QUASI SYMMETRIC CONJUGACY FROM CIRCLE DIFFEOMORPHISMS TO ROTATIONS AND APPLICATIONS TO SIEGEL SINGULAR DISKS, I (?)

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Very preliminary version

Translator's note: The source of this text from the middle of the 1980's is a photocopy of manuscripts of Herman, scanned and put online by Shishikura. I worked on them in june 2003, april 2005, june 2006 and june 2014. I take full responsibility for some modifications that I have made; they appear in green in the present document. They correspond either to minor corrections, or to omissions, or to unreadable parts of the scanned document. I may have introduced typographic or other kind of mistakes during the transcription. I rephrased some sentences to better fit in what I am used to see as written English in mathematical research articles. As I am not a native English speaker, some indulgence is asked for. I inserted personal comments in the form of footnotes starting with *TN* and indexed numerically, so as to make them easily distinguished from Herman's footnotes, which I indexed alphabetically. Last, I used the red color for parts of the text of which I am not certain.

Arnaud Chéritat.

Introduction

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From §1 to §13 we give a few generalities on the quasi symmetric conjugacy to rotations of circle diffeomorphisms. The main result is Theorem 1 in §1, which is essential for Theorem 2.

Theorem 2 allows to prove that if $f \in \mathcal{D}^\infty(\mathbb{T}^1)$ has property A_0 (defined in §7) then ther exists $\lambda \in \mathbb{R}$ such that $f_\lambda = f + \lambda$ is quasi symetrically conjugated to a translation $R_\alpha : x \mapsto x + \alpha$ ($\alpha \in \mathbb{R} - \mathbb{Q}$):

$$f = h \circ R_\alpha \circ h^{-1}, \quad h \in \mathcal{D}^{\rm qs}(\mathbb{T}^1)$$

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but h is not of class C^2 (equivalently, by [H,IV.4][1], $h \notin \mathcal{D}^2(\mathbb{T}^1)$).

Using in §22 the construction of Étienne Ghys [G] the theorem allows to prove that there exists many rational maps having Siegel singular disks whose boundary is a quasi circle that does not contain critical points, in particular there exists $\alpha \in \mathbb{R} - \mathbb{Q}$ such that this is the case for

$$z \mapsto e^{2\pi i \alpha}(z + z^2). \quad \text{(Theorem 3 §22.1)}$$

Of course, we also obtain singular rings (§23) with similar properties, and we leave to teratology enthusiast reader to build, using quasicircles, one's own fantastic zoology, for instance using the constructions of M. Shishikura [S].

The construction of Ghys proves that the result of E. Ghys [G] and those of [H2] require arithmetic conditions on the rotation numbers (§22.12 to §22.16).

For a survey on singular domain, see [H3].

Notations

We denote $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and translations (or rotations) of $\mathbb{R}$ or $\mathbb{T}^1$ by $R_\alpha(x) = x + \alpha$.

We denote by $\mathbb{S}^1 = \{z \in \mathbb{C},\ |z| = 1\}$ the circle and by $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ the unit disk in $\mathbb{C}$.

The universal cover group of C^r diffeomorphisms, $r \in \mathbb{N} \cup \{\infty, \omega\}$ is denoted

$$\mathcal{D}^r(\mathbb{T}^1) = \{f \in \mathrm{Diff}^r_+(\mathbb{R}),\ f \circ R_p = R_p \circ f,\ p \in \mathbb{Z}\}$$

where $\mathrm{Diff}^r_+(\mathbb{R})$ is the group of diffeomorphisms of $\mathbb{R}$, increasing and of class C^r (by a C^0 diffeomorphism we mean a homeomorphism and C^ω denotes the $\mathbb{R}$-analytic class).

If $r \in \mathbb{N} \cup \{\infty, \omega\}$, $C^r(\mathbb{T}^1)$ denotes the functions from $\mathbb{R}$ to $\mathbb{R}$ that are $\mathbb{Z}$-periodic and of class C^r.

If $r \in \mathbb{N}$ and $\phi \in C^r(\mathbb{T}^1)$ then $D^r\phi$ denotes the r^{th} derivative of ϕ with the convention that $D^0\phi = \phi$. We endow $\mathbb{R}$ with the standard metric and the properties Lipschitz continuous and Hölder continuous always refer to this metric. If $r \in \mathbb{N}$, $C^{1+\mathrm{Lip}}$ means that the r^{th} derivative is Lipschitz.

If $\phi \in C^0(\mathbb{T}^1)$,

$$\|\phi\|_{C^0} = \sup_{\theta \in \mathbb{R}} |\phi(\theta)|$$

and $L^\infty = L^\infty(\mathbb{T}^1, \mathbb{R}, d\theta)$ and $\|\ \|_{L^\infty}$ the norm defined by the essential supremum.

A number $\alpha \in \mathbb{R}$ is called of bounded type if there exists $\gamma > 0$ such that for all $p/q \in \mathbb{Q}$, we have

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{\gamma}{q^2}$$

and if $p/q \in \mathbb{Q}$, the convention is that $q \geqslant 1$ and p and q are mutually prime.

If $S \subset \mathbb{C}$ is a subset, we denote by ∂S its boundary.

1. Let $h \in \mathcal{D}^0(\mathbb{T}^1)$, h is called a quasi symmetric homeomorphism if there exists $M \geqslant 1$ such that for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$ we have

$$(1) \quad |h|_{\mathrm{qs}} = \sup_{x, t \neq 0} \left(\frac{h(x+t) - h(x)}{h(x) - h(x-t)}\right) \leqslant M$$

[1] TN : i.e. by the theorem of Gottschalk and Hedlund, but I think there is a simpler argument: if h' vanishes at one point then it must vanish on its orbit therefore on a dense set and thus everywhere by continuity, which is absurd.

or equivalently

$$\frac{1}{M} \leqslant \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqslant M \quad \forall x \in \mathbb{R},\ \forall t \neq 0.$$

2. Let $\mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1) = \{h \in \mathcal{D}^0(\mathbb{T}^1),\ h \text{ is a quasi symmetric homeomorphism}\}$.
The set $\mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ is a group because[2] (1) is equivalent to

For all $C \geqslant 1$, there exists $M(C) \geqslant 1$, non-decreasing as a function of C, such that all adjacent intervals $I_1 = [a,b]$, $I_2 = [b,c]$, $a < b < c$ satisfying

$$(2) \qquad \frac{1}{C} \leqslant \frac{|I_1|}{|I_2|} \leqslant C \qquad \textit{satisfy} \qquad \frac{1}{M(C)} \leqslant \frac{|f(I_1)|}{|f(I_2)|} \leqslant M(C)$$

where $|I_1| = |b-a|$ (its length).

Indeed, if we assume $t = b-a < c-b$ (the other case is analogous) we get, if

$$J_k = [b+(k-1)t, b+kt], \quad k = 1, 2, \ldots$$

then $|f(J_1)|/|f(I_1)| \leqslant M$, $|f(J_{k+1})|/|f(J_k)| \leqslant M$ thus

$$|f(J_k)|/|f(I_1)| \leqslant M^k$$

but

$$\bigcup_{k=1}^{l} J_k \supset I_2, \quad l = [C]+1$$

whence

$$\frac{|f(I_2)|}{|f(I_1)|} \leqslant M\frac{M^l-1}{M-1}.$$

We get $|f(I_2)|/|f(I_1)| \geqslant |f(J_1)|/|f(I_1)| \geqslant 1/M$ which proves (2).

3. If h verifies (1) then by [A] or [L]

$$h(x+t)-h(x) \leqslant \left(\frac{M}{M+1}\right)^n, \quad \text{when } 0 \leqslant t \leqslant 2^{-n}.$$

This implies that the set of homeomorphisms h that satisfy (1) with M fixed and $h(0) = 0$ is compact for the topology of uniform convergence and that each $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ is a Hölder-continous homeomorphism (i.e. $h - \mathrm{Id} \in C^\beta(\mathbb{T}^1)$ and $h^{-1} - \mathrm{Id} \in C^\beta(\mathbb{T}^1)$ where $0 < \beta < 1$ depends only on the constant M).

4. If h verifies (1) then it it the same for $S_1 \circ h \circ S_2$ where S_1 and S_2 are affine maps (i.e. $x \mapsto ax+b$).

5. We project $\mathbb{R} \to \mathbb{S}^1$ by $t \mapsto e^{2\pi i t}$ and $h \in \mathcal{D}^0(\mathbb{T}^1)$ induces a homeomorphism $\overline{h}$ on $\mathbb{S}^1$. The Ahlfors Beurling theorem ([A] or [L]) claims that $\overline{h}$ exteds to a quasi conformal homeomorphism of $\overline{\mathbb{D}}$ if and only if h is a quasi symmetric homeomorphism (which also implies that $\mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ is a group).

6. Let $f \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$, $\rho(f) = \alpha \in \mathbb{R}$ be its rotation number.

Proposition. *The following claims are equivalent:*

[2] TN : Stability by composition indeed naturally follows from (2). Stability by inversion seems to require a different argument.

(i) $f = h^{-1} \circ R_\alpha \circ h$ *with* $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$*;*

(ii) $\sup_{n \geqslant 1} \left| f^n \right|_{\mathrm{qs}} = M < +\infty.$

Moreover (ii) implies

$$\left| h \right|_{\mathrm{qs}} \leqslant M$$

Proof. The fact that (i) implies (ii) results from 2. and 4. By [H,IV.5], if $n \longrightarrow +\infty$,

$$h_n = \frac{1}{n} \sum_{i=0}^{n-1} \left(f^i - i\alpha \right)$$

converges uniformly to a map h such that $h - \mathrm{Id} \in C^0(\mathbb{T}^1)$, satisfying

$$h \circ f = R_\alpha \circ h$$

(i.e. a semi conjugacy to R_α). But $\left| h_n \right|_{\mathrm{qs}} \leqslant M$ and thus $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ with $\left| h \right|_{\mathrm{qs}} \leqslant M$. ■

By [H,IV.5], if $\alpha \in \mathbb{Q}$, (i) is equivalent to

(iii) $f^q = R_p$.

7. Let $f \in \mathcal{D}^0(\mathbb{T}^1)$. The map f is said to have property A_0 if for all $\lambda \in \mathbb{R}$ and all $p/q \in \mathbb{Q}$ we have $(R_\lambda \circ f)^q \not\equiv R_p$. The following examples are drawn from [H,III.3] and have property A_0.

- $f = \mathrm{Id} + \phi$ where ϕ extends to an entire map from $\mathbb{C}$ to $\mathbb{C}$ that is not constant (for instance $\phi(\theta) = \frac{a}{2\pi} \sin(2\pi\theta)$, $0 < |a| < 1$)
- The homeomorphism that $\overline{f}$ induces on $\mathbb{S}^1$ is the restriction of a rational map of degree $d \geqslant 2$, see also [H1,IV].

8. It follows from [H,III.5] that if f has property A_0, the the closure K_f of the set $\left\{ \lambda,\ \rho(R_\lambda \circ f) \in \mathbb{R} - \mathbb{Q} \right\}$ is modulo 1 a Cantor set.

By [H,XII.2], there exists a G_δ dense subset G of $\mathbb{R} - \mathbb{Q}$ such that if $\rho(R_\lambda \circ f) \in G$ then by the theorem of Denjoy, $f = h \circ R_\alpha \circ h^{-1}$ with $h \in \mathcal{D}^0(\mathbb{T}^1)$ but for all $0 < \beta < 1$, h is not a homeomorphism of class C^β and thus by §3, h is not a quasi symmetric homeomorphism. We can even replace C^β by any module of continuity that is fixed in advance. Examples in 7 show that even if $f \in \mathcal{D}^\omega(\mathbb{T}^1)$, $\rho(f) = \alpha \in \mathbb{R} - \mathbb{Q}$, Denjoy's theorem does not allow in full generality to get a quasi symmetric homeomorphism.

9. In the sequel, $\alpha \in (\mathbb{R} - \mathbb{Q}) \cap [0, 1/2]$ and $\alpha = [a_1, a_2, \ldots] = 1/(a_1 + 1/(a_2 + \ldots))$ denotes its continued fraction expansion and $(p_n/q_n)_{n \geqslant 0}$ its convergents: $q_0 = 1$, $p_0 = 0$, $q_1 = a_1 \geqslant 2$, $p_1 = 1$ and $q_n = a_n q_{n-1} + q_{n-2}$ if $n \geqslant 2$. We recall that (see for instance [H,V]) if $n \geqslant 0$,

$$(-1)^n \left(\alpha - \frac{p_n}{q_n} \right) > 0$$

$$\frac{1}{(a_{n+1} + 2) q_n^2} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{a_{n+1} q_n{}^2}$$

$$\text{et} \quad \left\| q_n \alpha \right\| = \left| q_n \alpha - p_n \right|$$

where we define

$$\|x\| = \inf_{p\in\mathbb{Z}} |x+p|,\ x\in\mathbb{R}.$$

Moreover

$$(3)\quad \|q_{n-1}\alpha\| = a_{n+1}\|q_n\alpha\| + \|q_{n+1}\alpha\|.$$

If $\alpha = [a_1,\ldots]$ satisfies $a_1 = 1$ then $1-\alpha$ satisfies $1-\alpha = [a_1, a_2, \ldots]$ with $a_1 \geqslant 2$.

10. We start from $f \in \mathcal{D}^0(\mathbb{T}^1)$, a homeomorphism of class P (cf [H,VI.4]): we assume f has everywhere a left and a right derivative and that $\log Df$ has bounded variation and we let

$$V = \mathrm{Var}(\log Df) = \text{ the measure norm of } D\log Df \text{ on } \mathbb{T}^1.$$

This implies that f and f^{-1} are absolutely continuous on every compact interval. If $\rho(f) = \alpha$ the we have Denjoy's inequality [H,VI.4]

$$(4)\quad \big\|\log Df^{\pm q_n}\big\|_{L^\infty} \leqslant V$$

which implies Denjoy's theorem [H,VI.5]. In the sequel, we will let

$$\widehat{f}^{q_n} = f^{q_n} - p_n$$
$$\text{and } I_n(x) = [x, \widehat{f}^{q_n}(x)]$$

where $[x, \widehat{f}^{q_n}(x)]$ denotes the compact interval determined by x and $\widehat{f}^{q_n}(x)$.

11.

Theorem 1. *We assume that f satisfies the hypotheses of §10 and that there exists $C > 0$ such that for all $n \geqslant 1$ we have*

$$(5)\quad \big\|\log Df^{q_n}\big\|_{L^\infty} \leqslant \frac{C}{(2+a_{n+1})},\ n\geqslant 0.$$

Then $f = h\circ R_\alpha \circ h^{-1}$, $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ and we have

$$(6)\quad \big|h\big|_{\mathrm{qs}} \leqslant 2e^{2C}.$$

Proof: (5) implies the following inequality almost everywhere for the Lebesgue measure

$$e^{-C} \underset{\text{a.e.}}{\leqslant} Df^{kq_n} \underset{\text{a.e.}}{\leqslant} e^{C},\quad k\in\mathbb{Z},\quad |k|\leqslant 2+a_{n+1}$$

and thus

$$(7)\quad e^{-C} \leqslant |\widehat{f}^{kq_n}(I_n(x))|/|I_n(x)| \leqslant e^{C},\quad |k|\leqslant 2+a_{n+1}.$$

Let h be the homeomorphism given by Denjoy's theorem, uniquely determined if we impose that $h(0)=0$ and satisfying

$$f = h\circ R_\alpha\circ h^{-1}.$$

Let y such that $h(y) = x$. Inequality (7) implies[3]

$$e^{-2C} \leqslant \frac{h\big(y+k\|q_n\alpha\|\big)-h(y)}{h(y)-h\big(y-k\|q_n\alpha\|\big)} \leqslant e^{2C},\quad 1\leqslant|k|\leqslant a_{n+1}+1,\quad k\in\mathbb{Z}^*.$$

[3]TN : Cut the interval $[y-k\|q_n\alpha\|, y+k\|q_n\alpha\|]$ in intervals of length $\|q_n\alpha\|$ and compare the length of their images to the length of the image of the one corresponding to $I_n(x)$. In the statement we can replace $2+a_{n+1}$ by $1+a_{n+1}$.

Since $\|q_n\alpha\|(a_{n+1}+1) \geqslant \|q_{n-1}\alpha\| \geqslant a_{n+1}\|q_n\alpha\|$ valid even if $n=0$ with the convention that $\|q_{-1}\alpha\|=1$, we deduce that[4] for all $n \geqslant 0$,

$$(8) \quad \frac{1}{2}e^{-2C} \leqslant \frac{h(y+t)-h(y)}{h(y)-h(y-t)} \leqslant 2e^{2C}, \quad \text{if } \|q_n\alpha\| \leqslant |t| \leqslant \|q_{n-1}\alpha\|.$$

It follows that (8) is true for all y and all $0<t<1$, which proves the theorem. ■

12. The following corollary immediately follows from (4).

Corollary. *Let f be a homeomorphism of class P such that $\rho(f)=\alpha$ is a bounded tyle number (i.e. $\sup_{i\geqslant 1} a_i = l < +\infty$). Then $f = h\circ R_\alpha \circ h^{-1}$ where $h \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1)$ and*

$$\big|h\big|_{\mathrm{qs}} \leqslant 2\exp(V(2l+4))$$

using the notations of §10, i.e. $V = \mathrm{Var}(\log Df)$.

13. Example Let $\lambda > 1$ and consider the piecewise linear homeomorphism $g \in \mathcal{D}^0(\mathbb{T}^1)$ defined by

$$\begin{aligned} g(x) &= \lambda x, \quad \text{if } 0 \leqslant x \leqslant a = (\lambda+1)^{-1}, \\ g(x) &= 1+\lambda^{-1}(x-1), \quad \text{if } a \leqslant x \leqslant 1, \\ g(x+p) &= p+g(x), \quad \text{if } p\in\mathbb{Z} \text{ and } 0 \leqslant x \leqslant 1. \end{aligned}$$

We choose $0<b<1$ so that $b+g=f$ satisfies $\rho(f)=\alpha$ where α is a bounded type number and thus, by the previous corollary, we obtain

$$f = h\circ R_\alpha\circ h^{-1}, \quad h\in\mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1), \quad h(0)=0.$$

By [H,VI.7], on $\mathbb{T}^1$, f mod 1 does not leave invariant any σ-finite measure that is absolutely continuous with respect to the Haar measure m of $\mathbb{T}^1$, hence the unique probability measure μ on $\mathbb{T}^1$ that is invariant by f mod 1 is singular with respect to m. But μ is the derivative of h in the sense of distributions, from which it follows that h and h^{-1} are singular with respect to the Lebesgue measure. For other examples of quasi symmetric homeomorphisms that are not absolutely continuous, c.f. [AB] and [P].

14. Choose $f \in \mathcal{D}^\infty(\mathbb{T}^1)$ and assume that f has property A_0, defined in §7. We let[5] $K = K_f \cap [0,1]$. Up to replacing f by $\lambda_1 + f$ where $\lambda_1 \in \mathbb{R}$, we may assyme that $K \subset (0,1)$ and

$$\big\{\rho(R_\lambda\circ f),\ \lambda\in K\big\} = \Big[-\frac{1}{2},\frac{1}{2}\Big]$$

We let $f_\lambda = \lambda + f$.

15. (Under the assumptions of the previous §)

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[4]TN : It is easy to get a bound that depends ont C, but the precise form that is stated here may need a few more explanations. To simplify let us assume that $t>0$ and let us bound the expression from above. Let k be the greatest integer so that $k\|q_n\alpha\| \leqslant |t|$. Let $y_j = y + j\|q_n\alpha\|$. Then $h(y+t)-h(y) \leqslant h(y_{k+1})-h(y)$ and $h(y)-h(y-t) \geqslant h(y)-h(y_{-k})$, whence $\frac{h(y+t)-h(y)}{h(y)-h(y-t)} \leqslant \frac{h(y_k)-h(y)}{h(y)-h(y_{-k})} + \frac{h(y_{k+1})-h(y_k)}{h(y)-h(y_{-k})}$. The first term of the sum is $\leqslant e^{2C}$. For the second, according to (7) the numerator is $\leqslant e^C|I_n(x)|$ and the denominator is $\geqslant e^{-C}|I_n(x)|$.

[5]TN : K_f was defined in §8.

Theorem 2. *There exists $\lambda \in K$ such that:*

(i) $f_\lambda = g \circ R_\alpha \circ g^{-1}, \quad g \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1), \quad g(0) = 0, \quad \alpha \in \mathbb{R} - \mathbb{Q}$;

(ii) *g is not of class C^2 (and thus by[6] [H,IV] not of class $C^{1+\mathrm{Lip}}$).*

15.1 To prove Theorem 2 we need some reminders and preliminary facts. We have

$$D(R_\lambda \circ f) = Df$$

and thus

$$\mathrm{Var}(\log Df_\lambda) = V$$

is independent of λ.

By [H,VI.6], if $y, z \in I_n(x) = [x, \widehat{f}^{q_n}(x)]$ then

$$(9) \quad e^{-V} \leqslant \frac{Df^j(y)}{Df^j(z)} \leqslant e^V \quad \text{pour } 0 \leqslant j < q_{n+1}.$$

This results from the fact that the following intervals, taken modulo 1

$$\Big(f^k(I_n(x))\Big)_{0 \leqslant k < q_{n+1}}$$

have pairwise disjoint interiors [H,V.8.3].

It follows[7] that if $\xi_k \in I_n(x)$ then

$$(10) \quad \sum_{k=0}^{q_{n+1}-1} Df^k(\xi_k) \leqslant \frac{e^V}{|I_n(x)|} = e^V \Big|\widehat{f}^{q_n}(x) - x\Big|^{-1}.$$

16. Let the irrational numbers $\alpha_{n,l} = [a_1, \ldots, a_n, a_{n+1} + l, a_{n+2}, \ldots]$ with $l = 0, 1, \ldots$. If we assume that

$$(11) \quad n \equiv 0 \bmod 2$$

then (see §9)

$$\frac{p_n}{q_n} < \alpha_{n,l+1} < \alpha_{n,l} < \alpha_{n,0}$$

and $\alpha_{n,l}$ has its n^{th} convergent equal to p_n/q_n.

By [H,III.4], there exists a unique $\lambda_l \in \mathbb{R}$ such that

$$\rho(f_{\lambda_l}) = \alpha_{n,l}.$$

For 1. *For $0 \leqslant j < q_n$*

$$(12) \quad f^j_{\lambda_{l+1}}(x) \in f^j_{\lambda_l}\Big(I_n\big(\widehat{f}^{-q_n}_{\lambda_l}(x)\big)\Big) \equiv \big[f^j_{\lambda_l} \circ \widehat{f}^{-q_n}_{\lambda_l}(x), f^j_{\lambda_l}(x)\big].$$

<u>Proof</u>. We have $\lambda_{l+1} < \lambda_l$, so for all $j \geqslant 1$ and all $x \in \mathbb{R}$:

$$f^j_{\lambda_{l+1}}(x) < f^j_{\lambda_l}(x).$$

Suppose by contradiction that for some $i < q_n$ and $x \in \mathbb{R}$ we have

$$f^i_{\lambda_{l+1}}(x) < f^i_{\lambda_l} \circ \widehat{f}^{-q_n}_{\lambda_l}(x)$$

thus

$$\widehat{f}^{q_n}_{\lambda_{l+1}}(x) = f^{q_n - i}_{\lambda_{l+1}} \circ f^i_{\lambda_{l+1}}(x) - p_n < f^{q_n - i}_{\lambda_l} \circ f^i_{\lambda_{l+1}}(x) - p_n < f^{q_n - i}_{\lambda_l}\big(f^i_{\lambda_l} \circ \widehat{f}^{-q_n}_{\lambda_l}(x)\big) - p_n = x.$$

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[6]TN : Gottschalk and Hedlund's theorem. It implies that if α is irrational and if ϕ is a L^∞ function on $\mathbb{T}$ such that $\phi \circ R_\alpha - \phi$ is continuous (has a continuous representative), then ϕ is continuous (has a continuous representative). Apply this to the derivative of $\log Dg$.

[7]TN : Explanations: let $I = I_n(x)$. The $f^k(I)$ are disjoint. The sum of their lengths is thus $\leqslant 1$. Now $|f^k(I)| = \int_I Df^k \geqslant e^{-V} Df^k(\xi_k)\,|I|$.

This contradicts

$$\widehat{f}^{q_n}_{\lambda_{l+1}}(x) > x \quad \text{for all } x$$

since p_n/q_n is the n^{th} convergent of $\alpha_{n,l}$.

Corollary.

$$(13) \qquad \sum_{j=0}^{q_n-1} \Big| - f^j_{\lambda_{l+1}}(x) + f^j_{\lambda_l}(x) \Big| \leqslant e^V \frac{\big|\widehat{f}^{-q_n}_{\lambda_l}(x) - x\big|}{\big|\widehat{f}^{q_{n-1}}_{\lambda_l}(x) - x\big|}.$$

Proof. By Lemma 1:

$$A(x) = \sum_{j=0}^{q_n-1} \big|f^j_{\lambda_l}(x) - f^j_{\lambda_{l+1}}(x)\big| \leqslant \sum_{j=0}^{q_n-1} \big|f^j_{\lambda_l}(\widehat{f}^{-q_n}_{\lambda_l}(x)) - f^j_{\lambda_l}(x)\big|$$

and by the mean value theorem

$$A(x) \leqslant \sum_{j=0}^{q_n-1} Df^j_{\lambda_l}(\xi_j)\big|\widehat{f}^{-q_n}_{\lambda_l}(x) - x\big|$$

where

$$\xi_l \in [\widehat{f}^{-q_n}(x), x] \subset [\widehat{f}^{q_{n-1}}(x), x]$$

and the corollary follows from (10). ■

If we replace (11) by

$$(11') \quad n \equiv 1 \bmod 2$$

then

$$f_{\lambda_l} < f_{\lambda_{l+1}}$$

and (13) becomes[8]

$$(13') \qquad \sum_{j=0}^{q_n-1} \big|f^j_{\lambda_{l+1}}(x) - f^j_{\lambda_l}(x)\big| \leqslant e^V \frac{|\widehat{f}^{-q_n}_{\lambda_l}(x) - x|}{|\widehat{f}^{q_{n-1}}_{\lambda_l}(x) - x|} \; .$$

17. If $f = h \circ R_\alpha \circ h^{-1}$ where $h \in \mathcal{D}^{1+\text{Lip}}(\mathbb{T}^1)$ then

$$(14) \qquad \big\| \log Df^{q_n} \big\|_{C^0} \leqslant \|D \log Dh\|_{L^\infty} \|q_n\alpha\| \leqslant \frac{\|D \log Dh\|_{L^\infty}}{q_{n+1}}$$

since

$$\log Df^{q_n} \circ h = \log Dh \circ R_{q_n\alpha - p_n} - \log Dh.$$

For $\lambda \in \mathbb{R}$ let[9]

$$\widetilde{H}_2(\lambda) \equiv \widetilde{H}_2(f_\lambda) = \sup\Big(\sup_{i\in\mathbb{Z}}\big(\big\|Df^i_\lambda\big\|_{C^0}\big), \sup_{i\in\mathbb{N}}\big(\big\|D^2 f^i_\lambda\big\|_{C^0}\big)\Big) \in \mathbb{R} \cup \{+\infty\}$$

We have $\widetilde{H}_2(\lambda) \geqslant 1$ and $\widetilde{H}_2(\lambda) = 1$ implies $f = R_\lambda$, $\lambda \in \mathbb{R}$.

By [H,IV.6], f_λ is C^2 conjugated to R_α if and only if

$$\widetilde{H}_2(\lambda) < +\infty$$

and if

$$\widetilde{H}_2(\lambda) \leqslant p + 1$$

then we have $f = h \circ R_\alpha \circ h^{-1}$ and h satisfies

$$(15) \qquad \frac{1}{p+1} \leqslant Dh \leqslant p+1 \qquad \|D^2(h^{-1})\|_{L^\infty} \leqslant p+1 \quad \text{(cf [H,IV.6.2])}.$$

[8]TN : This is identical to (13).

[9]TN : The use of the pair $(\mathbb{Z},\mathbb{N})$ is a shorthand to express that we take the smallest constant $C > 0$ such that $\underline{\forall n \geqslant 0}$, $\forall x$, $\frac{1}{C} \leqslant |Df^i_\lambda(x)| \leqslant C$ and $|D^2 f^i_\lambda(x)| \leqslant C$.

Ths implies: $\|D\log Dh\|_{L^\infty} \leqslant (p+1)^3$ and $\|D^2h\|_{L^\infty} \leqslant (p+1)^4$.

All we will use in the sequel is that

$$f_\lambda = h\circ R_\alpha \circ h^{-1} \quad \widetilde{H}_2(\lambda) \leqslant p+1 \quad \text{implies}$$

$h\in \mathcal{D}^{1+\mathrm{Lip}}(\mathbb{T}^1)$ (i.e. the diffeomorphisms h of class C^1 such that Dh are Lipschitz continuous) satisfies inequalities (15). This follows from Ascoli's theorem and from the fact that

$$h_n^{-1} = \frac{1}{n}\sum_{i=0}^{n-1}(f_\lambda^i - i\alpha), \quad \text{if } n \longrightarrow +\infty,$$

uniformly converges to

$$h^{-1} = \mathrm{Id} + \phi, \quad \phi \in C^0(\mathbb{T}^1)$$

satisfying

$$h^{-1}\circ f_\lambda = R_\alpha \circ h^{-1}.$$

The inequalities follow from the fact that $\widetilde{H}_2(\lambda) \leqslant p+1$ implies $\frac{1}{p+1} \leqslant Df_\lambda^i \leqslant p+1$ and $\|D^2 f_\lambda^i\|_{L^\infty} \leqslant p+1$, for all $i \geqslant 0$.

18. Proof of Theorem 2.

We will build numbers $\mu_1,\dots,\mu_p$ in K, with $\rho(f_{\mu_p}) = \alpha_p \in [0,\frac{1}{2}] - \mathbb{Q}$, associated to a sequence of integers

$$2 < k_1 < k_2 < \dots$$

such that if $\alpha_p = [a_{1,p}, a_{2,p}, \dots]$ denotes the continued fraction expansion of the numbers α_p then

$$a_{1,p} = 2 \qquad p \geqslant 1 \quad (\text{to get } \alpha_p \in [0,\frac{1}{2}])$$

$$(16) \quad a_{n,p} = a_{n,p-1} \quad \text{except if } n = k_p$$

and

$$(17) \quad a_{n,p} = 1 \quad \text{if } n \neq k_j \quad 1 \leqslant j \leqslant p \quad \text{and} \quad n>1.$$

We will prove that we can determine the sequence $k_1, \dots, k_p, \dots$, the numbers α_p and a constant satisfying

$$(18) \quad C_0 \geqslant 3V + 1\ ;$$

$$(18') \quad C_0 \geqslant 6\big\|D\log Df\big\|_{C^0} e^V + \frac{1}{3}$$

such that for all p we have by induction on $p \geqslant 1$:

$$(19)_p \quad \forall k \geqslant 0,\ \big\|\log Df_{\mu_p}^{q_k(\alpha_p)}\big\|_{C^0} \leqslant \frac{C_p}{2 + a_{k+1}(\alpha_p)}$$

where the integers $q_k(\alpha_p)$ and $a_{n+1}(\alpha_p)$ are associated to α_p ;

$$(20)_p \quad C_0 \leqslant C_p = C_{p-1} + \frac{1}{2^p}\ ;$$

$$(21)_p \quad \widetilde{H}_2(\mu_p) > p\ ;$$

$$(22)_p \quad \begin{array}{l} \mu_p \in l_p \cap K \quad \text{where} \quad l_p = [\mu_{p-1} - \varepsilon_p, \mu_{p-1} + \varepsilon_p] \subset l_{p-1}, \quad \varepsilon_p > 0 \\ \qquad\qquad \text{and we have} \quad \widetilde{H}_2(\mu) > p, \quad \text{if } \mu \in l_p \cap K. \end{array}$$

For $p = 1$, we choose $k_1 = 3$, $a_{1,3} = 1$, $k_2 > 10$, $l_1 = [-\frac{1}{2}, \frac{1}{2}]$ and we have $H_2(\mu) > 1$ if $\mu \in l_1$.

Denjoy's inequality (4) shows that[10] for all $p \geqslant 1$,

$$(22') \quad \big\|\log Df_{\mu_q}^{q_k(\alpha_p)}\big\|_{C^0} \leqslant \frac{C_0}{2 + a_{k+1}(\alpha_p)} \quad \text{if } k \neq k_2 - 1, \dots, k_p - 1.$$

[10] TN : . . . Assuming (17) and (18). Indeed, $a_{k+1}(\alpha_p) = 1$ for the values of k considered here. There were several equations numbered 22 in the original so I decided to renumber this one 22'.

19. We will show how to get to Step $p+1$.

For this we will perturb α_p into $\alpha_{p+1} = [a_{1,p+1}, \ldots, a_{k_{p+1},p+1}, 1, \ldots]$. We let $n+1 = k_{p+1}$ and

$$\beta_j = [b_1, \ldots, j, 1, 1, \ldots]$$

where

$$(23) \quad \begin{cases} b_k &= a_{k,p} \quad \text{if } k \neq n+1, \\ b_{n+1} &= j \geqslant 1. \end{cases}$$

By continuity there exists $\varepsilon'_{p+1} > 0$, $l'_{p+1} = [\mu_p - \varepsilon'_{p+1}, \mu_p + \varepsilon'_{p+1}] \subset l_p$ such that if $\mu \in l'_{p+1}$ then $\alpha = \rho(f_\mu)$ has the same convergents (p_k/q_k) as α_p for $k \leqslant k_p + 10$ and such that we have

$$(24) \quad \forall i \leqslant k_p, \quad \left\| \log Df_\mu^{q_i(\alpha)} \right\|_{C^0} \leqslant \frac{C_p + 2^{-(p+1)}}{a_{i+1}(\alpha) + 2}, \quad \text{if} \quad \mu \in l'_{p+1}.$$

(A) We will assume that k_{p+1} is big enough (see §9) so that for all $j \in \mathbb{N}^*$, if $\rho(f_{\lambda_j}) = \beta_j$, then

$$(25) \quad \lambda_j \in l'_{p+1} \cap K.$$

Since β_j is a bounded type number, by [H,IX] we get

$$f_{\lambda_j} = h_j \circ R_{\beta_j} \circ h_j^{-1}, \quad h_j \in \mathcal{D}^\infty(\mathbb{T}^1).$$

We also recall that for all $\alpha \in \mathbb{R} - \mathbb{Q}$,

$$(26) \quad q_n(\alpha) \geqslant 2^{(n-1)/2}.$$

(B) If k_{p+1} is big enough then by using (14) we get[11]

$$(27) \quad \left\| \log Df_{\lambda_1}^{q_n} \right\|_{C^0} \leqslant \frac{\|D \log Dh_1\|_{L^\infty}}{q_{n+1}} \leqslant \frac{1}{(a_{n+1}(\beta_1) + 2)(p+1)^2}$$

$(a_{n+1}(\beta_1) = 1)$.

Claim. *There is a biggest integer $1 < l < +\infty$ such that*[12]

$$(28) \quad \left\| \log Df_{\lambda_l}^{q_n} \right\|_{C^0} \leqslant \frac{1}{(l+2)(p+1)^2}$$

and thus

$$(29) \quad \left\| \log Df_{\lambda_{l+1}}^{q_n} \right\|_{C^0} > \frac{1}{(l+3)(p+1)^2}.$$

Proof. It follows from (27) that (28) holds for $l = 1$. If (28) were true for all l we would have $\lambda_l \longrightarrow \lambda_\infty$, $\rho(f_{\lambda_\infty}) = \beta_\infty = p_n/q_n$ and f satisfies

$$\log Df_{\lambda_\infty}^{q_n} \equiv 0$$

in other words $f_{\lambda_\infty}^{q_n} = R_{p_n}$, which contradicts the assumption that f has property A_0. ■

We will consider 2 possibilities:

$$(30) \quad \widetilde{H}_2(\lambda_l) > p+1,$$

or

$$(31) \quad \widetilde{H}_2(\lambda_l) \leqslant p+1.$$

19.1 If (30) holds then we choose $\alpha_{p+1} = \beta_l$. Since the map $\lambda \mapsto \widetilde{H}_2(\lambda)$ is lower semi continuous we can find an interval $l_{p+1} \subset l'_{p+1}$ such that we have $(22)_{p+1}$. From (22'), (24) and (28) it follows that $(19)_{p+1}$ is verified, which shows that, under Assumption (30), we can pass to Step $p+1$.

19.2 We assume that (31) holds. We choose $\alpha_{p+1} = \beta_{l+1}$.

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[11]TN : Indeed, h_1 does not depend on $k_{p+1} = n+1$ and $q_{n+1} \longrightarrow +\infty$.

[12]TN : In (27) and below, we can replace the factor $(1+p)^2$ by a constant. See §21.9.

Claim. *If k_{p+1} is big enough,*

$$(32) \quad \widetilde{H}_2(\lambda_{l+1}) > p+1.$$

Proof. If we assume that (32) does not hold then by (15) we get

$$\|D\log Dh_{l+1}\|_{L^\infty} \leqslant (p+1)^3$$

whence by (14)

$$\left\|\log Df^{q_n}_{\lambda_{l+1}}\right\|_{C^0} \leqslant \frac{(p+1)^3}{a_{n+1}(\beta_{l+1})q_n}.$$

From (29) we must have

$$\frac{1}{(l+3)(p+1)^2} \leqslant \frac{(p+1)^3}{(l+1)q_n}, \quad l \geqslant 1,$$

in particular

$$\frac{1}{2}q_n \leqslant (p+1)^5.$$

The integer p is fixed; by (26) this cannot hold

(C) if $k_{p+1} = n+1$ is big enough.

By contradiction if (C) holds then the claim follows. ■

Claim *if k_{p+1} is big enough*

$$(33) \quad \left\|\log Df^{q_n}_{\lambda_{l+1}}\right\|_{C^0} \leqslant \frac{C_0}{a_{n+1}(\beta_{l+1})+2} = \frac{C_0}{l+3}.$$

Proof. Using (28) we get

$$\left\|\log Df^{q_n}_{\lambda_{l+1}}\right\|_{C^0} \leqslant \left\|\log Df^{q_n}_{\lambda_{l+1}} - \log Df^{q_n}_{\lambda_l}\right\|_{C^0} + \frac{1}{4(l+2)}.$$

We obtain:

$$\begin{aligned} B &= \left\|\log Df^{q_n}_{\lambda_{l+1}} - \log Df^{q_n}_{\lambda_l}\right\|_{C^0} = \left\|\sum_{j=0}^{q_n-1} \log Df\circ f^j_{\lambda_{l+1}} - \log Df\circ f^j_{\lambda_l}\right\|_{C^0} \\ &\leqslant \left\|D\log Df\right\|_{C^0}\left\|\sum_{j=0}^{q_n-1}(f^j_{\lambda_{l+1}} - f^j_{\lambda_l})\right\|_{C^0}. \end{aligned}$$

Using (13) and (13') we deduce[13]

$$B \leqslant L\sup_\theta\left(\frac{|h_l(q_n\beta_l - p_n+\theta) - h_l(\theta)|}{|h_l(\theta - q_{n-1}\beta_l + p_{n-1}) - h_l(\theta)|}\right)$$

with

$$L = \left\|D\log Df\right\|_{C^0}e^V.$$

By the mean value theorem

$$\frac{h_l(q_n\beta_l - p_n+\theta) - h_l(\theta)}{h_l(\theta - q_{n-1}\beta_l + p_{n-1}) - h_l(\theta)} = \frac{Dh_l(\xi_1)}{Dh_l(\xi_2)}\ \frac{\|q_n\beta_l\|}{\|q_{n-1}\beta_l\|}$$

with

$$\xi_1 \in [\theta, \theta+q_n\beta_l - p_n], \quad \xi_2 \in [\theta, \theta - (q_{n-1}\beta_l - p_{n-1})]\ ;$$

whence

$$B_1 = \left|\frac{Dh_l(\xi_1)}{Dh_l(\xi_2)} - 1\right| \leqslant \|D^2h_l\|_{L^\infty}\left\|\frac{1}{Dh_l}\right\|_{C^0}\|q_{n-1}\beta_l\|.$$

[13]TN : There is a harmless mistake in some sign: one has to replace in the whole proof $q_n\beta_l - p_n + \theta$ by $p_n - q_n\beta_l + \theta$ and $\theta - q_{n-1}\beta_l + p_{n-1}$ by $\theta + q_{n-1}\beta_l - p_{n-1}$.

Using (31) and (15) we get to the conclusion that[14]

$$B_1 \leqslant (p+1)^5 \|q_{n-1}\alpha_p\| \leqslant (p+1)^5 \frac{1}{q_n}.$$

The integer p is fixed and thus

$$\text{(D)} \quad \text{if } k_{p+1} \text{ is big enough}$$

using (26) we get

$$B_1 \leqslant \frac{1}{2}$$

and

$$B \leqslant L\,\frac{3}{2}\,\frac{\|q_n\beta_l\|}{\|q_{n-1}\beta_l\|}.$$

It follows from (3) that

$$B \leqslant \frac{3L}{2}\,\frac{1}{a_{n+1}(\beta_l)} = \frac{3L}{2l}.$$

Finally, using (18'):

$$\left\| \log Df^{q_n}_{\lambda_{l+1}} \right\|_{C^0} \leqslant \frac{3}{2}\,\frac{L}{l} + \frac{1}{4(l+2)} \leqslant \frac{C_0}{l+3}.$$

■

We now conclude using (32) and §19.1 that there exists an interval $l_{p+1} \subset l'_{p+1}$ such that we have $(22)_{p+1}$. It follows from (22), (24) and (33) that $(19)_{p+1}$ is satisfied.

With the choices (A), (B), (C) and (D) on k_{p+1} we have shown how to construct α_{p+1} such that $f_{\mu_{p+1}}$ satisfies $(19)_{p+1}$ through $(22)_{p+1}$.

20. End of the proof of Theorem 2.

It follows from Theorem 1 and from $(19)_p$, that for all $p \geqslant 1$, we have

$$f_{\mu_p} = g_p \circ R_{\alpha_p} \circ g_p^{-1} \quad \text{with} \quad g_p \in \mathcal{D}^\infty(\mathbb{T}^1),\; g_p(0) = 0$$

and

$$(34) \quad \left|g_p\right|_{\mathrm{qs}} \leqslant 2e^{2C_\infty},$$

with

$$C_\infty = \sup_p C_p = C_0 + 1.$$

Let $l_\infty = \bigcap_{p \geqslant 1} l_p \cap K$, which is a non-empty compact set. By compactness, if $\lambda = \mu_\infty \in l_\infty \cap K$ and if g_∞ is a cluster value of the sequence $(g_p)_{p\geqslant 1}$ for the C^0 topology (c.f. §3) by passing to a uniform limit we get

$$f_{\mu_\infty} = g_\infty \circ R_{\alpha_\infty} \circ g_\infty^{-1} \quad \text{with} \quad g_\infty \in \mathcal{D}^{\mathrm{qs}}(\mathbb{T}^1),\; g_\infty(0) = 0$$

and

$$\left|g_\infty\right|_{\mathrm{qs}} \leqslant 2e^{2C_\infty}$$

(we use (34) and §3).

It follows from §7 that l_∞ is reduced to a point $\mu_\infty \in K$ and $\rho(f_\infty) = \alpha_\infty \in \mathbb{R} - \mathbb{Q}$ (since f has property A_0). This proves part (i) of Theorem 2.

To see that (ii) holds, notice that from $(22)_p$, since $\mu_\infty \in l_p$, we get

$$\widetilde{H}_2(\mu_\infty) > p, \quad \text{for all } p \geqslant 1,$$

and thus $\widetilde{H}_2(\mu_\infty) = +\infty$. Now (ii) follows from §17. ■

21. Remarks:

[14]TN : The number α_p must be replaced by $\frac{p_n}{q_n}$.

21.1 The crucial point of the whole proof is (33).

21.2 If $f \in \mathcal{D}^\omega(\mathbb{T}^1)$ then f_{μ_∞} is the limit of the sequence f_{μ_p} where each $\rho(f_{\mu_p})$ is a bounded type number and so by [H,IX] $f_{\mu_p} = g_p \circ R_{\alpha_p} \circ g_p^{-1}$ with $g_p \in \mathcal{D}^\omega(\mathbb{T}^1)$.

21.3 We could start from $\alpha_1 = [a_{1,1}, \ldots, a_{1,k}, \ldots] \in \mathbb{R} - \mathbb{Q}$ assuming only

$$a_{1,k} = 1 \quad \text{if} \quad k \geqslant k_0.$$

21.4 Without changing anything, we can replace (17) by

$$1 \leqslant a_{j,p} \leqslant t \quad \text{if} \quad j \neq k_q,\ q \leqslant p$$

where $t \in \mathbb{N}^*$ is given.

In fact, one can do much better using the following facts:[15]

$$\text{(34')} \qquad \left\| \widehat{f}^{q_n} - \mathrm{Id} \right\|_{C^0} \leqslant L_1 (1 + e^{-V})^{-n/2}$$

that follows from [H,VIII.2], with $L_1 = \sup(\left\| \widehat{f}^{q_1} - \mathrm{Id} \right\|_{C^0}, \left\| f - \mathrm{Id} \right\|_{C^0})$;

$$\text{(35)} \quad \left\| \log Df^{q_n} \right\|_{C^0} \leqslant L_2 \left\| \widehat{f}^{q_n} - \mathrm{Id} \right\|_{C^0}^{1/2}$$

that is Yoccoz's inequality [Y1], where L_2 is a constant that only depends on $\left\| D^2 \log Df \right\|_{C^0}$. If $f \in \mathcal{D}^r(\mathbb{T}^1)$, $r \geqslant 4$, we have even better if we use [Y2].

21.5

Conjecture. *Theorem 2 still holds if we replace (i) by*[16]

$$(i') \quad f_\lambda = g \circ R_\alpha \circ g^{-1} \quad \textit{with} \quad g \in \mathcal{D}^1(\mathbb{T}^1).$$

Using (34') and (35) it would be enough, by the same proof as in [H,IX.1.6], to ensure that[17]

$$\sum_{j=1}^{p} \left\| \log Df_{\mu_p}^{q_{k_j}(\alpha_p)} \right\|_{C^0} a_{k_j+1}(\alpha_p) \leqslant C_p$$

with $\sum_{p \geqslant 1} C_p < +\infty$.

For this, it might be possible to improve (33).

21.6 If $\varepsilon > 0$ is given then there exists $\eta > 0$ such that if $\left\| D^3 f \right\|_{C^0} \leqslant \eta$ then the homeomorphism g of (i) satisfies:

$$\text{(36)} \quad \left| g \right|_{\mathrm{qs}} \leqslant 1 + \varepsilon.$$

To see that, we choose k_1 very high, we use (34') and (35) and we replace in inequality (28) the factor $\frac{1}{(p+1)^2}$ by $\frac{1}{(p+t)^2}$ with t fixed but big.

We can choose C_0 small,

$$\text{(20')}_p \quad C_0 \leqslant C_{p+1} \leqslant C_p + \frac{1}{2^{p+t}}, \quad t \geqslant 1 \text{ big}$$

and we replace $(19)_p$ by

$$\text{(19')}_p \quad \sup_{p \geqslant j \geqslant 1} \left\| \log Df_{\mu_p}^{q_{k_j-1}(\alpha_p)} \right\|_{C^0} \leqslant \frac{C_0}{a_{k_j}(\alpha_p) + 2}.$$

To estimate $\left\| \log Df_{\mu_p}^{q_k} \right\|_{C^0}$, if $k < k_1$, we use that η is small, and if $k > k_1$, $k \neq k_j - 1$ we use (34') and (35).

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[15] TN : In the source there are two equations numbered 34.

[16] TN : $\mathcal{D}^1(\mathbb{T}^1)$ denotes the (orientation preserving) C^1 diffeomorphisms.

[17] TN : The author probably meant to write $\forall j$, $\exists C_j$, $\forall p$, $\left\| \log Df_{\mu_p}^{q_{k_j}(\alpha_p)} \right\|_{C^0} a_{k_j+1}(\alpha_p) \leqslant C_j$ with $\sum_1^\infty C_j < +\infty$.

Inequality (35) forces l to be very big, which allows in (6) to replace the factor $1/2$ by $1+\varepsilon/2$.

21.7

Proposition. *Let α be a Liouville number. Then there exists $f_\infty \in \mathcal{D}^\infty(\mathbb{T}^1)$, $\rho(f_\infty)=\alpha$ such that*

$$(i') \quad f_\infty \text{ is } C^1\text{-conjugated to } R_\alpha\,;$$
$$(ii) \quad f_\infty \text{ is not } C^2\text{-conjugated to } R_\alpha.$$

The proof is simpler than the proof of Theorem 2. We first need the following lemma.

Lemma. *Let $r\in\mathbb{N}^*$. For all $\varepsilon>0$ and $t>0$, there exists $f\in\mathcal{D}^\infty(\mathbb{T}^1)$ such that*

$$f = h\circ R_\alpha \circ h^{-1} \quad \text{where} \quad h\in\mathcal{D}^\infty(\mathbb{T}^1),\ h(0)=0$$

and satisfying

$$\begin{aligned} \big\|\log Df\big\|_{C^r} &\leqslant \varepsilon\ ;\\ \big\|\log Dh\big\|_{C^0} &\leqslant \varepsilon\ ;\\ \text{and}\quad \big\|D\log Dh\big\|_{C^0} &\geqslant t. \end{aligned}$$

Proof. Let $p/q\in\mathbb{Q}$ satisfying, $q\geqslant 2$, $\left|\dfrac{p}{q}-\alpha\right| < \dfrac{1}{q^k}$, q and k very big (this is possible since α is a Liouville number). Let

$$\log Dh = \frac{\varepsilon}{100}\cos(2\pi q\theta)+\lambda$$

where λ satisfies

$$e^\lambda \int e^{\frac{\varepsilon}{100}\cos(2\pi q\theta)}\,d\theta = 1.$$

We determine f by

$$\begin{aligned} \log Df\circ h &= \log Dh\circ R_\alpha - \log Dh\\ &= \frac{\varepsilon}{200}\Big[(e^{2i\pi q\alpha}-1)e^{2i\pi q\theta} + (e^{-2i\pi q\alpha}-1)e^{-2i\pi q\theta}\Big]. \end{aligned}$$

If q and k are big enough, one easily sees using

$$\left|e^{2i\pi q\alpha}-1\right| \leqslant \text{constant}\ \frac{1}{q^{k-1}}$$

that the inequalities of the lemma are satisfied. ■

Proof of the proposition. Let d_∞ be a complete metric defining the C^∞ topology of the Polish topological group $\mathcal{D}^\infty(\mathbb{T}^1)$. We will construct by induction on $p\geqslant 1$,

$$f_p = h_p\circ R_\alpha\circ h_p^{-1},\quad h_p\in\mathcal{D}^\infty(\mathbb{T}^1),\ h_p(0)=0$$

$$h_p = h_{p-1}\circ g_p,\quad g_p\in\mathcal{D}^\infty(\mathbb{T}^1),\ g_p(0)=0$$

such that for all $p\geqslant 1$:

$$(37)_p \quad d_\infty(f_p, f_{p+1}) < \frac{1}{2^p}\ ;$$

$$(38)_p \quad \big\|\log Dg_p\big\|_{C^0} < \frac{1}{2^p}\ ;$$

$$(39)_p \quad H_2(f_p) = \sup_{k\geqslant 1}\big\|D^2 f_p^k\big\|_{C^0} > p-1\ ;$$

$$(40)_p \quad f_p\in U_p\subset U_{p-1},$$

U_p in an open set with $\forall f \in U_p$, $H_2(f) > p-1$, $\mathrm{diam}(U_p) < \frac{1}{2^p}$ (diameter for the metric d_∞).

We choose $f_1 = R_\alpha$, $h_1 = \mathrm{Id}$ and $U_1 = \{f,\ d_\infty(R_\alpha, f) < \frac{1}{2}\}$.

We want to show how to pass to Step $p+1$.

If $H_2(f_p) \leqslant p$; using (15) and[18] the lemma, there exists g_{p+1} such that $h_{p+1} = h_p \circ g_{p+1}$, $f_{p+1} \in U_p$ and satisfies $(37)_{p+1}$, $(38)_{p+1}$ and $(39)_{p+1}$.

If $H_2(f_p) > p$ we choose $g_{p+1} = \mathrm{Id}$.

Since the map $f \mapsto H_2(f) \in \mathbb{R} \cup \{+\infty\}$ is lower semi continuous for the C^∞-topology, we get that $f_{p+1} \in U_p \cap H_2^{-1}([p, +\infty]) = V_p$ is a (non-empty) open set and we can find U_{p+1} satisfying $(40)_{p+1}$ and contained in V_p. This ends the construction by induction of the sequence $(f_p)_{p\geqslant 1}$.

If $p \longrightarrow +\infty$, $(f_p)_{p\geqslant 1}$ is a Cauchy sequence in $\mathcal{D}^\infty(\mathbb{T}^1)$ whence $f_p \longrightarrow f_\infty \in \mathcal{D}^\infty(\mathbb{T}^1)$ in the C^∞ topology. By $(40)_{p+1}$, we get

$$\bigcap_{p\geqslant 1} U_p = \{f_\infty\}.$$

If $p \longrightarrow +\infty$, it follows from $(38)_p$ that $h_p \longrightarrow h_\infty \in \mathcal{D}^1(\mathbb{T}^1)$ in the C^1-topology and we have $f_\infty = h_\infty \circ R_\alpha \circ h_\infty^{-1}$. Il follows from $(39)_p$ and $(40)_p$ that $H_2(f_\infty) = +\infty$ and it follows from[19] §17 that $h_\infty \notin \mathcal{D}^{1+\mathrm{Lip}}(\mathbb{T}^1)$. ■

21.8 Theorem 2 applies to $f \in \mathcal{D}^\omega(\mathbb{T}^1)$. Unfortunately, if $f \in \mathcal{D}^\omega(\mathbb{T}^1)$ the author of the present lines has not been ablet to adapt to this case the very simple argument that we have given just before[20] §21.7. The deep reason is related to the fact that with the C^ω-topology, $\mathcal{D}^\omega(\mathbb{T}^1)$ is not a Baire space nor even metrisable.[21]

21.9 We used the fundamental theorem[22] of [H] to prove (32) and (33) but it is not necessary if we use §17.

We have also used it to get (27) to conclude the existence of an integer l satisfying (28) and (29). We can avoid this by replacing (27), (28) and (29) with:

$$(27') \quad \left\| \log Df_{\lambda_1}^{q_n} \right\|_{C^0} \leqslant \frac{3V}{a_{n+1}(\beta_1)+2}\ ;$$

$$(28') \quad \left\| \log Df_{\lambda_l}^{q_n} \right\|_{C^0} \leqslant \frac{3V}{a_{n+1}(\beta_l)+2}\ ;$$

$$(29') \quad \left\| \log Df_{\lambda_{l+1}}^{q_n} \right\|_{C^0} \leqslant \frac{3V}{a_{n+1}(\beta_{l+1})+2}\ ;$$

[18]TN : We use (15) to get $(39)_{p+1}$, we do not use it on the hypothesis $H_2(f_p) \leqslant p$.

[19]TN : We use the easy implication.

[20]TN : I think the author meant "the very simple argument of §21.7 ».

[21]TN : We call today *Herman numbers* the class of rotation numbers rotations than ensure that any $f \in \mathcal{D}^\omega(\mathbb{T}^1)$ has a C^ω conjugacy h. Yoccoz proved that this class contains Liouville numbers. Even if one uses numbers α much closer to rationnals, the construction of the lemma seems to fail: the superficial reason is that, passing from $\log Df \circ h$ to $\log Df$, the composition with h^{-1} increases the successive derivatives of f at least like an exponential sequence of ratio $e^{\pi q}$. Herman chose to take h entire but if one takes h^{-1} entire then it becomes possible to carry the construction: even if $\log Dh$ has a small domain of holomorphy, the fonction f has a much bigger one, provided it is defined as the composition of the rational rotation p/q and of the time $\alpha - p/q$ map of the vector field defined as the pull-back by h^{-1} of the trivial field d/dz. Notice the connections between Herman's approach and the Anosov-Katok method.

[22]TN : The fundamenta theorem, stated in [H,IX], says that a C^∞ diffeomorphism whose rotation number "satisfies a condition A" is automatically C^∞-conjuguated to the rotation; the condition A is defined in [H,V], these numbers are of Roth type and form a class of full Lebesgue measure. However, the author seems to have only used the C^2 character of the conjugacies h_l, and uniquely for bounded type rotation numbers. If one of them ever turns out not being C^2 then we can stop the induction there: recall that Denjoy's inequality (4) ensures quasisymmetric conjugacy to the rotation when the rotation number has bounded type.

choosing C_0 big enough and modifying slighlty the proofs of (32) and (33).[23]

22. Application to Siegel singular disks, following E. Ghys [G].

This §22 must be considered as due to E. Ghys [G] up to a few small enhancements and a few supplementary details.

22.1 Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a rational map on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$, of degree $d > 1$, leaving invariant $\mathbb{S}^1 = \{z \in \mathbb{C},\ |z| = 1\}$ and such that $f\big|_{\mathbb{S}^1}$ is an $\mathbb{R}$-analytic diffeomorphism. We denote the open unit disk of $\mathbb{C}$ by $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$.

22.2 Examples:

$$g : z \mapsto z^2 \frac{1 - \overline{a}z}{z - a} \quad \text{with} \quad 0 < |a| < 1 \text{ and } |a| < \frac{1}{3}.$$

For all the examples, see [H1,IV]. We have $d = 2k + 1$, $k \in \mathbb{N}^*$.

22.3 If we lift $f\big|_{\mathbb{S}^1}$ to $\mathbb{R}$ into $\widetilde{f}$, the projection from $\mathbb{R}$ to $\mathbb{S}^1$ being given by $t \mapsto e^{2\pi i t}$, then $\widetilde{f}$ has property A_0 and μf, $\mu = e^{2\pi i \lambda}$, $\lambda \in \mathbb{R}$ gives the family $\widetilde{f} + \lambda \equiv R_\lambda \circ \widetilde{f}$.

22.4 Let $\mu \in \mathbb{S}^1$ such that $\alpha = \rho(\mu f) \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$, where $\rho(\mu f) = \rho(\widetilde{\mu f}) \bmod 1$([a]), but μf is not C^ω-conjugated to $r_\alpha : z \mapsto e^{2i\pi\alpha} z$.

Let $C_1 = \{c_1, \ldots, c_q \,\big|\, c_i \text{ is a critical point of } f \text{ and for all } j > 0\ f^j(c_i) \notin \mathbb{S}^1\}$. The following proposition is a small modification of an argument of P. Fatou.

22.5

Proposition. *With the hypotheses of §22.4 the set*

$$L = \omega_f(C_1) = \bigcap_{N \geqslant 1} \overline{\bigcup_{j \geqslant N} f^j(C_1)}$$

contains $\mathbb{S}^1$.

Proof. Since the closed set L is invariant by f, if $L \cap \mathbb{S}^1 \neq \varnothing$ then by Denjoy's theorem $L \supset \mathbb{S}^1$. If we assume by contradiction that $L \cap \mathbb{S}^1 = \varnothing$, then we can determine a sequence of determinations of the inverse of f^n such that $(f^{-n})_{n \geqslant 1}$ are defined for $n \geqslant 1$ on $A = \{\frac{1}{r} < |z| < r\}$ where $r > 1$ and satisfy

$$f^{-n}\big|_{\mathbb{S}^1} = (f\big|_{\mathbb{S}^1})^{-n}.$$

The family $(f^{-n}\big|_A)_{n \geqslant 1}$ is normal (if r is small enough then $f^{-n}(A)$ avoids for all $n \geqslant 1$ three distinct periodic cycles given in advance, and after conjugacy of f by an element of $\mathrm{PSL}(2, \mathbb{R})$ we can assume that two of these cycles contain the points 0 and ∞). We lift by $z \mapsto e^{2\pi i z} \in \mathbb{S}^2 - \{0, \infty\}$, $f^{-n}\big|_A$ into $\widetilde{f}^{-n}\big|_{\widetilde{A}}$ where

$$\widetilde{A} = \{z \in \mathbb{C},\ |\operatorname{Im} z| < \log r\}$$

[a] The rotation number of a homeomorphism of $\mathbb{S}^1$ depends of the choice of an orientation of $\mathbb{S}^1$ and we choose the one given by $t \mapsto e^{2\pi i t}$.

[23] TN : In reality, he already seems to have only applied §17 to justify (27), (28) and (29), and he could directly have used (27'), (28') and (29'). I was not able to find where he used the fundamental theorem, nor the factor $(p + 1)^2$.

and $\widetilde{f}^{-n}\big|_{\mathbb{R}} = (\widetilde{f}\big|_{\mathbb{R}})^{-n}$. The family

$$h_n = \frac{1}{n}\sum_{i=0}^{n-1}(\widetilde{f}^{-i}\big|_{\widetilde{A}} - i\widetilde{\alpha}), \quad \widetilde{\alpha} = \rho(\widetilde{f})$$

is normal from A to $\mathbb{S}^2$ (i.e. equicontinuous for the compact open topology on $C^0(\widetilde{A}, \mathbb{S}^2)$). Let $(h_{n_i})_{i\geqslant 0}$ be such that $h_{n_i} \longrightarrow h$ for the compact open topology where $1 < n_i < n_{i+1}$. On $\mathbb{R}$ we have

$$h \circ f = R_{-\alpha} \circ h, \quad h \in \mathcal{D}^0(\mathbb{T}^1)$$

and thus[24]

$$h \neq \{\infty\}.$$

On $\widetilde{A}$ we have

$$h \circ f(z) = R_{-\alpha} \circ h(z), \quad z \in \widetilde{A}$$

this implies that $h\big|_{\mathbb{R}}$ is C^ω and by[25] [H,IX.6.3] we conclude that $h \in \mathcal{D}^\omega(\mathbb{T}^1)$. This contradicts the hypothesis we made in §22.4 and shows that $L \cap \mathbb{S}^1 \neq \varnothing$. ∎

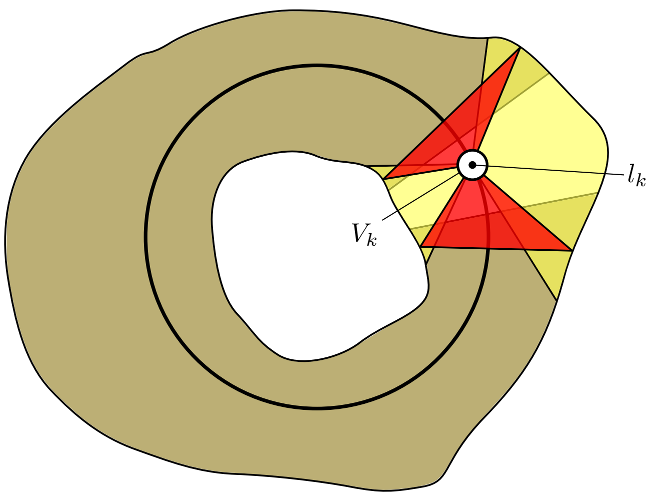

Figure: Example of a cover of $A - V_k$, in a with only one c_k. Two of the U_i are in red, two in yellow and one in pale brown.[26]

22.6 Remark:[b] even if c_k is a critical point of f and $l_k = f^j(c_k) \in \mathbb{S}^1$, we can still define $f\big|_A^{-n}$ if we suppose that $L \cap \mathbb{S}^1 = \varnothing$ and r is small enough. Indeed, if $C = \{\text{critical points of } f\}$, $VC = \{\text{critical values of } f\} = f(C)$, $VC_n = VC(f^n) = VC \cup f(VC) \cup \cdots \cup f^{n-1}(VC)$

$$f^n : \mathbb{S}^2 - f^{-n}(VC_n) \to \mathbb{S}^2 - VC_n \text{ is a covering.}$$

Since $f\big|_{\mathbb{S}^1}$ is a C^ω diffeomorphism, on a small neighborhood V_k of l_k, we can choose a determination of $f^{-n}\big|_V$ such that $f^{-n}\big|_{\mathbb{S}^1} = (f\big|_{\mathbb{S}^1})^{-n}$. The we can extend $(f\big|_{\mathbb{S}})^{-n}$ on a ring A avoiding L and $\bigcup_{j\geqslant 1} f^j(C - C_1) - \mathbb{S}^1 = L_1$ by covering $A - \bigcup V_k$ by a

[b]This argument is implicitly used several times by P. Fatou and G. Julia and we have also used it implicitly in [H2] pages [missing pages].[27]

[24]TN : In other words, he uses that the formula defining h_n is known to converge on $\mathbb{R}$ to the conjugacy to the rotation.

[25]TN : Is there a mistake in the reference? And why is the conclusion not immediate?

[26]TN : I took the liberty of completing Herman's sketch. It is possible to us fewer domains but that is not the point here.

[27]TN : I was not able to find where in [H2] this implicit arguement is used.

finite sequence of simply connected open sets U_j: in a neighborhood of l_k we choose a cover by sectors (see the figure).

22.7 The construction of Ghys.

Let f be as in §22.1 and choose $\mu \in \mathbb{S}^1$ such that $\mu f = h^{-1} \circ r_\alpha \circ h$, $r_\alpha(z) = e^{2\pi i\alpha} z$ where h is a quasi symmetric homeomorphism of $\mathbb{S}^1$ but so that μf is not C^ω-conjugated to R_α (it is possible by Theorem 2). By the Ahlfors-Beurling theorem [A] or [L] there exists a K-quasi conformal homeomorphism of $\mathbb{D}$ such that $H\big|_{\mathbb{S}^1} = h$. Let

$$t(z) = \mu f(z), \quad \text{if} \quad |z| \geqslant 1 \ ;$$
$$t(z) = H^{-1} \circ r_\alpha \circ H, \quad \text{if} \quad |z| \leqslant 1.$$

The continuous map $t : \mathbb{S}^2 \to \mathbb{S}^2$ leaves invariant the following Beltrami form u, which is mesurable, satisfies

$$\|u\|_{L^\infty} \leqslant \delta < 1 \quad \text{where} \quad (\delta+1)(1-\delta)^{-1} = K$$

and is defined as follows

$$u(z) = \frac{H_{\overline{z}}}{H_z} \quad \text{where} \quad H_{\overline{z}} = \overline{\partial} H,\ H_z = \partial H \quad \text{and} \quad \overline{\partial} = \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big)$$

$$u(t^n(z)) = u(z)\frac{\overline{(f^n)'(z)}}{(f^n)'(z)} \quad \text{if}$$

$z \in V_n = t^{-n}(\mathbb{D}) \cap \{|z| > 1\}$ $n \geqslant 1$ (we use the pairwise disjoint character of the open sets V_n[28]) ; and

$$u(z) = 0, \quad \text{if} \quad z \notin \mathbb{D} \cup \big(\bigcup_{n \geqslant 1} V_n\big).$$

Let G be the homeomorphism of $\mathbb{S}^2$ given by the Morrey-Ahlfors-Bers theorem [A]; it is K quasi conformal and thus absolutely continuous with respect to the Lebesgue measure and satisfies $G(\infty) = \infty$

$$G_{\overline{z}}/G_z = u(z).$$

The map $f_1 = G \circ t \circ G^{-1}$ is continuous, locally quasi-conformal except at a finite number of points, absolutely continuous on almost every line and almost everywhere conformal. The map f_1 is thus a rational map of $\mathbb{S}^2$.

22.8 The rational map f_1 has a Siegel singular disk $G(\mathbb{D})$. The open set $G(\mathbb{D})$ is indeed the connected component of $\mathbb{S}^2 - J(f_1)$ containing the linearizable elliptic fixed point $G \circ H^{-1}(0)$ of multiplier $e^{2\pi i\alpha}$, since $\mu f\big|_{\mathbb{S}^1}$ is not $\mathbb{R}$-analytically conjugated to R_α. Since t is injective on a neighborhood of $\mathbb{S}^1$, f_1 is injective on a neighborhood of the quasi circle $\partial G(\mathbb{D})$ and thus f_1 has no critical point on $\partial G(\mathbb{D})$.

22.9 The degree of f is of the form $2k+1$, $k \in \mathbb{N}^*$ and that of f_1 is $k+1$ (we removed the k poles or iverse images by g of $\{\infty\}$, contained in $\mathbb{D}$).

22.10 If we start from a $\mathbb{C}$-analytic map $R_\lambda \circ f : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$, $\lambda \in \mathbb{R}$ such that on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, $R_\lambda \circ f\big|_{\mathbb{T}^1}$ is a diffeomorphism then the same construction gives an entire map $f_1 : \mathbb{C} \to \mathbb{C}$ having a Siegel singular disk with the same properties as in 22.8 (we define f_1 only on $\mathbb{S}^2 - \{\infty\} \cong \mathbb{C}$).

22.11 If we start from G defined in §22.2 then for f_1, the point ∞ is super attracting and since f_1 has degree 2, the map f_1 is a degree 2 polynomial and thus is conjugated by an affine transformation (i.e. $z \mapsto b_1 z + b_2$, $b_1 \in \mathbb{C}^*$, $b_2 \in \mathbb{C}$) to

$$g_\alpha : z \mapsto e^{2\pi i\alpha}(z + z^2).$$

[28]TN : The definition of V_n has to be slightly modified for them to be pairwise disjoint: one should take $n =$ the first iterate that falls in $\mathbb{D}$.

From all this the following theorem follows:

Theorem 3. *There exists $\alpha \in \mathbb{R} - \mathbb{Q}$ such that g_α is lineazisable at the point 0 and such that its Siegel disk S satisfies:*

(i) ∂S *is a quasi circle;*

(ii) $c = \dfrac{-1}{2} \notin \partial S$ *(g_α is injective on a neighborhood of $\overline{S}$) ;*

(iii) $g_\alpha^n(c) \notin \partial S$, *for all $n \geqslant 1$.*

Indeed, (i) and (ii) follow from §22.8 and (iii) follows from the proof of §22.5 and §22.6.[29] ■

It is worth noticing that for g_α the orbit of the critical point $c = \frac{-1}{2}$ will be "very similar" to the orbit for μg of the critical point c, $c \neq \infty$, $|c| > 1$ (the critical points of μg are c, $1/\overline{c}$, 0 and ∞).

22.12 By the result of [G] or [H2] the number $\alpha \in \mathbb{R} - \mathbb{Q}$ does not satisfy a diophantine condition.

We recall that if α is a Brjuno number:

$$\sum_{k \geqslant 0} \frac{\log q_{k+1}}{q_k} < +\infty$$

where q_k are the denominators of the convergents of α and if $f_1(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$ is a germ of $\mathbb{C}$-analytic map at 0 then f is lineazisable at 0 (Siegel-Brjuno theorem [B]).

22.13

Proposition. *Let α be the number given by Theorem 3, then one of the following claims is true*
(i) α is not a Brjuno number ;
(ii) there exists an $\mathbb{R}$-analytic diffeomorphism f of $\mathbb{T}^1$ such that $\rho(f) = \alpha$ is a Brjuno number but f is not C^ω conjugated to R_α.

Proof. If both (i) and (ii) are false then the same proof as in [G] or [H2], using not-(ii), implies in Theorem 3 that $c \in \partial S$. ■

The author, as he writes the present lines, does not have any opinion on which of the claims of §22.13 is true[30] ((i) implies unexpected cancellations, see [Y3], and not-(ii) holds if f is a perturbation of R_α for the C^ω topology).

22.14

Proposition. *There exists a non-linear entire map $f_1 = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$, $z \longrightarrow 0$ such that*
(i) $f_1'(z) \neq 0$, for all $z \in \mathbb{C}$;
(ii) f_1 is lineazisable at 0 and the Siegel singular disk S of f_1 has compact closure in $\mathbb{C}$ and f_1 is injective in a neighborhood of $\overline{S}$.

Proof. We use the construction of 22.10 starting from f such that

$$Df(\theta) = e^{a \sin(2\pi\theta) + c}, \quad \text{where} \quad a \in \mathbb{R}^*$$

[29]TN : Point (iii) immediately follows from §22.5 for t, and thus for f_1 by the conjugacy. The proof of §22.5 (proof that includes the argument given in §22.6) can be adapted to deduce (iii) from (ii) without using t.

[30]TN : We know today that the true fact is point (ii). Yoccoz has indeed proved the optimality of the Brjuno for the quadratic family.

and $c \in \mathbb{R}$ satisfies

$$\int_0^1 e^{a\sin(2\pi\theta)+c} d\theta = 1.$$

■

22.15

Proposition. *There exists a univalent holomorphic map* $G : \mathbb{D} \to \mathbb{C}$ *with* $G(0) = 0$, $G'(0) = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$, G *is lineazisable at* 0 *and the maximal linearization domain of* G, S_1 *satisfies* $\overline{S}_1 \cap \partial\mathbb{D} = \varnothing$.

Proof.[31] We conjugate, using the conformal representation theorem, the map g_α given by Theorem 3, remarking that g_α is injective on a small simply connected open neighborhood V of $\overline{S}$ and satisfies $g_\alpha(\overline{S}) = \overline{S}$. ([32]) ■

From [G] or [H2] it follows that 22.14 and 22.15 are false if α satisfies a diophantine conditions.

22.16 Remark. In Ghys' construction 22.7, if $\rho(\mu f) = \alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$ then μf is not generally quasi symmetrically conjugated to R_α (cf. §8) but by the theorem of Denjoy μf is topologically conjugated to R_α: $\mu f = h^{-1} \circ R_\alpha \circ h$. We can extend h into a C^1 diffeomorphism of $\mathbb{D}$ and then define a continuous map $t : \mathbb{S}^2 \to \mathbb{S}^2$. Generally t is not topologically conjugated to a rational map for otherwise f_1 would have a lineazisable elliptic fixed point of multiplier $e^{2\pi i\alpha}$ yet this does not hold for α belonging to a G_δ dense subset G of $\mathbb{T}^1$ (the set G does not depend on the rational map f) cf. [H1,VIII.15].

Question. *Find a necessary and sufficient condition for* t *to be topologically conjugated to a rational map on the Riemann sphere.*

23. Generalisations of the construction of E. Ghys.

Let f be like in §22.1 and μ such that μf satisfies the conclusions of Theorem 2 and $\alpha = \rho(\mu f|_{\mathbb{S}^1})$.

Proposition. *There exists a rational map* f_1 *of the same degree as* f, *leaving* $\mathbb{S}^1$ *invariant, with* $\rho(f_1|_{\mathbb{S}^1}) = \alpha$, *having a singular ring* A *that contains* $\mathbb{S}^1$ *(*A *is a connected component of* $\mathbb{S}^2 - J(f_1)$*) and such that* f_1 *has no critical point on* ∂A *and* ∂A *is the union of two disjoint quasi circles.*

Proof. Let $0 < t < 1$ be given, we define

$$g_1(z) = \frac{1}{t}\mu f(tz) \qquad |z| \geqslant \frac{1}{t}$$

$$g_1(z) = \frac{1}{\overline{g_1(1/\overline{z})}} \qquad |z| \leqslant t.$$

Let $H : \{t \leqslant |z| \leqslant \frac{1}{t}\} \to \{t \leqslant |z| \leqslant \frac{1}{t}\} = B$ be a quasi conformal homeomorphism such that:

[31]TN : There is a problem in this proof, see the next footnote.

[32]TN : There is no reason for the conformal map from V to $\mathbb{D}$ to have an extension to $V \cup g_\alpha(V)$ taking values in $\mathbb{C}$. A priori we only get a $G : U \to \mathbb{D}$ with $U \subset \mathbb{D}$, not a $G : \mathbb{D} \to \mathbb{C}$. It is not clear that one can ensure that the domain of G is $\mathbb{D}$. One would like to send $V \cup g_\alpha(V)$ in $\mathbb{C}$ so that V is sent to $\mathbb{D}$ and 0 to 0. Thus at least ∂V should be analytic on the part inside $g_\alpha(V)$. We may as well take the whole boundary ∂V analytic but this is not sufficient.

$$H \text{ commutes with } z \mapsto 1/\overline{z}$$
$$H \circ r_\alpha \circ H^{-1}(z) = g_1(z), \quad z \in \partial B.$$

This is possible since $(\mu f)\big|_{\mathbb{S}^1}$ is quasi symmetrically conjugated to $r_\alpha : z \mapsto e^{2\pi i\alpha} z$.

We define
$$T_1(z) = g_1(z), \quad \text{if } |z| \geqslant \frac{1}{t} \text{ or } |z| \leqslant t$$
and
$$T_1(z) = H \circ r_\alpha \circ H^{-1}(z), \quad \text{if } t \leqslant |z| \leqslant \frac{1}{t}.$$

By construction, T_1 commutes with the conformal orientation reversing involution: $z \mapsto 1/\overline{z}$. By the same argument as in §22.7, T_1 leaves invariant a Beltrami form u. We can therefore conjugate T_1 by a quasi conformal homeomorphism to a rational map g and we can choose u so that g commutes with an involution j, which is conjugated to $z \mapsto 1/\overline{z}$, conformal and thus ~~$j \in \mathrm{PGL}(2,\mathbb{C})$~~ we can conjugate j by an element $h \in \mathrm{SL}(2,\mathbb{C})$ to $h^{-1} \circ j \circ h(z) = 1/\overline{z}$ and $f_1 = h \circ g \circ h^{-1}$ satisfies all the conclusions of the proposition. ∎

24. If we started from $\mu f = \mu z^2 \dfrac{1-\overline{a}z}{z-a}$, f_1 would have the same form for some μ_1, $\mu_1 \in \mathbb{S}^1$ and $a_1 \in \mathbb{C}$, $0 < |a_1| < \frac{1}{3}$.

Conclusion. *The existence of a singluar ring for a rational map does not only depend on the arithmetical properties of the rotation number but also on the rational map (i.e. on the values of the coefficients of P and Q such that $f = P/Q$ where P and Q are relatively prime polynomials of degree $\leqslant d$).*

25. Let f be as in §22.1 and let $\mu \in \mathbb{S}^1$ such that $\rho(\mu f)$ satisfies a diophantine condition. Then by the result of J.C. Yoccoz [Y2] the map $\mu f\big|_{\mathbb{S}^1}$ is C^ω conjugated to $r_\alpha(z) = e^{2\pi i\alpha} z$. From this, it follows that μf has a singular ring containing $\mathbb{S}^1$. This ring will disappear if $\mu f\big|_{\mathbb{S}^1}$ is only an (analytic $\mathbb{S}^1$) homeomorphism i.e. μf has a critical point on $\mathbb{S}^1$. For many examples the reader is invited to see [H1,IV].

<u>For instance</u> if $f = z^2 \dfrac{1-\overline{a}z}{z-a}$, if $a\overline{a} = \frac{1}{9}$ then $f\big|_{\mathbb{S}^1}$ is a homeomorphism having a double critical point on $\mathbb{S}^1$. J.C. Yoccoz proved [Y4] that if $\mu f\big|_{\mathbb{S}^1}$ is an (analytic) homeomorphism and $\rho(\mu f\big|_{\mathbb{S}^1}) \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$ then $\mu f\big|_{\mathbb{S}^1}$ is topologically conjugated to R_α. Generally $f\big|_{\mathbb{S}^1}$ is not quasi symmetrically conjugated to a rotation (cf. §8).

Question. [33] *If $\rho(\mu f\big|_{\mathbb{S}^1}) = \alpha$ is a bounded type number, is $\mu f\big|_{\mathbb{S}^1}$ quasi symmetrically conjugated to R_α ?*

26. The following proposition has been obtained independently by Adrien Douady:

Proposition. *Let $\mu f\big|_{\mathbb{S}^1}$ be an analytic homeomorphism having a critical point on $\mathbb{S}^1$ and such that $(\mu f)\big|_{\mathbb{S}^1}$ is quasi symmetrically conjugated to R_α, $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$ then there exists a rational map g having Siegel disk S associated to a lineazisable fixed point of g, of multiplier $e^{2\pi i\alpha}$, such that ∂S is a quasi circle and such that there exists a critical point of g on ∂S.*

[33] TN : The answer is positive, as proved by Herman and Świątek in subsequent work. These works were possible thanks to the introduction of the schwarzian derivative, an analogue of higher order to the distortion derivative $D \log Df$.

The proof is almost identical to that of §22.7 and §22.8. ■

Sequel in the next issue.

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