one point, this gives the so called *geometric Satake correspondence*, which is a theorem.

Let G be a connected complex reductive group. The geometric Satake correspondence says that there exists a natural equivalence between the category of finite dimensional representations of the Langlands dual ${}^LG$ of G and a certain tensor category of perverse sheaves (or $\mathcal{D}$-modules) on the (infinite) affine grassmannian of G. Proofs, or outline of proofs, using the perverse sheaves point of view, have been given by V. Ginzburg ([18]), by I. Mirković and K. Vilonen ([32]); in [6], §5.3, A. Beilinson and V. Drinfeld described the proof using $\mathcal{D}$-modules. We will give in § 2.3 the precise statement of this last formulation of the Satake correspondence.

The aim of our work it to give an analog of the Satake correspondence, when $\mathcal{D}$-modules are replaced by $\mathcal{D}_\sigma$-modules, that means when the usual derivation is replaced by σ-derivations, for automorphisms σ of the affine grassmannian. More precisely, let K be a "big" field admitting k as subfield, let $\mathcal{C}$ be a smooth, projective and geometrically irreducible curve over the field K, admitting a smooth model over k, let P be a k-rational point of $\mathcal{C}$ and σ be an automorphism of $\mathcal{C}$, defined over k, which leaves fixed the point P. The affine grassmannian $\underline{\mathcal{G}r}$ of G can be interpreted as the stack of isomorphy classes of G-bundles on the infinitesimal disk around P ([3] and [21], see also our review § 2), then σ acts naturally on $\underline{\mathcal{G}r}$, which permits to define σ-derivations and $\mathcal{D}^\sigma$-modules (i.e. $\mathcal{D}$-modules for σ-derivations) on $\underline{\mathcal{G}r}$. We give *a common formalism* for an analog of the Satake correspondence, for certain automorphisms σ, mainly

- when $k = \mathbb{F}$ is a finite field, $K = \mathbb{F}^{\mathrm{alg}}$ is an algebraic closure of $\mathbb{F}$ and σ is the geometric Frobenius morphism of $\mathcal{C}$ coming from the Frobenius morphism of K with fixed field k,
- or when $k = K = \mathbb{C}$ and σ is induced on the curve by the q-difference morphism on its field of rational functions.

This work has began following discussions with Jean-Pierre Ramis, we have had many stimulated conversations with Vadim Schechtman during its preparation. We would like to thank both here.

1.1. Notations, conventions.

All rings, algebras... will be commutative with units and their morphisms will map unit to unit (the morphism inclusion also). Let k be any field, $K = k^{\text{alg}}$ be an algebraic closure of k and $\mathbb{O} = K[[t]]$ be the ring of formal power series with coefficients in K, $\mathbb{K} = \text{Quot}(\mathbb{O}) = K((t))$.

Let G be a connected reductive group defined over K. For any commutative (with unit) K-algebra R, we set $G(R[[t]])$ and $G(R((t)))$ to be respectively the $R[[t]]$-valued and $R((t))$-valued points of G.

Let $\mathcal{C}$ be a smooth connex projective curve defined over K, we suppose that $\mathcal{C}$ has a model over k, i.e. that there exists a curve $\mathcal{C}_k$ defined and smooth over k such that $\mathcal{C}_k \times_{\text{Spec}k} \text{Spec}K = \mathcal{C}$.

For all algebra R as before, we set

$$\mathcal{C}_R = \mathcal{C} \times_{\text{Spec}K} \text{Spec}R \quad \text{and} \quad \mathcal{C}_R^* = (\mathcal{C} - \{P\}) \times_{\text{Spec}K} \text{Spec}R;$$

we set also

$$D_R = \text{Spec}R[[t]] \quad \text{and} \quad D_R^* = \text{Spec}R((t)),$$

(we delete the subscript when $R = K$). We have the cartesian diagram

$$\begin{array}{ccc} D_R^* & \hookrightarrow & D_R \\ \downarrow & & \downarrow \\ \mathcal{C}_R^* & \hookrightarrow & \mathcal{C}_R \end{array} \tag{2}$$

the image of the closed point of D_R being P and the horizontal arrows being inclusions.

2. The affine grassmannian

Our main referencs are [3] and [21], but some facts or informations need [35] and [30].

2.1. The functors $\underline{G(\mathbb{O})}$ and $\underline{G(\mathbb{K})}$.— Let $(K-\text{alg})$ be the category of K-algebras, associative, commutative, with unit and $(\text{aff}-K-\text{Sch})$ be the opposite category of affine K-scheme. We denote also by $K-\text{Sch}$

the category of K-schemes. Let $\mathcal{E}$ be the category of sets or groups, the two functors

$$\begin{array}{rcl} \underline{G(\mathbb{O})} : (K-\mathrm{alg}) & \to & \mathcal{E} \\ R & \mapsto & G(R[[t]]) \end{array} \tag{3}$$

$$\begin{array}{rcl} \underline{G(\mathbb{K})} : (K-\mathrm{alg}) & \to & \mathcal{E} \\ R & \mapsto & G(R((t))) \end{array} \tag{4}$$

are K-spaces (see [28]), this means that they are sheaves for the faithfully flat topology.

2.1.1. The K-space $\underline{G(\mathbb{O})}$ is a representred by an affine group scheme, indeed

$$\underline{G(\mathbb{O})} = \mathrm{Hom}_{(K-\mathrm{Sch})}(\,\cdot\,, G(\mathbb{O}))$$

and $G(\mathbb{O})$ is an affine group scheme, for instance, if $G = \mathrm{GL}_n$,

$$\mathrm{GL}_n(\mathbb{O}) = \mathrm{GL}_n(K) \times \prod_{i\in\mathbb{N}^*} M_n(K),$$

where the products of the different schemes are over $\mathrm{Spec}K$.

2.1.2. The K-space $\underline{G(\mathbb{K})}$ is represented by an ind-affine group, more precisely, there exists an increasing sequence $(G(\mathbb{K})_N)_{N\in\mathbb{N}}$ of affine scheme such that the inclusions $G(\mathbb{K})_N \subseteq G(\mathbb{K})_{N+1}$ are closed immersions, and, if $\underline{G(\mathbb{K})}_N$ denote the functor

$$\begin{array}{rcl} \underline{G(\mathbb{K})}_N : (K-\mathrm{alg}) & \to & \mathcal{E} \\ R & \mapsto & \mathrm{Hom}_{(K-\mathrm{Sch})}(\mathrm{Spec}R, G(\mathbb{K})_N) \end{array},$$

such that

$$\underline{G(\mathbb{K})} = \varinjlim(\underline{G(\mathbb{K})}_N)_{N\in\mathbb{N}} = \varinjlim(\mathrm{Hom}_{(K-\mathrm{Sch})}(\,\cdot\,, G(\mathbb{K})_N))_{N\in\mathbb{N}},$$

then such that for all R

$$\underline{G(\mathbb{K})}(R) = \varinjlim(\underline{G(\mathbb{K})}_N(R))_{N\in\mathbb{N}} = \varinjlim(G(R((t)))_N) = G(R((t))).$$

For instance, if $G = \mathrm{GL}_n$, for any non negative integer N let

$$M_n(\mathbb{K})_N = \prod_{i\geq -N} t^i M_n(K) \subseteq M_n(K((t))),$$

then

$$\mathrm{GL}_n(\mathbb{K})_N = \{P, Q \in M_n(\mathbb{K})_N \ / \ PQ = \mathrm{Id}\}.$$

In the general case, we can suppose that G is a closed subscheme of GL_n, for some n, and we have

$$G(\mathbb{K})_N = G(\mathbb{K}) \cap \mathrm{GL}_n(\mathbb{K})_N.$$

2.1.3. For any algebra R, let $F_1(R)$ be the isomorphy classes of triples (E, φ, ψ), where E is a G-bundle on $\mathcal{C}_R$, $\varphi : \mathcal{C}_R^* \times_{\mathrm{Spec}K} G \simeq E_{|\mathcal{C}_R^*}$ is a trivialisation of E over $\mathcal{C}_R^*$ and $\psi : D_R \times_{\mathrm{Spec}K} G \simeq E_{|D_R}$ is a trivialisation of the pull back $E_{|D_R} = E \times_{\mathcal{C}_R} D_R$ of E on D_R.

Proposition 2.1. — *The two functors F_1 and $\underline{G(\mathbb{K})}$, from $(R - \mathrm{alg})$ to the category of sets, are isomorphic.*

Proof. — (sketch) Let (E, φ, ψ) be as above. With the diagram (2) one can define the two maps

$$f : D_R^* \times_{\mathrm{Spec}K} G \to \mathcal{C}_R^* \times G \xrightarrow{\varphi} E \times_{\mathcal{C}_R} \mathcal{C}_R^* \to E \times_{\mathcal{C}_R} D_R^*,$$

$$g : D_R^* \times_{\mathrm{Spec}K} G \to D_R \times G \xrightarrow{\psi} E \times_{\mathcal{C}_R} D_R \to E \times_{\mathcal{C}_R} D_R^*.$$

One sees that f and g are two trivializations of E over D_R^*, then $g^{-1} \circ f$ is an automorphism of the trivial G-bundle $D_R^* \times_{\mathrm{Spec}K} G$, which is given by an element of $G(R((t))) = \underline{G(\mathbb{K})}(R)$, as every $R((t))$-linear automorphism of the algebra of $D_R^* \times_{\mathrm{Spec}K} G$. This gives a map $F_1(R) \to \underline{G(\mathbb{K})}(R)$ which is obviously functorial in R. □

2.2. The functor $\underline{\mathcal{G}r}$ (i.e. the affine grassmannian of G).— Let T be the fpqc site of K-schemes, i.e. T is a category which objects are K-schemes and which arrows are faithfully flat, quasi compact morphisms of K-schemes. The functors $\underline{G(\mathbb{O})}$ and $\underline{G(\mathbb{K})}$ extend naturally to T. Let S be the category of scheaves on T.

Definition 2.2. — The affine grassmannian $\underline{\mathcal{G}r}$ of G is, in the abelian category S, the cokernel of the inclusion morphism $\underline{G(\mathbb{O})} \hookrightarrow \underline{G(\mathbb{K})}$.

Proposition 2.3. — *The affine grassmannian $\underline{\mathcal{G}r}$ of G is representable by an ind-projective K-scheme: there exists an increasing sequence $(\mathcal{G}r_N)_{N\in\mathbb{N}}$ of projective K-schemes, such that the inclusion morphisms $\mathcal{G}r_N \hookrightarrow \mathcal{G}r_{N+1}$ are closed immersions and such that, if $\mathcal{G}r := \cup_{N\in\mathbb{N}}\mathcal{G}r_N$, then for any K-scheme U, $\underline{\mathcal{G}r}(U) = \mathrm{Hom}_{K-\mathrm{Sch}}(U, \mathcal{G}r)$.*

Moreover the ind-scheme $\mathcal{G}r$ is equipped with a left action of $G(\mathbb{O})$ such that each $\mathcal{G}r_N$ is $G(\mathbb{O})$-stable and consists of finitely many orbits of the form $G(\mathbb{O})\chi$, where χ is (the canonical image in $\mathcal{G}r$ of) a one parameter *subgroup $\chi : \mathbb{G}_m \to G$. This decomposition of $\mathcal{G}r$ in $G(\mathbb{O})$-orbits is a stratification.*

The proof is given, at least partially, in many papers (see [3], [4], [18]), it is complete in [6], § 4.5 and 7.11.2.

Remark 2.4. — The ind-scheme $\mathcal{G}r$ is, in the fpqc site of ind-K-schemes, the fpqc quotient

$$\mathcal{G}r = G(\mathbb{K})/G(\mathbb{O})$$

and with the notations of 2.1.2 one has for all N

$$\mathcal{G}r_N = G(\mathbb{K})_N/G(\mathbb{O}).$$

2.2.1. The group $G(\mathbb{K})$ acts by the left on $\mathcal{G}r$, for any $\gamma \in G(\mathbb{K})$ let $(\gamma(N))_{N\in\mathbb{N}}$ be an increasing sequence of integers such that for any N

$$\gamma(\mathcal{G}r_N) \subset \mathcal{G}r_{\gamma(N)},$$

and we suppose moreover that $\gamma(N) \geq N$ and that $\gamma(N)$ is chosen as small as possible. If $\gamma \in G(\mathbb{O})$, we have $\gamma(N) = N$.

2.2.2. By restriction on objects of T which are affine K-schemes, one can consider $\underline{\mathcal{G}r}$ as a functor on (aff-K-Sch) (or (K-alg)).

For any algebra R, let $T_2(R)$ be the isomorphy classes of pairs (E, φ), where E is a G-bundle on $\mathcal{C}_R$ and $\varphi : \mathcal{C}_R^* \times_{\mathrm{Spec}K} G \simeq E_{|\mathcal{C}_R^*}$ is a trivialisation of E over $\mathcal{C}_R^*$.

Proposition 2.5. — *The two functors T_2 and $\underline{\mathcal{G}r}$, from $(K - \mathrm{alg})$ to the category of sets, are isomorphic.*

The proof uses same arguments as in proposition 2.1.

2.3. The geometric Satake correspondence.— In this section, we have $K = \mathbb{C}$. This correspondence is given in [18], [32] and [6] § 5.3, we follow the last reference, where it is given in terms of $\mathcal{D}$-modules.

The geometric Satake correspondance is between on one hand the category $\mathrm{Rep}^L G$ of finite representations the langlands dual ${}^L G$ and on the other hand the category of coherent holonomic $\mathcal{D}$-modules on $\underline{\mathcal{G}r}$. This functor $\underline{\mathcal{G}r}$ in indeed a stack (and the notion of $\mathcal{D}$-modules on a stack has been clarified in [6]), it is represented by the ind-projective K-scheme $\mathcal{G}r$.

Let χ be a one parameter subgroup of G, any smooth $\mathcal{D}$-module on the orbit $G(\mathbb{O})\chi$ (see proposition 2.3) is constant ([6], lemma 5.3.2). Let

$\mathcal{P}$ be the category of holonomic $\mathcal{D}$-modules on $\mathcal{G}r$ smooth on each $G(\mathbb{O})$-orbit, it is semisimple (op. cit. proposition 5.3.3) and carries a canonical tensor structure (op. cit. 5.3.5, [32]). Consider $G(\mathbb{K}) \times \mathcal{G}r$ equipped with the following action of $G(\mathbb{O})$: for any $(g, x) \in G(\mathbb{K}) \times \mathcal{G}r$ and any $\gamma \in G(\mathbb{O})$, we set $\gamma \cdot (g, x) = (g\gamma^{-1}, \gamma x)$. Denote by $G(\mathbb{K}) \times_{G(\mathbb{O})} \mathcal{G}r$ the quotient of $G(\mathbb{K}) \times \mathcal{G}r$ by this action. Consider the following diagram

$$\mathcal{G}r \times \mathcal{G}r \xleftarrow{u} G(\mathbb{K}) \times \mathcal{G}r \xrightarrow{v} G(\mathbb{K}) \times_{G(\mathbb{O})} \mathcal{G}r \xrightarrow{m} \mathcal{G}r \tag{5}$$

where u is the product of the canonical map $G(\mathbb{K}) \to \mathcal{G}r$ and of the identity map on $\mathcal{G}r$, where v is the quotient morphism and where m is the multiplication map. Let P_1 and P_2 be two $\mathcal{D}$-modules on $\mathcal{G}r$, as u and v are locally trivial for the Zariski topology (op. cit. theorem 4.5.1), there exists a $\mathcal{D}$-module P on $G(\mathbb{K}) \times_{G(\mathbb{O})} \mathcal{G}r$ such that $v^* P = P_1 \boxtimes P_2$. We set

$$P_1 \circledast P_2 = m_* P \tag{6}$$

and a very important result says that $P_1 \circledast P_2$ is a $\mathcal{D}$-module on $\mathcal{G}r$, that (6) defines a tensor product on $\mathcal{P}$ ([18], [32] and [6] theorem 5.3.6).

For P in $\mathcal{P}$ set $h^{\cdot}(P) = H^{\cdot}_{DR}(P)$ (to define the De Rham cohomology we use the fact that $\mathcal{G}r$ is an ind-projective scheme). Let $\underline{\mathrm{Vect}}_{\mathbb{C}}$ be the category of gradued finite dimensional $\mathbb{C}$-vector spaces, then we have a functor

$$h^{\cdot} : \mathcal{P} \longrightarrow \underline{\mathrm{Vect}}_{\mathbb{C}}, \tag{7}$$

indeed $h^{\cdot}$ is a tensor functor and $\mathcal{P}$ is a tannakian category, the isomorphism group of $h^{\cdot}$ is isomorphic with the Langlands dual ${}^L G$ of G. Then it follows from the general tannakian formalism ([10],[9] and [8]) an equivalence of categories

$$\mathcal{P} \simeq \mathrm{Rep}^L G,$$

where $\mathrm{Rep}^L G$ is viewed as a tensor category (of finite representations of ${}^L G$). All these facts can be found in [18], or [32], or [6] § 5.3.

3. Topology and sheaves

The notations are now those of § 1.1.

3.1. Actions on the affine grassmannian.—

3.1.1. Two automorphisms on $\mathcal{C}$.— First we suppose that the curve $\mathcal{C}$ is defined over $\mathbb{C}$, i.e. that $k = K = \mathbb{C}$. Let q be a complex number, $q \neq 0$. Suppose that the field $K(\mathcal{C})$ of rational functions over $\mathcal{C}$ is equipped with a q-difference automorphism $\sigma^\sharp$, it means that there exists a field extension

$$K(X) \hookrightarrow K(\mathcal{C}),$$

which is finite (where X is an indeterminate), and an automorphism $\sigma^\sharp$ of $K(\mathcal{C})$ which retricted to $K(X)$ gives the q-difference automorphism sending $f(X) \in K(X)$ to $f(qX)$. Then $\sigma^\sharp$ induces an automorphism of the curve $\mathcal{C}$, denoted by σ. We suppose that this automorphism leaves fixed a closed point P of $\mathcal{C}$.

Now k is a finite field of characteristic p, with p^r elements, and K is an algebraic closure of k. Let τ be the absolute Frobenius morphism of $\mathcal{C}_k$ and denote by σ the endomorphism $\tau^r \times \mathrm{Id}_K$ on $\mathcal{C}_k \times_{\mathrm{Spec}k} \mathrm{Spec}K = \mathcal{C}$. This morphism σ is a Frobenius morphism of $\mathcal{C}$, which leaves fixed all k-rational points of $\mathcal{C}$. Let P be a k-rational point of $\mathcal{C}$.

3.1.2. Actions on $\underline{\mathcal{G}r}$.— Let P be a closed point of $\mathcal{C}$ fixed under σ and denote by t_P a local parameter at P. We have the isomorphisms

$$K((t_P)) \simeq \mathbb{K} \text{ and } K[[t_P]] \simeq \mathbb{O}$$

given by $t_P \mapsto t$, this implies an action of σ on D and D^*; we denote again by σ this last action. By scalar extension, σ acts also on $\mathcal{C}_R$, D_R and D^*_R, for any K-algebra R (on which σ acts trivially).

All these σ-action permit to construct an action on $\underline{\mathcal{G}r}$.

Lemma 3.1. — *Let R be a K-algebra and Y an R-scheme equipped by a R-morphism $Y \to \mathcal{C}_R$ and by an automorphism Σ compatible with σ, i.e. such that the following diagram is commutative*

$$\begin{array}{ccc} Y & \xrightarrow{\Sigma} & Y \\ \downarrow & & \downarrow \\ \mathcal{C}_R & \xrightarrow{\sigma} & \mathcal{C}_R \end{array}$$

Let E be a G-bunble on $\mathcal{C}_R$ and set

$$E_\sigma = E \times_{\mathcal{C},\sigma} \mathcal{C},$$

where the morphism $\mathcal{C} \to \mathcal{C}$ used in the product is σ. Suppose that E is trivial on Y, i.e. that $E \times_{\mathcal{C}_R} Y \simeq G \times_{\mathrm{Spec}K} Y$, then E_σ is also trivial on

Y; more precisely, we have a commutative diagram

$$\begin{array}{ccc} E_\sigma \times_{\mathcal{C}_R} Y & \simeq & G \times_{\mathrm{Spec}K} Y \\ \rho \downarrow & & \downarrow \\ E \times_{\mathcal{C}_R} Y & \simeq & G \times_{\mathrm{Spec}K} Y \end{array}$$

where the left vertical arrow comes from the projection $E_\sigma \to E$ and the right is $\mathrm{Id}_G \times \Sigma^{-1}$.

Proof. — It is easy, it suffices to remark that

$$E_\sigma \times_{\mathcal{C}_R} Y \simeq (E \times_{\mathcal{C}_R} Y) \times_{Y,\Sigma} Y.$$

□

It follows from this lemma that σ *induces an automorphism of $\underline{\mathcal{G}r}$, denoted always σ.* We do not give more details because the proof is straightforward (see § 2.2.2).

As in § 2.2.2, consider (E, φ), where E is a G-bundle on $\mathcal{C}_R$ and $\varphi : \mathcal{C}_R^* \times_{\mathrm{Spec}K} G \simeq E_{|\mathcal{C}_R^*}$ is a trivialisation of E over $\mathcal{C}_R^*$; the pair (E, φ) represents an element of $\underline{\mathcal{G}r}(R)$ (see prop. 2.5), it follows from the lemma 3.1 (with $\mathcal{C}_R^*$ in place of Y) that its image by σ is the isomorphy class of

$$(E_\sigma, ((\mathrm{Id}_G \times \sigma) \circ \varphi \circ \rho)^{-1})$$

(note that, as before, we denote Σ on $\mathcal{C}_R^*$ by σ).

3.1.2.1. The q-difference case.— Let

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{K} \\ \sum_{-\infty \ll i} a_i t^i & \mapsto & \sum_{-\infty \ll i} a_i (qt)^i \end{array}$$

This map induces endomorphisms (all denoted by σ) on $G(\mathbb{K})$, on $\mathcal{G}r$ and the expected σ on $\underline{\mathcal{G}r}$.

Note that any orbit $G(\mathbb{O})\chi$ (see proposition 2.3) is maped on itself by σ.

3.1.2.2. The Frobenius case.— Let

$$\begin{array}{ccc} \mathbb{K} & \to & \mathbb{K} \\ \sum_{-\infty \ll i} a_i t^i & \mapsto & \sum_{-\infty \ll i} a_i t^{ip^r} \end{array}$$

(recall that the fixed point P is rational over k). This map induces endomorphisms (all denoted by σ) on $G(\mathbb{K})$, on $\mathcal{G}r$ and the expected σ on $\underline{\mathcal{G}r}$.

The image by σ of a $G(\mathbb{O})$-orbit is a $G(\mathbb{O})$-orbit, but in general it is not the same. We have for any integer $N \geq 0$ (see proposition 2.3 and §

2.1.2) $\sigma(\mathcal{G}r_N) \subset \mathcal{G}r_{p^r N}$; this is a closed embedding of degree p^r, as the inclusion $\mathcal{G}r_N \subset \mathcal{G}r_{p^r N}$.

3.1.3. Notations.— Let $(\sigma(N))_{N\in\mathbb{N}}$ be an increasing sequence of integers such that for any N

$$\sigma(\mathcal{G}r_N) \subset \mathcal{G}r_{\sigma(N)},$$

and we suppose moreover that $\sigma(N) \geq N$ and that $\sigma(N)$ is chosen as small as possible. *In the rest of this paper we suppose that such a sequence exists*, as for the two examples above.

3.2. The topology T.— We want to define $\mathcal{D}$-modules for σ-derivations in place of the usual one. As in the Satake correspondance, we will consider these $\mathcal{D}$-modules restricted to the $G(\mathbb{O})$-orbits, but in our situation, at least in the Frobenius morphism case, we cannot consider a $G(\mathbb{O})$-orbit alone, indeed they have to be considerd with all their images by the σ^n, $n \in \mathbb{N}$; moreover the main property of these orbits is that any morphism

$$G(\mathbb{O}) \to \mathcal{G}r,$$

which maps $G(\mathbb{O})$ onto an orbit, is fpqc (faithfully flat and quasi compact). These two remarks maybe explain why we define the following Grothendieck topology.

Let T be the following category:

The *objects* of T are the

$$U = \big((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U = (\sigma_{U,N})_{N\in\mathbb{N}}\big),$$

where

- for any $N \in \mathbb{N}$, U_N is a *smooth* K-scheme of finite type and $\varphi_N : U_N \to U_{N+1}$ is a closed immersion,

- for any $N \in \mathbb{N}$, $f_N : U_N \to \mathcal{G}r_N$ is a fpqc morphism, compatible with the φ_N, i.e. such that

$$\begin{array}{ccc} U_{N+1} & \stackrel{f_{N+1}}{\to} & \mathcal{G}r_{N+1} \\ \varphi_N \uparrow & & \cup \\ U_N & \stackrel{f_N}{\to} & \mathcal{G}r_N \end{array}$$

is a commutative diagram,

- for any $N \in \mathbb{N}$, $\sigma_{U,N} : U_N \to U_{\sigma(N)}$ is a morphism of K-schemes such that

$$\begin{array}{ccccccc} U_N & \stackrel{\sigma_{U,N}}{\to} & U_{\sigma(N)} & \quad & U_N & \stackrel{\sigma_{U,N}}{\to} & U_{\sigma(N)} \\ f_N \downarrow & & \downarrow f_{\sigma(N)} & & \varphi_N \downarrow & & \downarrow \varphi_N^\sigma \\ \mathcal{G}r_N & \stackrel{\sigma}{\to} & \mathcal{G}r_{\sigma(N)} & & U_{N+1} & \stackrel{\sigma_{U,N+1}}{\to} & U_{\sigma(N+1)} \end{array}$$

where $\varphi_N^\sigma = \varphi_{\sigma(N+1)-1} \circ \cdots \circ \varphi_{\sigma(N)}$, are commutative diagrams (see 3.1.3),

- the ind-group $G(\mathbb{K})$ acts on U and this action is compatible with that of σ, this meams that for each N and each $\gamma \in G(\mathbb{K})$ we have the commutative diagrams (see 2.2.1)

$$\begin{array}{ccccccc} U_N & \stackrel{\gamma\cdot}{\to} & U_{\gamma(N)} & \quad & U_N & \stackrel{\gamma\cdot}{\to} & U_{\gamma(N)} \\ f_N \downarrow & & \downarrow f_{\gamma(N)} & & \varphi_N \downarrow & & \downarrow \varphi_N^\gamma \\ \mathcal{G}r_N & \stackrel{\gamma\cdot}{\to} & \mathcal{G}r_{\gamma(N)} & & U_{N+1} & \stackrel{\gamma\cdot}{\to} & U_{\gamma(N+1)} \end{array}$$

where $\varphi_N^\gamma = \varphi_{\gamma(N+1)-1} \circ \cdots \circ \varphi_{\gamma(N)}$ and where $\gamma\cdot$ is the left action by γ.

A *morphism* in T is the datum

$$u = (u_N)_{N\in\mathbb{N}} : U \longrightarrow V,$$

where $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ and $V = ((V_N, \psi_N, g_N)_{N\in\mathbb{N}}, \sigma_V)$ are objects of T and each u_N is a morphism fpqc morphism $u_N : U_N \to V_N$, compatible with all previous data, this means that we have commutative diagrams

$$\begin{array}{ccccccc} U_N & \stackrel{u_N}{\to} & V_N & \quad & U_N & \stackrel{u_N}{\to} & V_N \\ \varphi_N \downarrow & & \downarrow \psi_N & & f_N \downarrow & & \downarrow g_N \\ U_{N+1} & \stackrel{u_{N+1}}{\to} & V_{N+1} & & \mathcal{G}r_N & = & \mathcal{G}r_N \end{array}$$

and

$$\begin{array}{ccccccc} U_N & \stackrel{u_N}{\to} & V_N & \quad & U_N & \stackrel{u_N}{\to} & V_N \\ \sigma_{U,N} \downarrow & & \downarrow \sigma_{V,N} & & \gamma\cdot \downarrow & & \downarrow \gamma\cdot \\ U_{\sigma(N)} & \stackrel{u_{\sigma(N)}}{\to} & V_{\sigma(N)} & & U_{\gamma(N)} & \stackrel{u_{\gamma(N)}}{\to} & V_{\gamma(N)} \end{array}$$

where in the last diagram $\gamma \in G(\mathbb{K})$. We will omit often the subscript N in u_N, then for instance we will write now $u : U_N \to V_N$ in the preceeding diagrams.

Note that $\mathcal{G}r$ is an object of T ($\mathcal{G}r = ((\mathcal{G}r_N, \subset, =)_{N\in\mathbb{N}}, \sigma)$).

Let U be an object of T, a covering of U is the datum of a family $(U^i, u^i)_{i\in I}$ such that each U^i is an object of T and each $u^i : U^i \to U$ a morphism in T, we suppose moreover that for each N, $\{u^i(U^i_N)\}_{i\in I}$ is a (set) covering of U_N.

Proposition 3.2. — *The category T equipped wih the preceeding notion of coverings is a Grothendieck topology.*

Axioms of such a topology (and accessible perequisites) can be found for instance in [37]. The proof of the proposition is immediate, we just mention how are fiber products in T.

Lemma 3.3. — *Let*

$$U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U) \text{ and } V = ((V_N, \psi_N, g_N)_{N\in\mathbb{N}}, \sigma_V)$$

be two objects of T, then their fiber product exists in T, it is

$$U \times_T V = \Big(((U_N \times_{\mathcal{G}r_N} V_N, \varphi_N \times \psi_N, f_N \times g_N)_{N\in\mathbb{N}}, \sigma_{U\times_T V}), \mathrm{pr}_1, \mathrm{pr}_2 \Big),$$

where $U_N \times_{\mathcal{G}r_N} V_N$ is the fiber product in the category of schemes over $\mathcal{G}r_N$, with projections $\mathrm{pr}_{1,N}$ and $\mathrm{pr}_{2,N}$, where $\mathrm{pr}_1 = (\mathrm{pr}_{1,N})_{N\in\mathbb{N}}$ and $\mathrm{pr}_2 = (\mathrm{pr}_{2,N})_{N\in\mathbb{N}}$, and where $\sigma_{U\times_T V} = (\sigma_{U,N} \times \sigma_{V,N})_{N\in\mathbb{N}}$.

3.3. Sheaves on the topology T.—

3.3.1. The structure sheaf of T.— Let $\mathcal{O}$ be the following abelian sheaf on T: for any object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T we set

$$\mathcal{O}(U) = ((\mathcal{O}_N(U), \varphi_N^\sharp)_{N\in\mathbb{N}}, \sigma_U^\sharp),$$

where $\mathcal{O}_N(U) = \mathcal{O}_{U_N}(U_N)$ is the ring of global sections of the structural sheaf of the scheme U_N and where $\varphi_N^\sharp$ and $\sigma_U^\sharp = (\sigma_{U,N}^\sharp)$ are the comorphisms of φ_N and $\sigma_U = (\sigma_{U,N})$. Note that the sheaf $\mathcal{O}$ can be viewed as a sheaf with values in the category of inverse systems of K-algebras, note also that it is equipped by a $G(\mathbb{K})$-action, coming from the same action on the objects U of T. The sheaf $\mathcal{O}$ is called *the structure sheaf on T*, this is an $\mathcal{O}^p$-module, in the sense of [6], § 7.11.3 and 4 (see 3.3.4).

3.3.2. The sheaf $\widehat{\mathcal{O}}$.— Let $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ be an object of T, we set

$$\widehat{\mathcal{O}}(U) \stackrel{\text{def}}{=} \varprojlim_{N\in\mathbb{N}} (\mathcal{O}_N(U), \varphi_N^\sharp). \tag{8}$$

On the left hand side we take the pro-discrete topology, i.e. the topology induces by the discrete topology on each $\mathcal{O}_N(U)$, it is equiped with a σ-action (denoted again σ), which is induces by the one of $\mathcal{O}$ (see 3.3.1). Then (8) defines a sheaf on T, with values in topological K-algebras equipped by a continuous $\sigma^\sharp$-action and by a continuous $G(\mathbb{K})$- action, these two actions coming from the hypotheses satisfied by the topology T.

An element a of $\widehat{\mathcal{O}}(U)$ will be usually written $a = (a_N)$ with a_N in $\mathcal{O}_N(U)$.

3.3.3. $\mathcal{O}$-modules.— An $\mathcal{O}$-module on T is a sheaf S on T such that for any object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T,

$$S(U) = (S_N(U_N))_{N\in\mathbb{N}},$$

where for each N $S_N(U_N)$ is an $\mathcal{O}_N(U)$-module, these structures of modules being compatible with the morphisms in T (indeed each S_N is a sheaf of $\mathcal{O}$-modules on the fpqc site of K-schemes).

We will write often $S_N(U)$ in place of $S_N(U_N)$.

The sheaf $\mathcal{O}_N$ is itself an $\mathcal{O}$-module and also for each N the sheaf $\mathcal{O}_N$ (for any object U of T, the N'-component of $\mathcal{O}_N(U)$ is $\{0\}$ if $N' \neq N$ and the N-th one is $\mathcal{O}_{U_N}(U_N)$).

3.3.4. $\mathcal{O}^p$-modules.— An $\mathcal{O}^p$-module on T is a sheaf which associates to each object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T a sequence

$$P(U) = (P_N(U), P(\varphi_N))_{N\in\mathbb{N}},$$

such that $(P_N(U))_{N\in\mathbb{N}}$ is an $\mathcal{O}$-module and where $P(\varphi_N) : P_{N+1}(U) \to P_N(U)$ is an $\mathcal{O}_{N+1}(U)$-morphism ($P_N(U)$ is an $\mathcal{O}_{N+1}(U)$-module via $\varphi_N^\sharp$). We suppose moreover that we have the isomorphisms of $\mathcal{O}_{N+1}(U)$-modules

$$\mathcal{O}_N(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U) \overset{\mathrm{Id}\otimes P(\varphi_N)}{\longrightarrow} P_N(U). \tag{9}$$

We suppose also that the ind-group $G(\mathbb{K})$ acts on P, more precisely that for each integer $N \geq 0$ we have a $\mathcal{O}(U_{\gamma(N)})$-morphism (see 2.2.1)

$$P(\gamma(N)) : P(U_{\gamma(N)}) \longrightarrow P_N(U), \tag{10}$$

$P_N(U)$ being a $\mathcal{O}(U_{\gamma(N)})$-module via $\gamma^\sharp : \mathcal{O}(U_{\gamma(N)}) \to \mathcal{O}_N(U)$.

Let U be an object of T and let $T_{|U}$ be the category whose objects are the pairs (V, v), V being an object of T and $v : V \to U$ a morphism in T, whose morphisms $w : (V, v) \to (V', v')$ are morphism in T such that $v' \circ w = v$, whose coverings $(V^i, v^i)_{I\in I}$ are coverings in T with morphisms v^i in $T_{|U}$. If S is a sheaf on T, we denote by $S_{|U}$ the sheaf that it induces on $T_{|U}$ (i.e. its restriction to $T_{|U}$). An $\mathcal{O}^p$-module P is said quasicoherent if it is locally presented, i.e. if for each object U of

T their exists a covering $(U^i, u^i)_{i\in I}$ such that, for all i, their exists an exact sequence of sheaves on $T_{|U_i}$

$$L_i \to L'_i \to P_{|U_i} \to 0,$$

which is a sequence of $\mathcal{O}_{|U_i}$-modules, L_i and L'_i being free $\mathcal{O}_{|U_i}$-modules.

We denote by $\mathcal{M}^p$ the category of quasicoherent $\mathcal{O}^p$-modules.

3.3.5. $\mathcal{O}^!$-modules.— We continue to define $\mathcal{O}$-modules as in [6], § 7.11.3 and 4. An $\mathcal{O}^!$-module is a sheaf which associates to each object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T a sequence

$$M(U) = (M_N(U), M(\varphi_N))_{N\in\mathbb{N}},$$

where $(M_N(U))_{N\in\mathbb{N}}$ is an $\prime$-module and where $M(\varphi_N) : M_N(U) \to M_{N+1}(U)$ is an $\mathcal{O}_{N+1}(U)$-morphism ($M_N(U)$ is an $\mathcal{O}_{N+1}(U)$-module via $\varphi_N^\sharp$) such that we have for all N the isomorphism of $\mathcal{O}_{N+1}(U)$-modules

$$\theta_N(U) : M_N(U) \simeq \mathrm{Hom}_{\mathcal{O}_{N+1}(U)}(\mathcal{O}_N(U), M_{N+1}(U)), \tag{11}$$

which maps $m \in M_N(U)$ to $u : \mathcal{O}_N(U) \to M_{N+1}(U)$ defined by $u(1) = M(\varphi_N)(m)$ (recall that $\varphi_N^\sharp$ is surjective).

It follows from these hypothesises

Lemma 3.4. — *For all integer N, the map $M(\varphi_N)$ is injective and its image is*

$$\mathrm{Im} M(\varphi_N) = \{m \in M_{N+1}(U) \ / \ am = 0 \ \forall a \in \ker \varphi_N^\sharp\}.$$

Denote by $\mathcal{M}$ be the category of $\mathcal{O}^!$-modules, it is abelian.

3.3.6. Tensor product.— Let M and P be objects of $\mathcal{M}$ and $\mathcal{M}^p$ respectively. let $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ be an object of T. We have for any integer $N \geq 0$

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_{N+1}(U)}\big(\mathcal{O}_N(U), M_{N+1}(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U)\big)\\
\simeq\ &\mathrm{Hom}_{\mathcal{O}\left(U_{N+1}\right)}\big(\mathcal{O}_N(U), M_{N+1}(U)\big) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U)\\
\simeq\ &\qquad M_N(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U)\\
\simeq\ &\qquad M_N(U) \otimes_{\mathcal{O}_N(U)} P_N(U)
\end{aligned}$$

the first isomorphism uses the fact that $\varphi_N^\sharp$ is surjective, the two others comes from (3.3.5) and (3.3.4). It follows that the rule

$$U \mapsto \big(M \otimes_{\mathcal{O}} P\big)(U) \stackrel{\mathrm{def}}{=} (M_N(U) \otimes_{\mathcal{O}_N(U)} P_N(U), \rho_N)_{N\in\mathbb{N}},$$

where ρ_N is the composition of the two morphisms

$$M_N(U) \otimes_{\mathcal{O}_N(U)} P_N(U) \xrightarrow{\sim} M_N(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U)$$
$$M_N(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U) \xrightarrow{M(\varphi_N)\otimes \mathrm{Id}} M_{N+1}(U) \otimes_{\mathcal{O}_{N+1}(U)} P_{N+1}(U)$$

defines a tensor product

$$\mathcal{M} \times \mathcal{M}^p \longrightarrow \mathcal{M}.$$

4. Derivations and differentials.

4.1. σ-derivations.—

4.1.1. σ-derivations and $\mathcal{O}^!$-modules.— let M be an $\mathcal{O}^!$-module and $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ be an object of T. Let $N \geq 0$ be an integer. A σ-derivation

$$d_N(U) : \widehat{\mathcal{O}}(U) \longrightarrow M_N(U)$$

defined on $\widehat{\mathcal{O}}(U)$ and with values in $M_N(U)$ is a k-linear continuous morphism, $M_N(U)$ being equipped with the discrete topology (the field k has been introduced in 1.1 and 3.1), satisfying the following rule: for all a and b in $\widehat{\mathcal{O}}(U)$,

$$d_N(U)(ab) = \sigma(a)_N d_N(U)(b) + b_N d_N(U)(a)$$

(for the notations see 3.3.2).

Let $\underline{\mathrm{Der}}_{\sigma,N}(M)(U)$ be the set of these derivations, this is a $\mathcal{O}(U)$-module, via the canonical morphism $\mathcal{O}(U) \to \mathcal{O}_N(U)$ and the $\mathcal{O}_N(U)$-structure on $M_N(U)$.

Let M be a $\mathcal{O}^!$-module. For any object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T let

$$\underline{\mathrm{Der}}_\sigma(M)(U) = (\underline{\mathrm{Der}}_{\sigma,N}(M)(U), \zeta_N)_{N\in\mathbb{N}}, \tag{12}$$

where

$$\zeta_N : \underline{\mathrm{Der}}_{\sigma,N}(M)(U) \to \underline{\mathrm{Der}}_\sigma(M)(U)_{N+1}$$

maps δ to $M(\varphi_N) \circ \delta$. This defines a sheaf on T, called *the derivation sheaf with values in M*, its sections are the σ-derivations.

Proposition 4.1. — *Let M be a $\mathcal{O}^!$-module, then $\underline{\mathrm{Der}}_\sigma(M)$ is also a $\mathcal{O}^!$-module.*

Proof. — Let $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ be an object of T. During this proof we set $\mathcal{O}_N = \mathcal{O}_N(U)$, $M_N = M_N(U)$, $D_N = \underline{\mathrm{Der}}_{\sigma,N}(M)(U)$, etc. We have to prove that for all N we have the isomorphism

$$D_N \simeq \mathrm{Hom}_{\mathcal{O}_{N+1}}(\mathcal{O}_N, D_{N+1}),$$

given by the map which sends $\delta \in D_N$ to the map u such that $u(1) = M(\varphi_N)\circ\delta$ (see 12). The lemma 3.4 shows that this map is well defined. We construct the inverse map: let $u \in \mathrm{Hom}_{\mathcal{O}_{N+1}}(\mathcal{O}_N, D_{N+1})$, then $au(1) = 0$ for all $a \in \ker\varphi_N^\sharp$, it follows that $u(1)$ takes values in $\mathrm{Im}M(\varphi_N)$ (op. cit.), as $M(\varphi_N)$ is injective, this means that their exists $\delta \in D_N$ such that $u(1) = M(\varphi_N)\circ\delta$. □

Let M be an $\mathcal{O}^!$-module, for any object $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ of T, we set

$$\mathrm{Der}_\sigma(M)(U) = \varinjlim_{N\in\mathbb{N}}(\underline{\mathrm{Der}}_{\sigma,N}(M)(U), \zeta_N),$$

this defines the sheaf $\mathrm{Der}_\sigma(M)$ on T, which in an obvious sense an $\widehat{\mathcal{O}}$-module. Sections of this sheaf are called global σ-derivations with values in M.

4.1.2. σ-derivations and $\mathcal{O}^p$-modules.— The definition of a σ-derivation with values in an $\mathcal{O}^p$-module P and its main properties are the same as before. Let $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ be an object of T, we have for any N a map

$$\eta_N : \underline{\mathrm{Der}}_{\sigma,N+1}(P) \to \underline{\mathrm{Der}}_{\sigma,N}(M),$$

which is the composition on the left by $P_N(\varphi_N^\sharp)$, then

PROPOSITION 4.2. — *Let M be a $\mathcal{O}^p$-module, then $\underline{\mathrm{Der}}_\sigma(P)$ is also a $\mathcal{O}^p$-module.*

4.2. σ-differentials.—

4.2.1. The sheaves $\mathcal{I}^r$. — Consider the sheaf $\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}$ (see 3.3.2) associated to the presheaf on T

$$U \mapsto \widehat{\mathcal{O}}(U) \otimes_K \widehat{\mathcal{O}}(U)$$

(U is an object of T), it is an $\widehat{\mathcal{O}}$-module by the right.

Let r be a non negative integer and

$$\Phi^r : \widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$$

be the $\widehat{\mathcal{O}}$-linear map associated to the morphism of presheaves defined by

$$\begin{array}{rcl}\widetilde{\Phi}^r(U) : \widehat{\mathcal{O}}(U) \otimes_K \widehat{\mathcal{O}}(U) & \to & \widehat{\mathcal{O}}(U) \\ a \otimes b & \mapsto & \sigma^r(a)b\end{array}$$

Let $\mathcal{I}^r$ be the kernel of Φ^r, it is a ideal of $\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}$. Note that $\mathcal{I}^r$ is locally generated by the $a \otimes 1 - 1 \otimes \sigma^i(a)$, where a is a local section of $\widehat{\mathcal{O}}$.

4.2.2. The sheaf Ω^1_σ. — We denote by Ω^1_σ the quotient sheaf

$$\Omega^1_\sigma = \mathcal{I}^0 / \mathcal{I}^0 \mathcal{I}^1$$

and let

$$d^0 : \widehat{\mathcal{O}} \longrightarrow \Omega^1_\sigma$$

be the map induces by $a \mapsto a \otimes 1 - 1 \otimes a$. The structure of $\widehat{\mathcal{O}}$-module by the right on $\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}$ induces such a structure on Ω^1_σ, it is a topological $\widehat{\mathcal{O}}$-module, for the quotient topology. Note that Ω^1_σ is as $\widehat{\mathcal{O}}$-module generated by the image of d.

Let M be an $\mathcal{O}^!$-module and let $\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)$ be the sheaf on T

(13)

$$U \mapsto \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)(U) \stackrel{\mathrm{def}}{=} \left(\mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), M_N(U))\right)_{N \in \mathbb{N}},$$

where elements of $\mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), M_N(U))$ are continuous $\widehat{\mathcal{O}}(U)$-morphisms, for the discrete topology on $M_N(U)$. Moreover this sheaf is equiped with morphisms

$$\mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), M_N(U)) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), M_{N+1}(U))$$

induced by the $M(\varphi_N)$ (see 3.3.5) and then this is an $\mathcal{O}^!$-module.

PROPOSITION 4.3. — *Let M be an $\mathcal{O}^!$-module, then,*

$$\underline{\mathrm{Der}}_\sigma(M) \simeq \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)$$

functorially in M.

The proof is straightforward. It follows from the propositions 4.3 and 4.1 that for any $\mathcal{O}^!$-module, $\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)$ is also a $\mathcal{O}^!$-module.

REMARK 4.4. — The sheaf Ω^1_σ is equipped with an action of σ, coming from that of $\mathcal{O}$ (and denoted again by $\sigma^\sharp$). Indeed, Ω^1_σ is locally generated by the set $\{(d^0a)b\}$, where a and b run through local sections of $\mathcal{O}$, and we have $\sigma^\sharp(d^0a) = d^0\sigma^\sharp(a)$.

REMARK 4.5. — Let a be a local section of the sheaf $\widehat{\mathcal{O}}$, we set $d'(a) = a \otimes 1 - 1 \otimes a$, this is a local section of the sheaf $\mathcal{I}^0$. Let a and b be local section of $\widehat{\mathcal{O}}$, we have in $\mathcal{I}^0$

$$(a \otimes 1)d'(b) = d'(ab) - d'(a)(1 \otimes b),$$

it follows that Ω^1_σ has a left structure of $\widehat{\mathcal{O}}$-module, which is locally given by

$$(a, \omega) \mapsto \omega\sigma^\sharp(a),$$

where a and ω are local sections of respectively $\widehat{\mathcal{O}}$ and Ω^1_σ. Then our notions of derivations and differentials are closed for those of Y. André (see [2]).

REMARK 4.6. — Let U be an object of T, M be an $\mathcal{O}^!$-module and $u \in \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)(U)$, $u = (u_N)_{N\in\mathbb{N}}$ (see 13). It follows from 4.5 that we have locally, for sections a and b of $\Omega^1_\sigma(U)$:

$$u_N((a \otimes 1)d(b)) = \sigma(a)_N u_N(b)$$

(see 4.3) and recall that by definition u is $\widehat{\mathcal{O}}$-linear "by the right", indeed this means that sections of $\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M)$ are sequilinear, in the sense of [2].

4.2.3. The sheaf Θ. — Consider the sheaf

$$\Theta = \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, \mathcal{O}) \tag{14}$$

on T, defined by

$$U \mapsto \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, \mathcal{O})(U) \overset{\mathrm{def}}{=} \left(\mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), \mathcal{O}_N(U))\right)_{N\in\mathbb{N}},$$

where the topology on $\mathcal{O}_N(U)$ is the discrete one (recall that $\mathcal{O}_N$ is the $\mathcal{O}$-module defined by $\mathcal{O}_N(U) = \mathcal{O}_{U_N}(U_N))$. Set, for any object U of T and any N,

$$\Theta_N(U) = \mathrm{Hom}_{\widehat{\mathcal{O}}(U),\mathrm{cont}}(\Omega^1_\sigma(U), \mathcal{O}_N(U)),$$

This sheaf Θ_N is naturally an $\mathcal{O}_N$-module and we have

$$\Theta = \left(\Theta_N\right)_{N\in\mathbb{N}}, \tag{15}$$

the maps $\varphi_N : U_N \to U_{N+1}$ induce maps $\Theta_N(U) \to \Theta_{N+1}(U)$, then

PROPOSITION 4.7. — *The sheaf* Θ *is an* $\mathcal{O}^p$*-module.*

The $G(\mathbb{K})$-action is induced by this one on $\mathcal{O}$. The proof is obvious.

The sheaf Θ looks like "the tangent sheaf at T".

PROPOSITION 4.8. — *The sheaf Θ is a torsion free $\mathcal{O}$-module of profinite type, more precisely, each sheaf θ_N is a torsion free $\mathcal{O}_N$-module of finite type.*

Proof. — For any non negative integer N let $\mathfrak{A}_N$ be the kernel of the projection $\widehat{\mathcal{O}} \to \mathcal{O}_N$. Let $\mathcal{I}_N^0$ be the subsheaf of $\mathcal{I}^0$ generated locally by the $a \otimes 1 - 1 \otimes a$, where a is a local section of $\mathfrak{A}_N$, let s be the canonical morphism $\mathcal{I}^0 \to \Omega_\sigma^1$ and set

$$\Omega^1_{\sigma,N} = \frac{\Omega^1_\sigma}{\Omega^1_\sigma \mathfrak{A}_N + s(\mathcal{I}^0_{\sigma(N)})}$$

($\sigma(N)$ is defined in §3.1.3), this sheaf can be viewed as an $\mathcal{O}_N$-module.

LEMMA 4.9. — *There is a natural isomorphism of $\mathcal{O}_N$-modules*

$$\Theta_N \simeq \mathrm{Hom}_{\mathcal{O}_N}(\Omega^1_{\sigma,N}, \mathcal{O}_N).$$

Proof. — Let $u \in \Theta_N$, we have to prove that the kernel of u contains $\Omega^1_\sigma \mathfrak{A}_N + s(\mathcal{I}^0_{\sigma(N)})$. Let a and b be local sections of $\widehat{\mathcal{O}}$, we have in Ω^1_σ

$$(d^0 a)(b - \sigma^\sharp(b)) = (d^0 b)(a - \sigma^\sharp(a)),$$

then if $a \in \mathfrak{A}_{\sigma(N)}$

$$(d^0 a)(b - \sigma^\sharp(b)) \in \Omega^1_\sigma \mathfrak{A}_N,$$

as U_N is a locally integral scheme, this means that $u(d^0(a) = 0$. □

It follows from this lemma that Θ_N is locally finitely generated over $\mathcal{O}_N$. Let $U = (U_N)_{N\in\mathbb{N}}$ be an object of T, one can suppose that $U_{\sigma(N)}$ is affine, $U_{\sigma(N)} = \mathrm{Spec} K[t'_1, \cdots, t'_r]$, where $t'_1, \cdots, t'_r$ are images of sections $t_1, \cdots, t_r$ of $\widehat{\mathcal{O}}(U)$. It is clear that $\Omega_{\sigma,N}(U)$ is generated over $\mathcal{O}_N(U)$ by $d^0 t_1, \cdots, d^0 t_r$.

Now we prove that Θ_N is torsion free over $\mathcal{O}_N$. Let $U = (U_N)_{N\in\mathbb{N}}$ be an object of T, up to the lemma we can consider Θ_N as a sheaf on U_N and we have to prove that any section of Θ_N over U_N does not vanish on a whole open subset of U_N, but this is a consequence of the fact that U_N is smooth: let W be an open subset of U_N, the restriction $\mathcal{O}_N(U_N) \to \mathcal{O}_N(W)$ is injective, then we have an injective map $\Theta_N(U_N) \hookrightarrow \Theta_N(W)$. □

Proposition 4.10. — *Let M be a torsion free $\mathcal{O}^!$-module of pro-finite type (i.e. $M = (M_N)_{N\in\mathbb{N}}$ and each M_N is torsion free and of finite type over $\mathcal{O}_N$), then we have the isomorphism of $\mathcal{O}^!$-modules*

$$\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M) \simeq M \otimes_{\mathcal{O}} \Theta$$

(on the right hand side this is the tensor product defined in § 3.3.6).

Proof. — We have a canonical map

$$\Psi : M \otimes_{\mathcal{O}} \Theta \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\Omega^1_\sigma, M),$$

which is defined locally by (the morphism of presheaves)

$$\begin{array}{ccc} M_N(U) \otimes_{\mathcal{O}_N(U)} \Theta(U)_N & \longrightarrow & \mathrm{Hom}_{\widehat{\mathcal{O}}(U)_N,\mathrm{cont}}(\Omega^1_\sigma(U), M_N(U)) \\ m \otimes v & \longmapsto & (\omega \mapsto v(\omega)m) \end{array} \tag{16}$$

We prove that it is an isomorphism.

Let $U = (U_N)_{N\in\mathbb{N}}$ be an object of T, we can suppose that U_N is affine, let x be a closed point of U_N and denote by $K(x)$ the residue field at x, i.e. the field K equipped with the canonical structure $\mathcal{O}_N(U) \to K = K(x)$. The same arguments as in lemma 4.9 show that

$$\mathrm{Hom}_{\widehat{\mathcal{O}}(U)_N,\mathrm{cont}}(\Omega^1_\sigma(U), M_N(U)) = \mathrm{Hom}_{\mathcal{O}_N(U)}(\Omega^1_{\sigma,N}(U), M_N(U))$$

is of finite type over $\mathcal{O}_N(U) = \mathcal{O}_N(U_N)$, then, for any closed point x of U_N, the map of 16 gives an isomorphim

$$M_N(U)\otimes_{\mathcal{O}_N(U)}\Theta(U)_N\otimes_{\mathcal{O}_N(U)}K(x) \simeq \mathrm{Hom}_{\mathcal{O}_N(U)}(\Omega^1_{\sigma,N}(U), M_N(U))\otimes_{\mathcal{O}_N(U)}K(x).$$

This implies the expected isomorphism, because all $\mathcal{O}_N(U)$-modules appearing here are of finite type.

$\square$

4.3. The De Rham complex. — Condider the tensor algebra $\bigotimes_{\widehat{\mathcal{O}}} \Omega^1_\sigma$ and let $\mathcal{J}$ be its ideal locally generated by

$$\{d^0\sigma(a) \otimes d^0 b + d^0 b \otimes d^0 a\},$$

where a and b are local sections of Ω^1_σ. Let

$$\Omega^{\cdot}_\sigma \overset{\mathrm{def}}{=} \Big(\bigotimes_{\widehat{\mathcal{O}}} \Omega^1_\sigma\Big)/\mathcal{J},$$

it is a gradued algebra, we denote by Ω^r_σ the group of its homogeneous elements of degree r.

PROPOSITION 4.11. — *There exists a map of graded ring*

$$d^{\cdot} : \Omega^{\cdot}_{\sigma} \longrightarrow \Omega^{\cdot+1}_{\sigma}$$

which satisfies locally the following rule: let ω_1 and ω_2 be local sections of Ω^r_σ and Ω^s_σ respectively, then

$$d^{r+s}(\omega_1\omega_2) = (d^r\omega_1)\omega_2 + (-1)^r\sigma(\omega_1)d^s\omega_2$$

(see remark 4.4); $(\Omega^{\cdot}_{\sigma}, d^{\cdot})$ is a complex.

Proof. — We use the method of [7], Ch. 3, § 10, n°9.

(1) There exists a map

$$d^1 : \Omega^1_\sigma \longrightarrow \Omega^2_\sigma$$

locally defines by

$$ad^0b \longmapsto (d^0a)(d^0b),$$

where a and b are local sections of $\widehat{\mathcal{O}}$. The proof of this fact is easy.

(2) Let A be an $\widehat{\mathcal{O}}$-algebra and

$$f : \Omega^1_\sigma \longrightarrow A$$

be an $\widehat{\mathcal{O}}$-linear map such that, locally, for local sections a and b of $\widehat{\mathcal{O}}$,

$$f(d^0\sigma(a))f(d^0b) + f(d^0b)f(d^0a),$$

then there exists a morphism of $\widehat{\mathcal{O}}$-algebras

$$g : \Omega^{\cdot}_{\sigma} \longrightarrow A$$

which extends f.

The proof is also easy, g is locally defined on Ω^r_σ by

$$g(\omega_1 \cdots \omega_r) = f(\omega_1) \cdots f(\omega_r)$$

where $\omega_1, \cdots, \omega_r$ are local sections of Ω^1_σ.

Consider on $A = \Omega^{\cdot}_{\sigma} \oplus \Omega^{\cdot}_{\sigma}$ the multiplication defined locally by the following formula: let α, β, α' and β' be local sections of $\Omega^{\cdot}_{\sigma}$, α being homogeneous,

$$(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', (-1)^{\deg\alpha}\sigma(\alpha)\beta' + \beta\alpha')$$

($\sigma(\alpha)$ is defined in the remark 4.4), then A is a ring, with $(1, 0)$ as unit. Let $h : \widehat{\mathcal{O}} \to A$ be the map locally defined by $a \mapsto (a, d^0a)$ (a is a local section of $\widehat{\mathcal{O}}$), this is a morphism of rings with units. Let $f : \Omega^1_\sigma \to A$ be the map locally defined by $f(\omega) = (\omega, d^1\omega)$ (see part (1) of this proof, ω is a local section of Ω^1_σ). The map h makes A an $\widehat{\mathcal{O}}$-algebra and a direct calculation proves that f is then a morphism of $\widehat{\mathcal{O}}$-modules, which satisfies all the hypotheses of the part (2) above. Let $g : \Omega^{\cdot}_{\sigma} \to A$ be the map given by (2), a direct calculation again proves that $\mathrm{pr}_1 \circ g$ is the identity map on $\Omega^{\cdot}_{\sigma}$, we set $g = (\mathrm{Id}, d^{\cdot})$, then $d^{\cdot}$ is the expected map. □

5. $\mathcal{D}_{\sigma}$-modules

In this section we want to define a notion of $\mathcal{D}$-modules for σ-derivations and a notion of holonomy for these modules. Holonomic $\mathcal{D}$-modules for this sort of derivations has been previously defined by Loeser and Sabbah ([31]), our definitions work also in the situation they have studied.

5.1. The sheaves $\mathcal{P}^{\cdot}_{\sigma}$. — We use the same method as in [1]. Let $r \geq 1$ be a non negative integer, set

$$\mathcal{P}^r_{\sigma} = \widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}} / (\mathcal{I}^0 \mathcal{I}^1 \cdots \mathcal{I}^r)$$

and let

$$\delta^r : \widehat{\mathcal{O}} \longrightarrow \mathcal{P}^r_{\sigma}$$

be the map locally defined by $a \mapsto a \otimes 1$. We say that $\mathcal{P}^r_{\sigma}$ is the sheaf on T of *σ-differential principal parts of degree r*, it is a topological $\widehat{\mathcal{O}}$-module, for the quotient topology (recall that, as $\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}$, $\mathcal{I}^r_{\sigma}$ is an $\widehat{\mathcal{O}}$-modules "by the right").

Let $r \geq 1$ and $s \geq 1$ be two integers, consider the tensor product

$$\mathcal{P}^r_{\sigma} \otimes_{\widehat{\mathcal{O}}} \mathcal{P}^s_{\sigma},$$

it is an $\widehat{\mathcal{O}}$-module via its right factor, but the tensor product is for the $\widehat{\mathcal{O}}$-module structure of $\mathcal{P}^r_{\sigma}$ and the $\widehat{\mathcal{O}}$-module structure through δ^s on $\mathcal{P}^s_{\sigma}$.

Lemma 5.1. — *There exists an $\widehat{\mathcal{O}}$-morphism*

$$\mathcal{P}^{r+s}_{\sigma} \longrightarrow \mathcal{P}^r_{\sigma} \otimes_{\widehat{\mathcal{O}}} \mathcal{P}^s_{\sigma},$$

given locally by $a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b)$.

Proof. — Consider the map

$$\varphi : \widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}} \longrightarrow (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}})$$

locally defined by $a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b)$, we prove now that

(17)
$$\varphi(\mathcal{I}^0 \cdots \mathcal{I}^{r+s}) \subseteq (\mathcal{I}^0 \cdots \mathcal{I}^r) \otimes_{\widehat{\mathcal{O}}} (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) + (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} (\mathcal{I}^0 \cdots \mathcal{I}^s).$$

All following computations are local (or with sections of tensor product presheaves...), we omit to write objects of T on which sections are defined. For any integer i, $0 \leq i \leq r+s$, set

$$\Delta_i = \{a \otimes 1 - 1 \otimes \sigma^i(a) \ / \ a \in \widehat{\mathcal{O}}\},$$

this is a set of generators of $\mathcal{I}^i$ and we have for all j, $0 \leq j \leq i$,

$$\varphi(\Delta_i) \subseteq \Delta_j \otimes_{\widehat{\mathcal{O}}} (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) + (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} \Delta_{i-j} \tag{18}$$

as it is proved by the following formulae

$$\begin{array}{rcl} \varphi(a \otimes 1 - 1 \otimes \sigma^i(a)) & = & (a \otimes 1) \otimes (1 \otimes 1) - (a \otimes 1) \otimes (1 \otimes \sigma^i(a)) \\ & = & (a \otimes 1 - 1 \otimes \sigma^j(a)) \otimes (1 \otimes 1) \\ & + & (1 \otimes 1) \otimes (\sigma^j(a) \otimes 1 - 1 \otimes \sigma^i(a)). \end{array}$$

Let n be a non negative integer and for $0 \leq i \leq n$ let

$$\mathcal{I}(n-i, i-1) = (\prod_{0 \leq j \leq n-i} \mathcal{I}^j) \otimes_{\widehat{\mathcal{O}}} (\prod_{0 \leq j \leq i-1} \mathcal{I}^j)$$

where $\prod \mathcal{I}^j = \widehat{\mathcal{O}} \otimes_{\widehat{\mathcal{O}}} \widehat{\mathcal{O}}$ when j describes the empty set; let w_i be an element of $\varphi(\Delta_i)$, we prove

$$\prod_{0 \leq i \leq n} w_i \in \sum_{0 \leq i \leq n+1} \mathcal{I}(n-i, i-1),$$

this formula implies (17), it is proved by induction: we write, following (18), $w_{n+1} = u_{n+1-i} + v_i$, with $u_{n+1-i} \in \Delta_{n+1-i} \otimes_{\widehat{\mathcal{O}}} (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}})$ and $v_i \in (\widehat{\mathcal{O}} \otimes_K \widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} \Delta_i$, we have

$$\prod_{1 \leq i \leq n+1} w_i \in \sum_{0 \leq i \leq n+1} \mathcal{I}(n-i, i-1) w_{n+1},$$

then

$$\prod_{1 \leq i \leq n+1} w_i \in \sum_{0 \leq i \leq n+1} \mathcal{I}(n-i, i-1)(u_{n+1-i} + v_i)$$

$$\prod_{1 \leq i \leq n+1} w_i \in \sum_{0 \leq i \leq n+1} \mathcal{I}(n+1-i, i-1) + \sum_{0 \leq i \leq n+1} \mathcal{I}(n-i, i)$$

which gives the expected formula. $\square$

5.2. The sheaves $\mathcal{D}^{\cdot}_{\sigma}$. — We continue to use the same method as in [1]. As before, for any non negative integer, let $\mathcal{O}_N$ be the sheaf such that $\mathcal{O}_N(U) = \mathcal{O}_{U_N}(u_N)$, where $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ is an object of T. For any integer $r \geq 0$, let

$$\mathcal{D}^r_{\sigma,N} = \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, \mathcal{O}_N)$$

and

$$\mathcal{D}^r_\sigma = \Big(\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, \mathcal{O}_N)\Big)_{N\in\mathbb{N}} = \Big(\mathcal{D}^r_{\sigma,N}\Big)_{N\in\mathbb{N}},$$

where for any object U of T, $\mathcal{O}_N(U)$ is equipped with the discrete topology. The maps $\varphi^\sharp_N$ induce morphisms

$$\mathcal{D}^r_{\sigma,N+1} \longrightarrow \mathcal{D}^r_{\sigma,N}.$$

The sheaf $\mathcal{D}^r_\sigma$, equipped with these morphisms, is called *the sheaf of σ-differential operators of degree $\leq r$.*

Proposition 5.2. — *The sheaves $\mathcal{D}^r_\sigma$ are $\mathcal{O}^p$-modules.*

The proof is clear. Let $s \geq r \geq 0$ be integers, the canonical map $\mathcal{P}^s_\sigma \to \mathcal{P}^r_\sigma$ induces a canonical injective morphism $\mathcal{D}^r_\sigma \hookrightarrow \mathcal{D}^s_\sigma$, these map permit to define

$$\mathcal{D}_\sigma = \varinjlim_{r\in\mathbb{N}} \mathcal{D}^r_\sigma$$

which is, as a consequence of lemma 5.1, a $\widehat{\mathcal{O}}$-algebra:

Proposition 5.3. — *(Composition of σ-differential operators) Let r and s be non negative integers, then there is a morphism*

$$\mathcal{D}^r_\sigma \otimes_{\widehat{\mathcal{O}}} \mathcal{D}^r_\sigma \longrightarrow \mathcal{D}^{r+s}_\sigma.$$

In other words, $\mathcal{D}_\sigma$ is a $\widehat{\mathcal{O}}$-algebra.

5.2.1. The sheaves $\mathcal{D}^{\cdot}_\sigma$ and the $\mathcal{O}^!$-modules;— Let r and N as before and let M be an $\mathcal{O}^!$-module. Consider the sheaf

$$\underline{\mathcal{D}}^r_\sigma(M) = \mathrm{Hom}_{\mathcal{O},\mathrm{cont}}(\mathcal{P}^r_\sigma, M) = \Big(\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, M_N)\Big)_{N\in\mathbb{N}}$$

(see 13) where, for each object U of T, elements of $\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, M_N)(U)$ are continuous $\widehat{\mathcal{O}}(U)$-morphisms, for the discrete topology on $M_N(U)$. Moreover this sheaf is equiped with morphisms

$$\mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, M_N)(U) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{O}},\mathrm{cont}}(\mathcal{P}^r_\sigma, M_{N+1})(U)$$

induced by the $M(\varphi_N)$ (see 3.3.5) and we see that

PROPOSITION 5.4. — *Let M be an $\mathcal{O}^!$-module, then the sheaf $\underline{\mathcal{D}}^r_\sigma(M)$ is also an $\mathcal{O}^!$-module.*

We have also a property closed to proposition 4.10, the proof being almost the same:

PROPOSITION 5.5. — *Let M be a torsion free $\mathcal{O}^!$-module, then we have the isomorphism of $\mathcal{O}^!$-modules*

$$\underline{\mathcal{D}}^r_\sigma(M) \simeq M \otimes_{\mathcal{O}} \mathcal{D}^r_\sigma$$

(on the right hand side this is the tensor product defined in § 3.3.6).

5.2.2. The sheaves $\mathcal{D}^{\cdot}_\sigma$ and the $\mathcal{O}^p$-modules.— In this paragraph we use a method coming from [5]. Let P be an $\mathcal{O}^p$-module. Set $\underline{\mathcal{D}}^{-1}_\sigma(P) = \{0\}$ and for any integer $r \geq 0$ let $\underline{\mathcal{D}}^r_\sigma(P)$ be the subsheaf of $\mathrm{End}_k(P)$ of sections f which locally satisfy the rule

$$f(a\cdot) - (\sigma^\sharp)^r(a)f(\cdot) \in \underline{\mathcal{D}}^{r-1}_\sigma(P)$$

for any section a of $\mathcal{O}$; more precisely, for any object U of T, $\underline{\mathcal{D}}^r_\sigma(P)(U)$ is the set of $f \in \mathrm{End}_k(P)(U)$ such that, for each $V \to U$ in T and for each $a \in \mathcal{O}(V)$,

$$f\mid_V(a\cdot) - (\sigma^\sharp)^r_V(a) f\mid_V(\cdot) \in \underline{\mathcal{D}}^{r-1}_\sigma(P)(V).$$

PROPOSITION 5.6. — *For any non negative integer r we have an isomorphism of sheaves $\mathcal{D}^r_\sigma \simeq \underline{\mathcal{D}}^r_\sigma(\mathcal{O})$.*

Proof. — We make a confusion between the sheaf $\widehat{\mathcal{O}}$ and the inverse system of sheaves $\mathcal{O}$. The map $\mathcal{D}^r_\sigma \to \underline{\mathcal{D}}^r_\sigma(\mathcal{O})$ is $D \mapsto D \circ \delta^r$ (see § 5.1). By the inverse map, the image of a section f of $\underline{\mathcal{D}}^r_\sigma(\mathcal{O})$ is locally defined by $\delta^r(a) \mapsto f(a)$ (a is a section of $\mathcal{O}$). □

5.3. $\mathcal{D}_\sigma$-modules. — *A $\mathcal{D}_\sigma$-module P is an $\mathcal{O}^p$-module with a structure of module over the ring $\mathcal{D}_\sigma$.* This means that, if we denote by P_N the sheaf $U \mapsto P_N(U)$, P_N is equipped with an action of the additive group $\mathcal{D}^r_{\sigma,N}$, for all N and r, which is compatible with the inverse systems $\mathcal{D}^r_\sigma$ and P, and, with the proposition 5.3, which gives a $\mathcal{D}_\sigma$-module structure on P.

5.3.1. Some words about holonomy.— We do not have a complete notion of "σ-holonomy", we explain now how maybe one can find it.

Consider the analog for σ-derivations of the Weyl algebra, namely

$$W_\sigma = K[z_1, \cdots, z_n, \partial_1, \cdots, \partial_n],$$

with for all i and j, $i \neq j$,

$$\partial_i z_i = \sigma(z_i)\partial_i + 1 \quad \text{and} \quad \partial_i z_j = z_j \partial_i,$$

σ being an endomorphism of the algebra $K[z_1, \cdots, z_n]$. For any $D \in W_\sigma$ and any $a \in K[z_1, \cdots, z_n]$ set $[D, a]_r = Da - \sigma^r(a)D$. With our previous definition of σ-differential operators (see § 5.2.2), we see that $D \in \mathcal{D}_\sigma^r$ if and only if for all $a_0, \cdots, a_r \in K[z_1, \cdots, z_n]$,

$$\Big[\big[\cdots[[D, a_r]_r, a_{r-1}]_{r-1}, \cdots]_2, a_1\big]_1, a_0\Big]_0 = 0,$$

then $\mathcal{D}_\sigma^r$ is the set of (finite sums) $\sum_{\alpha,\beta} a_{\alpha,\beta} z^\alpha \partial^\beta$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ go through $\mathbb{N}^r$, and $\mid \beta \mid= \beta_1 + \cdots + \beta_n \leq r$. Set

$$\mathrm{Gr}(W_\sigma) = \oplus_{r \in \mathbb{N}} \mathcal{D}_\sigma^{r+1}/\mathcal{D}_\sigma^r,$$

it is not a commutative algebra and we cannot define a notion of characteristic variety for a W_σ-module. We define holonomic W_σ-module whithout using such a variety.

Let P be a W_σ-module of finite type and let $\mathrm{dp}_{W_\sigma}(P)$ be the projective dimension of P, set

$$j(P) = \max\{j \;/\; \mathrm{Ext}^j_{W_\sigma}(M, W_\sigma) \neq 0\},$$

the following lemma is wellknown

Lemma 5.7. — $\mathrm{dp}_{W_\sigma}(P) = j(P)$.

Definition 5.8. — Let P be a W_σ-module of finite type, the defect of P is $d(P) := 1 - j(P)/n$.

We expect the following result

Proposition 5.9. — *Let P be a W_σ-module of finite type, then $d(P) \geq 0$.*

Definition 5.10. — Let P be a W_σ-module of finite type, it is said holonomic if $d(P) = 0$.

All these properties and definitions correspond to the notion of holonomy in the classical case.

5.3.2. The De Rham complexes of some $\mathcal{D}_\sigma$-module.— In the classical case any smooth $\mathcal{D}$-module on an orbit $G(\mathbb{O})\chi$ (see proposition 2.3) is constant (it is proved in [6], lemma 5.3.2), this explain the following definitions.

5.3.2.1. Elementary objects.— Recall that the orbits in $\mathcal{G}r$ under the left action by $G(\mathbb{O})$ are of the form $G(\mathbb{O})\chi$, where χ describes a subset Ξ of the subgroup of one parameter subgroups of G (see proposition 2.3). Let $U^\omega = ((U^\omega_N, \varphi^\omega_N, f^\omega_N)_{N\in\mathbb{N}}, \sigma_{U^\omega})$ be the following object of T: for any integer $N \geq 0$ let Ξ_N be the set of $\chi \in \Xi$ such that $G(\mathbb{O})\chi \subset \mathcal{G}r_N$, for any $\chi \in \Xi_N$ let $U^\omega_{N,\chi} = G(\mathbb{O})$, and set

$$U^\omega_N = \coprod_{\chi\in\Xi_N} U^\omega_{N,\chi} \ ,$$

let φ^ω_N be the inclusion $U^\omega_N \subset U^\omega_{N+1}$ and $f^\omega_N : U^\omega_N \to \mathcal{G}r_N$ be the morphism which maps canonically $U^\omega_{N,\chi}$ to $G(\mathbb{O})\chi$. Left actions by $G(\mathbb{O})$, $G(\mathbb{K})$ and the map σ_{U^ω} are obvious.

Let V be an object of T which admits a covering $(V^i \to V)_{i\in I}$ where each $V^i = ((V^i_N, \psi^i_N, g^i_N)_{N\in\mathbb{N}}, \sigma_{V^i})$ is such that V^i_N is Zariski open in U^ω_N, ψ^i_N, g^i_N and σ_{V^i} are the restriction of φ^ω_N, f^ω_N and σ_{U^ω} respectively. We say that V *is an elementary object of* T.

Let P be an $\mathcal{O}^p$-module on T, let V be an object of T and denote by $T\mid_V$ the restricted topology T to V, by $P\mid_V$ the sheaf on $T\mid_V$ induces by P. We say that the sheaf P is $\mathcal{O}$-free of pro-finite type on V if $P\mid_V$ is $\mathcal{O}$-free on $T\mid_V$ and if each $(P\mid_V)_N$ is of finite type over $\mathcal{O}_N$.

Definition 5.11. — We say that a coherent $\mathcal{D}_\sigma$-module P satisfies the property (H) if it is $\mathcal{O}$-free of pro-finite type on every elementary object of T.

Proposition 5.12. — *Let P be a $\mathcal{D}_\sigma$-module sastisfying the property (H), then for each elementary object V of T there exists a map of sheaves of graded groups on $T\mid_V$, of degree* 1,

$$\nabla^\cdot\mid_V : P\otimes_{\widehat{\mathcal{O}}}\Omega^\cdot_\sigma \longrightarrow P\otimes_{\widehat{\mathcal{O}}}\Omega^{\cdot+1}_\sigma$$

such that, for any integer $r \geq 0$, for local sections z and ω of P and Ω^r_σ respectively,

$$\nabla^r(\omega\otimes z) = d^r(\omega)\otimes z + (-1)^r\sigma(\omega)\nabla^0(z);$$

$\nabla^\cdot\mid_V$ is functorial in V, it is an integrable connection (i.e. $\nabla^1\mid_V \circ\nabla^0\mid_V = 0$).

This is a consequence of the definition 5.11 and of the De Rham complex for $\mathcal{O}$, see proposition 4.11.

6. A tannakian category

6.1. The symmetric product.— Let U be an elementary object of T, we consider the ind-scheme

$$(G(\mathbb{K}) \times_{\mathcal{G}r} U) \times_{\mathrm{Spec}K} U$$

on which we consider the topology

$$T^{(U)} \overset{\mathrm{def}}{=} \big(G(\mathbb{K}) \times_{\mathcal{G}r} (T\,|_U)\big) \times_{\mathrm{Spec}K} (T\,|_U)$$

and the two morphisms of topology, with values in $T\,|_U$, coming from the projections (over Spec$K$). This topology is equipped with the action of $G(\mathbb{O})$ defined by the formula,

$$\gamma\big((g,x),y\big) = \big((g\gamma^{-1},x),\gamma y\big),$$

where $\gamma \in G(\mathbb{O})$, $g \in G(\mathbb{K})$ and $x, y \in V$ and V is an object of $T\,|_U$.

Let P and Q be two $\mathcal{D}_\sigma$-modules, then on the topology $T^{(U)}$, we have the sheaf

$$(P \boxtimes Q)^{G(\mathbb{O})}$$

of $G(\mathbb{O})$-invariant sections of the sheaf $P \boxtimes Q$. Let

$$m_U : (G(\mathbb{K}) \times_{\mathcal{G}r} U) \times_{\mathrm{Spec}K} U \longrightarrow U$$

be the map defined (with the previous notations) by $m((g,x),y) = gy$. Set $(P \circledast Q)\,|_U = (m_U)_*((P \boxtimes Q)^{G(\mathbb{O})})$, these sheaves glue, to give a sheaf on T, denoted by $P \circledast Q$.

Proposition 6.1. — *Let $\mathfrak{P}$ be the category of $\mathcal{D}_\sigma$-modules satisfying the hypothesis (H) of definition 5.11, then $\circledast$ is a symmetric product on $\mathfrak{P}$.*

6.2. The fiber functor.— let P be a $\mathcal{D}_\sigma$-module satisfying the hypothesis *(H)* of definition 5.11 and let

$$P \to P \otimes_{\mathcal{O}} \Omega^1_\sigma \to P \otimes_{\mathcal{O}} \Omega^2_\sigma \cdots$$

be its De Rham complex on elementary objects (see 5.12), this is a complex of inverse systems of sheaves on which we apply the functor

$\Gamma(\mathcal{G}r, \cdot)$ of global sections on $\mathcal{G}r$. We obtain a complex of inverse systems of $\mathcal{O}(\mathcal{G}r)$-modules

$$\Gamma(\mathcal{G}r, P) \to \Gamma(\mathcal{G}r, P \otimes_{\mathcal{O}} \Omega^1_\sigma) \to \Gamma(\mathcal{G}r, P \otimes_{\mathcal{O}} \Omega^2_\sigma) \cdots ,$$

take an injective resolution in the category of such complexes, thus we obtain a bicomplex of inverse systems of $\mathcal{O}(\mathcal{G}r)$-modules, which cohomology $\mathbb{H}^{\cdot}(\mathcal{G}r, P)$ is *the De Rham cohomology on $\mathcal{G}r$ of the $\mathcal{D}_\sigma$-module P*. Note that each $\mathbb{H}^r(\mathcal{G}r, P)$ is an inverse system of $\mathcal{O}(\mathcal{G}r)$-modules.

Remark 6.2. — We have seen that the complex $(\Omega^{\cdot}_\sigma, d^{\cdot})$ is equipped with a morphism $\sigma^\sharp$ of degree 0 (for instance locally defined on Ω^1_σ by $\sigma^\sharp(d^0 a) = d^0\sigma^\sharp(a)$, where a is a local section of $\mathcal{O}$), this implies the existence of a morphism of degree 0 on the De Rham complex of P, thus of a morphism, denoted again by $\sigma^\sharp$, on each inverse system $\mathbb{H}^r(\mathcal{G}r, P)$.

Definition 6.3. — Let $\mathfrak{P}_c$ be the category of $\mathcal{D}_\sigma$-modules which satisfy the hypothesis (H) of proposition 5.11 and which moreover are submited to the following condition: if P is in $\mathfrak{P}_c$ and if $U = ((U_N, \varphi_N, f_N)_{N\in\mathbb{N}}, \sigma_U)$ is an object of T, then the support of P on U is determined by finitely many U_N, more precisely, there exists an integer $N_0 = N(P, U)$ such that on U

$$\operatorname{supp}(P) \subseteq U_0 \times \cdots \times U_{N_0} \times \prod_{N \geq N_0} \varphi_{N_0,N}(U_{N_0}),$$

where $\varphi_{N_0,N} = \varphi_N \circ \cdots \circ \varphi_{N_0} : U_{N_0} \to U_{N+1}$. We say that P has compact support.

Let $\mathbb{V}^K_\sigma$ be the category objects of which being couples $(V, \sigma^\sharp)$, where V is a finite dimensional K-vector spaces and $\sigma^\sharp$ an automorphism of V, then

Lemma 6.4. — *Restricted on the category $\mathfrak{P}_c$, the functor $\mathbb{H}(\mathcal{G}r, \cdot)$ takes values in $\mathbb{V}^K_\sigma$.*

Proposition 6.5. — *Equipped with the symmetric product $\circledast$ and with the fiber functor $\mathbb{H}(\mathcal{G}r, \cdot)$, the category $\mathfrak{P}_c$ is a tannakian category.*

7. The main result

Theorem 7.1. — *Let A be the automorphism group of the fiber functor $\mathbb{H}(\mathcal{G}r, \cdot)$ from $\mathfrak{P}_c$ to $\mathbb{V}^K_\sigma$, then A is isomorphic to the Langlands dual ${}^L G$*

of G and we have an equivalence of categories

$$\mathfrak{P}_c \simeq \mathrm{Rep}_\sigma^K$$

between $\mathfrak{P}_c$ and the category of finite dimensional K-representaton of ${}^LG$ compatible with the σ-actions.

The end of the theorem means that if $\rho : {}^LG \to V$ is an object of Rep_σ^K, ρ commutes with the actions of σ on V and on ${}^LG$, this last one being also the set of K-points of an algebraic linear group.

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