THE MEASURE OF QUADRATIC JULIA-LAVAURS SETS WITH A BOUNDED TYPE VIRTUAL SIEGEL DISK IS ZERO

by

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We shall prove here what was called hypothesis 3 in the author's thesis [**Ch1**], page 159.

1. Theorem

1.1. In short. — Let p/q be an irreducible fraction ($q > 0$). Let us define the polynomial $P(z) = e^{i2\pi p/q}z + z^2$. It is well known that the parabolic fixed point $z = 0$ is the only non repelling periodic point of P, and moreover that it has precisely q repelling petals. The action of P on the (repelling) Écalle-Voronin cylinders is therefore transitive. We will choose one end of these cylinders and call it ν. Let $\omega \in \mathbb{R}$ be any bounded type irrational. The theory of *parabolic enrichment* then associates to this data a unique Julia-Lavaurs set L with a virtual Siegel disk with center ν and virtual multiplier $\exp(i2\pi\omega)$.

Theorem 1. —

$$\operatorname{leb} L = 0.$$

It shall be noted that we will not make use of the classification of Fatou components of Julia-Lavaurs sets.

1.2. More details. — For typesetting system reasons, we will note $\operatorname{int}(X)$ the interior of a set X.

Let K be the filled-in Julia set, and J the Julia set, of

$$P(z) = e^{i2\pi p/q}z + z^2.$$

The theory of parabolic enrichment defines Lavaurs maps $g_\sigma : \operatorname{int}(K) \to \mathbb{C}$, where $\sigma \in \mathbb{C}$ is a complex parameter. They commute with P. For simplification, the Lavaurs map we will use is a modification of the usual Lavaurs map, that is described in [**Ch2**]. It is less symmetric (the relation $g_\sigma \circ P = P \circ g_\sigma$ is slightly modified, but $g_\sigma \circ P^q = P^q \circ g_\sigma$ still holds; moreover they define the same Julia-Lavaurs sets).

The filled-in Julia-Lavaurs set N consists in the complement of preimages of the basin of infinity by iterates of the Lavaurs maps:

$$\mathbb{C} \setminus N = \bigcup_{n\in\mathbb{N}} g_\sigma^{-n}(\mathbb{C} \setminus K)$$

One of many equivalent definitions of the Julia-Lavaurs set is the boundary (in $\mathbb{C}$) of the filled-in Julia-Lavaurs set:

$$L = \partial N.$$

This set is compact, completely invariant by P (meaning $P(L) = L$ and $P^{-1}(L) = L$), and contains J.

These sets depend only on $\sigma \bmod \mathbb{Z}$, as follows for instance from the property: $g_{\sigma+1} = P^q \circ g_\sigma$ and the fact K is completely invariant by P.

A closely related and more easily understandable function is the *horn map* h_σ. Let T_σ be the translation by σ in the complex plane. The Lavaurs and horn maps are related to each other by the following relations:

$$\begin{aligned} g_\sigma &= \psi_+ \circ T_\sigma \circ \phi_\div \\ h_\sigma &= T_\sigma \circ \phi_\div \circ \psi_+ \end{aligned}$$

Where $\phi_\div : \mathrm{int}(K) \to \mathbb{C}$ and $\psi_+ : \mathbb{C} \to \mathbb{C}$ are analytic functions. This implies for instance that ψ_+ is a semi-conjugacy from h_σ to g_σ: $\psi_+ \circ h_\sigma = g_\sigma \circ \psi_+$.

These relations are better explained on the following *non commuting* diagram:

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\phi_\div"] & \\ \mathbb{C}_+ \arrow[ur, "\psi_+"] & & \mathbb{C}_- \arrow[ll, "T_\sigma"] \end{tikzcd}$$

In this diagram, $\mathbb{C}$ is the complex plane where P lives, $\mathbb{C}_-$ is the plane of the attracting Fatou coordinates, $\mathbb{C}_+$ the plane of the repelling Fatou coordinates, $\phi_\div : \mathrm{int}(K) \to \mathbb{C}_-$ is the extended attracting Fatou coordinate, and $\psi_+ : \mathbb{C}_+ \to \mathbb{C}$ is the extended repelling Fatou coordinate. See [**Do**], then [**Ch2**] for their definitions. Then g_σ and h_σ are defined by following the arrows according to the illustration below:

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\phi_\div"] \arrow[d, phantom, "g_\sigma"] & \\ \mathbb{C}_+ \arrow[ur, "\psi_+"] & {} & \mathbb{C}_- \arrow[ll, "T_\sigma"] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C} \arrow[dr, "\phi_\div"] \arrow[d, phantom, "h_\sigma"] & \\ \mathbb{C}_+ \arrow[ur, "\psi_+"] & {} & \mathbb{C}_- \arrow[ll, "T_\sigma"] \end{tikzcd}$$

The preimage of L in repelling Fatou coordinates is a closed set, $\psi_+^{-1}(L)$, that we will note L'. The preimage of N will be noted N'. And we set $J' = \psi_+^{-1}(J)$ and $K' = \psi_+^{-1}(K)$, which are also closed. Then L' and N' can be characterized in terms of h_σ and K':

1. $\operatorname{def} h_\sigma = \operatorname{int}(K')$,
2. $\mathbb{C} \setminus N' = \bigcup_{n\in\mathbb{N}} h_\sigma^{-1}(\mathbb{C} \setminus K')$,
3. $L' = \partial N'$.

Proof. — Obviously, $\operatorname{def} h_\sigma = \psi_+^{-1}(\operatorname{int}(K))$. Claim (1) then follows from ψ_+ being a continuous and open map: $\psi_+^{-1}(\operatorname{int}(K)) = \operatorname{int}(\psi_+^{-1}(K))$. Claim (2) is trivial. Claim (3) also follows from ψ_+ being continuous and open. □

From the definition of h_σ, it follows that

$$h_\sigma^{-1}(N') = N' \cap \operatorname{def} h_\sigma.$$

Since h_σ is an open and continuous map, this yields

$$h_\sigma^{-1}(L') = L' \cap \operatorname{def} h_\sigma.$$

Let $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the canonical projection. The map h_σ turns out to commute with T_1. It therefore defines a quotient map from $\pi(\operatorname{int}(K'))$ to $\mathbb{C}/\mathbb{Z}$. The cylinder is identified to $\mathbb{C}^*$ via the exponential map $z \mapsto \exp(i2\pi z)$. Let $\mathbb{S}^2$ be the Riemann sphere: the embedding $\mathbb{C}^* \subset \mathbb{S}^2$, enables to consider the cylinder $\mathbb{C}/\mathbb{Z}$ completed by its two ends as a Riemann surface isomorphic to $\mathbb{S}^2$. We will note $+i\infty$ the upper end (corresponding to 0 in $\mathbb{S}^2$) and $-i\infty$ the lower end (corresponding to ∞ in $\mathbb{S}^2$).

The set $\operatorname{int}(K')$ is a neighborhood of both ends of the cylinder, and it turns out that h_σ has an analytic extension there, fixing each end, with non zero multiplier (see [**DoHu**], [**La**], [**Sh**]). These two multipliers are called the *virtual multipliers*.

Now, remind that the h_σ are defined as the following composition: $T_\sigma \circ \phi_{\div} \circ \psi_+$. Therefore, given $t \in \mathbb{C}$, changing h_σ into $h_{\sigma+t}$ changes the virtual multipliers by multiplying them by $\exp(i2\pi t)$ for the upper end and $\exp(-i2\pi t)$ for the lower end. This is why, given a rotation number ω and one of the two ends of the cylinder, there is only one value of σ mod 1 such that h_σ has virtual multiplier $\exp(i2\pi\omega)$ at this end. From now on, we will fix σ to this value.

2. Proof

The question of measure of the Julia set in the case of a quadratic polynomial with a bounded type Siegel disk has been first settled by Petersen and Lyubich. They use a quasiconformal model constructed from a Blaschke fraction. Then McMullen proved this Julia set has Hausdorff dimension < 2. These works have been further generalised.

In our case, points in the Julia set will be separated into classes, according to their behavior: P-type, J-type, R-type and F-type points. The first class will be handled by adapting McMullen's techniques to our setting: it turns out to be quite straightforward. The other classes are easier (in the author's opinion).

We will make use of the following notations. If $I \subset \mathbb{R}$ is an open interval, $\mathbb{C}_I$ is defined as $C_I = \mathbb{C} \setminus (\mathbb{R} \setminus I)$. It is an open subset of $\mathbb{C}$. If $U \subsetneq \mathbb{C}$ is a connected and simply connected open subset of $\mathbb{C}$, let d_U be the hyperbolic distance on U and $B_U(z,r)$ the hyperbolic ball of center z and radius r.

2.1. Remark about critical circle maps. — Let us recall the

Definition 1. — *Here, a* critical circle map *is a map $f : \mathbb{T} \to \mathbb{T}$ that is analytic, bijective, orientation preserving, and has one and only one critical point, of local degree* 3.

A reference on this is [**FaMe**], that generalise the work of [**Mm**], using [**Ya**].

Here, for simplicity, we chose to adapt [**Mm**] to prove that the set of P-type points has measure equal to 0. However, the techniques of Lyubich explained in [**Ya**] probably directly give it. The author ignores if [**FaMe**] does.

2.2. Reduction to a model. — To fix ideas, we will now assume that the end of the cylinder at which we put the Siegel disk is the upper end. All the discussion below works for the other case.

In [**Ch2**], the author defined a pre-model map β. It plays for h_σ the role that the Blaschke fraction $B = z^2 \frac{z-3}{1-3z}$ plays for the quadratic map $e^{i2\pi\omega}z + z^2$ in the Douady, Ghys, Herman (and others) surgery: in both cases one can modify by surgery the pre-model to obtain a model $\widetilde{\beta}$ (resp. $\widetilde{B}$). See [**Ch2**] for details.

We will use the following properties of β and $\widetilde{\beta}$: (the following list has redundancies)

1. $\operatorname{def}(\widetilde{\beta})$ is an open set that contains the closed upper half plane "$\operatorname{Im} z \geq 0$",
2. $\widetilde{\beta}$ is a continuous open map,
3. β is analytic, and real-symmetric (i.e. $\operatorname{def}(\beta)$ is invariant by the reflection $s : z \mapsto \overline{z}$, and β commutes with s),
4. $\mathbb{R} \subset \operatorname{def}(\beta)$,
5. the restrictions of $\widetilde{\beta}$ and β to the half-plane "$\operatorname{Im} z \leq 0$" coincide (i.e. they have the same set of definition and are equal on this set),
6. in particular, $\widetilde{\beta}$ is analytic below $\mathbb{R}$,
7. the restriction of $\widetilde{\beta}$ to $\mathbb{R}$ induces a critical circle map, with rotation number ω,
8. $\widetilde{\beta}$ induces a homeomorphism from $\mathbb{H}$ to $\mathbb{H}$,
9. $\operatorname{def}(\widetilde{\beta})$ is T_1 invariant and $\widetilde{\beta}$ commutes with T_1, and the same hold for β,
10. there is a quasiconformal map S, commuting with T_1, and conjugating $\widetilde{\beta}$ to h_σ, sending the upper half cylinder to the Siegel disk of h_σ associated to the upper end,
11. 0 is a critical value of $\widetilde{\beta}$.

The maps h_σ and $\widetilde{\beta}$ commute with T_1 and therefore induce dynamics on the cylinder. The map S commutes with T_1 and therefore induces a (quasiconformal) conjugacy between them, that fixes both ends of the cylinder. Let us call $\widehat{h}_\sigma$ the continuation (of the map induced by h_σ) that fixes both ends. Thus the map induced by $\widetilde{\beta}$ has also a continuation $\widehat{\beta}$ that fixes both ends. It is analytic at the lower end.

Since quasiconformal maps preserve sets of zero Lebesgue measure, it is enough to work on the model $\widetilde{\beta}$. If we define sets J'', K'', L'', N'' as the preimages of J', K', L', N' by S, we have an analogous characterization in terms of $\widetilde{\beta}$.

1. $\operatorname{def} \widetilde{\beta} = \operatorname{int}(K'')$,
2. $\mathbb{C} \setminus N'' = \bigcup_{n\in\mathbb{N}} \widetilde{\beta}^{-1}(\mathbb{C} \setminus K'')$,
3. $L'' = \partial N''$.

We will note b be the restriction of $\widetilde{\beta}$ to the half plane $H =$ "$\operatorname{Im} z < 0$": it is an analytic map from an open subset of H to $\mathbb{C}$.

2.3. Covering properties. — Let X, Y be Riemann surfaces. For an analytic function $f : X \to Y$, let $V_a(f)$ be its set of asymptotic values, and $V_c(f)$ its set of critical values. We recall that asymptotic values are points $y \in Y$ for which there exists a continuous function $\gamma : [0, +\infty[\to X$ going to X's infinity (i.e. eventually avoiding all compact) with $g(\gamma(t)) \underset{n\to+\infty}{\longrightarrow} y$. Note that the set of asymptotic values depends on which set Y is being considered: replacing Y by a Riemann surface that contains it happens to increase the set of asymptotic values of f. Taking $Y = \mathbb{C}$, we have

1. $V_a(h_\sigma) = \varnothing$,
2. $V_c(h_\sigma) = (\sigma + z_0) + \mathbb{Z}$.

and therefore,

1. $V_a(\beta) = \varnothing$,
2. $V_c(\beta) = \mathbb{Z}$.

We will use the following well known property:

Proposition 1. — *Assume $f : X \to Y$ is an analytic map between Riemann surfaces. If V is a simply connected connected open subset of Y that contains no asymptotic value and no critical value, then for all connected component U of $f^{-1}(V)$, $f : U \to V$ is an analytic isomorphism.*

2.4. About the cycles of $\widehat{\beta}$. — Apart from the upper end of the cylinder, the cycles of $\widehat{\beta}$ are all analytic (meaning $\widehat{\beta}$ is analytic in a neighborhood of the cycle) since they are either equal to the lower end or contained in the half plane "$\operatorname{Im} z < 0$".

Lemma 1. — *The lower end of the cylinder is repelling for $\widehat{\beta}$.*

Proof 1. — It is known that the product of multipliers of both ends of $\widehat{h}_\sigma$ (which is independent of σ) has modulus > 1 (see for instance [**BuEp**]). Therefore, since the upper end is neutral, the lower end is repelling. Now, since there is a topological characterization of repelling points, this implies $\widehat{\beta}$ is also repelling at this point. $\square$

Proof 2. — We can give a specific proof in our case: Let U be the connected component of the preimage of the half plane $H =$ "$\operatorname{Im} z$" < 0 whose projection by $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is a neighborhood of the lower end (hence $T_1(U) = U$). Then according to proposition 1, $\widetilde{\beta} : U \to L$ is an isomorphism. Therefore, $\widehat{\beta} : \pi(U) \cup \{-i\infty\} \to \pi(L) \cup \{-i\infty\}$

is an analytic isomorphism between two simply connected sets, fixing $-i\infty$. The first being a strict subset of the second, $-i\infty$ is repelling. $\square$

In fact, A. Epstein adapted Fatou-Shishikura's inequality to a class of maps including the horn maps h_τ, and since they possess only one critical value, they can have at most one non repelling cycle. We will not use this result here, but since in the case of $\widetilde{\beta}$ there is a very simple proof, we will mention it here:

Lemma 2. — *All the cycles of $\widehat{\beta}$ that are not the upper fixed point are repelling. Therefore the only non repelling cycle of $\widehat{h}_\sigma$ is the upper end.*

Proof. — Let n be the period, and $z \in H$ a point in the cycle. As in proof 2, the set H contains no critical value of $\widetilde{\beta}$, and proposition 1 implies $\widetilde{\beta}^n$ is an isomorphism from a simply connected set U with $z \in U \subsetneq H$, to H. $\square$

2.5. Definition of a piece: P_0. — Remind that 0 is a critical value of β, and therefore not a critical point (otherwise β would have integer rotation number on $\mathbb{R}$). Let $m \in \mathbb{Z}$ be such that $\beta(0) \in]m, m+1[$. Let $V = \mathbb{C}_{]m,m+1[}$. Let U be the connected component of $\beta^{-1}(V)$ containing 0. By covering properties of β, the restriction $\beta : U \to V$ is an analytic isomorphism. Let $H =$ "$\mathrm{Im}(z) < 0$". Let $P_0 = H \cap U$. From $0 \in U$ we know than $P_0 \neq \varnothing$. Therefore,

$$\widetilde{\beta} : P_0 \to H$$

is an isomorphism.

Lemma 3. — *The set $\widetilde{\beta}^{-1}(H) \setminus (P_0 + \mathbb{Z})$ is at positive distance to $\mathbb{R}$.*

Proof. — Recall that the picture is invariant by T_1. Now, β being analytic and having local degree 1 or 3 at points in $\mathbb{R}/\mathbb{Z}$, every point in $\mathbb{R}/\mathbb{Z}$ has a neighborhood W on which $W \cap \beta^{-1}(H) \cap H$ is contained in U mod $\mathbb{Z}$. The claim follows by compactness of $\mathbb{R}/\mathbb{Z}$. $\square$

Lemma 4. —

$$\sup_{z \in P_0} |\mathrm{Im}(z)| < +\infty$$

Proof. — (first proof) Since $\widehat{\beta}$ is defined at $-i\infty$, it implies there is a component U' of $\widetilde{\beta}^{-1}(H)$ that contains a lower half plane "$\mathrm{Im}\, z < -h_1$" for some $h_1 > 0$. It is therefore enough to prove that $U' \cap P_0 = \varnothing$. Since P_0 is also a component of $\widetilde{\beta}^{-1}(H)$, if it intersected U', it would be equal to U'. Therefore, P_0 would be invariant by T_1. Studying the situation at the real critical point $z = m+1$ of β shows that there would be points $z_0 \in P_0$ and $z_1 \in T_1(P_0)$ such that $\beta(z_0) = \beta(z_1)$, which contradicts injectivity of $\beta : P_0 \to H$.
(second proof) It is well known that $\pi(\mathbb{C} \setminus K'')$ is an annulus, separating the two ends of the cylinder. It does not intersect the closure of $\mathbb{H}$. It therefore separates P_0 from the lower end of the cylinder. The lemma follows. $\square$

2.6. Classification of points of L''. — What can happen to a point $z \in L''$ when it is iterated by $\widetilde{\beta}$? By definition, z never gets to $\mathbb{C} \setminus K''$. So either it is eventually mapped to $\partial K''$, in which case we will say z is of type J (Julia). Or all its iterates are defined. It also never gets to $\mathbb{H}$, otherwise it would belong to the interior of N'', and thus not to $L'' = \partial N''$. If it is eventually mapped to $\mathbb{R}$, then it must remain there, and we will say z is of type R (real). If it belongs infinitely many times to $H \setminus P_0 \bmod \mathbb{Z}$, we will say z is of type F (far). Otherwise, it eventualy lands *and* remains in $P_0 \bmod \mathbb{Z}$: it will be called of type P.

Therefore, we have decomposed

$$L'' = J_t \sqcup R_t \sqcup F_t \sqcup P_t$$

($\sqcup$ means a disjoint union), where X_t means the set of points $z \in L''$ of type X.

Let us recall that we note b be the restriction of $\widetilde{\beta}$ to the half plane $H =$ "$\operatorname{Im} z < 0$", and that b is analytic on its domain of definition.

2.7. Type J. — The following fact is now well known and is for instance a consequence of [**DeUr**] or [**Ly**]

Lemma 5. —

$$\operatorname{leb} J = 0.$$

Since ψ_+ is analytic, $J' = \psi_+^{-1}(J)$ has measure 0. Since S is quasiconformal, $J'' = S^{-1}(J')$ also has measure 0. Now,

$$J_t = \bigcup_{n \in \mathbb{N}} b^{-n}(J'').$$

Since b is analytic,

Proposition 2. —

$$\operatorname{leb} J_t = 0.$$

2.8. Type R. — Of course, $\mathbb{R}$ has measure 0. From

$$R_t = \bigcup_{n \in \mathbb{N}} b^{-n}(\mathbb{R})$$

(with the convention that $b^0(\mathbb{R}) = \mathbb{R}$) it follows that

Proposition 3. —

$$\operatorname{leb} R_t = 0.$$

2.9. Type F. — For rational maps, there is a theorem of Lyubich [**Ly**] stating that the set of points in the Julia set that do not tend to the postcritical set has Lebesgue measure equal to 0. Here we will give a simple adaptation to the specific case we are studying.

To see the relation with F_t, note that the postcritical set of $\widetilde{\beta} \bmod \mathbb{Z}$ is $\mathbb{R}/\mathbb{Z}$, and that there exists $\varepsilon > 0$ such that points of type F must infinitely often visit $H_\varepsilon =$ "$\operatorname{Im} z \leq -\varepsilon$" (as follows from lemma 3).

Let $H =$ "$\operatorname{Im} z < 0$" and let us call U the connected component of $\widetilde{\beta}^{-1}(H)$ that is a neighborhood of $-i\infty$ (when projected to $\mathbb{C}/\mathbb{Z}$). Let g be the restriction of $\widetilde{\beta}$ to U.

Let $z \in F_t$ and let us prove that z is not a Lebesgue density point of F_t, by providing arbitrarily small balls that do not intersect F_t, and whose radii are commensurable to the distance of their centers to z. They are provided by the classical method: taking univalent branches and applying Koebe's distortion theorem.

Let us introduce the *nest* of z: this is the sequence of *pieces* $P_n(z)$ where $P_n(z)$ is the connected component containing z of $\widetilde{\beta}^{-n}(H) = b^{-n}(H)$. This sequence is decreasing for inclusion. According to proposition 1, the sets $P_n(z)$ are open and simply connected and $b^n : P_n(z) \to H$ is an analytic isomorphism.

Lemma 6. — *For all $z \in L''$ not of type J, the distance from z to the boundary of $P_n(z)$ tends to 0 when $n \longrightarrow +\infty$.*

Proof. — Otherwise, the intersection A of the sets $P_n(z)$ would be a neighborhood of z, included in N'' (since all points $z' \in A$ has infinite orbit that never goes to the upper half plane). Therefore z would belong to $\operatorname{int}(N'')$ and thus not to $L'' = \partial N''$, which leads to a contradiction. □

Claim. — *Every point of U is eventually mapped out of U under iteration of b.*

Proof. — We defined g as the restriction of $\widetilde{\beta}$ to U. The univalent map $g^{-1} : H \to U$ commutes with T_1, and its quotient, completed by the lower end of the cylinder, is conjugated by $z \mapsto \exp(-2i\pi z)$ to an analytic isomorphism f from the unit disk $\mathbb{D}$ to an open subset W with compact closure in $\mathbb{D}$, with $f(0) = 0$. In that case, it is well known that the intersection of the images $f^n(\mathbb{D})$ is reduced to $\{0\}$. Therefore, $\bigcap_{n \in \mathbb{N}} g^{-n}(\mathbb{H}) = \varnothing$. □

Let now $z \in F_t$. The set $\mathbb{C} \setminus K''$ is open an contained in H. Its preimages by b^n are of course disjoint from F_t. Therefore, by Koebe's distortion theorem and lemma 6, it is enough to find some M such that, infinitely many times, the orbit of z passes at hyperbolic distance to $\mathbb{C} \setminus K''$ in H less than M:

$$(\forall z \in F_t)\ (\exists M > 0)\ (\forall N \in \mathbb{N})\ (\exists n \geq N)\ \operatorname{dist}_H(b^n(z), \mathbb{C} \setminus K'') < M.$$

The constant M will here be independent of z and be equal to the supremum of the hyperbolic distance in H to $\mathbb{C} \setminus K''$, of points in the set

$$B = H_\varepsilon \setminus g^{-1}(H_\varepsilon),$$

where $H_\varepsilon = \{z \in \mathbb{C} \mid \operatorname{Im} z \leq -\varepsilon\}$. This supremum is finite, because $\mathbb{C} \setminus K''$ is T_1-invariant and B has imaginary part bounded away from 0 and $-\infty$.

Now, let $m \geq N$ such that $b^m(z) \in H_\varepsilon$. By the claim(1), there is some $n \geq m$ such that $b^n(z) \in B$. This yields:

Proposition 4. —

$$\operatorname{leb} F_t = 0.$$

Remark. Since this works for all $\varepsilon > 0$, the measure of the set of points which do not tend to $\mathbb{R}$ is equal to 0: this is Lyubich's theorem in our particular case.

2.10. Type P**.** — Let

$$C = \{z \in P_0 \bmod \mathbb{Z} \mid \text{all iterates of } z \text{ belong to } P_0 \bmod \mathbb{Z}\}$$

then, by definition

$$P_t = L'' \cap \bigcup_{n\in\mathbb{N}} b^{-n}(C).$$

From $\widetilde{\beta}^{-1}(L'') = L'' \cap \operatorname{def} \widetilde{\beta}$, it follows that

$$b^{-1}(L'') = L'' \cap \operatorname{def} b.$$

Therefore(2),

$$P_t = \bigcup_{n\in\mathbb{N}} L'' \cap b^{-n}(C) = \bigcup_{n\in\mathbb{N}} b^{-n}(L'' \cap C).$$

Let us postpone to the next section the proof of

Lemma 7. —

$$\operatorname{leb} L'' \cap C = 0,$$

from which follows that

Proposition 5. —

$$\operatorname{leb} P_t = 0,$$

which will end the proof of theorem 1.

(1) The claim reads $\bigcap_{n\in\mathbb{N}} g^{-n}(H) = \varnothing$, which implies $\bigcap_{n\in\mathbb{N}} g^{-n}(H_\varepsilon) = \varnothing$. Therefore $H_\varepsilon = \bigcup_{n\in\mathbb{N}} g^{-n}(B)$.

(2) In fact, more is true (even if we will not use it): Lavaurs' versions of Sullivan's non-wandering theorem and of the classification of components, applied to h_σ, imply $C \subset L''$ (an therefore $\operatorname{int}(C) = \varnothing$). Indeed, by the absence of non-repelling cycles of $\widehat{h}$ except from the upper end, every component of $\operatorname{int}(N')$ is eventually mapped to the Siegel disk at the upper end. By the quasiconformal transformation S, this turns into an analogous statement for $\widehat{\beta}$.

2.11. Proof of lemma 7. — We will endow the lower half-plane H with the hyperbolic metrics given by

$$\frac{|dz|}{|\operatorname{Im} z|}.$$

Let us state the main lemma

Lemma 8. — *There exists $M > 1$ such that $\forall x \in \mathbb{R}$, $\forall s \in]0,1[$, there is a euclidean ball $B = B(z,r)$ with*

1. *$B \subset H$,*
2. *B eventually falls in $\mathbb{H}$ under iteration of b, and thus $B \cap C = \varnothing$,*
3. *$r > s/M$,*
4. *$|z - x| < sM$.*

Asumme we have fixed some height $h_0 > 0$. In terms of hyperbolic metrics in H, lemma 8 implies that for every point $z' \in H$ with $\operatorname{Im} z' > -h_0$, there is a hyperbolic ball B' which is eventually mapped to $\mathbb{H}$ under iteration of b, with hyperbolic diameter C_2 and contained in the hyperbolic ball of center z' and radius C_3. The constant C_2 depends on M. The constant C_3 depends on M and h_0.

Now let $h_0 = \sup_{z \in P_0} |\operatorname{Im}(z)|$. By lemma 4, $h_0 < +\infty$. For $z \in C$, let r_n be the distance from z to the boundary of the piece $P_n(z)$. For a given n, set $z' = b^n(z)$ and let B' be as above. If we pull back B' by the branch of b^{-n} mapping H to $P_n(z)$, then according to Koebe's distortion theorem, B' is mapped to a set which contains a euclidean ball B'' with center within distance $< M' r_n$ to z and diameter $> r_n/M'$. The constant M' depends on C_2, C_3, but not on n, z. Note that these balls B'' are eventually mapped to $\mathbb{H}$ under iteration of b, and thus do not intersect C. Now if $z \in C \cap L''$, then $r_n \longrightarrow 0$ (by lemma 6). The balls B'' then prevent z from being a density point of C. This proves lemma 7.

2.12. Proof of lemma 8. — For an interval I, $|I|$ will denote its length. Intevals I and J are called K-commensurable if and only if the quotient of their lengths lies between $1/K$ and K.

Let us introduce the *dynamical partition*: let f be a critical circle map with irrationnal rotation number θ and let $c \in \mathbb{R}/\mathbb{Z}$ be its critical point. Let p_n/q_n be the convergents of θ. The partition of $\mathbb{R}/\mathbb{Z}$ into intervals separated by the points $f^{-k}(c)$, $0 \leq k < q_n + q_{n+1}$ is called the *dynamical partition at level n*. Let us fix by convention that the intervals are of the form $[a, b[$. For $x \in \mathbb{R}/\mathbb{Z}$, we will note $I_n(x)$ the unique element of the dynamical partition at level n that contains x.

Le us recall the following (see [**Yo**])

Theorem (**Yoccoz**). — *The map $f : \mathbb{R}/Z \to \mathbb{R}/\mathbb{Z}$ is conjugated to the rotation $T_\theta : \mathbb{R}/Z \to \mathbb{R}/\mathbb{Z}$ (by an orientation preserving homeomorphism of $\mathbb{R}/\mathbb{Z}$).*

In particular, the backward orbit $f^{-k}(c)$ is dense in $\mathbb{R}/\mathbb{Z}$.

We will make use of the following theorem of Herman and Swiatek:

***Theorem* (Herman, Swiatek, real bounds).** — *For all critical circle map f with irrationnal rotation number, there exists $K > 1$ such that for all $n \in \mathbb{N}$, any two consecutive intervals of the dynamical partition at level n are K-commensurable.*

A well known corollary is the following:

Corollary 1. — *If, furthermore, the rotation number has bounded type, then there exists $K' > 1$ such that $\forall x \in \mathbb{R}$, $\forall \ell \in]0,1[$, $\exists n \in \mathbb{N}$ such that $|I_n(x)|$ is K'-commensurable to ℓ.*

The following lemma is purely geometric

Lemma 9. — *For all $K > 1$ there exists $M_0 > 0$ and $r_0 > 0$ such that the following holds. Assume we are given three intervals $[a,b[$, $[b,c[$ and $[c,d[$ in $\mathbb{R}$, and that touching intervals are K-commensurable. Let $U = \mathbb{C}_{]a,d[}$. Then,* for the hyperbolic metrics on *U, every cone based on a point $x \in [a,d]$, with central direction $-90°$ and opening $30°$, contains a hyperbolic ball $B = B_U(z,r)$, with radius $r = r_0$, and such that the hyperbolic diameter* $\mathrm{diam}_U([b,c] \cup B) < M_0$.

The proof is elementary. This would hold if the opening angle is replaced by any number between $0°$ and $180°$.

Lemma 10. — *There is some $h_1 > 0$ such that the set $P_0 + \mathbb{Z}$ avoids the interior of the triangle of vertices c, $c - a - ih_1$, and $c + a - ih_1$, where $c \in \mathbb{R}$ is a real critical point of β, and a is chosen so that the angle at vertex c is equal to $30°$.*

Proof. — This immediately follows from the critical point of β having local degree 3 and β sending $\mathbb{R}$ to $\mathbb{R}$ increasingly. □

Corollary 2. — *For the map β, there exists $M_2 > 0$ and $r_2 > 0$ such that $\forall n \geq 2$, for all intervals I of the dynamical partition at level n, there is a euclidean ball $B = B(z,r)$ eventually mapped to $\mathbb{H}$ under iteration of b, with $r > r_2|I|$ and $d(z,I) < M_2|I|$.*

Proof. — Let us note $I = [b,c[$ and let $[a,b[$ be the previous interval and $[c,d[$ the next one, in the dynamical partition at level n. Let us note $a = \beta^{-m_1}(u)$, $b = \beta^{-m_2}(u)$, $c = \beta^{-m_3}(u)$ and $d = \beta^{-m_4}(u)$, where u is the critical point of β on $\mathbb{R}/\mathbb{Z}$. We took $n \geq 2$, therefore $q_n + q_{n+1} \geq 5$, so there are at least 4 distinct points defining the dynamical partition at level n, so the points a, b, c, d are distinct. Thus the m_j are distinct. Let $m \in \mathbb{N}$ be the least(3) of m_1, m_2, m_3 and m_4. Let $[a',b'[$, $[b',c'[$ and $[c',d'[$ be the image of the three intervals by β^m. These are still three consecutive intervals of the dynamical partition at level n. (Indeed, the image by β of an interval I in the partition $\mathcal{P}$ created by the preimages of the critcal point, from order 0 up to $k-1$, is still in $\mathcal{P}$, unless the k-th preimage of the critical point belongs to $\mathrm{int}(I)$, which for $k = q_n + q_{n+1}$ implies that the critical point bounds I.) Therefore, $[a',b'[$ and $[c',d'[$ are K-commensurable to $[b',c'[$, where K is given by the real bounds. Let us consider, for the hyperbolic metrics on $U = \mathbb{C}_{]a',d'[}$, the hyperbolic ball B provided

(3) in fact, we could also take $m = \min(m_2, m_3)$

by lemma 9 for the cone based on the critical point. Now, either B is contained in the triangle provided by lemma 10, and we set $B' = B$. If it is not, then n must be less than some $n_0 \in \mathbb{N}$ (indeed, the size of the three intervals is bounded from above by a sequence that tends to 0 as n tends to infinity). We then take B' = the biggest hyperbolic disk contained in the triangle. Then, there exist constants M_1 and r_1 independant of n such that, in both cases, B' has hyperbolic radius (in U) $\geq r_1$ and $\mathrm{diam}_U(B' \cup [b', c']) < M_1$. Let us now apply the branch g of β^{-m} defined on U and sending $]a', d'[$ to $]a, d[$. This branch exists because U does not contain any singular value of β^m (otherwise, $]a, d[$ would contain a point of the form $f^{-k}(c)$ for some $0 \leq k < m$, which is not the case). The lemma follows using bounded distortion of g in $B_U(b', M_1)$, which contains both B' and $[b', c']$. □

Corollaries 1 and 2 give lemma 8.

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