COMPUTATIONS OF NAMBU-POISSON COHOMOLOGIES

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Abstract. In this paper, we want to associate to a n-vector on a manifold of dimension n a cohomology which generalizes the Poisson cohomology of a 2-dimensional Poisson manifold. Two possibilities are given here. One of them, the Nambu-Poisson cohomology, seems to be the most pertinent. We study these two cohomologies locally, in the case of germs of n-vectors on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

1. Introduction

A way to study a geometrical object is to associate to it a cohomology. In this paper, we focus on the n-vectors on a n-dimensional manifold M.
If $n = 2$, the 2-vectors on M are the Poisson stuctures thus, we can consider the Poisson cohomology. In dimension 2, this cohomology has three spaces. The first one, H^0, is the space of functions whose Hamiltonian vector field is zero (Casimir functions). The second one, H^1, is the quotient of the space of infinitesimal automorphisms (or Poisson vector fields) by the subspace of Hamiltonian vector fields. The last one, H^2, describes the deformations of the Poisson structure. In a previous paper ([Mo]), we have computed the cohomology of germs at 0 of Poisson structures on $\mathbb{K}^2$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).
In order to generalize this cohomology to the n-dimensional case ($n \geq 3$), we can follow the same reasoning. These spaces are not necessarily of finite dimension and it is not always easy to describe them precisely.

Recently, a team of Spanish researchers has defined a cohomology, called Nambu-Poisson cohomology, for the Nambu-Poisson structures (see [I2]). In this paper, we adapt their construction to our particular case. We will see that this cohomology generalizes in a certain sense the Poisson cohomology in dimension 2. Then we compute locally this cohomology for germs at 0 of n-vectors $\Lambda = f\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with the assumption that f is a quasihomogeneous polynomial of finite codimension ("most of" the germs of n-vectors have this form). This computation is based on a preliminary result that we have shown, in the formal case and in the analytical case (so, the $\mathcal{C}^\infty$ case is not entirely solved). The techniques we use in this paper are quite the same as in [Mo].

2. Nambu-Poisson cohomology

Let M be a differentiable manifold of dimension n ($n \geq 3$), admitting a volume form ω. We denote $\mathcal{C}^\infty(M)$ the space of $\mathcal{C}^\infty$ functions on M, $\Omega^k(M)$ ($k = 0, \ldots, n$) the $\mathcal{C}^\infty(M)$-module of k-forms on M, and $\mathcal{X}^k(M)$ ($k = 0, \ldots, n$) the $\mathcal{C}^\infty(M)$-module of k-vectors on M.

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We consider a n-vector Λ on M. Note that Λ is a Nambu-Poisson structure on M. Recall that a **Nambu-Poisson structure** on M of order r is a skew-symmetric r-linear map $\{\,,\dots,\,\}$

$$\mathcal{C}^\infty(M)\times\ldots\times\mathcal{C}^\infty(M)\longrightarrow\mathcal{C}^\infty(M),\quad (f_1,\ldots,f_r)\longmapsto\{f_1,\ldots,f_r\},$$

which satifies

$$\{f_1,\ldots,f_{r-1},gh\}=\{f_1,\ldots,f_{r-1},g\}h+g\{f_1,\ldots,f_{r-1},h\}\quad (L)$$

$$\{f_1,\ldots,f_{r-1},\{g_1,\ldots,g_r\}\}=\sum_{i=1}^{r}\{g_1,\ldots,g_{i-1},\{f_1,\ldots,f_{r-1},g_i\},g_{i+1},\ldots,g_r\}\quad (FI)$$

for any $f_1,\ldots,f_{r-1},g,h,g_1,\ldots,g_r$ in $\mathcal{C}^\infty(M)$. It is clear that we can associate to such a bracket a r-vector on M. If $r=2$, we rediscover Poisson structures. Thus, Nambu-Poisson structures can be seen as a kind of generalization of Poisson structures. The notion of Nambu-Poisson structures was introduced in [T] by Takhtajan in order to give a formalism to an idea of Y. Nambu ([Na]).

Here, we suppose that the set $\{x\in M\,;\,\Lambda_x\neq 0\}$ is dense in M. We are going to associate a cohomology to (M,Λ).

2.1. **The choice of the cohomology.** If M is a differentiable manifold of dimension 2, then the Poisson structures on M are the 2-vectors on M. If Π is a Poisson structure on M, then we can associate to (M,Π) the complex

$$0\longrightarrow\mathcal{C}^\infty(M)\xrightarrow{\partial}\mathcal{X}^1(M)\xrightarrow{\partial}\mathcal{X}^2(M)\longrightarrow 0$$

with $\partial(g)=[g,\Pi]=X_g$ (Hamiltonian of g) if $g\in\mathcal{C}^\infty(M)$ and $\partial(X)=[X,\Pi]$ ($[\,,\,]$ indicates Schouten's bracket) if $X\in\mathcal{X}^1(M)$. The cohomology of this complex is called the Poisson cohomology of (M,Π). This cohomology has been studied for instance in [Mo], [N1] and [V].

Now if M is of dimension n with $n\geq 3$, we want to generalize this cohomology. Our first approach was to consider the complex

$$0\longrightarrow\big(\mathcal{C}^\infty(M)\big)^{n-1}\xrightarrow{\partial}\mathcal{X}^1(M)\xrightarrow{\partial}\mathcal{X}^n(M)\longrightarrow 0$$

with $\partial(X)=[X,\Lambda]$ and $\partial(g_1,\ldots,g_{n-1})=i_{dg_1\wedge\ldots\wedge dg_{n-1}}\Lambda=X_{g_1,\ldots,g_{n-1}}$ (Hamiltonian vector field) where we adopt the convention $i_{dg_1\wedge\ldots\wedge dg_{n-1}}\Lambda=\Lambda(dg_1,\ldots,dg_{n-1},\bullet)$. We denote $H^0_\Lambda(M)$, $H^1_\Lambda(M)$ and $H^2_\Lambda(M)$ the three spaces of cohomology of this complex. With this cohomology, we rediscover the interpretation of the first spaces of the Poisson cohomology, i.e. $H^2_\Lambda(M)$ describes the infinitesimal deformations of Λ and $H^1_\Lambda(M)$ is the quotient of the algebra of vector fields which preserve Λ by the ideal of Hamiltonian vector fields.

In [I2], the authors associate to any Nambu-Poisson structure on M a cohomology. The second idea is then to adapt their construction to our particular case.
Let $\#_\Lambda$ be the morphism of $\mathcal{C}^\infty(M)$-modules $\Omega^{n-1}(M)\longrightarrow\mathcal{X}^1(M):\ \alpha\mapsto i_\alpha\Lambda$. Note that $\ker\#_\Lambda=\{0\}$ (because the set of regular points of Λ is dense). We can define (see [I1]) a $\mathbb{R}$-bilinear operator $[\![\,,\,]\!]:\Omega^{n-1}(M)\times\Omega^{n-1}(M)\longrightarrow\Omega^{n-1}(M)$ by

$$[\![\alpha,\beta]\!]=\mathcal{L}_{\#_\Lambda\alpha}\beta+(-1)^n(i_{d\alpha}\Lambda)\beta\quad .$$

The vector space $\Omega^{n-1}(M)$ equiped with $[\![\ ,\ ]\!]$ is a Lie algebra (for any Nambu-Poisson structure, it is a Leibniz algebra). Moreover this bracket verifies $\#_\Lambda[\![\alpha,\beta]\!] = [\#_\Lambda\alpha, \#_\Lambda\beta]$ for any α,β in $\Omega^{n-1}(M)$. The triple $(\Lambda^{n-1}(T^*(M)), [\![\ ,\ ]\!], \#_\Lambda)$ is then a Lie algebroid and the Nambu-Poisson cohomology of (M,Λ) is the Lie algebroid cohomology of $(\Lambda^{n-1}(T^*(M))$ (for any Nambu-Poisson structure, it is more elaborate see [I2]). More precisely, for every $k \in \{0,\dots,n\}$, we consider the vector space $C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M))$ of the skew-symmetric and $\mathcal{C}^\infty(M)$-k-multilinear maps $\Omega^{n-1}(M)\times\ldots\times\Omega^{n-1}(M) \longrightarrow \mathcal{C}^\infty(M)$. The cohomology operator $\partial : C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M)) \longrightarrow C^{k+1}(\Omega^{n-1}(M);\mathcal{C}^\infty(M))$ is defined by

$$\begin{aligned}\partial c(\alpha_0,\dots,\alpha_k) &= \sum_{i=0}^{k}(-1)^i(\#_\Lambda\alpha_i).c\big(\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_k\big)\\ &\quad+\sum_{0\le i<j\le k}(-1)^{i+j}c([\![\alpha_i,\alpha_j]\!],\alpha_0,\dots,\hat{\alpha_i},\dots,\hat{\alpha_j},\dots,\alpha_k)\end{aligned}$$

for all $c\in C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M))$ and $\alpha_0,\dots,\alpha_k$ in $\Omega^{n-1}(M)$.
The **Nambu-Poisson cohomology** of (M,Λ), denoted by $H^\bullet_{NP}(M,\Lambda)$, is the cohomology of this complex.

2.2. **An equivalent cohomology.** So defined, the Nambu-Poisson cohomology is quite difficult to manipulate. We are going to give an equivalent cohomology which is more accessible.
Recall that we assume that M admits a volume form ω.
Let $f\in\mathcal{C}^\infty(M)$, we define the operator

$$\begin{aligned} d_f : \Omega^k(M) &\longrightarrow \Omega^{k+1}(M)\\ \alpha &\longmapsto f d\alpha - k df\wedge\alpha.\end{aligned}$$

It is easy to prove that $d_f\circ d_f = 0$. We denote $H^\bullet_f(M)$ the cohomology of this complex. Let $\flat$ be the isomorphism $\mathcal{X}^1(M)\longrightarrow\Omega^{n-1}(M)\quad X\longmapsto i_X\omega$.

Lemma 2.1. *1- If $X\in\mathcal{X}(M)$, then $\#_\Lambda\big(\flat(X)\big) = (-1)^{n-1}fX$ where $f = i_\Lambda\omega$.*
2- If X and Y are in $\mathcal{X}(M)$, then

$$(-1)^{n-1}[\![\flat(X),\flat(Y)]\!] = f\flat([X,Y]) + (X.f)\flat(Y) - (Y.f)\flat(X)\,.$$

Proof : 1- Obvious.
2- We have $\#_\Lambda\big([\![\flat(X),\flat(Y)]\!]\big) = [\#_\Lambda(\flat(X)), \#_\Lambda(\flat(Y))]$ (property of the Lie algebroid), which implies that

$$\begin{aligned}\#_\Lambda\big([\![\flat(X),\flat(Y)]\!]\big) &= f(X.f)Y - f(Y.f)X + f^2[X,Y]\\ &= (-1)^{n-1}\#_\Lambda\big((X.f)\flat(Y) - (Y.f)\flat(X) + f\flat([X,Y])\big)\,.\end{aligned}$$

The result follows via the injectivity of $\#_\Lambda$. □

Proposition 2.2. *If we put $f = i_\Lambda\omega$, then $H^\bullet_{NP}(M,\Lambda)$ is isomorphic to $H^\bullet_f(M)$.*

Proof : For every k, we consider the application $\varphi : C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M)) \longrightarrow \Omega^k(M)$ defined by

$$\varphi(c)\big(X_1,\dots,X_k\big) = c\big((-1)^{n-1}\flat(X_1),\dots,(-1)^{n-1}\flat(X_k)\big)\,,$$

where $c\in C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M))$ and $X_1,\dots,X_k\in\mathcal{X}(M)$. It is easy to see that φ is an isomorphism of vector spaces. We show that it is an isomorphism of complexes.

Let $c \in C^k(\Omega^{n-1}(M); \mathcal{C}^\infty(M))$. We put $\alpha = \varphi(c)$. If $X_0, \ldots, X_k$ are in $\mathcal{X}(M)$ then $\varphi(\partial c)(X_0, \ldots, X_k) = (-1)^{(n-1)(k+1)} \partial c(\flat(X_0), \ldots, \flat(X_k)) = A + B$ where
$A = (-1)^{(n-1)(k+1)} \sum_{i=0}^k (-1)^i \#_\Lambda\big(\flat(X_i)\big).c\big(\flat(X_0), \ldots, \widehat{\flat(X_i)}, \ldots, \flat(X_k)\big)$
$B = (-1)^{(n-1)(k+1)} \sum_{0 \le i < j \le k} (-1)^{i+j} c\big([\![\flat(X_i), \flat(X_j)]\!], \flat(X_0), \ldots, \widehat{\flat(X_i)}, \ldots, \widehat{\flat(X_j)}, \ldots, \flat(X_k)\big)$.
We have $A = f \sum_{i=0}^k (-1)^i X_i.\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)$ and

$$\begin{aligned} B &= f \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} (X_i.f) \alpha(X_j, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\ &\quad - \sum_{0 \le i < j \le k} (-1)^{i+j} (X_j.f) \alpha(X_i, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\ &= f \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k) \\ &\quad -k \sum_{i=0}^k (-1)^i (X_i.f) \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k) \,. \end{aligned}$$

Consequently, $\varphi(\partial c) = d_f \alpha = d_f\big(\varphi(c)\big)$. $\square$

Remark 2.3. We claim that this cohomology is a "good" generalisation of the Poisson cohomology of a 2-dimensional Poisson manifold. Indeed, if (M, Π) is an orientable Poisson manifold of dimension 2, we consider the volume form ω on M and we put

$$\phi^2 : \mathcal{X}^2(M) \longrightarrow \Omega^2(M) \;\text{ and }\; \phi^1 : \mathcal{X}^1(M) \longrightarrow \Omega^1(M)$$

defined by

$$\phi^2(\Gamma) = (i_\Gamma \omega)\omega \quad \text{ and } \quad \phi^1(X) = -i_X \omega$$

for every 2-vector Γ and vector field X.
We also put $\phi^0 = id : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$.
If we denote ∂ the operator of the Poisson cohomology, and $f = i_\Pi \omega$, it is quite easy to see that

$$\phi : (\mathcal{X}^\bullet(M), \partial) \longrightarrow (\Omega^\bullet(M), d_f)$$

is an isomorphism of complexes.

Remark 2.4. 1- The definitions we have given make sense if we work in the holomorphic case or in the formal case.
2- **Important :** If h is a function on M which doesn't vanish on M, then the cohomologies $H_f^\bullet(M)$ and $H_{fh}^\bullet(M)$ are isomorphic.
Indeed, the applications $\big(\Omega^k(M), d_{fh}\big) \longrightarrow \big(\Omega^k(M), d_f\big) \quad \alpha \longmapsto \frac{\alpha}{h^k}$ give an isomorphism of complexes.
In particular, if f doesn't vanish on M then $H_f^\bullet(M)$ is isomorphic to the de Rham's cohomology.

2.3. **Other cohomologies.** We can construct other complexes which look like $(\Omega^\bullet(M), d_f)$. More precisely we denote, for $p \in \mathbb{Z}$,

$$\begin{aligned} d_f^{(p)} : \Omega^k(M) &\longrightarrow \Omega^{k+1}(M) \\ \alpha &\longmapsto f d\alpha - (k-p) df \wedge \alpha. \end{aligned}$$

We will denote $H^\bullet_{f,p}(M)$ the cohomology of these complexes. We will see in the next section some relations between these different cohomologies.

Using the contraction $i_\bullet\omega$, it is quite easy to prove the following proposition.

Proposition 2.5. *The spaces $H^1_\Lambda(M)$ and $H^2_\Lambda(M)$ are isomorphic to $H^{n-1}_{f,n-2}(M)$ and $H^n_{f,n-2}(M)$.*

Remark 2.6. The two properties of remark 2.4 are valid for $H^\bullet_{f,p}(M)$ with $p \in \mathbb{Z}$.

3. Computation

Henceforth, we will work **locally**. Let Λ be a germ of n-vectors on $\mathbb{K}^n$ ($\mathbb{K}$ indicates $\mathbb{R}$ or $\mathbb{C}$) with $\underline{n \geq 3}$. We denote $\mathcal{F}(\mathbb{K}^n)$ ($\Omega^k(\mathbb{K}^n), \mathcal{X}(\mathbb{K}^n)$) the space of **germs** at 0 of (holomorphic, analytic, $\mathcal{C}^\infty$, formal) functions (k-forms, vector fields). We can write Λ (with coordinates $(x_1,\dots,x_n)$) $\Lambda = f\frac{\partial}{\partial x_1}\wedge\dots\wedge\frac{\partial}{\partial x_n}$ where $f \in \mathcal{F}(\mathbb{K}^n)$. We assume that the volume form ω is $dx_1\wedge\dots\wedge dx_n$.
We suppose that $f(0) = 0$ (see remark 2.4) and that f is of **finite codimension**, which means that $Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$ (I_f is the ideal spanned by $\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}$) is a finite dimensional vector space.

Remark 3.1. It is important to note that, according to Tougeron's theorem (see for instance [AGV]), if f is of finite codimension, then the set $f^{-1}(\{0\})$ is, from the topological point of view, the same as the set of the zeros of a polynomial.
Therefore, if g is a germ at 0 of functions which satisfies $fg = 0$, then $g = 0$.

Moreover we suppose that f is a **quasihomogeneous** polynomial of degree N (for a justification of this additional assumption, see section 3). We are going to recall the definition of the quasi-homogeneity.

3.1. **Quasi-homogeneity.** Let $(w_1,\dots,w_n) \in (\mathbb{N}^*)^n$. We denote W the vector field $w_1x_1\frac{\partial}{\partial x_1}+\dots+w_nx_n\frac{\partial}{\partial x_n}$ on $\mathbb{K}^n$. We will say that a tensor T is quasihomogeneous with weights $w_1,\dots,w_n$ and of (quasi)degree $\mathrm{N} \in \mathbb{Z}$ if $\mathcal{L}_W T = \mathrm{N}T$ ($\mathcal{L}$ indicates the Lie derivative operator). Note that T is then polynomial.
If f is a quasihomogeneous polynomial of degree N then $\mathrm{N} = k_1w_1+\dots+k_nw_n$ with $k_1,\dots,k_n \in \mathbb{N}$; so, an integer is not necessarily the quasidegree of a polynomial. If $f \in \mathbb{K}\big[[x_1,\dots,x_n]\big]$, we can write $f = \sum_{i=0}^\infty f_i$ with f_i quasihomogeneous of degree i (we adopt the convention that $f_i = 0$ if i is not a quasidegree); f is said to be of order d ($\mathrm{ord}(f) = \mathrm{d}$) if all of its monomials have a degree d or higher. For more details, consult [AGV].
Since $\mathcal{L}_W$ and the exterior differentiation d commute, if α is a quasihomogeneous k-form, then $d\alpha$ is a quasihomogeneous $(k+1)$-form of degree $\deg\alpha$. In particular, it is important to notice that dx_i is a quasihomogeneous 1-form of degree w_i (note that $\frac{\partial}{\partial x_i}$ is a quasihomogeneous vector field of degree $-w_i$). Thus, the volume form $\omega = dx_1\wedge\dots\wedge dx_n$ is quasihomogeneous of degree $w_1+\dots+w_n$. Note that a quasihomogeneous non zero k-form ($k \geq 1$) has a degree strictly positive.
Note that if f is a quasihomogeneous polynomial of degree N, then the n-vector $\Lambda = f\frac{\partial}{\partial x_1}\wedge\dots\wedge\frac{\partial}{\partial x_n}$ is quasihomogeneous of degree $\mathrm{N}-\sum_i w_i$.
In the sequel, the degrees will be quasidegrees with respect to $W = w_1x_1\frac{\partial}{\partial x_1}+\dots+w_nx_n\frac{\partial}{\partial x_n}$.
We will need the following result.

Lemma 3.2. *Let $k_1, \dots, k_n \in \mathbb{N}$ and put $p = \sum k_i w_i$. Assume that $g \in \mathcal{F}(\mathbb{K}^n)$ and $\alpha \in \Omega^i(\mathbb{K}^n)$ verify $\operatorname{ord}(j_0^\infty(g)) > p$ and $\operatorname{ord}(j_0^\infty(\alpha)) > p$ (j_0^∞ indicates the ∞-jet at 0). Then*

1. *there exists $h \in \mathcal{F}(\mathbb{K}^n)$ such that $W.h - ph = g$,*
2. *there exists $\beta \in \Omega^i(\mathbb{K}^n)$ such that $\mathcal{L}_W\beta - p\beta = \alpha$.*

proof : The first claim is only a generalisation of a lemma given (in dimension 2) in [Mo] and it can be proved in the same way. The second claim is a consequence of the first.

Now we are going to compute the spaces $H_f^k(\mathbb{K}^n)$ (i.e $H_{NP}^k(\mathbb{K}^n, \Lambda)$) for $k = 0, \dots, n$. We will denote $Z_f^k(\mathbb{K}^n)$ and $B_f^k(\mathbb{K}^n)$ the spaces of k-cocycles and k-cobords. We will also compute some spaces $H_{f,p}^k(\mathbb{K}^n)$ with particular interest in the spaces $H_{f,n-2}^n(\mathbb{K}^n)$ (i.e. $H_\Lambda^2(\mathbb{K}^n)$) and $H_{f,n-2}^{n-1}(\mathbb{K}^n)$ (i.e. $H_\Lambda^1(\mathbb{K}^n)$). We will denote $Z_{f,p}^k(\mathbb{K}^n)$ $\left(B_{f,p}^k(\mathbb{K}^n)\right)$ the spaces of k-cocycles (k-cobords) for the operator $d_f^{(p)}$.

3.2. **Two useful preliminary results.** In the computation of these spaces of cohomology, we will need the two following propositions. The first is only a corollary of the de Rham's division lemma (see [dR]).

Proposition 3.3. *Let $f \in \mathcal{F}(\mathbb{K}^n)$ of finite codimension. If $\alpha \in \Omega^k(\mathbb{K}^n)$ ($1 \le k \le n-1$) verifies $df \wedge \alpha = 0$ then there exists $\beta \in \Omega^{k-1}(\mathbb{K}^n)$ such that $\alpha = df \wedge \beta$.*

Proposition 3.4. *Let $f \in \mathcal{F}(\mathbb{K}^n)$ of finite codimension. Let α be a k-form ($2 \le k \le n-1$) which verifies $d\alpha = 0$ and $df \wedge \alpha = 0$ then there exists $\gamma \in \Omega^{k-2}(\mathbb{K}^n)$ such that $\alpha = df \wedge d\gamma$.*

Proof : We are going to prove this result in the formal case and in the analytical case.
Formal case: Let α be a quasihomogeneous k-form of degree p which verifies the hypotheses. Since $df \wedge \alpha = 0$, we have $\alpha = df \wedge \beta_1$ where β_1 is a quasihomogeneous $(k-1)$-form of degree $p - \mathrm{N}$. Now, since $d\alpha = 0$, we have $df \wedge d\beta_1 = 0$ and so $d\beta_1 = df \wedge \beta_2$, where β_2 is a quasihomogeneous $(k-1)$-form of degree $p - 2\mathrm{N}$. This way, we can construct a sequence (β_i) of quasihomogeneous $(k-1)$-forms with $\deg \beta_i = p - i\mathrm{N}$ which verifies $d\beta_i = df \wedge \beta_{i+1}$. Let $q \in \mathbb{N}$ such that $p - q\mathrm{N} \le 0$. Thus, we have $\beta_q = 0$ and so $d\beta_{q-1} = 0$ i.e. $\beta_{q-1} = d\gamma_{q-1}$ where γ_{q-1} is a $(k-2)$-form. Consequently, $d\beta_{q-2} = df \wedge d\gamma_{q-1}$ which implies that $\beta_{q-2} = -df \wedge \gamma_{q-1} + d\gamma_{q-2}$, where γ_{q-2} is a $(k-2)$-form. In the same way, $d\beta_{q-3} = df \wedge d\gamma_{q-2}$ so $\beta_{q-3} = -df \wedge \gamma_{q-2} + d\gamma_{q-3}$ where γ_{q-3} is a $(k-2)$-form. This way, we can show that $\beta_1 = -df \wedge \gamma_2 + d\gamma_1$ where γ_1 and γ_2 are $(k-2)$-forms. Therefore, $\alpha = df \wedge d\gamma_1$
Analytical case : In [Ma], Malgrange gives a result on the relative cohomology of a germ of an analytical function. In particular, he shows that in our case, if β is a germ at 0 of analytical r-forms ($r < n-1$) which verifies $d\beta = df \wedge \mu$ (μ is a r-form) then there exists two germs of analytical $(r-1)$-forms γ and ν such that $\beta = d\gamma + df \wedge \nu$.
Now, we are going to prove our proposition. Let α be a germ of analytical k-forms ($2 \le k \le n-1$) which verifies the hypotheses of the proposition. Then there exists a $(k-1)$-form β such that $\alpha = df \wedge \beta$ (proposition 3.3). But since $0 = d\alpha = -df \wedge d\beta$, we have $d\beta = df \wedge \mu$ and so ([Ma]) $\beta = d\gamma + df \wedge \nu$ where γ and ν are analytical $(k-2)$-forms. We deduce that $\alpha = df \wedge d\gamma$ where γ is analytic. $\square$

Remark 3.5. **Important:**
In fact, some results which appear in [R] lead us to think that this proposition is not true in the real $\mathcal{C}^\infty$ case.
The computation of the spaces $H^n_{f,p}(\mathbb{K}^n)$, $H^{n-1}_{f,p}(\mathbb{K}^n)$ $(p \neq n-2)$ and $H^0_{f,p}(\mathbb{K}^n)$ doesn't use this proposition so, it still holds in the $\mathcal{C}^\infty$ case.
The results we find on the other spaces should be the same in the $\mathcal{C}^\infty$ case as in the analytical case but another proof need to be found.

3.3. **Computation of $H^0_{f,p}(\mathbb{K}^n)$.** We consider the application $d_f^{(p)} : \Omega^0(\mathbb{K}^n) \longrightarrow \Omega^1(\mathbb{K}^n) \quad g \longmapsto fdg + pdf \wedge g$.

Theorem 3.6. *1- If $p > 0$ then $H^0_{f,p}(\mathbb{K}^n) = \{0\}$*
2- If $p \leq 0$ then $H^0_{f,p}(\mathbb{K}^n) = \mathbb{K}.f^{-p}$

Proof : 1- If $g \in \mathcal{F}(\mathbb{K}^n)$ verifies $d_f^{(p)} g = 0$ then $d(f^p g) = 0$, and so $f^p g$ is constant. But as $f(0) = 0$, $f^p g$ must be 0 i.e. $g = 0$ (because f is of finite codimension; see remark 3.1).
2- We will use an induction to show that for any $k \geq 0$, if g satisfies $fdg = kgdf$ then $g = \lambda f^k$ where $\lambda \in \mathbb{K}$.
For $k = 0$ it is obvious.
Now we suppose that the property is true for $k \geq 0$. We show that it is still valid for $k+1$. Let $g \in \mathcal{F}(\mathbb{K}^n)$ be such that $fdg = (k+1)gdf$ $(*)$. Then $df \wedge dg = 0$ and so there exists $h \in \mathcal{F}(\mathbb{K}^n)$ such that $dg = hdf$ (proposition 3.3). Replacing dg by hdf in $(*)$, we get $fhdf = (k+1)gdf$ i.e. $g = \frac{1}{k+1} fh$. Now, this former relation gives on the one hand $fdg = \frac{1}{k+1}(f^2 dh + fhdf)$ and on the other hand, using $(*)$, $fdg = fhdf$. Consequently, $fdh = khdf$ and so $h = \lambda f^k$ with $\lambda \in \mathbb{K}$. □

3.4. **Computation of $H^k_f(\mathbb{K}^n)$ $1 \leq k \leq n-2$.**

Lemma 3.7. *Let $\alpha \in Z^k_{f,p}(\mathbb{K}^n)$ with $1 \leq k \leq n-2$. Then α is cohomologous to a closed k-form.*

Proof : We have $fd\alpha - (k-p)df \wedge \alpha = 0$. If $k = p$ then α is closed. Now we suppose that $k \neq p$. We put $\beta = d\alpha \in \Omega^{k+1}(\mathbb{K}^n)$. We have

$$0 = df \wedge (fd\alpha - (k-p)df \wedge \alpha) = fdf \wedge \alpha$$

so, $df \wedge \alpha = 0$.
Now, since $d\beta = 0$ and $df \wedge \beta = 0$, proposition 3.4 gives $\beta = df \wedge d\gamma$ with $\gamma \in \Omega^{k-1}(\mathbb{K}^n)$. Then, if we consider $\alpha' = \alpha - \frac{1}{k-p}\big(fd\gamma - (k-p-1)df \wedge \gamma\big)$, we have $d\alpha' = 0$ and $fd\gamma - (k-p-1)df \wedge \gamma \in B^k_{f,p}(\mathbb{K}^n)$. □

Theorem 3.8. *If $k \in \{2, \ldots, n-2\}$ then $H^k_f(\mathbb{K}^n) = \{0\}$.*

Proof : Let $\alpha \in Z^k_f(\mathbb{K}^n)$. Then $\alpha \in \Omega^k(\mathbb{K}^n)$ and verifies $fd\alpha - kdf \wedge \alpha = 0$.
According to lemma 3.7 we can assume that α is closed. Now we show that $\alpha \in B^k_f(\mathbb{K}^n)$.
Since $d\alpha = 0$ and $df \wedge \alpha = 0$, there exists $\beta \in \Omega^{k-2}(\mathbb{K}^n)$ such that $\alpha = df \wedge d\beta$ (proposition 3.4). Thus, $\alpha = d_f\big(\frac{-1}{k-1} d\beta\big)$. □

Remark 3.9. It is possible to adapt this proof to show that $H^k_{f,p}(\mathbb{K}^n) = \{0\}$ if $k \in \{2, \ldots, n-2\}$ and $p \neq k, k-1$.

Lemma 3.10. *Let $\alpha \in Z^1_f(\mathbb{K}^n)$. If $\mathrm{ord}\big(j^\infty_0(\alpha)\big) > \mathrm{N}$ then $\alpha \in B^1_f(\mathbb{K}^n)$.*

Proof : According to lemma 3.7, we can assume that $d\alpha = 0$.
Since $df \wedge \alpha = 0$ we have $\alpha = gdf$ (proposition 3.3) where g is in $\mathcal{F}(\mathbb{K}^n)$ and verifies $\mathrm{ord}\big(j^\infty_0(g)\big) > 0$. We show that f divides g.
Let $\bar{g} \in \mathcal{F}(\mathbb{K}^n)$ be such that $W.\bar{g} = g$ (lemma 3.2); note that $\mathrm{ord}\big(j^\infty_0(\bar{g})\big) > 0$.
We have $\mathcal{L}_W(df \wedge d\bar{g}) = \mathrm{N}df \wedge d\bar{g} + df \wedge dg$, and since $df \wedge dg = -d\alpha = 0$, $df \wedge d\bar{g}$ verifies

$$\mathcal{L}_W(df \wedge d\bar{g}) = \mathrm{N}df \wedge d\bar{g}\,,$$

which means that $df \wedge d\bar{g}$ is either 0 or quasihomogeneous of degree N.
But since $\mathrm{ord}\big(j^\infty_0(df \wedge d\bar{g})\big) > \mathrm{N}$, $df \wedge d\bar{g}$ must be 0.
Consequently, there exists $\nu \in \mathcal{F}(\mathbb{K}^n)$ such that $\frac{\partial \bar{g}}{\partial x_i} = \nu \frac{\partial f}{\partial x_i}$ for any i. Thus, $W.\bar{g} = \nu W.f$ and so $g = \nu f$.
We deduce that $\alpha = f\beta$ with $\beta \in \Omega^1(\mathbb{K}^n)$.
Now, we have

$$0 = d\alpha = df \wedge \beta + fd\beta \quad \text{and} \quad 0 = df \wedge \alpha = fdf \wedge \beta\,,$$

which implies that $d\beta = 0$.
Therefore, $\alpha = fdh = d_f(h)$ with $h \in \mathcal{F}(\mathbb{K}^n)$. □

Theorem 3.11. *The space $H^1_f(\mathbb{K}^n)$ is of dimension 1 and spanned by df.*

Proof : Let $\alpha \in Z^1_f(\mathbb{K}^n)$. According to lemma 3.10 we only have to study the case where α is quasihomogeneous with $\deg(\alpha) \leq \mathrm{N}$. We have $fd\alpha - df \wedge \alpha = 0$ so, $df \wedge d\alpha = 0$. We deduce that $d\alpha = df \wedge \beta$ where β is a quasihomogeneous 1-form of degree $\deg(\alpha) - \mathrm{N} \leq 0$. But since dx_i is quasihomogeneous of degree $w_i > 0$ for any i, every quasihomogeneous non zero 1-form has a strictly positive degree. We deduce that $\beta = 0$ and so $d\alpha = 0$. Therefore, $df \wedge \alpha = 0$ which implies that $\alpha = gdf$ where g is a quasihomogeneous function of degree $\deg(\alpha) - \mathrm{N} \leq 0$. Consequently, if $\deg(\alpha) < \mathrm{N}$ then $g = 0$; otherwise g is constant. To conclude, note that df is not a cobord because f doesn't divide df. □

3.5. **Computation of $H^n_{f,p}(\mathbb{K}^n)$.** We are going to compute the spaces $H^n_{f,p}(\mathbb{K}^n)$ for $p \neq n-1$. We consider the application

$$d_f^{(n-q)} : \Omega^{n-1}(\mathbb{K}^n) \longrightarrow \Omega^n(\mathbb{K}^n) \quad \alpha \longmapsto fd\alpha - (q-1)df \wedge \alpha$$

with $\underline{q \neq 1}$ (note that if $q = n$ then we obtain the space $H^n_{NP}(M, \Lambda)$ and if $q = 2$ then we have $H^2_\Lambda(\mathbb{K}^n)$).
We denote $\mathcal{I}^n = \{df \wedge \alpha; \alpha \in \Omega^{n-1}(\mathbb{K}^n)\}$. It is clear that $\mathcal{I}^n \simeq I_f$ (recall that I_f is the ideal of $\mathcal{F}(\mathbb{K}^n)$ spanned by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1}$) and that $\Omega^n(\mathbb{K}^n)/\mathcal{I}^n \simeq Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$.
We put $\sigma = i_W\omega$ (recall that $W = w_1x_1\frac{\partial}{\partial x_1} + \ldots + w_nx_n\frac{\partial}{\partial x_n}$ and that ω is the standard volume form on $\mathbb{K}^n$). Note that σ is a quasihomogeneous $(n-1)$-form of degree $\sum_i w_i$ and that $dg \wedge \sigma = (W.g)\omega$ if $g \in \mathcal{F}(\mathbb{K}^n)$.
If $\alpha \in \Omega^{n-1}(\mathbb{K}^n)$, we will use the notation $\underline{\mathrm{div}(\alpha)}$ for $d\alpha = \mathrm{div}(\alpha)\omega$; for example, $\mathrm{div}(\sigma) = \sum_i w_i$. Note that if α is quasihomogeneous then $\mathrm{div}(\alpha)$ is quasihomogeneous of degree $\deg\alpha - \sum_i w_i$.

Lemma 3.12. *1- If the ∞-jet at 0 of γ doesn't contain a component of degree $q\mathrm{N}$ (in particular if $q \leq 0$) then $\gamma \in B^n_{f,n-q}(\mathbb{K}^n) \Leftrightarrow \gamma \in \mathcal{I}^n$.*

2- If γ is a quasihomogeneous n-form of degree $q\mathrm{N}$ then $\gamma \in B^n_{f,n-q}(\mathbb{K}^n) \Rightarrow \gamma \in \mathcal{I}^n$.

Proof : If $\gamma = fd\alpha - (q-1)df \wedge \alpha \in B^n\big(d_f^{(n-q)}\big)$ where $\alpha \in \Omega^{n-1}$ then $\gamma = df \wedge \beta$ with $\beta = -(q-1)\alpha + \frac{\mathrm{div}(\alpha)}{\mathrm{N}}\sigma$. This shows the second claim and the first part of the first one.
Now we prove the reverse of the first claim.
Formal case : Let $\gamma = \sum_{i>0} \gamma^{(i)}$ and $\beta = \sum \beta^{(i-\mathrm{N})}$ (with $\gamma^{(i)}$ of degree i, $\gamma^{(q\mathrm{N})} = 0$ and $\beta^{(i-\mathrm{N})}$ of degree $i-\mathrm{N}$) such that $\gamma = df \wedge \beta$. If we put $\alpha = \frac{-1}{q-1}\beta + \sum_i \frac{\mathrm{div}(\beta^{(i-\mathrm{N})})}{(q-1)(i-q\mathrm{N})}\sigma$, we have $d_f^{(n-q)}(\alpha) = \gamma$.
Analytical case : If β is analytic at 0, the function $\mathrm{div}(\beta)$ is analytic too and since $\lim_{i\to+\infty} \frac{1}{i-q\mathrm{N}} = 0$, the (n-1)-form defined above is also analytic at 0.
$\mathcal{C}^\infty$ case : We suppose that $\gamma = df \wedge \beta$. If we denote $\tilde{\gamma} = j_0^\infty\big(\gamma\big)$ then there exists a formal (n-1)-form $\tilde{\alpha}$ such that $\tilde{\gamma} = fd\tilde{\alpha} - (q-1)df \wedge \tilde{\alpha}$. Let α be a $\mathcal{C}^\infty$-(n-1)-form such that $\tilde{\alpha} = j_0^\infty(\alpha)$. This form verifies $fd\alpha - (q-1)df \wedge \alpha = \gamma + \varepsilon$ where ε is flat at 0. Since $B^n_{f,n-q}(\mathbb{K}^n) \subset \mathcal{I}^n$, $\varepsilon \in \mathcal{I}^n$ so that $\varepsilon = df \wedge \mu$ where μ is flat at 0. Let $g \in \mathcal{F}(\mathbb{K}^n)$ be such that $W.g - \big((q-1)\mathrm{N} - \sum w_i\big)g = \frac{\mathrm{div}(\mu)}{q-1}$ (lemma 3.3). Then the form $\theta = \frac{-1}{q-1}\mu + g\sigma$ verifies $d_f^{(n-q)}(\theta) = \varepsilon$. □

Remark 3.13. 1- This lemma gives $B^n_{f,n-q}(\mathbb{K}^n) \subset \mathcal{I}^n$. Thus, there is a surjection from $H^n_{f,n-q}(\mathbb{K}^n)$ onto Q_f. Therefore, if f is not of finite codimension then $H^n_{f,n-q}(\mathbb{K}^n)$ is a infinite-dimensional vector space.
2- According to this lemma, if γ is in $\mathcal{I}^n$ then there exits a quasihomogeneous n-form θ, of degree $q\mathrm{N}$, such that $\gamma + \theta \in B^n_{f,n-q}(\mathbb{K}^n)$.

The first claim of this lemma allows us to state the following theorem.

Theorem 3.14. *If $q \le 0$ then $H^n_{f,n-q}(\mathbb{K}^n) \simeq Q_f$.*

Now we suppose that $\underline{q > 1}$.

Lemma 3.15. *Let $\alpha \in \Omega^k(\mathbb{K}^n)$ and $p \in \mathbb{Z}$. Then $fd_f^{(p)}(\alpha) = d_f^{(p-1)}(f\alpha)$.*

Proof : Obvious.

Lemma 3.16. *1- Let $q > 2$. If $\alpha \in \Omega^n(\mathbb{K}^n)$ is quasihomogeneous of degree $(q-1)\mathrm{N}$ and verifies $f\alpha \in B^n_{f,n-q}(\mathbb{K}^n)$ then $\alpha \in B^n_{f,n-q+1}(\mathbb{K}^n)$.*
2- If α is quasihomogeneous of degree N with $f\alpha \in B^n_{f,n-2}(\mathbb{K}^n)$ then $\alpha = 0$.

Proof : 1- We suppose that $\alpha = g\omega$ with $g \in \mathcal{F}(\mathbb{K}^n)$ quasihomogeneous of degree $(q-1)\mathrm{N} - \sum w_i$. We have $fg\omega = fd\beta - (q-1)df \wedge \beta$ where β is a quasihomogeneous (n-1)-form of degree $(q-1)\mathrm{N}$.
If we put $\theta = -(q-1)\beta + \frac{\mathrm{div}(\beta)-g}{\mathrm{N}}\sigma$ then $df \wedge \theta = 0$, and so $\theta = df \wedge \gamma$ where γ is a quasihomogeneous (n-2)-form of degree $(q-2)\mathrm{N}$. Consequently $\beta = \frac{-1}{q-1}df \wedge \gamma + \frac{\mathrm{div}(\beta)-g}{(q-1)\mathrm{N}}\sigma$. Now, a computation shows that $fd\beta - (q-1)df \wedge \beta = \frac{1}{q-1}fdf \wedge d\gamma$ i.e. $f\alpha = \frac{1}{q-1}fdf \wedge d\gamma$.
Therefore, $\alpha = \frac{1}{q-1}df \wedge d\gamma = \frac{1}{q-1}d_f^{(n-q+1)}\big(\frac{-1}{q-2}d\gamma\big)$.
2- As in 1- (with $q = 2$), we have $f\alpha = fg\omega = d_f^{(n-2)}(\beta)$ with $\deg g = \mathrm{N}$ and $\deg \beta = \mathrm{N}$. We put $\theta = -\beta + \frac{\mathrm{div}(\beta)-g}{\mathrm{N}}\sigma$.

If $\theta \neq 0$ then $\theta = df \wedge \gamma$ where γ is a quasihomogeneous (n-2)-form of degree 0 which is not possible. So, $\theta = 0$ i.e. $\beta = \frac{\mathrm{div}(\beta)-g}{\mathrm{N}}\sigma$.
We deduce that $fd\beta - df \wedge \beta = 0$ i.e. $\alpha = 0$. $\square$

Let $\mathcal{B}$ be a monomial basis of Q_f (for the existence of such a basis, see [AGV]). We denote r_j $(j = 2, \dots, q-1)$ the number of monomials of $\mathcal{B}$ whose degree is $j\mathrm{N} - \sum w_i$ (this number doesn't depend on the choice of $\mathcal{B}$). We also denote s the dimension of the space of quasihomogeneous polynomials of degree $\mathrm{N} - \sum w_i$ and c the codimension of f.

Theorem 3.17. *Let $\alpha \in \Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials $h_1, \dots, h_q$ (possibly zero) such that*
- *h_1 is quasihomogeneous of degree $\mathrm{N} - \sum w_i$,*
- *$h_j (2 \leq j \leq q-1)$ is a linear combination of monomials of $\mathcal{B}$ of degree $j\mathrm{N} - \sum w_i$,*
- *h_q is a linear combination of monomials of $\mathcal{B}$ and*

$$\alpha = (h_q + fh_{q-1} + \dots + f^{q-1}h_1)\omega \mod B^n_{f,n-q}(\mathbb{K}^n)\,.$$

In particular, the dimension of $H^n_{f,n-q}(\mathbb{K}^n)$ is $c + r_{q-1} + \dots + r_2 + s$.

Proof : *Existence* : We suppose that $\alpha = g\omega$ with $g \in \mathcal{F}(\mathbb{K}^n)$. There exists h_q, a linear combination of the monomials of $\mathcal{B}$, such that $g = h_q \mod I_f$. So, according to lemma 3.12 (see the former remark), $g\omega = h_q\omega + df \wedge \beta \mod B^n_{f,n-q}(\mathbb{K}^n)$ where β is a quasihomogeneous (n-1)-form of degree $(q-1)\mathrm{N}$.
Consequently, $g\omega = h_q\omega + \frac{1}{q-1}fd\beta - \frac{1}{q-1}[fd\beta - (q-1)df \wedge \beta] \bmod B^n_{f,n-q}(\mathbb{K}^n)$ so, we can write

$$g\omega = h_q\omega + fg_{q-1}\omega \mod B^n_{f,n-q}(\mathbb{K}^n)$$

with $\deg g_{q-1} = (q-1)\mathrm{N} - \sum w_i$.
In the same way,

$$g_{q-1}\omega = h_{q-1}\omega + fg_{q-2}\omega \mod B^n_{f,n-q+1}(\mathbb{K}^n)$$

where h_{q-1} is a linear combination of the monomials of $\mathcal{B}$ of degree $(q-1)\mathrm{N} - \sum w_i$ and g_{q-2} is quasihomogeneous of degree $(q-2)\mathrm{N} - \sum w_i \dots$

$$\dots \qquad \dots$$

... and

$$g_2\omega = h_2\omega + fh_1\omega \mod B^n_{f,n-2}(\mathbb{K}^n)$$

where h_2 is a linear combination of the monomials of $\mathcal{B}$ of degree $2\mathrm{N} - \sum w_i$ and h_1 is quasihomogeneous of degree $\mathrm{N} - \sum w_i$.
Using lemma 3.15, we get

$$\alpha = g\omega = h_q + h_{q-1} + f^2h_{q-2} + \dots + f^{q-1}h_1\omega \mod B^n\big(d_f^{(n-q)}\big)\,.$$

Unicity : Let $g = h_q + fh_{q-1} + \dots + f^{q-1}h_1$ with $h_1, \dots, h_q$ as in the statement of the theorem. We suppose that $g\omega \in B^n_{f,n-q}(\mathbb{K}^n)$. Then $g\omega \in \mathcal{I}^n$ i.e. $g \in I_f$. But since $fh_{q-1} + \dots + f^{q-1}h_1 \in I_f$ (because $f \in I_f$) we have $h_q \in I_f$ and so $h_q = 0$.
Now , according to lemma 3.16, $(h_{q-1} + fh_{q-2} + \dots + f^{q-2}h_1)\omega$ is in $B^n_{f,n-q+1}(\mathbb{K}^n)$ and so, in the same way, $h_{q-1} = 0$.
This way, we get $h_q = h_{q-1} = \dots = h_2 = 0$ and $fh_1\omega \in B^n_{f,n-2}(\mathbb{K}^n)$. Lemma 3.16 gives $h_1 = 0$. $\square$

This theorem allows us to give the dimension of the spaces $H^n_{NP}(\mathbb{K}^n,\Lambda)$ and $H^2_\Lambda(\mathbb{K}^n)$.

Corollary 3.18. *Let $\alpha\in\Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials $h_1,\dots,h_n$ (possibly zero) such that*
- *h_1 is quasihomogeneous of degree $\mathrm{N}-\sum w_i$,*
- *$h_j (2\le j\le n-1)$ is a linear combination of monomials of $\mathcal{B}$ of degree $j\mathrm{N}-\sum w_i$,*
- *h_n is a linear combination of monomials of $\mathcal{B}$ and*

$$\alpha=(h_n+fh_{n-1}+\ldots+f^{n-1}h_1)\omega \mod B^n_f(\mathbb{K}^n)\,.$$

In particular, the dimension of $H^n_{NP}(\mathbb{K}^n,\Lambda)$ is $c+r_{n-1}+\ldots+r_2+s$.

Corollary 3.19. *Let $\alpha\in\Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials h_1,h_2 (possibly zero) such that*
- *h_1 is quasihomogeneous of degree $\mathrm{N}-\sum w_i$,*
- *h_2 is a linear combination of monomials of $\mathcal{B}$ and*

$$\alpha=(h_2+fh_1)\omega \mod B^n_{f,n-2}(\mathbb{K}^n)\,.$$

In particular, the dimension of $H^2_\Lambda(\mathbb{K}^n)$ is $c+s$.

Remark 3.20. If $q=1$ then the space $H^n_{f,n-1}(\mathbb{K}^n)$ is $\Omega^n(\mathbb{K}^n)/f\Omega^n(\mathbb{K}^n)$ which is of infinite dimension.

3.6. **Computation of $H^{n-1}_{f,p}(\mathbb{K}^n)$.** We are going to compute the spaces $H^{n-1}_{f,p}(\mathbb{K}^n)$ with $p\neq n-1$. We consider the piece of complex

$$\Omega^{n-2}(\mathbb{K}^n)\longrightarrow\Omega^{n-1}(\mathbb{K}^n)\longrightarrow\Omega^n(\mathbb{K}^n)$$

with $d_f^{(n-q)}(\alpha)=fd\alpha-(q-2)df\wedge\alpha$ if $\alpha\in\Omega^{n-2}(\mathbb{K}^n)$,
and $d_f^{(n-q)}(\alpha)=fd\alpha-(q-1)df\wedge\alpha$ if $\alpha\in\Omega^{n-1}(\mathbb{K}^n)$ with $q\neq 1$.// Remember that if $q=n$ we obtain $H^{n-1}_{NP}(K^n,\Lambda)$ and if $q=2$ we have $H^1_\Lambda(\mathbb{K}^n)$.

Lemma 3.21. *If $\alpha\in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ then $\alpha=\frac{\mathrm{div}(\alpha)}{(q-1)\mathrm{N}}\sigma+df\wedge\beta$ with $\beta\in\Omega^{n-2}(\mathbb{K}^n)$ and so, $d\alpha$ verifies $\mathcal{L}_W(d\alpha)-(q-1)\mathrm{N}d\alpha=(q-1)\mathrm{N}df\wedge d\beta$.*

Proof : It is sufficient to notice that $df\wedge\big(\alpha-\frac{\mathrm{div}(\alpha)}{(q-1)\mathrm{N}}\sigma\big)=0$ (proposition 3.3). For the second claim, we have $(q-1)\mathrm{N}d\alpha=\big(W.\mathrm{div}(\alpha)+(\sum w_i)\mathrm{div}(\alpha)\big)\omega-(q-1)\mathrm{N}df\wedge d\beta$ and the conclusion follows. □

Lemma 3.22. *If $\alpha\in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ with $\mathrm{ord}\big(j_0^\infty(\alpha)\big)>(q-1)\mathrm{N}$ then α is cohomologous to a closed (n-1)-form. In particular, if $q\le 0$ then every $(n-1)$-cocycle for $d_f^{(n-q)}$ is cohomologous to a closed $(n-1)$-form.*

Proof : We have $\alpha=\frac{\mathrm{div}(\alpha)}{(q-1)\mathrm{N}}\sigma+df\wedge\beta$ (lemma 3.21) with

$$\mathcal{L}_W(d\alpha)-(q-1)\mathrm{N}d\alpha=(q-1)\mathrm{N}df\wedge d\beta\quad(*)\,.$$

Now, let $\gamma\in\Omega^{n-2}(\mathbb{K}^n)$ such that $\mathcal{L}_W\gamma-(q-2)\mathrm{N}\gamma=(q-1)\mathrm{N}\beta$ (γ exists because $\mathrm{ord}\big(j_0^\infty(\beta)\big)>(q-2)\mathrm{N}$, see lemma 3.2).
We have $\mathcal{L}_Wd\gamma-(q-2)\mathrm{N}d\gamma=(q-1)\mathrm{N}d\beta$. Thus $df\wedge d\gamma$ verifies

$$\mathcal{L}_W(df\wedge d\gamma)-(q-1)\mathrm{N}df\wedge d\gamma=(q-1)\mathrm{N}df\wedge d\beta\quad(**)\,.$$

¿From $(*)$ and $(**)$ we get $d\alpha = df \wedge d\gamma$.
Indeed, $\mathcal{L}_W(d\alpha - df \wedge d\gamma) = (q-1)\mathrm{N}(d\alpha - df \wedge d\gamma)$ but $d\alpha - df \wedge d\gamma$ is not quasi-homogeneous of degree $(q-1)\mathrm{N}$.
Now, if we put $\theta = \alpha - \frac{1}{q-1}\big(fd\gamma - (q-2)df \wedge \gamma\big)$, we have $d\theta = 0$ and $\theta = \alpha \mod B^{n-1}_{f,n-q}(\mathbb{K}^n)$.□

This lemma allows us to state the following theorem.

Theorem 3.23. *If we suppose that $q \leq 0$ then $H^{n-1}_{f,n-q}(\mathbb{K}^n) = \{0\}$.*

Proof : Let $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$. We can suppose (according to the former lemma) that $d\alpha = 0$. Thus we have $df \wedge \alpha = 0$. Proposition 3.4 gives then, $\alpha = df \wedge d\gamma$ with $\gamma \in \Omega^{n-3}(\mathbb{K}^n)$. Therefore, $\alpha = d_f^{(n-q)}\big(-\frac{1}{q-2}d\gamma\big)$. □

Now, we assume that $\underline{q > 1}$.

Lemma 3.24. *If $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ is a quasihomogeneous (n-1)-form whose degree is strictly lower than $(q-1)\mathrm{N}$ then α is cohomologous to a closed (n-1)-form.*

Proof : According to lemma 3.21, we have $\alpha = \frac{\mathrm{div}(\alpha)}{(q-1)\mathrm{N}}\sigma + df \wedge \beta$ and so,

$$d\alpha = \frac{(q-1)\mathrm{N}}{\deg(\alpha) - (q-1)\mathrm{N}} df \wedge d\beta\,.$$

We deduce that, if we put $\theta = \alpha - d_f^{(n-q)}\big(\frac{\mathrm{N}}{\deg(\alpha)-(q-1)\mathrm{N}}d\beta\big)$, we have $d\theta = 0$. □

Remark 3.25. A consequence of lemmas 3.22 and 3.24 is that, if $q > 1$, every cocycle $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ is cohomologous to a cocycle $\eta + \theta$ where η is in $Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ and is closed, and θ is quasihomogeneous of degree $(q-1)\mathrm{N}$.

Lemma 3.26. *Let $\alpha = g\sigma$ where g is a quasihomogeneous polynomial of degree $(q-1)\mathrm{N} - \sum w_i$. Then*
1- If $q > 2$ then, $\alpha \in B^{n-1}_{f,n-q}(\mathbb{K}^n) \Longleftrightarrow g\omega \in B^n_{f,n-q+1}(\mathbb{K}^n)$.
2- If $q = 2$, $\alpha \in B^{n-1}_{f,n-2}(\mathbb{K}^n) \Longleftrightarrow \alpha = 0$.

Proof :1- • We suppose that $\alpha \in B^{n-1}_{f,n-q}(\mathbb{K}^n)$ i.e. $\alpha = fd\beta - (q-2)df \wedge \beta$ with $\beta \in \Omega^{n-2}(\mathbb{K}^n)$. Then $d\alpha = (q-1)df \wedge d\beta$.
On the other hand, $d\alpha = (q-1)\mathrm{N}g\omega$ so $g\omega = \frac{1}{\mathrm{N}}df \wedge d\beta = d_f^{(n-q+1)}\big(-\frac{d\beta}{(q-2)\mathrm{N}}\big)$.
• Now we suppose that $g\omega \in B^n_{f,n-q+1}(\mathbb{K}^n)$ i.e. $g\omega = fd\beta - (q-2)df \wedge \beta$ where β is a quasihomogeneous (n-1)-form of degree $(q-2)\mathrm{N}$. We put $\gamma = i_W\beta \in \Omega^{n-2}(\mathbb{K}^n)$. We have

$$\begin{aligned}
d_f^{(n-q)}(\gamma) &= fd\gamma - (q-2)df \wedge \gamma \\
&= fd(i_W\beta) - (q-2)df \wedge (i_W\beta) \\
&= f\big(\mathcal{L}_W\beta - i_W d\beta\big) - (q-2)\big[-i_W(df \wedge \beta) + (i_W df) \wedge \beta\big] \\
&= f(q-2)\mathrm{N}\beta - i_W\big[fd\beta - (q-2)df \wedge \beta\big] - (q-2)(W.f)\beta \\
&= -i_W\big[fd\beta - (q-2)df \wedge \beta\big]\,.
\end{aligned}$$

Consequently, $d_f^{(n-q)}(\gamma) = -i_W(g\omega) = -g\sigma$.
2- If $\alpha = fd\beta$ where β is a quasihomogeneous (n-2)-form of degree $\deg\alpha - \mathrm{N} = 0$

then $\beta = 0$ and so $\alpha = 0$. $\square$

We recall that $\mathcal{B}$ indicates a monomial basis of Q_f. We adopt the same notations as for theorem 3.17.

Theorem 3.27. *We suppose that $q > 2$. Let $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$. There exist unique polynomials $h_1, \ldots, h_{q-1}$ (possibly zero) such that*

- *h_1 is quasihomogeneous of degree $\mathrm{N} - \sum w_i$,*
- *h_k $(k \geq 2)$ is a linear combination of monomials of $\mathcal{B}$ of degree $k\mathrm{N} - \sum w_i$ and*

$$\omega = (h_{q-1} + f h_{q-2} + \ldots + f^{q-2} h_1)\sigma \mod B^{n-1}_{f,n-q}(\mathbb{K}^n)\,.$$

In particular, the dimension of the space $H^{n-1}_{f,n-q}(\mathbb{K}^n)$ is $r_{q-1} + \ldots + r_2 + s$.

Proof : If $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ then α is cohomologous to $\eta + \theta$, where η is in $Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ and is closed, and θ is quasihomogeneous of degree $(q-1)\mathrm{N}$ (see remark 3.25).
The same proof as in theorem 3.23 shows that η is a cobord.
Now, we have to study θ. According to lemma 3.21, we can write $\theta = \frac{\mathrm{div}(\theta)}{(q-1)\mathrm{N}}\sigma + df \wedge \beta$ $(\beta \in \Omega^{n-2}(\mathbb{K}^n))$ with $\mathcal{L}_W(d\theta) - (q-1)\mathrm{N}d\theta = (q-1)\mathrm{N} df \wedge d\beta$. Since θ is quasihomogeneous of degree $(q-1)\mathrm{N}$, the former relation gives $df \wedge d\beta = 0$. Consequently, if we put $\gamma = df \wedge \beta$, proposition 3.4 gives $\gamma = df \wedge d\xi$.
Therefore, $\gamma = d_f^{(n-q)}\big(-\frac{1}{q-2} d\xi\big)$ and so $\theta = \frac{\mathrm{div}(\theta)}{(q-1)\mathrm{N}}\sigma \mod B^{n-1}_{f,n-q}(\mathbb{K}^n)$. The conclusion follows using lemma 3.26 and theorem 3.17. $\square$

Corollary 3.28. *We suppose that $q = n$. Let $\alpha \in Z^{n-1}_f(\mathbb{K}^n)$. There exist unique polynomials $h_1, \ldots, h_{n-1}$ (possibly zero) such that*

- *h_1 is quasihomogeneous of degree $\mathrm{N} - \sum w_i$,*
- *h_k $(k \geq 2)$ is a linear combination of monomials of $\mathcal{B}$ of degree $k\mathrm{N} - \sum w_i$ and*

$$\omega = (h_{n-1} + f h_{n-2} + \ldots + f^{n-2} h_1)\sigma \mod B^{n-1}_f(\mathbb{K}^n)\,.$$

In particular, the dimension of the space $H^{n-1}_{NP}(\mathbb{K}^n, \Lambda)$ is $r_{n-1} + \ldots + r_2 + s$.

Remark 3.29. If $q = 2$, the description of the space $H^{n-1}_{f,n-2}(\mathbb{K}^n)$ (and so $H^1_\Lambda(\mathbb{K}^n)$) is more difficult. It is possible to show that this space is not of finite dimension. Indeed, let us consider the case $n = 3$ in order to simplify (but it is valid for any $n \geq 3$). We put $\alpha = g\big(\frac{\partial f}{\partial x} dx \wedge dz + \frac{\partial f}{\partial y} dy \wedge dz\big)$ where g is a function which depends only on z. We have $d\alpha = 0$ and $df \wedge \alpha = 0$ so $\alpha \in Z^{n-1}_{f,n-2}(\mathbb{K}^n)$ but $\alpha \notin B^n_{f,n-2}(\mathbb{K}^n)$ because f doesn't divide α.

We can yet give more precisions on the space $H^{n-1}_{f,n-2}(\mathbb{K}^n)$.

Theorem 3.30. *Let E be the space of (n-1)-forms $h\sigma$ where h is a quasihomogeneous polynomial of degree $\mathrm{N} - \sum w_i$, and F the quotient of the vector space $\{df \wedge d\gamma; \gamma \in \Omega^{n-3}(\mathbb{K}^n)\}$ by the subspace $\{df \wedge d(f\beta); \beta \in \Omega^{n-3}(\mathbb{K}^n)\}$.
Then $H^{n-1}_{f,n-2}(\mathbb{K}^n) = E \oplus F$.*

Proof : Let α in $Z^{n-1}_{f,n-2}(\mathbb{K}^n)$.
According to remark 3.25, there exixt a closed $(n-1)$-form η with $\eta \in Z^{n-1}_{f,n-2}(\mathbb{K}^n)$ and a quasihomogeneous $(n-1)$-form θ, such that α is cohomologous to $\eta + \theta$.
We have (lemma 3.21) $\theta = \frac{\mathrm{div}(\theta)}{\mathrm{N}}\sigma + df \wedge \beta$ with β quasihomogeneous of degree

0 which is possible only if $\beta \neq 0$. So, $\theta = g\sigma$ where g is a quasihomogeneous polynomial of degree $\mathrm{N} - \sum w_i$. Lemma 3.26 says that $\theta \in B^{n-1}_{f,n-2}(\mathbb{K}^n)$ if and only if $\theta = 0$.

Now we study η. Proposition 3.4 gives $\eta = df \wedge d\gamma$ where γ is a $(n-3)$-form. If we suppose that $\eta \in B^{n-1}_{f,n-2}(\mathbb{K}^n)$ then $df \wedge d\gamma = fd\xi$ with $\xi \in \Omega^{n-2}(K^n)$ and so, $df \wedge d\xi = 0$. Now we apply proposition 3.4 to $d\xi$ and we obtain $d\xi = df \wedge d\beta$ with $\beta \in \Omega^{n-3}(\mathbb{K}^n)$. Consequently, $df \wedge d\gamma = fdf \wedge d\beta$ which implies that $d\gamma = fd\beta + df \wedge \mu$ with $\mu \in \Omega^{n-3}(\mathbb{K}^n)$, and so $d\gamma = d(f\beta) + df \wedge \nu$ with $\nu \in \Omega^{n-3}(\mathbb{K}^n)$. Therefore, $\eta \in B^{n-1}_{f,n-2}(\mathbb{K}^n) \Leftrightarrow \eta = df \wedge d(f\beta)$. $\square$

3.7. **Summary.** It is time to sum up the results we have found.

The cohomology $H^\bullet_f(\mathbb{K}^n)$ (and so the Nambu-Poisson cohomology $H^\bullet_{NP}(\mathbb{K}^n, \Lambda)$) has been entirely computed (see theorems 3.6, 3.8, 3.11, and corollaries 3.18 and 3.28) :

The spaces of this cohomology are of finite dimension and only the "extremal" ones (i.e H^0, H^1, H^{n-1} and H^n) are possibly different to $\{0\}$. The spaces $H^0_{NP}(\mathbb{K}^n, \Lambda)$ and $H^1_{NP}(\mathbb{K}^n, \Lambda)$ are always of dimension 1. The dimensions of the spaces $H^{n-1}_{NP}(\mathbb{K}^n, \Lambda)$ and $H^n_{NP}(\mathbb{K}^n, \Lambda)$ depend on the one hand on the type of the singularity of Λ (via the role played by Q_f), and on the other hand, on the "polynomial nature" of Λ.

Concerning the cohomology $H^\bullet_{f,n-2}(\mathbb{K}^n)$, we have computed H^n, i.e. $H^n_\Lambda(\mathbb{K}^n)$ (see corollary 3.19) and we have given a sketch of description of H^{n-1} (see theorem 3.30). We have also computed the spaces $H^0_{f,n-2}(\mathbb{K}^n)$ (theorem 3.6) and $H^k_{f,n-2}(\mathbb{K}^n)$ (theorem 3.8) for $k \neq n-2, n-1$, but these spaces are not particularly interesting for our problem.

The space $H^2_\Lambda(\mathbb{K}^n)$, which describes the infinitesimal deformations of Λ is of finite dimension and its dimension has the same property as the dimension of $H^n_{NP}(\mathbb{K}^n, \Lambda)$. On the other hand, the space $H^1_\Lambda(\mathbb{K}^n)$ which is the space of the vector fields preserving Λ modulo the Hamiltonian vector fields, is not of finite dimension.

It is interesting to compare the results we have found on these two cohomologies with the ones given in [Mo] on the computation of the Poisson cohomology in dimension 2.

Finally, if $p \neq 0, n-2, n-1$ we have computed the spaces $H^0_{f,p}(\mathbb{K}^n)$, $H^{n-1}_{f,p}(\mathbb{K}^n)$, $H^n_{f,p}(\mathbb{K}^n)$ and $H^k_{f,p}(\mathbb{K}^n)$ with $k \neq p, p+1$.

If $p = n-1$ we have computed the spaces $H^0_{f,n-1}(\mathbb{K}^n)$ and $H^k_{f,n-1}(\mathbb{K}^n)$ for $2 \leq k \leq n-2 \quad k \neq p, p+1$ (the space $H^n_{f,n-1}(\mathbb{K}^n)$ is of infinite dimension).

4. Examples

In this section, we will explicit the cohomology of some particular germs of n-vectors.

4.1. **Normal forms of n-vectors.** Let $\Lambda = f\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ be a germ at 0 of n-vectors on $\mathbb{K}^n$ ($\underline{n \geq 3}$) with f of finite codimension (see the beginning of section 3) and $f(0) = 0$ (if $f(0) \neq 0$, then the local triviality theorem, see [AlGu], [G] or [N2], allows us to write, up to a change of coordinates, that $\Lambda = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$).

Proposition 4.1. *If 0 is not a critical point for f then there exist local coordinates $y_1, \ldots, y_n$ such that*

$$\Lambda = y_1 \frac{\partial}{\partial y_1} \wedge \ldots \wedge \frac{\partial}{\partial y_n} .$$

Proof : A similar proposition is shown for instance in [Mo] in dimension 2. The proof can be generalized to the n-dimensional ($n \geq 3$) case.

Now we suppose that 0 is a critical point of f. Moreover, we suppose that the germ f is **simple**, which means that a sufficiently small neighbourhood (with respect to Whitney's topology; see [AGV]) of f intersects only a finite number of R-orbits (two germs g and h are said R-equivalent if there exits φ, a local diffeomorphism at 0, such that $g = h \circ \varphi$). Simple germs are those who present a certain kind of stability under deformation.
The following theorem can be found in [A] with only sketches of the proofs. In [Mo], a similar theorem (in dimension 2) is proved and the demonstration can be adapted here.

Theorem 4.2. *Let f be a simple germ at 0 of finite codimension. Suppose that f has at 0 a critical point with critical value 0. Then there exist local coordinates $y_1, \ldots, y_n$ such that the germ $\Lambda = f \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ can be written, up to a multiplicative constant, $g \frac{\partial}{\partial y_1} \wedge \ldots \wedge \frac{\partial}{\partial y_n}$ where g is in the following list.*

$$\begin{array}{lll} A_k & : & y_1^{k+1} \pm y_2^2 \pm \ldots \pm y_n^2 \quad k \geq 1 \\ D_k & : & y_1^2 y_2 \pm y_2^{k-1} \pm y_3^2 \pm \ldots \pm y_n^2 \quad k \geq 4 \\ E_6 & : & y_1^3 + y_2^4 \pm y_3^2 \pm \ldots \pm y_n^2 \\ E_7 & : & y_1^3 + y_1 y_2^3 \pm y_3^2 \pm \ldots \pm y_n^2 \\ E_8 & : & y_1^3 + y_2^5 \pm y_3^2 \pm \ldots \pm y_n^2 \end{array}$$

Proposition 4.1 and theorem 4.2 describe most of the germs at 0 of n-vectors on $\mathbb{K}^n$ vanishing at 0.
We can notice that the models given in the former list are all quasihomogeneous polynomials; which justifies the assumption we made in section 2.

4.2. **Some examples.** 1- The regular case : $f(x_1, \ldots, x_n) = x_1$.
It is easy to see that $Q_f = \{0\}$ and that f is quasihomogeneous of degree $\mathrm{N} = 1$, with respect to $w_1 = \ldots = w_n = 1$. We have $\mathrm{N} - \sum w_i < 0$, so $H_f^0(\mathbb{K}^n) \simeq \mathbb{K}$, $H_f^1(\mathbb{K}^n) = \mathbb{K}.dx_1$ and $H_f^k(\mathbb{K}^n) = \{0\}$ for any $k \geq 2$.

2- Non degenerate singularity: $f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$ with $n \geq 3$.
We have $\mathrm{N} = 2$ and $w_1 = \ldots = w_n = 1$. The space Q_f is isomorphic to $\mathbb{K}$ and is spanned by the constant germ 1, which is of degree 0.
We deduce that $H_f^0(\mathbb{K}^n) \simeq \mathbb{K}$, $H_f^1(\mathbb{K}^n) = \mathbb{K}.(x_1 dx_1 + \ldots + x_n dx_n)$ and $H_f^k = \{0\}$ for $2 \leq k \leq n-2$.
In order to describe the spaces $H_f^{n-1}(\mathbb{K}^n)$ and $H_f^n(\mathbb{K}^n)$, we look for an integer $k \in \{1, \ldots, n-1\}$ such that $k\mathrm{N} - \sum w_i = \deg 1$ i.e. $2k - n = 0$.
Therefore,

if n is even then $\{\omega, f^{\frac{n}{2}}\omega\}$ is a basis of $H^n_f(\mathbb{K}^n)$ and $H^{n-1}_f(\mathbb{K}^n)$ is spanned by $\{f^{\frac{n}{2}-1}\sigma\}$

if n is odd then $H^{n-1}_f(\mathbb{K}^n) = \{0\}$ and the space $H^n_f(\mathbb{K}^n)$ is spanned by $\{\omega\}$.

We recall that $\omega = dx_1 \wedge \ldots \wedge dx_n$ and

$$\sigma = i_W\omega = \sum_{i=1}^{n}(-1)^{i-1}x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\,.$$

3- The case A_2 with $n = 3$: $f(x_1, x_2, x_3) = x_1^3 + x_2^2 + x_3^2$.
Here, $w_1 = 2$, $w_2 = w_3 = 3$ and $\mathrm{N} = 6$. Thus, $\mathrm{N} - \sum w_i = -2$, $2\mathrm{N} - \sum w_i = 4$ and $3\mathrm{N} - \sum w_i = 10$.
Moreover, $\mathcal{B} = \{1, x_1\}$ is a monomial basis of Q_f. But as $\deg 1 = 0$ and $\deg x_1 = 3$, we have:

$$H^0_f(\mathbb{K}^3) \simeq \mathbb{K}\,, H^1_f(\mathbb{K}^3) = \mathbb{K}.(3x_1dx_1 + 2x_2dx_2 + 2x_3dx_3)$$
$$\text{and} \quad H^2_f(\mathbb{K}^3) = H^3_f(\mathbb{K}^3) = \{0\}\,.$$

4- The case D_5 with $n = 4$: $f(x_1, x_2, x_3, x_4) = x_1^2x_2 + x_2^4 + x_3^2 + x_4^2$.
We have $w_1 = 3$, $w_2 = 2$, $w_3 = w_4 = 4$ and $\mathrm{N} = 8$ then $\mathrm{N} - \sum w_i = -5$, $2\mathrm{N} - \sum w_i = 3$, $3\mathrm{N} - \sum w_i = 11$ and $4\mathrm{N} - \sum w_i = 19$.
Now, $\mathcal{B} = \{1, x_1, x_2, x_2^2, x_2^3\}$ is a monomial basis of Q_f. Here, $\deg 1 = 0$, $\deg x_1 = 3$, $\deg x_2 = 2$, $\deg x_2^2 = 4$ and $\deg x_2^3 = 6$. Thus, the only element of $\mathcal{B}$ whose degree is of type $k\mathrm{N} - \sum w_i$ is x_1.
Consequently,

$$H^0_f(\mathbb{K}^4) \simeq \mathbb{K}\,, H^1_f(\mathbb{K}^4) = \mathbb{K}\ .\big(2x_1x_2dx_1 + (x_1^2 + 4x_2^3)dx_2 + 2x_3dx_3 + 2x_4dx_4\big)\,,$$
$$H^2_f(\mathbb{K}^4) = \{0\}\,, H^3_f(\mathbb{K}^4) = \mathbb{K}.(x_1\sigma)$$

and $\{\omega, x_1\omega, x_2\omega, x_2^2\omega, x_2^3\omega, x_1f\omega\}$ is a basis of $H^4_f(\mathbb{K}^4)$.
Here, we have $W = 3x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + 4x_3\frac{\partial}{\partial x_3} + 4x_4\frac{\partial}{\partial x_4}$ and
$\sigma = 3x_1dx_2\wedge dx_3\wedge dx_4 - 2x_2dx_1\wedge dx_3\wedge dx_4 + 4x_3dx_1\wedge dx_2\wedge dx_4 - 4x_4dx_1\wedge dx_2\wedge dx_3$.

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