Siegel disks

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Definition of Siegel disks

$0 \in U$ open subset of $\mathbb{C}$
$f : U \longrightarrow \mathbb{C}$ holomorphic

f defines a dynamical system: $z_{n+1} = f(z_n)$

$f(0) = 0$ (the origin is a fixed point)
the multiplier $f'(0)$ has modulus one: $f'(0) = e^{2i\pi\theta}$
the multiplier is aperiodic: $\theta \notin \mathbb{Q}$

This is called an
irrationally indifferent fixed point of a complex one dimensional holomorphic dynamical system.

Near 0, f is close to an aperiodic rotation. Does the dynamics behave likewise?

Definition of Siegel disks

What would it mean to behave like a rotation? That near 0,

- f is analytically conjugated to $R_\theta : z \mapsto e^{2i\pi\theta} z$? (meaning $\exists \phi$ analytic bijection defined near 0 mapping 0 to 0 and $\phi \circ f \circ \phi^{-1} = R_\theta$)
- f is topologically conjugated to R_θ? (now ϕ is only required to be a homeomorphism)
- the orbits stay bounded? (Lyapunov stability)

It turns out that all three are equivalent. We say that f is *linearizable*.

The quantity θ is unique and is called the *rotation number*.

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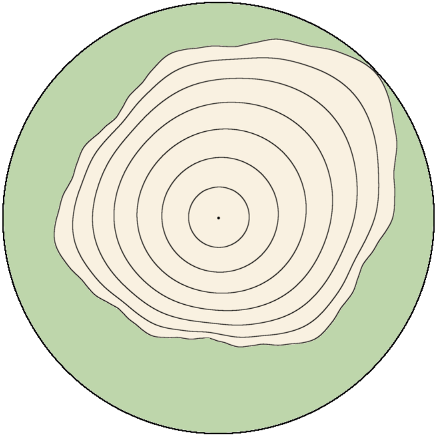
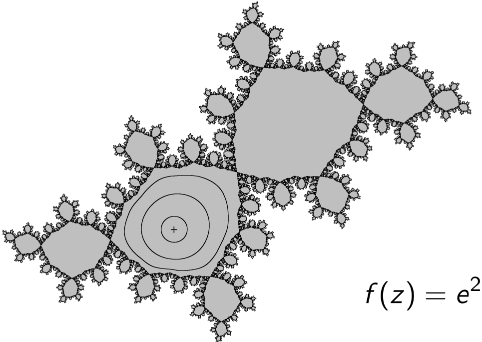
Definition

The Siegel disk Δ is the biggest open subset of U containing 0 on which f is analytically conjugated to a rotation. If f is not linearizable, then we set $\Delta = \emptyset$.



Examples

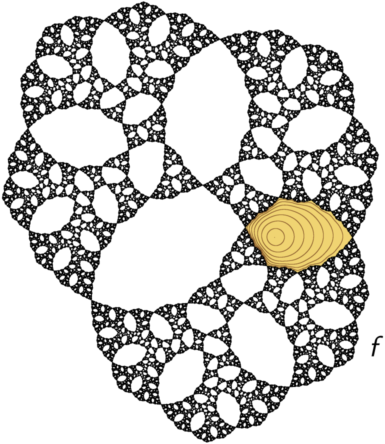


$$f(z) = e^{2i\pi \frac{\sqrt{5}-1}{2}} z + z^2$$

Polynomials. If f is a polynomial, ∞ is superattracting. Let $A(\infty)$ be the basin of attraction of ∞. Let $K = \mathbb{C} \setminus A(\infty)$ be its complement. Then K is also the set of points that have a bounded orbit.

- f is linearizable if and only if $0 \in \operatorname{int} K$.
- Δ is the connected component containing 0 of $\operatorname{int} K$.

Examples



$$f(z) = \frac{ze^{2i\pi\frac{\sqrt{5}-1}{2}}}{(1+z)^2}$$

Rational maps. If f is rational then Δ is a connected component of the Fatou Set. (Fatou set=Riemann sphere minus the Julia set)

Mañe's theorem

Note that, appart from its center, a Siegel disk cannot contain any periodic point, critical point, nor any iterated preimage of a critical or periodic point. On the other hand it can contain an iterated image of a critical point. However:

Let $\omega(z)$ be the omega-limit set of z, i.e. the set of accumulation points of its orbit.

Theorem: (Mañe) *For all polynomial with a Siegel disk Δ, there exists at least one critical point c such that c is recurrent (i.e. $c \in \omega(c)$) and $\partial\Delta \subset \omega(c)$.*

Tools of the proof: inverse branches, normal families and bounded distortion theorems à la Koebe.

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Bjruno's theorem

For some values of the rotation number θ, all functions f are linearizable. This was discovered by Siegel in 1942: he proved it when θ is diophantine.

Definition: $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is diophantine if $\exists C > 0$ and $\exists \nu \in \mathbb{R}$ such that $\forall p \in \mathbb{Z}$, $\forall q \in \mathbb{N}^*$,

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^\nu}$$

(Necessarily, $\nu \geq 2$).

Theorem: (Brjuno, 1965) *If $\theta \in \mathcal{B}$ then f is linearizable.*

The set $\mathcal{B}$ is the set of $\theta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\sum \frac{\log q_{n+1}}{q_n} < +\infty$, where $p_n/q_n \longrightarrow \theta$ is the sequence of approximants given by the continued fraction expansion of θ. It contains the diophantine numbers.

Method: estimation of the convergence radius of the power series expansion of the linearizer ϕ.

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Yoccoz's theorems

In 1988, Yoccoz proved that Brjuno's condition is optimal:

Theorem: (Yoccoz) *If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\theta \notin \mathcal{B}$, then there exist functions f with rotation number θ that are not linearizable.*

He also gave an optimal lower bound on the inner size of Siegel disks: let

$$B(\theta) = \sum \frac{\log q_{n+1}}{q_n}$$

Theorem: (Yoccoz) *There exist $C > 0$ and $C' > 0$ such that if $\theta \in \mathcal{B}$, then*

- *for all f injective in $B(0,1)$ then Δ contains $B(0, Ce^{-B(\theta)})$,*
- *there exist an f injective in $B(0,1)$ such that Δ does not contain $B(0, C'e^{-B(\theta)})$.*

His method is very different: he uses a more geometric approach, called sector renormalization.

Universality of the quadratic family

Yoccoz was able to transfer some of his results to Siegel disks of period one of quadratic polynomials, using a trick due to Ill'Yashenko. Let

$$P_\theta(z) = e^{2i\pi\theta} z + z^2.$$

Theorem: (Yoccoz) *If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and if there exists a non-linearizable f with rotation number θ then $P_\theta(z)$ is not linearzable (at the origin).*

He also gave an upper bound on the inner size of the Siegel disk of P_θ, but he could not prove the following conjecture:

Conjecture: (Yoccoz) *There exists $C > 0$ such that $\forall \theta \in \mathcal{B}$, the Siegel disk of P_θ does not contain $B(0, Ce^{-B(\theta)})$.*

This was proved by Buff and Chéritat in 2003.

Are there critical points on the boundary?

Subtle question...

Herman proved partial results. He defined a set $\mathcal{H}$ called the set of Herman numbers, and proved that a Siegel disk with rotation number in $\mathcal{H}$, for a unicritical polynomial ($f(z) = z^d + c$), necessarily has a critical point on its boundary. He proved that $\mathcal{H}$ is non empty.

Method: we reduce the problem to a question on analytic circle diffeomorphisms. It is then refinements of Denjoy's theorem.

Yoccoz characterized $\mathcal{H}$ arithmetically. In particular Dioph $\subset \mathcal{H} \subset \mathcal{B}$.

Is this true for all polynomials? All rational maps? What about the converse?

For polynomials, J.T. Rogers was able to prove that if there exists a counterexample, its Siegel disk necessarily has an *indecomposable* boundary: it is not the union of two closed connected strict subsets. In particular it is not locally connected.

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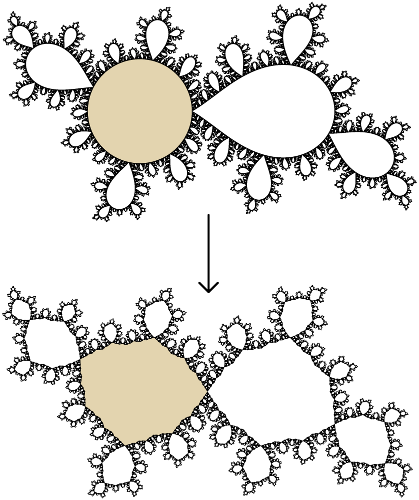
The boundary of quadratic Siegel disks

Herman also proved that there are quadratic polynomial with Siegel disks whose boundary is a Jordan curve and *does not contain* the critical point. Tool: quasiconformal model due to Ghys, constructed by holomorphic surgery.

Independently, Perez-marco constructed maps $f : B(0,1) \to \mathbb{C}$ (not polynomial, nor entire) whose Siegel disks are compactly contained in $B(0,1)$ and have a *smooth boundary* (C^∞). There can be no critical points on $\partial\Delta$. Tool : a modification with tube-log Riemann surfaces of Yoccoz's inverse renormalization.

A variation of Ghys's model has proved to be extremely useful:

Models of Quadratic Siegel disks



Theorem (Herman, Swiatek, Ghys, Douady, Petersen, Lyubich, McMullen, ...): when θ is a bounded type number then the Siegel disk's boundary is a Jordan curve, better: a quasicircle, it contains the critical point, the Julia set is locally connected, its Lebesgue measure is $= 0$, its Hausdorff dimension is < 2. The filled-in Julia set is asymptotically dense at the boundary of Δ. If θ is a quadratic algebraic number over $\mathbb{Q}$, the Julia set is asymptotically self-similar at the critical point. [... + universality properties ...]

Omited in this talk

There are a lot of other important results (Perez Marco, Petersen-Zakeri, Graczyk-Swiatek, Graczyk-Jones, Rodin, Rogers, Shishikura, Zhang, . . .)

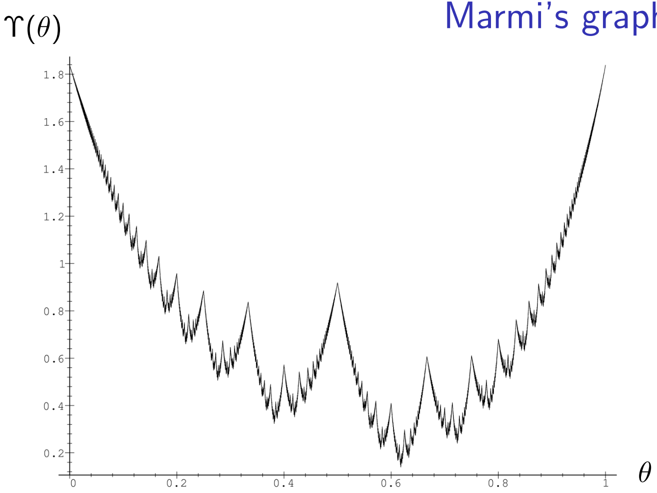
More on quadratic Siegel disks

Buff and Chéritat introduced two tools in the subject: control on parabolic explosion, relative Schwarz lemma, semi-cntinuity methods. This yielded, in the family P_θ of quadratic polynomials:

- the proof of the existence of smooth Siegel disks
- the proof of Yoccoz's conjecture on the size of Siegel disks
- the proof of a conjecture of Marmi refining Yoccoz's: the (uniform) continuity of the function $\Upsilon(\theta) = \log r(\theta) + Y(\theta)$ where $r(\theta)$ is the conformal radius of the Siegel disk and Y is a variant of the Brjuno sum introduced by Yoccoz

among other results.

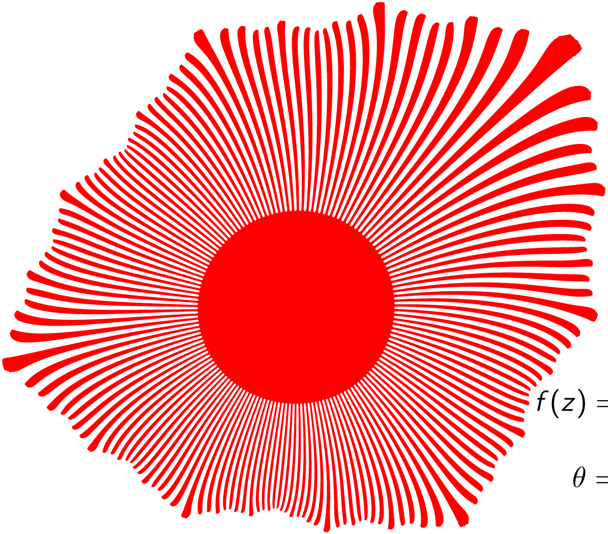
Marmi's graph



Conjecture (Marmi, Moussa, Yoccoz)

The function Υ is $1/2$-Hölder continuous.

Digitated Siegel disks

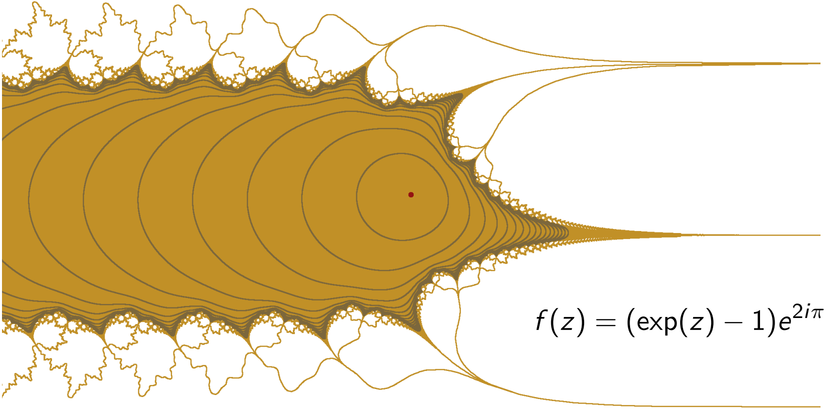


$$f(z) = P_\theta(z) = e^{i2\pi\theta} z + z^2$$

$$\theta = [0; 1^{[11]}, 10^{60}, 1^{[\infty]}]$$

Time for conjectures

An abysmal monster



$$f(z) = (\exp(z) - 1)e^{2i\pi \frac{\sqrt{5}-1}{2}}$$

Bounded Siegel disks

Conjecture

If a Siegel disk is compactly contained in the domain of definition of f then its boundary is a Jordan arc.

This includes rational maps and polynoms

Conjecture (Douady)

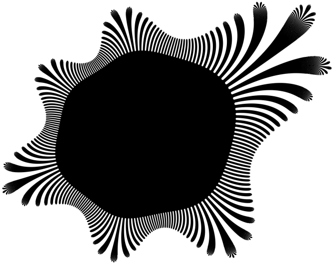
If a polynomial has a Siegel disk, then its rotation number is necessarily in $\mathcal{B}$.

Some particular cases of this conjecture have been proved by Lukas Geyer.

Ongoing work

Shishikura and Inou have build a renormalization operator that allows a very fine study of Siegel disks of quadratic polynomials, in particular of the critical orbit. This should allow to prove a lot of conjectures on quadratic Siegel disks. Their method does not yet cover all rotation numbers.

It also allows to study *Hedgehogs*. These are objects invented by Perez-Marco, that generalize Siegel disks. We are developing toy models of hedgehogs of quadratic polynomials (Buff, Rempe, Chéritat).



Siegel disks of degree 3 polynomials

A refinement of Douady's conjecture:

Conjecture (Buff)

There exists $C_d > 0$ such that for all polynomial f of degree d having an irrationnally indifferent linearizable fixed point at the origin:

$$\frac{\text{conformal radius}(\Delta)}{\text{distance}(0, \text{crit. pts. of } f)} \leq C_d e^{-\frac{B(\theta)}{d-1}}$$

It is known that we cannot get better than $-\frac{B(\theta)}{d-1}$.

A slice of the parameter space