A NOTE ABOUT RATIONAL REPRESENTATIONS OF DIFFERENTIAL GALOIS GROUPS.

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ABSTRACT. We give a description of the rational representations of the differential Galois group of a Picard-Vessiot extension.

CONTENTS

We give a description of the rational representations of the differential Galois group of a Picard-Vessiot extension (theorems 1.1 and 2.1). This gives a new description of the differential Galois correspondence. More results will be given for abelian differential extensions in a forthcoming paper [1], especially the analog of the Artin correspondence.

1. On representations of Galois groups of Picard-Vessiot extensions.

In all this note we consider differential fields with algebraically closed fields of constants, denoted by C. The derivative will be denoted by a dash.

Let K be a differential field. Let $n \geq 1$ be an integer, $\mathrm{M}_n(K)$ and $\mathrm{GL}_n(K)$ are the usual notations for algebra and group of $n \times n$ matrices with entries in K. The group $\mathrm{GL}_n(K)$ acts on $\mathrm{M}_n(K)$ by the following rule:

$$\begin{array}{ccc} \mathrm{GL}_n(K) \times \mathrm{M}_n(K) & \longrightarrow & \mathrm{M}_n(K) \\ (U, A) & \longmapsto & U'U^{-1} + UAU^{-1} \end{array}$$

where if $U = (u_{i,j})_{1 \leq i,j \leq n}$, then $U' = (u'_{i,j})_{1 \leq i,j \leq n}$. This action can be defined in an other way, maybe more comprehensible. Consider the group

$$H_n(K) := \mathrm{M}_n(K) \times \mathrm{GL}_n(K),$$

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the law being defined by the following formula: for all A, B in $\mathrm{M}_n(K)$ and all $F, G \in \mathrm{GL}_n(K)$

$$(A, F)(B, G) = (A + FBF^{-1}, FG).$$

It admits the subgroups

$$\Delta_n(K) := \left\{(U'U^{-1}, U) \ / \ U \in \mathrm{GL}_n(K)\right\},$$

$\{0\} \times \mathrm{GL}_n(K)$ and $\mathrm{M}_n(K) \times \{1\}$, this last one being normal. We set

$$Z_n(K) := \Delta_n(K) \backslash H_n(K) / \left(\{0\} \times \mathrm{GL}_n(K)\right).$$

The group $\mathrm{GL}_n(K)$ acts on $\mathrm{M}_n(K) \times \{1\}$ by the following rule:

$$\begin{array}{ccc} \mathrm{GL}_n(K) \times (\mathrm{M}_n(K) \times \{1\}) & \longrightarrow & (\mathrm{M}_n(K) \times \{1\}) \\ (U, (A, 1)) & \longmapsto & (U'U^{-1}, U)(A, 1)(0, U^{-1}) \\ & & = (U'U^{-1} + UAU^{-1}, 1) \end{array}$$

With the identification $\mathrm{M}_n(K) = (\mathrm{M}_n(K) \times \{1\})$ and inclusion $(\mathrm{M}_n(K) \times \{1\}) \subset H_n(K)$, it induces a canonical bijection

$$\mathrm{GL}_n(K) \backslash \mathrm{M}_n(K) \simeq \Delta_n(K) \backslash H_n(K) / \left(\{0\} \times \mathrm{GL}_n(K)\right) = Z_n(K). \tag{1}$$

We will use below these two definitions of $Z_n(K)$.

Let A be in $\mathrm{M}_n(K)$ and F be in $\mathrm{GL}_n(K)$, we denote by $[(A, F)]$, resp. $[A]$, the class in $Z_n(K)$ of $(A, F) \in H_n(K)$, resp. of $A \in \mathrm{M}_n(K)$.

Let L/K be a Picard-Vessiot extension. The inclusion $H_n(K) \subseteq H_n(L)$ gives rise to a map $\alpha(L/K) : Z_n(K) \to Z_n(L)$. We set

$$Z_n(L/K) := \{a \in Z_n(K) \ / \ \alpha(L/K)(a) = [0]\}.$$

For any group G we denote by $\mathrm{Rep}_n(G)$ the set of equivalent classes of representations of G in $\mathrm{GL}_n(C)$, if $G = dGal(L/K)$ is the differential group of L over K, we set $\mathrm{Rep}_n(L/K) := \mathrm{Rep}_n(G)$.

Theorem 1.1. *Let L/K be a Picard-Vessiot extension, then there exists a natural bijection between $Z_n(L/K)$ and $\mathrm{Rep}_n(L/K)$.*

Proof. First of all we recall some facts that are of main importance in our proof.

1.0.1. *The representation c_A.* Consider the differential equation $Y' = AY$, with $A \in \mathrm{M}_m(K)$, and let E/K be a corresponding Picard-Vessiot extension, i.e. E is generated over K by the coefficients of a fundamental matrix F_A of the equation. The rational representation c_A is

$$\begin{array}{ccc} dGal(E/K) & \longrightarrow & \mathrm{GL}_m(C) \\ \sigma & \longmapsto & c_A(\sigma) \end{array}$$

where $c_A(\sigma)$ is such that $\sigma(F_A) = F_A c_A(\sigma)$. Note that c_A depends only on the class $[A]$ of $(A, 1)$ in $Z_m(K)$, because if $B = U'U^{-1} + UAU^{-1}$ with $U \in \mathrm{GL}_m(K)$, a fundamental matrix of the equation $Y' = BY$ is UF_A and we see that $c_B = c_A$. Note also that an other fundamemtal matrix is of the form $F_A\gamma$, with $\gamma \in \mathrm{GL}_n(C)$, then it gives the representation $\gamma^{-1} c_A \gamma$ equivalent to c_A.

We will write equivalently c_A, $c_{[A]}$ or c_E for this class of representations.

1.0.2. *The Galois group $dGal(E/K)$.* Let $A \in \mathrm{M}_m(K)$ and

$$R = K[(X_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}]/q = K[(x_{i,j})_{1\leq i,j\leq m}]$$

be a Picard-Vessiot ring over K for the equation $Y' = AY$. In these formulas the $X_{i,j}$ are indeterminates, the ring $K\left[(X_{i,j})_{1\leq i,j\leq m}\right]$ is equipped by the derivation satisfying $(X'_{i,j})_{1\leq i,j\leq m} = A(X_{i,j})_{1\leq i,j\leq m}$, "det" is the determinant of the matrix $(X_{i,j})_{1\leq i,j\leq m}$, q is a maximal differential ideal and $x_{i,j}$ is the image of $X_{i,j}$. Let $E = \mathrm{Quot}(R)$ and $\mathfrak{U} = dGal(E/K)$, consider

$$K\left[(X_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right] \subseteq E\left[(X_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right]$$
$$= E\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right] \supseteq C\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right],$$

where $(Y_{i,j})_{1\leq i,j\leq m}$ is defined by $(X_{i,j})_{1\leq i,j\leq m} = (x_{i,j})_{1\leq i,j\leq m}(Y_{i,j})_{1\leq i,j\leq m}$. Note that $Y'_{i,j} = 0$. We know that

$$\mathfrak{U} = dGal(E/K) = \mathrm{Spec}C\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right]/J$$

where $J = qE\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right] \cap C\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right]$ ([2] proof of prop. 1.24 or the beginning of §1.5). We denote by $y_{i,j}$ the image of $Y_{i,j}$, then we have

$$\mathfrak{U} = dGal(E/K) = \mathrm{Spec}C\left[(y_{i,j})_{1\leq i,j\leq m}\right]$$

1.0.3. *$\mathfrak{U} = dGal(E/K)$ as a torsor.* We continue with the previous notations. Set $\mathfrak{T} = \mathrm{Spec}R$, we know that $\mathfrak{T}$ is an $\mathfrak{U}$-torsor over K ([2] theorem 1.30), moreover, we know that there exists a finite extension $\widetilde{K}$ of K such that $\mathfrak{T} \times_K \widetilde{K} = \mathrm{Spec}\left(R \otimes_K \widetilde{K}\right)$ is a trivial $\mathfrak{U}$-torsor over $\widetilde{K}$ ([2] cor. 1.31), this means that there exists $\underline{b} \in \mathfrak{T}(\widetilde{K})$ such that the following map is an isomorphism of $\widetilde{K}$-schemes

$$\begin{array}{cccc} \psi: & \mathfrak{U} \times_C \widetilde{K} & \longrightarrow & \mathfrak{T} \times_K \widetilde{K} \\ & (c_{i,j})_{1\leq i,j\leq m} & \longmapsto & \underline{b}(c_{i,j})_{1\leq i,j\leq m} \end{array}$$

($\underline{b}$ can be seen as a matrix, on the right this is a product of matrices; see the definition of R above).

1.0.4. *Galois actions.* Let σ be an element of $\mathfrak{U} = dGal(E/K)$, the action of σ on R is given by the images of the $x_{i,j}$, which are defined by the matrix formula $(\sigma(x_{i,j})) = (x_{i,j})c_E(\sigma)$. We denote by $\sigma^\flat$ the morphism induces by σ on $\mathfrak{T}$ or on $\mathfrak{T} \times_K \widetilde{K}$, this is the action of $\mathfrak{U}$ which defines the torsor structure. An element of $\mathfrak{T}(\widetilde{K})$ can be represented by a matrix $\underline{a} = (a_{i,j})_{1\leq i,j\leq m}$ with $a_{i,j}$ in $\widetilde{K}$, its image is $\sigma^\flat(\underline{a}) = \underline{a}c_E(\sigma)$. For any σ in $\mathfrak{U}$ denote by λ_σ the right translation on $\mathfrak{U}$ by σ, i.e.

$$\begin{array}{cccc} \lambda_\sigma: & \mathfrak{U} & \rightarrow & \mathfrak{U} \\ & \tau & \mapsto & \tau\sigma \end{array}$$

Write again λ_σ for $\lambda_\sigma \times \mathrm{Id}_{\widetilde{K}} : \mathfrak{U} \times_C \widetilde{K} \to \mathfrak{U} \times_C \widetilde{K}$, then the morphism ψ of (1.0.3) is equivariant, this means that for any $\sigma \in \mathfrak{U}$ the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{U} \times_C \widetilde{K} & \stackrel{\psi}{\to} & \mathfrak{T} \times_K \widetilde{K} \\ \lambda_\sigma \downarrow & & \downarrow \sigma^\flat \\ \mathfrak{U} \times_C \widetilde{K} & \stackrel{\psi}{\to} & \mathfrak{T} \times_K \widetilde{K} \end{array}$$

Proof of the theorem, the map $Z_n(L/K) \to R_n(L/K)$.

Let $A \in \mathrm{M}_n(K)$ such that $[A] \in Z_n(L/K)$, then there exists $U \in \mathrm{GL}_n(L)$ such that

$$A = U'U^{-1},$$

this means that U is a fundamental matrix of the equation $Y' = AY$, as it is with entries in L, it exists a differential subextension E of L which is a Picard-Vessiot extension for the equation $Y' = AY$. Denote by ρ_A the representation

$$\rho_A : dGal(L/K) \stackrel{\text{restriction}}{\longrightarrow} dGal(E/K) \stackrel{c_A}{\longrightarrow} \mathrm{GL}_n(C). \tag{2}$$

Now we prove that this representation ρ_A does not depend on the class of A in $Z_n(K)$ and of the choice of $U \in \mathrm{GL}_n(L)$ such that $A = U'U^{-1}$.

Let $B \in \mathrm{M}_n(K)$ such that $[B] = [A]$ in $Z_n(K)$, then there exists $W, T \in \mathrm{GL}_n(K)$ such that

$$(B, 1) = (W'W^{-1}, W)(A, 1)(0, T),$$

it follows that $B = W'W^{-1} + WAW^{-1}$, this means that WU is a fundamental matrix of the equation $Y' = BY$. We see that $\rho_A = \rho_B$, and we denote this representation by $\rho_{[A]}$.

Let $V \in \mathrm{GL}_n(L)$ such that $A = U'U^{-1} = V'V^{-1}$, then we see that $(V^{-1}U)' = 0$, this means that there exists $\gamma \in \mathrm{GL}_n()$ such that $U = V\gamma$ and the two representations define as before in (2) are conjugate.

Then to each element $[A]$ of $Z_n(L/K)$ we have associated the element $\rho_{[A]}$ of $\mathrm{Rep}_n(L/K)$.

Proof of the theorem, the map $R_n(L/K) \to Z_n(L/K)$.

Let $\rho : dGal(L/K) \to \mathrm{GL}_n(C)$ be a rational representation. Let E be the fixed field of $\ker \rho$, we set $\mathfrak{U} = dGal(E/K)$ and we denote again by ρ the representation $\mathfrak{U} \hookrightarrow \mathrm{GL}_n(C)$ coming from the given one. The field E is a Picard-Vessiot extension corresponding to an equation $Y' = AY$, with $A \in \mathrm{M}_m(K)$. Our aim is to prove that one can chose A in $\mathrm{M}_n(K)$, i.e. $m = n$, and that this gives the inverse map of $[A] \mapsto \rho_{[A]}$.

We use the previous notations and descriptions of E, R, $\mathfrak{U}$, $\mathfrak{T}$ etc. We set $\mathrm{GL}_n(C) = \mathrm{Spec} C[(T_{r,s})_{1 \le r,s \le n}, (\det)^{-1}]$, let

$$\rho^\sharp : C\left[(T_{r,s})_{1 \le r,s \le n}, (\det)^{-1}\right] \longrightarrow C\left[(Y_{i,j})_{1 \le i,j \le m}, (\det)^{-1}\right]/J$$

be the comorphism of $\rho : \mathfrak{U} \hookrightarrow \mathrm{GL}_n(C)$; $\rho^\sharp$ is onto. Set $I = \ker(\rho^\sharp)$, then we have an isomorphism induces by $\rho^\sharp$

$$\bar{\rho} : C\left[(T_{r,s})_{1\leq r,s\leq n}, (\det)^{-1}\right]/I \simeq C\left[(Y_{i,j})_{1\leq i,j\leq m}, (\det)^{-1}\right]/J.$$

Let $t_{r,s}$ be the image of $T_{r,s}$ in the quotient on the left, and recall that $y_{i,j}$ are that of $Y_{i,j}$ in the quotient on the right, then the preceding formula can be written

$$\bar{\rho} : C\left[(t_{r,s})_{1\leq r,s\leq n}\right] \simeq C\left[(y_{i,j})_{1\leq i,j\leq m}\right].$$

Set $\mathfrak{V} = \mathrm{Spec}\left(C\left[(t_{r,s})_{1\leq r,s\leq n}\right]\right)$, this is an algebraic subgroup of $\mathrm{GL}_n(C)$, it is isomorphic to $\mathfrak{U}$ via the morphism induces by $\bar{\rho}$, denoted by abuse of language $\rho : \mathfrak{U} \simeq \mathfrak{V}$.

The composed morphism (see (1.0.3))

$$\varphi : \mathfrak{T} \otimes_K \widetilde{K} \xrightarrow{\psi^{-1}} \mathfrak{U} \times_C \widetilde{K} \xrightarrow{\rho\times \mathrm{Id}_{\widetilde{K}}} \mathfrak{V} \times_C \widetilde{K} \tag{3}$$

is an isomorphism of $\widetilde{K}$-schemes, equivariant for the actions of $\mathfrak{U}$ and $\mathfrak{V}$, this means that for any σ in $\mathfrak{U}$ we have $\varphi \circ \sigma^\flat = \lambda_{\rho(\sigma)} \circ \varphi$, where, as before, $\lambda_{\rho(\sigma)}$ is the endomorphism of $\mathfrak{V} \times_C \widetilde{K}$ coming from the right translation by $\rho(\sigma)$ on $\mathfrak{V}$ (1.0.4).

Lemma 1.2. *Let $\varphi^\sharp$ be the comorphism of φ (see (2))and for any $r, s = 1, \cdots, n$ set $z_{r,s} = \varphi^\sharp(t_{r,s})$ (recall that $\mathfrak{V} = \mathrm{Spec} C[(t_{r,s})_{1\leq r,s\leq n}]$). Then, for all $\sigma \in \mathfrak{U}$, there exists a matrix $a(\sigma) \in \mathrm{GL}_n(C)$ such that we have the equality of matrices: $(\sigma(z_{r,s}))_{1\leq r,s\leq n} = (z_{r,s})_{1\leq r,s\leq n} a(\sigma)$.*

Proof. Denote by $\lambda^\sharp_{\rho(\sigma)}$ the comorphism of the right translation by $\rho(\sigma)$ on $\mathfrak{V} \times_C \widetilde{K}$, we have the equalities of matrices

$$\begin{aligned}(\sigma(z_{r,s}))_{1\leq r,s\leq n} &= \left(\sigma\left(\varphi^\sharp(t_{r,s})\right)\right)_{1\leq r,s\leq n} \\ &= \left(\varphi^\sharp\left(\lambda^\sharp_{\rho(\sigma)}(t_{r,s})\right)\right)_{1\leq r,s\leq n},\end{aligned}$$

because φ is equivariant, and

$$\left(\lambda^\sharp_{\rho(\sigma)}(t_{r,s})\right)_{1\leq r,s\leq n} = (t_{r,s})_{1\leq r,s\leq n} a(\rho(\sigma))$$

where for any $\tau \in \mathfrak{V}$ the matrix $a(\tau)$ is in $\mathrm{GL}_n(C)$ and is such that the formula $(\tau(t_{r,s}))_{1\leq r,s\leq n} = (t_{r,s})_{1\leq r,s\leq n}\, a(\tau)$ defines the images of the $t_{r,s}$ by the comorphism $\lambda^\sharp_\tau$ of the right translation on $\mathfrak{V}$ by τ. We have find

$$(\sigma(z_{r,s}))_{1\leq r,s\leq n} = (z_{r,s})_{1\leq r,s\leq n}\, a(\rho(\sigma))$$

with $a(\rho(\sigma))$ in $\mathrm{GL}_n(C)$. □

The fact that φ is an isomorphism implies that $R \otimes_K \widetilde{K}$ is generated over $\widetilde{K}$ by the $z_{r,s}$, $1 \leq r, s \leq n$, indeed $R \otimes_K \widetilde{K}$ is generated over $\widetilde{K}$ by the C-space $V := \sum_{1\leq r,s\leq n} C z_{r,s}$ and the lemma shows that this space V is (globally) invariant under the action of the Galois group $\mathfrak{U}$. The (ordinary) Galois group $Gal(\widetilde{K}/K)$ acts as usual on the right hand factor of $R \otimes_K \widetilde{K}$ and trivially on the left one, then we see that R is generated over K by the $z_{r,s}$, $1 \leq r, s \leq n$.

Another consequence of the previous lemma is that the matrix

$$D \stackrel{\text{def}}{=} \left(z'_{r,s}\right)_{1\leq r,s\leq n} \left(z_{r,s}\right)^{-1}_{1\leq r,s\leq n},$$

is in $\mathrm{M}_n(K)$, then, because $\varphi^\sharp$ is an isomorphism, the ring R is generated by the entries of a fundamental matrix of the equation $Y' = DY$, we know also that R is a simple differential ring. It follows that R, resp. E, is the Picard-Vessiot ring, resp. field, over K of this equation.

To a rational representation $\rho : dGAl(L/K) \to \mathrm{GL}_n(C)$ we have associated an element $[D]$ of $Z_n(L/K)$, this is clearly the inverse map of $[A] \mapsto \rho_{[A]}$. □

2. A correspondance.

Let K^{diff} be a universal Picard-Vessiot extension of K and set $G^{\mathrm{diff}} = dGal(K^{\mathrm{diff}}/K)$.We choose once of all an identification $\mathrm{GL}_n(C) = \mathrm{GL}(C^n)$.

Let $\underline{\mathrm{Rep}}_n(G^{\mathrm{diff}})$ be the category of representations of G^{diff} in $\mathrm{GL}_n(C)$: the objects are morphisms $\rho : G^{\mathrm{diff}} \to \mathrm{GL}_n(C)$, an arrow $f : \rho_1 \to \rho_2$ is a C-linear map from C^n into itself such that, for any $g \in G^{\mathrm{diff}}$, the following diagram is commutative

$$\begin{array}{ccc} C^n & \stackrel{\rho_1(g)}{\longrightarrow} & C^n \\ f\downarrow & & \downarrow f \\ C^n & \stackrel{\rho_2(g)}{\longrightarrow} & C^n \end{array}$$

To define the category $\underline{Z}_n(K)$ we need the following remarks. Let M and N be two elements of $\mathrm{M}_n(K)$, we say that they are equivalent if there exists U and V in $\mathrm{GL}_n(K)$ such that $N = VMU$. We denote by $\overline{M}$ the equivalent class of M. Let $A_i \in \mathrm{M}_n(K)$, $i = 1, 2$ and let $M \in \mathrm{M}_n(K)$ such that

$$M' = A_2M - MA_1. \tag{4}$$

Let $B_i \in [A_i]$, let $U_i \in \mathrm{GL}_n(K)$ such that

$$A_i = U_i'U_i^{-1} + U_iB_iU_i^{-1},$$

then an easy calculation shows that

$$(U_2^{-1}MU_1)' = B_2(U_2^{-1}MU_1) - (U_2^{-1}MU_1)B_2.$$

Suppose that $M \in \mathrm{GL}_n(K)$ and satisfies (4), then

$$(M^{-1})' = A_1M^{-1} - M^{-1}A_2.$$

Now we can define the category $\underline{Z}_n(K)$. Its objects are elements of $Z_n(K)$ (see (1)), an arrow $[A_1] \to [A_2]$, where A_1 and A_2 are elements of $\mathrm{M}_n(K)$, is an equivalence class $\overline{M}$ in $M_n(K)$ such that there exists $M \in \overline{M}$ satisfying (4). The two preceding formulas show that this definition does not depend on the choice of A_i in $[A_i]$, $i = 1, 2$, and that invertible arrows in $\underline{Z}_n(K)$ correspond to equivalence classes of invertible matrices. We explain the composition of arrows. Let $\overline{M} : [A_1] \to [A_2]$ and $\overline{N} : [A_2] \to [A_3]$ two arrows of $\underline{Z}_n(K)$, choose $M \in \overline{M}$, $N \in \overline{N}$ such that

$$M' = A_2M - MA_1 \quad \text{and} \quad N' = A_3N - NA_2,$$

then we see that

$$(NM)' = A_3NM - NMA_1.$$

The composed arrow is $\overline{N} \circ \overline{M} = \overline{NM}$, for a good choice of representing elements of the different classes of matrices.

Then $\underline{Z}_n(K)$ is a category, indeed it is easily to see that it is an additive category.

Theorem 2.1. *The two categories $\underline{Z}_n(K)$ and $\underline{\mathrm{Rep}}_n(G^{\mathrm{diff}})$ are equivalent. On objects, this equivalence is $[A] \mapsto c_{[A]}$ (see (1.0.1)).*

Proof. Note that here to write $c_{[A]}$ is an abuse of notation, if L/K is the Picard-Vessiot extension (contained in K^{diff}) associated to the equation $Y' = AY$, we denote always $c_{[A]}$ the representation

$$G^{\mathrm{diff}} \overset{\text{restriction}}{\longrightarrow} dGal(L/K) \overset{c_{[A]}}{\longrightarrow} \mathrm{GL}_n(C).$$

The map $[A] \mapsto c_{[A]}$ on objects of the categories has been constructed in the previous theorem, it is one to one. Let $[A_1]$ and $[A_2]$ be two objects of $\underline{Z}_n(K)$ and $\overline{M} : [A_1] \to [A_1]$ be an arrow, select $M \in \overline{M}$ such that $M' = A_2M - MA_1$. Let $F_1, F_2 \in \mathrm{GL}_n(K^{\mathrm{diff}})$ be fundamental matrices for respectively the equations $Y' = A_1Y$ and $Y' = A_2Y$. Then $F_i' = A_iF_i$, $i = 1, 2$. Let $f = F_2^{-1}MF_1$, a priori f is in $\mathrm{GL}_n(K)$, but

$$\begin{aligned} f' &= (F_2^{-1})'MF_1 + F_2^{-1}M'F_1 + F_2^{-1}MF_1' \\ &= (-F_2^{-1}A_2)MF_1 + F_2^{-1}(A_2M - MA_1)F_1 \\ &\quad +F_2^{-1}MA_1F_1 = 0. \end{aligned}$$

Then $f = F_2^{-1}MF_1$ is in $\mathrm{GL}_n(C)$. Now we prove that f is a morphism from $c_{[A_1]}$ to $c_{[A_2]}$. Let g be an element of G^{diff}. Applying g to the relation $f = F_2^{-1}MF_1$ we find

$$f = g(F_2^{-1})Mg(F_1) = g(F_2^{-1})F_2fF_1^{-1}g(F_1) = c_{[A_2]}(g)^{-1}fc_{[A_1]}(g),$$

(see (1.0.1)) for all g. This means that $f : c_{[A_1]} \to c_{[A_2]}$ is a map in $\underline{\mathrm{Rep}}_n(G^{\mathrm{diff}})$.

Conversely let $f : \rho_1 \to \rho_2$ be an arrow of $\underline{\mathrm{Rep}}_n(G^{\mathrm{diff}})$, then we can see f as a matrix with coefficient in C. We know that there exists A_i in $\mathrm{M}_n(K)$ such that $\rho_i = c_{[A_i]}$, $i = 1, 2$. Let as before F_i be a fundamental matrix for the equation $Y' = A_iY$. Set $M = F_2fF_1^{-1}$.

- We prove that M is in $\mathrm{M}_n(K)$. The fact that f is a morphism of representations means that for all g in G^{diff} we have

$$fc_{[A_1]}(g) = c_{[A_1]}(g)f,$$

which is equivalent to

$$fF_1^{-1}g(F_1) = F_2^{-1}g(F_2)f,$$

then

$$F_2fF_1^{-1} = g(F_2)fg(F_1^{-1}) = g(F_2fF_1^{-1}).$$

This prove that the entries of M are in K.
- We prove the formula $M' = A_2M - MA_1$. We have

$$M' = F_2'fF_1^{-1} + F_2f(F_1^{-1})' = A_2F_2fF_1^{-1} + F_2f(-F_1^{-1}A_1)$$

which is the expected formula. □

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