UPPER BOUND FOR THE SIZE OF QUADRATIC SIEGEL DISKS.

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Abstract. If α is an irrational number, we let $\{p_n/q_n\}_{n\geq 0}$, be the approximants given by its continued fraction expansion. The Bruno series $B(\alpha)$ is defined as

$$B(\alpha) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}.$$

The quadratic polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$ has an indifferent fixed point at the origin. If P_α is linearizable, we let $r(\alpha)$ be the conformal radius of the Siegel disk and we set $r(\alpha) = 0$ otherwise. Yoccoz proved that if $B(\alpha) = \infty$, then $r(\alpha) = 0$ and P_α is not linearizable. In this article, we present a different proof and we show that there exists a constant C such that for all irrational number α with $B(\alpha) < \infty$, we have

$$B(\alpha) + \log r(\alpha) < C.$$

Together with former results of Yoccoz (see [Y]), this proves the conjectured boundedness of $B(\alpha) + \log r(\alpha)$.

1. Introduction.

In this article, we are interested in the dynamics of quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$, $\alpha \in \mathbb{C}$. When α is real, the quadratic polynomial P_α has an indifferent fixed point at 0 and it is *linearizable* if it is conjugate to the rotation $z \mapsto e^{2i\pi\alpha}z$ in a neighborhood of 0. The arithmetic nature of α will play a central role. We denote by $\{p_n/q_n\}_{n\geq 0}$ the approximants to α given by its continued fraction expansion (see appendix A).

Remark. Every time we use the notation p/q for a rational number, we mean that $q > 0$ and p and q are coprime.

The first result of linearizability is due to C.L. Siegel [Si] in 1942. He proved that when

$$\log q_{n+1} = \mathcal{O}(\log q_n),$$

every germ $z \mapsto e^{2i\pi\alpha}z + \mathcal{O}(z^2)$ is linearizable. Around 1965, following Siegel's ideas, Bruno [Bru] proved that every germ $z \mapsto e^{2i\pi\alpha}z + \mathcal{O}(z^2)$ is linearizable under the weaker assumption:

$$\sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n} < +\infty.$$

An irrational number α satisfying this condition is called a Bruno number, and the sum on the left-hand side of the inequality is noted $B(\alpha)$. In 1987, Yoccoz [Y] has completely solved the problem, showing that when $\alpha \in \mathbb{R}$ is not a Bruno number, the quadratic polynomial P_α is not linearizable. Yoccoz first proved that if there were a non linearizable germ $z \mapsto e^{2i\pi\alpha}z + \mathcal{O}(z^2)$, then the quadratic polynomial P_α would not be linearizable. He then proved the existence of such a germ.

More recently, in [C], the second author proved directly the non linearizability of P_α when α is not a Bruno number. His proof consists in proving that when α is not a Bruno number, 0 is accumulated by periodic points of P_α (Pérez-Marco [PM] in fact proved that not only periodic points, but whole cycles, accumulate 0).

Let us now introduce the notion of conformal radius.

Definition 1. *If $U \subset \mathbb{C}$ is a hyperbolic domain containing 0, the conformal radius $\mathrm{rad}(U)$ of U at 0 is equal to $|\pi'(0)|$ where $\pi : (\mathbb{D}, 0) \to (U, 0)$ is a universal covering.*

Remark. When U is simply connected, for example in the case of a Siegel disk, π is the Riemann mapping, and $\mathrm{rad}(U)$ is the classical conformal radius.

Assume that $0 \in V \subset U$. Then the universal covering $\pi_V : (\mathbb{D}, 0) \to (V, 0)$ lifts to a mapping $f : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ such that $\pi_V = \pi_U \circ f$. By Schwarz's lemma, we have $|f'(0)| \le 1$, and thus, $\mathrm{rad}(V) = |\pi_V'(0)| \le |\pi_U'(0)| = \mathrm{rad}(U)$:

$$\mathrm{rad}(V) \le \mathrm{rad}(U).$$

The work by Yoccoz [Y] already provides a control of the conformal radius $r(\alpha)$ of the Siegel disk of P_α. First, if $B(\alpha) < \infty$, then

$$-B(\alpha) + C \le \log r(\alpha)$$

for some universal constant C. Second, there exists a function $C'(\varepsilon)$, such that $\forall \varepsilon > 0$,

$$\log r(\alpha) \le -(1-\varepsilon)B(\alpha) + C'(\varepsilon).$$

In other words,

$$C \le \log r(\alpha) + B(\alpha) \le \varepsilon B(\alpha) + C'(\varepsilon).$$

In this article, we will prove the following results.

Theorem 1. *Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational number and let p_n/q_n be the approximants to α. For $N \ge 0$, let Ω_N be the complement in $\mathbb{C}$ of the external ray of P_α of argument 0 and the periodic points of period less than or equal to q_N. Then,*

$$\log \mathrm{rad}(\Omega_N) + \sum_{n=0}^{N} \frac{\log q_{n+1}}{q_n} < 16.$$

If P_α has a Siegel disk, it must be contained in the intersection of those sets Ω_N. Thus, we have the following corollary.

Corollary 1. *If $B(\alpha) = \infty$, then $r(\alpha) = 0$. If $B(\alpha) < \infty$, then*

$$B(\alpha) + \log r(\alpha) < 16.$$

Note that we did not try to get the best possible constants. The proof we give is a quantification of the proof given in [C] (with some minor modifications), together with a big improvement in one inequality.

2. Sketch of the proof.

Let us first present the main steps of the proof. Note that the functions $\alpha \mapsto B(\alpha)$ and $\alpha \mapsto r(\alpha)$ are even and periodic of period 1: for $B(\alpha)$ it is proved in appendix A, and for $r(\alpha)$, the periodicity comes from $P_{\alpha+1} = P_\alpha$, and the other claim from the fact that $P_{-\alpha}$ and P_α are conjugated by an isometry (namely $z \mapsto \bar{z}$). Thus, without loss of generality, we may assume $\alpha \in]0, 1/2[\setminus\mathbb{Q}$.

Step 1. For each rational number p/q, the polynomial $P_{p/q}$ has a parabolic fixed point at 0. When α is sufficiently close to p/q, this parabolic point splits into a simple fixed point at 0 and a periodic cycle of period q which is close to 0. The first step consists in studying the dependence of this cycle on $\alpha \in \mathbb{C}$. Roughly speaking, as long as the cycle does not collide with another cycle, it is possible to follow it holomorphically. More precisely, we have the following two statements. The proofs are given in section 3 below.

Definition 2. *For each rational number p/q, let $R(p/q)$ be the largest real number such that $P_\alpha^{\circ q}$ has no multiple fixed point for $\alpha \in B\big(\frac{p}{q}, R(\frac{p}{q})\big) \setminus \big\{\frac{p}{q}\big\}$. Moreover, set $r(p/q) = [R(p/q)]^{1/q}$.*

The proofs of the two following propositions are detailed in [C] and [BC], but for completeness, we sketch them in section 3.

Proposition 1. *Let p/q be a rational number, and $\zeta = e^{2i\pi p/q}$. There exists an analytic function $\chi : B(0, r(p/q)) \to \mathbb{C}$ such that $\chi(0) = 0$ and for any $\delta \in B(0, r(p/q)) \setminus \{0\}$, $\chi(\delta) \neq 0$ and the set*

$$\Big\langle \chi(\delta), \chi(\zeta\delta), \chi(\zeta^2\delta), \ldots, \chi(\zeta^{q-1}\delta) \Big\rangle$$

forms a cycle of period q of $P_{p/q+\delta^q}$. We will note $\chi = \chi_{p/q}$, since it depends on p/q.

The proof is a simple application of the implicit function theorem.

Proposition 2 (Key inequality)**.** *For any rational number p/q, we have*

$$R(p/q) \geq \frac{1}{q^3}.$$

The proof relies on the Yoccoz inequality and on a combinatorial theorem. Note it is probably not optimal : the correct order is conjectured to be equal to 2.

Step 2. The polynomial P_α is a monic polynomial. It is affinely conjugate to the polynomial $z \mapsto z^2 + c$ with $c = e^{2i\pi\alpha}/2 - e^{4i\pi\alpha}/4$. As long as $c \notin [1/4, +\infty[$, there is a well-defined external ray $\mathcal{R}_0(\alpha)$ of argument 0 which does not bifurcate and lands at a repelling fixed point. Note that $c \in [1/4, +\infty[\iff \mathrm{Re}(e^{2i\pi\alpha}) = 1$.

Definition 3. *Denote by $\mathcal{B}$ the set of parameters $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(e^{2i\pi\alpha}) = 1$. For each rational number p/q with $q \geq 2$, let $R'(p/q)$ be the largest real number such that*

$$B\left(\frac{p}{q}, R'(p/q)\right) \subset \mathbb{C} \setminus \mathcal{B}.$$

Proposition 3. *For each rational number p/q with $q \geq 2$, when α ranges in $B(p/q, R'(p/q))$, the external ray $\mathcal{R}_0(\alpha)$ does not bifurcate and lands at a repelling fixed point located at $1 - e^{2i\pi\alpha}$.*

Proof. Since $\mathcal{R}_0(\alpha)$ does not bifurcate, it moves holomorphically together with its landing point. This landing point must be a fixed point of P_α and it cannot be 0 since for $\alpha = p/q$, and $q \geq 2$, the external rays landing at 0 are not fixed whereas $\mathcal{R}_0(\alpha)$ is fixed. ∎

Proposition 4. *For any rational number p/q with $q \geq 2$, we have*

$$R'(p/q) \geq \frac{1}{q^2}.$$

The proof is given in section 4 below. Note that in particular, $R'(p/q) \geq 1/q^3$.

Step 3. Let us now assume that $\alpha_0 \in]0, 1/2[\setminus\mathbb{Q}$ is an irrational number and let $\{p_n/q_n\}_{n\geq 0}$ be the approximants given by its continued fraction expansion. Then, for $n \geq 0$, q_n is bounded from below by the n-th Fibonacci number F_n ($F_0 = 1$, $F_1 = 2$, $F_{n+1} = F_n + F_{n-1}$).

The next definition will be better understood if the reader keeps in mind that, according to classical properties of approximants, for all $n \in \mathbb{N}$,

$$\frac{1}{2q_n q_{n+1}} < \left|\alpha_0 - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

Definition 4 (Good approximants)**.** *Let $\mathcal{N}$ be the set of integers $n \geq 1$ such that $q_{n+1} > 2q_n^2$. Let $\{n_i\}_{i\geq 1}$, be the sequence of those integers n ordered increasingly.*
For $i \geq 1$, let

- *B_i be the disk centered at p_{n_i}/q_{n_i} with radius $1/q_{n_i}^3$,*
- *B_i^* be the punctured disk $B_i \setminus \{p_{n_i}/q_{n_i}\}$,*
- *D_i be the disk centered at p_{n_i}/q_{n_i} with radius $1/q_{n_i}^2$ and*
- *U_i be the disk centered at 0 with radius $(1/q_{n_i}^3)^{1/q_{n_i}}$.*

Remark. Note that the set $\mathcal{N}$ may be finite, or even empty, for example if $\alpha = (3-\sqrt{5})/2$.

Remark. The choice of the condition $q_{n+1} > 2q_n^2$ and of the radius $1/q_{n_i}^3$ of B_i are related to the term $1/q^3$ in proposition 2.

The set $\mathcal{N}$ has been chosen so that the following two propositions hold.

Proposition 5. *We have $B_1 \subset D_1 \subset \mathbb{C} \setminus \mathcal{B}$ and for all $i \geq 1$, $\alpha_0 \in B_i$ and*

$$B_{i+1} \subset D_{i+1} \subset B_i^*.$$

Proposition 6. *Moreover, for any $N \geq 1$*

$$\sum_{n=1}^{N} \frac{\log q_{n+1}}{q_n} < \sum_{\substack{i\geq 1\\ n_i \leq N}} \frac{\log q_{1+n_i}}{q_{n_i}} + \sum_{n\in[1,N]\setminus\mathcal{N}} \frac{\log 2F_n^2}{F_n}.$$

The proofs are given in section 5 below. An important point is that the Fibonacci numbers grow exponentially fast and so, for any constant C, we have

$$\sum_{n\geq 1} \frac{\log CF_n^2}{F_n} < \infty.$$

Thus, proposition 6 tells us that the contribution of "bad approximants" to the sum defining $B(\alpha)$ is universally bounded.

Remark. Here, it is not critical to have an optimal bound in proposition 2. Having c/q^2 instead of $1/q^3$ would only replace $\sum \frac{\log F_n^2}{F_n}$ by $\sum \frac{\log(cF_n)}{F_n}$.

Definition 5. *For each $\alpha \in B_i$, set*

$$S_i(\alpha) = \left\{ \delta \in U_i \;\middle|\; \frac{p_{n_i}}{q_{n_i}} + \delta^{q_{n_i}} = \alpha \right\} \quad \text{and} \quad \mathcal{C}_i(\alpha) = \chi_{p_{n_i}/q_{n_i}}(S_i(\alpha)).$$

Moreover, define by induction

$$V_0(\alpha) = \mathbb{C} \setminus \mathcal{R}_0(\alpha) \quad \text{and} \quad V_i(\alpha) = V_{i-1}(\alpha) \setminus \mathcal{C}_i(\alpha)$$

and set

$$S_i = S_i(\alpha_0), \quad \mathcal{C}_i = \mathcal{C}_i(\alpha_0) \quad \text{and} \quad V_i = V_i(\alpha_0).$$

For $i \geq 1$, the sets $\mathcal{C}_i(\alpha)$ form periodic cycles for P_α of period $q_{n_i} \geq 2$. They are clearly contained in $V_0(\alpha)$ since no periodic cycle which is not fixed can belong to the closure of the fixed external ray $\mathcal{R}_0(\alpha)$.

Step 4. The conformal radius of V_0 may be estimated as follows.

Proposition 7. *We have*

$$\log \operatorname{rad}(V_0) < -\frac{\log q_1}{q_0} + \log(8\pi).$$

Proof. Since $V_0(\alpha)$ is simply connected and avoids the fixed point $1-e^{2i\pi\alpha}$, Koebe one-quarter theorem yields

$$\log \operatorname{rad}(V_0) \leq \log|1 - e^{2i\pi\alpha_0}| + \log 4.$$

We have $q_0 = 1$ and since $q_1 = \lfloor 1/\alpha_0 \rfloor$, we have

$$\log\left|1 - e^{2i\pi\alpha_0}\right| < \log(2\pi\alpha_0) < -\frac{\log q_1}{q_0} + \log(2\pi).$$

■

Step 5. Our goal is then to show the following inequality. It is the main estimate of the article.

Proposition 8. *For $i \geq 1$, we have*

$$\log \frac{\operatorname{rad}(V_i)}{\operatorname{rad}(V_{i-1})} \leq -\frac{\log q_{1+n_i}}{q_{n_i}} + \frac{\log 24 F_{n_i}^2}{F_{n_i}} + \frac{\log 16}{1 + F_{n_i}/1.5}.$$

Let us recall that for $N \geq 0$, Ω_N stands for the complement in $\mathbb{C}$ of the external ray of P_α of argument 0 and the periodic points of period less than or equal to q_N. In particular, for $i \geq 1$, $\Omega_{n_i} \subset V_i$. Combining the results from propositions 6, 7 and 8, we get (using $2 < 24$)

$$\begin{aligned}
\log \operatorname{rad}(\Omega_N) &< \log \operatorname{rad}(V_0) + \sum_{\substack{i \geq 1 \\ n_i \leq N}} \left(-\frac{\log q_{1+n_i}}{q_{n_i}} + \frac{\log 24 F_{n_i}^2}{F_{n_i}} + \frac{\log 16}{1 + F_{n_i}/1.5} \right) \\
&< -\sum_{n=0}^{N} \frac{\log q_{n+1}}{q_n} + \log(8\pi) + \sum_{n=1}^{N} \left(\frac{\log 24 F_n^2}{F_n} + \frac{\log 16}{1 + F_n/1.5} \right) \\
&< -\sum_{n=0}^{N} \frac{\log q_{n+1}}{q_n} + 16.
\end{aligned}$$

The proof is then completed.

Let us now explain how we get proposition 8. First, $\mathcal{C}_i \subset V_{i-1}$ is the image of S_i by the holomorphic function $\chi_{p_{n_i}/q_{n_i}}$. Proposition 8 is therefore almost a consequence of the following two propositions.

Proposition 9. *We have the inequality*

$$\log \frac{\mathrm{rad}(U_i \setminus S_i)}{\mathrm{rad}(U_i)} \leq -\frac{\log q_{1+n_i}}{q_{n_i}} + \frac{\log 24F_{n_i}^2}{F_{n_i}}.$$

This proposition only consists in estimating the conformal radius of the unit disk minus q points equidistributed on a circle of radius $r < 1$. The proof is given in section 6 below.

Proposition 10. *Assume $U, V \subset \mathbb{C}$ are two hyperbolic domains containing 0 and $\chi : U \to V$ is a holomorphic map fixing 0. Let S be a finite subset of U avoiding 0, such that $\chi(S)$ avoids 0. Then,*

$$\frac{\mathrm{rad}(V \setminus \chi(S))}{\mathrm{rad}(V)} \leq \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)}.$$

The proof of this inequality is a refinement of Schwarz's lemma and is based on the use of ultrahyperbolic metrics (see section 7 below).

Combining those two inequalities would yield proposition 8 if $\chi_{p_{n_i}/q_{n_i}} : U_i \to \mathbb{C}$ took its values in V_{i-1}, which is almost the case. In fact, let $\alpha(\delta) = p_{n_i}/q_{n_i} + \delta^{q_{n_i}}$. As δ varies in U_i, $\chi_{p_{n_i}/q_{n_i}}(\delta)$ belongs to $V_{i-1}(\alpha(\delta))$, which depends on δ. In section 8, using Slodkowski's theorem and the straightening of Beltrami forms, we define for $\alpha \in D_i$ an analytic family of universal coverings $\pi_\alpha : \widetilde{V}_\alpha \to V_{i-1}(\alpha)$, where $\widetilde{V}_\alpha$ are open subsets of $B(0,4)$, and $\widetilde{V}_{\alpha_0} = \mathbb{D}$. The map $\phi : \delta \mapsto \chi_{p_{n_i}/q_{n_i}}(\delta)$ "lifts" to a map $\delta \mapsto \widehat{\phi}(\delta)$ such that $\widehat{\phi}(\delta) \in \widetilde{V}_{\alpha(\delta)}$. It follows from the definitions that,

$$\log \frac{\mathrm{rad}(V_i)}{\mathrm{rad}(V_{i-1})} = \log \frac{\mathrm{rad}(\widetilde{V}_{\alpha_0} \setminus \pi_{\alpha_0}^{-1}(\phi(S_i)))}{\mathrm{rad}(\widetilde{V}_{\alpha_0})}.$$

Now $\widetilde{V}_{\alpha_0} = \mathbb{D}$ and $\widehat{\phi}(S_i) \subset \pi_{\alpha_0}^{-1}(\phi(S_i))$, thus

$$\log \frac{\mathrm{rad}(V_i)}{\mathrm{rad}(V_{i-1})} \leq \log \mathrm{rad}(\mathbb{D} \setminus \widehat{\phi}(S_i)).$$

We would like to apply proposition 10 to $U = B_i$ and $\chi = \widehat{\phi}$. But we cannot take $V = \mathbb{D}$ because $\widehat{\phi}$ does not necessarily take its values in $\mathbb{D}$. However, in section 8, we prove the following estimate.

Proposition 11. *For $\alpha \in B_i$, the sets $\widetilde{V}_\alpha$ are all contained in some ball $B(0, \rho_2)$ with*

$$\log \rho_2 = \frac{\log 16}{1 + q_{n_i}/1.5}$$

We can therefore take $V = B(0, \rho_2)$ in proposition 10 and we obtain

$$\log \frac{\mathrm{rad}(B(0,\rho_2) \setminus \widehat{\phi}(S_i))}{\mathrm{rad}(B(0,\rho_2))} \leq \log \frac{\mathrm{rad}(U_i \setminus S_i)}{\mathrm{rad}(U_i)}.$$

Now, by inclusion

$$\begin{aligned}\log \operatorname{rad}(\mathbb{D} \setminus \widehat{\phi}(S_i)) &\leq \log \operatorname{rad}(B(0,\rho_2) \setminus \widehat{\phi}(S_i)) \\ &\leq \log \frac{\operatorname{rad}(U_i \setminus S_i)}{\operatorname{rad}(U_i)} + \log \operatorname{rad}(B(0,\rho_2)) \\ &\leq -\frac{\log q_{1+n_i}}{q_{n_i}} + \frac{\log 24 F_{n_i}^2}{F_{n_i}} + \frac{\log 16}{1+q_{n_i}/1.5}\end{aligned}$$

according to propositions 9 and 11.

3. Parabolic Explosion.

The proofs of propositions 1 and 2 may be found in [C] or [BC], but for completeness, we sketch them here.

Proof of Proposition 1. It is well known that, when α varies, periodic points with multiplier different from 1 can be locally followed holomorphically in terms of α. To prove this, one applies the implicit function theorem (complex-analytic version) to the equation "$P_\alpha^{\circ k}(z) - z = 0$" where k is the period: at a point (α, z) in the surface defined by, the derivative with respect to z is equal to $m-1$ where m is the multiplier. In our case, on $B' = B(\frac{p}{q}, R(\frac{p}{q})) \setminus \{\frac{p}{q}\}$, no points of period dividing q has multiplier 1.

Since B' is not simply connected, the holomorphic dependence in terms of α may have a monodromy when α makes one turn around p/q. Let $B = B(\frac{p}{q}, R(\frac{p}{q}))$. Let us consider the subset of $B \times \mathbb{C}$ defined by

$$\mathcal{M} = \left\{(\alpha, z) \middle| P_\alpha^{\circ q}(z) - z = 0\right\}.$$

For α at the center of B (i.e. $\alpha = p/q$), only $z = 0$ is parabolic. Thus the fixed points of $P_{p/q}^{\circ q}$ that are different from 0 have no monodromy: they can be followed holomorphically as a function of α on all of B. The graphs of these functions $\alpha \mapsto z(\alpha)$ are connected components of $\mathcal{M}$. There is only one other component. It contains $(\alpha, z) = (p/q, 0)$, and it is singular. To study it, one looks at the expansion of the equation at this point. First, it is known (see [DH], chapter IX) that there exists a complex number $A \in \mathbb{C}^*$ such that

$$P_{p/q}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2}).$$

This means there are $q+1$ fixed points of $P_{p/q}^{\circ q}$ at $z = 0$. Then,

$$P_{p/q+\varepsilon}^{\circ q}(z) - z = z \cdot \big(2i\pi q\varepsilon + Az^q + O(z\varepsilon) + O(z^{q+1})\big).$$

To get rid of the singularity, one considers a new variable $\delta \in D = B(0, R(p/q)^{1/q})$ related to α by $\alpha = \delta^q + p/q$. This transforms the component of $\mathcal{M}$ containing $(p/q, 0)$ into the union of $1+q$ graphs of functions from D to $\mathbb{C}$ that are transversal and meet only at $(0,0)$. This can be proved by blowing-up $D \times \mathbb{C}$ at $(0,0)$, i.e. by introducing a new variable, the slope $\lambda = z/\delta$. One of the graphs corresponds to the fixed point $z = 0$ of P_α which does not move, and the others are graphs of functions $\phi_1(\delta)$, $\phi_2(\delta)$, ..., $\phi_q(\delta)$ passing through $(0,0)$ with slopes λ equal to the q-th roots of $-2i\pi q/A$.

The function χ of Proposition 1 is any of these functions ϕ_i. The points $P_\alpha(\chi(\delta))$ and $\chi(\zeta\delta)$ are both fixed points of $P_\alpha^{\circ q}$. Since there graphs pass through $(0,0)$ they coincide with functions ϕ_i and ϕ_j for some i and j. By comparing the derivatives at $\delta = 0$, one gets $i = j$ and so

$$P_\alpha(\chi(\delta)) = \chi(\zeta\delta).$$

This shows that the set

$$\Big\langle \chi(\delta), \chi(\zeta\delta), \chi(\zeta^2\delta), \dots, \chi(\zeta^{q-1}\delta) \Big\rangle$$

forms a cycle of period q of $P_{p/q+\delta^q}$. ■

Proof of proposition 2. The only values of the parameter α for which P_α has a multiple fixed point are the integers. Thus, the result is trivial for $q = 1$. Let us now assume that $q \geq 2$. In that case, the proof relies on Douady's landing theorem and the Pommerenke-Levin-Yoccoz inequality (see [H] or [P1]).

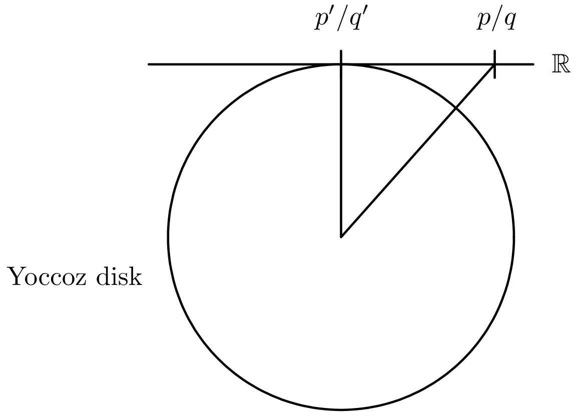

FIGURE 1. The complex number α lies somewhere in the Yoccoz disk

Let us choose a rational number p/q, and assume that $\alpha \neq p/q$ and $P_\alpha^{\circ q}$ has a multiple fixed point z_0. Then, P_α has a parabolic cycle $\langle z_0, z_1, \dots, z_{q_1-1}\rangle$ of period q_1 dividing q, and the immediate basin of this parabolic cycle contains the critical point $\omega_0 = -e^{2i\pi\alpha}/2$ of P_α. As a consequence, the Julia set $J(P_\alpha)$ is connected and all other periodic cycles of P_α are repelling.

If 0 is parabolic, then $\alpha = p'/q$ with p' not necessarily prime to q. So, the distance between α and p/q is bounded from below by $1/q$.

Otherwise, 0 must be repelling, thus α belongs to the lower half-plane $\{\mathrm{Im}(\alpha) < 0\}$. Since the Julia set is connected, Douady's landing theorem asserts that there are finitely many rays landing at 0, let's say q'. Those q' rays can be ordered cyclically by their arguments. They are permuted by P_α and each ray is mapped to the one which is p' further counter-clockwise for some $p' < q'$, p' prime to q'. Then, the Yoccoz inequality implies that α belongs to the closed disk of radius $(\log 2)/(2\pi q')$ tangent to the real axis at p'/q' (see for example [H]). A key combinatorial lemma that is proved below is that we necessarily have $q > q'$.

The Pythagoras theorem then gives

$$\left|\alpha - \frac{p}{q}\right| \geq \sqrt{\left(\frac{p'}{q'} - \frac{p}{q}\right)^2 + \left(\frac{\log 2}{2\pi q'}\right)^2} - \frac{\log 2}{2\pi q'}.$$

Since $q > q'$, and $q \geq 2$, an elementary computation gives $|\alpha - p/q| > 1/q^3$. ■

Lemma 1 (Key combinatorial lemma)**.**

$$q' < q$$

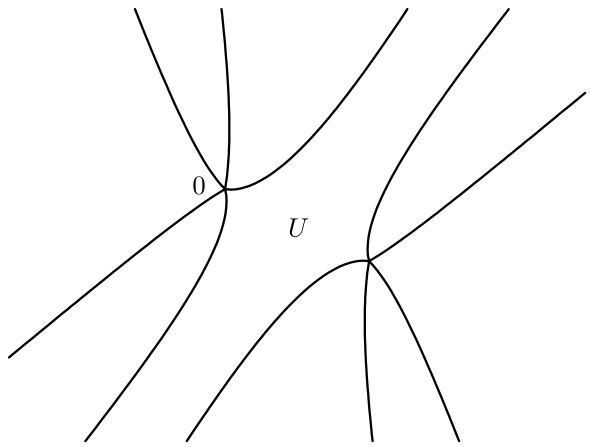

FIGURE 2. Schematic example for $q' = 5$.

Proof. By assumption, $q > 1$. Consider the complement in $\mathbb{C}$ of the q' external rays landing at 0 together with this point. It has q' connected components. Let V be the one containing the critical point. The orbit of the critical point of P_α must first visit each connected component of this complement, before first falling back somewhere in V. Since the critical point belongs to the immediate basin of the parabolic cycle, this implies the period is $\geq q'$, and thus $q \geq q'$.

Let us assume by contradiction that $q = q'$. The point 0 has two distinct preimages: 0 and another point. Consider the union of these two points and the $2q$ external rays landing at them. Let U be the component of the complement of this union containing the critical point. It is known that $P_\alpha^{\circ q}(U) = V$ and $P_\alpha^{\circ q} : U \to V$ is a proper ramified covering of degree 2. Let f be the restriction $P_\alpha^{\circ q} : \overline{U} \to \overline{V}$. Note that $\overline{U} \subset \overline{V}$. The contradiction follows from a version of the Lefschetz fixed point formula (see lemma 3.7 in [GM]): the point $z = 0$ is fixed, and the point of the parabolic cycle whose immediate basin contains the critical point is a multiple fixed point of f. Thus the sum of Lefschetz indices is ≥ 3, whereas according to the Lefschetz formula is should be equal to the degree of f, i.e., 2. This leads to contradiction and thus $q > q'$. ■

4. PROOF OF PROPOSITION 4.

The set $\mathcal{B}$ is contained in the union of $\mathbb{Z}$ and the lower half-plane. It is the graph of the function

$$f : x \mapsto \frac{1}{2\pi} \log\cos(2\pi x)$$

which is periodic of period 1 and defined for $x \in]-1/4+k, 1/4+k[$, $k \in \mathbb{Z}$. We will now show that for $x \in]0, 1/2[$, $f(x - x^2) < -x^2$. Since f is decreasing on $[0, 1/2[$,

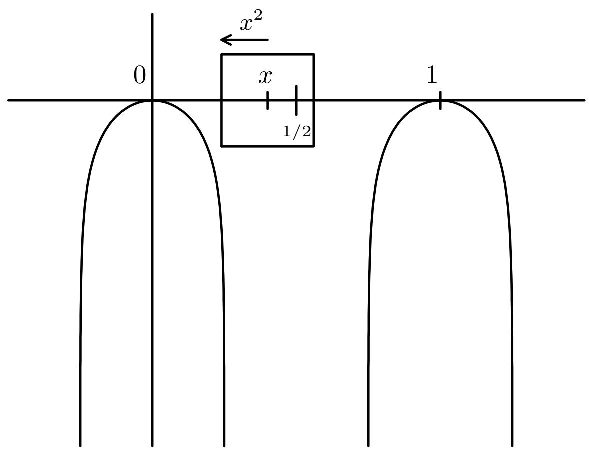

FIGURE 3. The graph of the function f. The inequality $f(x-x^2) < -x^2$ means the lower left corner of the square is above the curve.

proposition 4 follows. We want to show that the function $g(x) = x^2 + f(x - x^2)$ is negative on $]0, 1/2[$. Since $g(0) = 0$, it is sufficient to show that $g'(x) < 0$ on $]0, 1/2[$. This is equivalent to proving that

$$\tan(2\pi(x - x^2)) > \frac{2x}{1 - 2x}.$$

Let us make the change of variable $u = 1/2 - x$. Then, the previous becomes

$$\tan(2\pi u^2) < \frac{2u}{1 - 2u}.$$

We are done since for all $u \in]0, 1/2[$, we have

$$\sin(2\pi u^2) < 2\pi u^2 \quad \text{and} \quad \cos(2\pi u^2) > \frac{\pi/2 - 2\pi u^2}{\pi/2 - 0} = 1 - 4u^2 > \pi u(1 - 2u).$$

■

5. GOOD APPROXIMANTS.

Note that we made the assumption $\alpha_0 \in]0, 1/2[$ and so $q_1 \geq 2$. In particular, for all $i \geq 1$, we have $q_{n_i} \geq 2$.

The inclusion $D_1 \subset \mathbb{C} \setminus \mathcal{B}$ follows from proposition 4. For $i \geq 1$, the inclusion $B_i \subset D_i$ is immediate since the two disks have the same center and the radius of D_i is q_{n_i} times the radius of B_i.

The classical estimate we will use is that for all $n \geq 0$, we have

$$\frac{1}{2q_n q_{n+1}} < \left|\alpha_0 - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}.$$

Thus, we have

$$\left|\alpha_0 - \frac{p_{n_i}}{q_{n_i}}\right| < \frac{1}{q_{n_i} q_{1+n_i}} < \frac{1}{2q_{n_i}^3}$$

by definition of $\mathcal{N}$. In particular α_0 belongs to B_i and is closer to the center than to the boundary.

Moreover, for $i \geq 1$, $q_{n_{i+1}} \geq q_{1+n_i} > 2q_{n_i}^2$ and $q_{1+n_{i+1}} > 2q_{n_{i+1}}^2 \geq 2q_{1+n_i}^2$. So

$$\left|\alpha_0 - \frac{p_{n_{i+1}}}{q_{n_{i+1}}}\right| < \frac{1}{q_{n_{i+1}}q_{1+n_{i+1}}} < \frac{1}{4q_{n_i}^2 q_{1+n_i}^2} < \frac{1}{4q_{n_i}q_{1+n_i}} < \frac{1}{2}\left|\alpha_0 - \frac{p_{n_i}}{q_{n_i}}\right|.$$

In other words, the distance from α_0 to the center of B_{i+1} is less than half the distance from α_0 to the center of B_i. It follows from these two claims that in B_i^*, one can fit a disk centered at $p_{n_{i+1}}/q_{n_{i+1}}$ with radius at least equal to

$$\frac{1}{2}\left|\alpha_0 - \frac{p_{n_i}}{q_{n_i}}\right| > \frac{1}{4q_{n_i}q_{1+n_i}} > \frac{1}{q_{n_{i+1}}^2}.$$

Indeed, $q_{n_{i+1}} \geq q_{1+n_i} > 2q_{n_i}^2 \geq 4q_{n_i}$. In particular, for all $i \geq 1$, $D_{i+1} \subset B_i^*$. We proved proposition 5.

Finally, if $n \geq 1$ and $n \notin \mathcal{N}$, we have $q_{n+1} \leq 2q_n^2$. It follows that

$$\sum_{n=1}^{N} \frac{\log q_{n+1}}{q_n} < \sum_{\substack{i\geq 1\\ n_i \leq N}} \frac{\log q_{1+n_i}}{q_{n_i}} + \sum_{n\in[1,N]\setminus\mathcal{N}} \frac{\log 2q_n^2}{q_n}.$$

Since q_n is bounded from below by the n-th Fibonacci number F_n, according to the lemma 6 in the appendix, we have

$$\sum_{n\in[1,N]\setminus\mathcal{N}} \frac{\log 2q_n^2}{q_n} \leq \sum_{n\in[1,N]\setminus\mathcal{N}} \frac{\log 2F_n^2}{F_n}.$$

We proved proposition 6.

6. An estimate for a conformal radius.

Here, we prove proposition 9.

Definition 6. *Given an integer $q \geq 1$, set*

$$\mathbb{U}_q = \left\{e^{2i\pi k/q} \mid k = 0, \ldots, q-1\right\}.$$

The following estimate was explained to us by Douady.

Proposition 12. *There exists a constant $C > 0$ such that for $q \geq 2$ and $r < 1$, we have*

$$\log \operatorname{rad}(\mathbb{D} \setminus r\mathbb{U}_q) \leq \log r + \frac{C}{q}.$$

one can take $C = \log 4 + 2\log(1+\sqrt{2})$.

Proof. By inclusion, we have

$$\operatorname{rad}(\mathbb{D} \setminus r\mathbb{U}_q) \leq \operatorname{rad}(\mathbb{C} \setminus r\mathbb{U}_q) = r \cdot \operatorname{rad}(\mathbb{C} \setminus \mathbb{U}_q).$$

Let $\pi : \mathbb{D} \to \mathbb{C} \setminus \mathbb{U}_q$ be a universal covering which sends 0 to 0. By symmetry, for $k = 0, \ldots, q-1$, the half lines $L_k = \{\rho e^{2i\pi k/q} \mid \rho > 1\}$ are geodesics in $\mathbb{C} \setminus \mathbb{U}_q$ for the hyperbolic metric. Set $\Omega_q = \mathbb{C} \setminus \bigcup L_k$. There is a formula for the conformal representation $\phi_q : \mathbb{D} \to \Omega_q$:

$$\phi_q(z) = z\left(\frac{4}{(1-z^q)^2}\right)^{1/q}.$$

In particular, we have

$$\operatorname{rad}(\Omega_q) = 4^{1/q}.$$

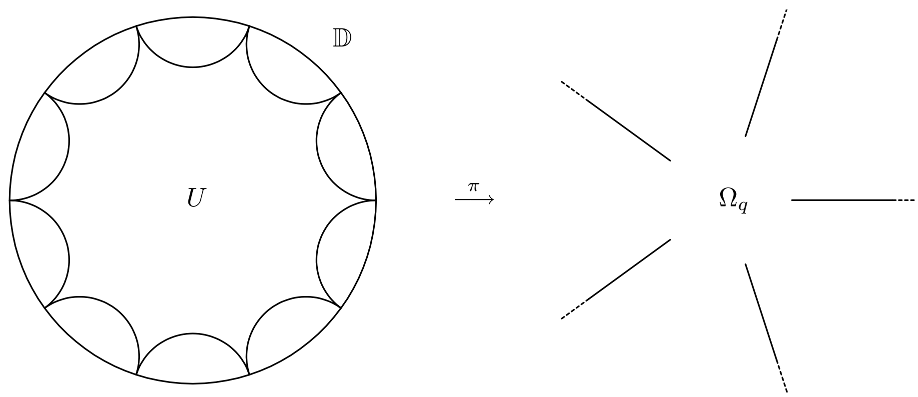

FIGURE 4. The map π maps U to the slit plane Ω_q

Now, the connected component U of $\pi^{-1}(\Omega_q)$ which contains 0 is bounded by $2q$ geodesic arcs of circles in $\mathbb{D}$ whose endpoints are equidistributed on S^1. An elementary computation shows that U contains the disk centered at 0 with radius

$$\rho_q = \frac{1-\tan(\pi/4q)}{1+\tan(\pi/4q)}$$

Since the image by π of this disk is contained in Ω_q, it follows from Schwarz's lemma that

$$\operatorname{rad}(\mathbb{C}\setminus \mathbb{U}_q) \leq \frac{\operatorname{rad}(\Omega_q)}{\rho_q}.$$

thus

$$\log \operatorname{rad}(\mathbb{C}\setminus \mathbb{U}_q) \leq \frac{\log 4}{q} + \log\frac{1+\tan(\pi/4q)}{1-\tan(\pi/4q)}.$$

By convexity of $f(x) = \log\dfrac{1+\tan x}{1-\tan x}$ on $[0,\pi/8]$, we have $f(\pi/4q) \leq \dfrac{2}{q} f(\pi/8)$. The result now follows easily. ■

We can now estimate the conformal radius of $U_i \setminus S_i$ for $i \geq 1$. The radius of the ball U_i is $(1/q_{n_i}^3)^{1/q_{n_i}}$ and the set S_i consists of q_{n_i} points equidistributed on a circle of radius

$$\left|\alpha_0 - \frac{p_{n_i}}{q_{n_i}}\right|^{1/q_{n_i}} < \left(\frac{1}{q_{n_i}q_{1+n_i}}\right)^{1/q_{n_i}}.$$

So, we have

$$\begin{aligned}\log\frac{\operatorname{rad}(U_i\setminus S_i)}{\operatorname{rad}(U_i)} &< \log\frac{(1/q_{n_i}q_{1+n_i})^{1/q_{n_i}}}{(1/q_{n_i}^3)^{1/q_{n_i}}} + \frac{C}{q_{n_i}}\\ &= -\frac{\log q_{1+n_i}}{q_{n_i}} + 2\frac{\log q_{n_i}}{q_{n_i}} + \frac{C}{q_{n_i}}.\end{aligned}$$

Since q_n is bounded from below by the n-th Fibonacci number F_n, we have

$$2\frac{\log q_{n_i}}{q_{n_i}} + \frac{C}{q_{n_i}} \leq \frac{\log 24 q_{n_i}^2}{q_{n_i}} \leq \frac{\log 24 F_{n_i}^2}{F_{n_i}}$$

according to lemma 6 in the appendix.

7. Comparison between conformal radii.

Our goal in this section is to prove proposition 10. The proof relies on a relative Schwarz's lemma.

7.1. **A relative Schwarz's lemma.**

Definition 7. *A metric $\rho|dz|$, $\rho \geq 0$ is said to be ultrahyperbolic in a Riemann surface X if it has the following properties:*

(i) *ρ is upper semicontinuous.*
(ii) *At every $x_0 \in X$ with $\rho(x_0) > 0$ there exists a "supporting metric" ρ_0, defined and of class C^2 in a neighborhood V of x_0, such that $\Delta \log \rho_0 \geq \rho_0^2$ and $\rho \geq \rho_0$ in V, while ρ coincides with ρ_0 at x_0.*

In a hyperbolic Riemann surface X, there exists a unique maximal ultrahyperbolic metric ρ_X, and this metric has constant curvature -1. It is maximal in the sense that every ultrahyperbolic metric ρ on X satisfies $\rho \leq \rho_X$ throughout X. This maximal metric is called the *Poincaré metric* on X.

For example, the Poincaré metric $\rho_{\mathbb{D}}$ on the unit disk $\mathbb{D}$ is

$$\rho_{\mathbb{D}} = \frac{2}{1-|z|^2}|dz|.$$

More generally, if $\pi : \mathbb{D} \to X$ is a universal covering, the Poincaré metric ρ_X coincides with the unique metric such that $\pi^*\rho_X = \rho_{\mathbb{D}}$.

Now, if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $f^*\rho_{\mathbb{D}}$ is ultrahyperbolic on $\mathbb{D}$, and thus $f^*\rho_{\mathbb{D}} \leq \rho_{\mathbb{D}}$. This may be written as the Schwarz-Pick theorem:

$$(\forall z \in \mathbb{D}) \quad |f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}.$$

More generally, if X and Y are two hyperbolic Riemann surfaces, then every holomorphic map $f : (X, \rho_X) \to (Y, \rho_Y)$ is contracting:

$$f^*\rho_Y \leq \rho_X.$$

In particular, if $X \subset Y$, then $\rho_Y \leq \rho_X$.

In this subsection, we are interested in comparing the relative contraction of a holomorphic map $f : X \to Y$ for several Poincaré metrics. We will show that if $Y' \subset Y$ is an arbitrary open subset and $X' = f^{-1}(Y')$, then $f : (X', \rho_{X'}) \to (Y', \rho_{Y'})$ is less contracting than $f : (X, \rho_X) \to (Y, \rho_Y)$.

Lemma 2. (Relative Schwarz's Lemma) *Let $f : X \to Y$ be an analytic map between two hyperbolic Riemann surfaces. Let $Y' \subset Y$ be an arbitrary open subset and set $X' = f^{-1}(Y')$. Then, on X',*

$$\frac{f^*\rho_Y}{\rho_X} \leq \frac{f^*\rho_{Y'}}{\rho_{X'}} \leq 1.$$

The main tool of the proof is the use of Ahlfors's ultrahyperbolic metrics (see[A] for example). This was suggested to us by McMullen.

Proof. Let us first consider the case where $y_0 \in Y$ is an arbitrary point and $Y' = Y \setminus \{y_0\}$. We will show that the metric σ defined on $X' = X \setminus f^{-1}\{y_0\}$ by

$$\sigma = \frac{f^*\rho_Y}{f^*\rho_{Y'}}\rho_{X'}$$

extends continuously to a ultrahyperbolic metric on X. It will then follow from the definition of the Poincaré metric ρ_X that

$$\frac{f^*\rho_Y}{f^*\rho_{Y'}}\rho_{X'} \le \rho_X,$$

which is the required result.

Step 1. The metric σ is a priori only defined on $X' \setminus \mathrm{Crit}(f)$, where $\mathrm{Crit}(f)$ is the set of critical points of f. But since

$$\sigma = \left(\frac{\rho_Y}{\rho_{Y'}} \circ f\right) \cdot \rho_{X'},$$

we see that σ is positive and C^2 on X'. We will now show that $\Delta \log \sigma \ge \sigma^2$ on $X' \setminus \mathrm{Crit}(f)$. Since $\mathrm{Crit}(f)$ is discrete in X', this inequality holds on X'. Therefore, σ is ultrahyperbolic on X'.

On $X' \setminus \mathrm{Crit}(f)$ we have

$$\Delta \log \sigma = \Delta \log f^*\rho_Y + \Delta \log \rho_{X'} - \Delta \log f^*\rho_{Y'} = [f^*\rho_Y]^2 + [\rho_{X'}]^2 - [f^*\rho_{Y'}]^2.$$

The second equality comes from the fact that the three metrics have curvature -1. Since $Y' \subset Y$, we have $\rho_Y \le \rho_{Y'}$ on Y', and so, $f^*\rho_Y \le f^*\rho_{Y'}$ on X'. And since, $f^*\rho_{Y'}$ is ultrahyperbolic on X', we have $f^*\rho_{Y'} \le \rho_{X'}$ on X'. Now, if a, b and c are three positive numbers such that $a \le b \le c$, then

$$(c-b)(b-a) \ge 0 \quad \Longrightarrow \quad cb - b^2 + ba \ge ac \quad \Longrightarrow \quad a + c - b \ge \frac{ac}{b}.$$

Therefore,

$$\Delta \log \sigma = [f^*\rho_Y]^2 + [\rho_{X'}]^2 - [f^*\rho_{Y'}]^2 \ge \left[\frac{f^*\rho_Y \cdot \rho_{X'}}{f^*\rho_{Y'}}\right]^2 = \sigma^2.$$

Step 2. We claim that we may extend σ continuously to $X \setminus X'$ by setting $\sigma = f^*\rho_Y$ there. Indeed, let x_0 be an arbitrary point in $X \setminus X'$. It is sufficient to show that

$$\lim_{x \to x_0} \frac{\rho_{X'}}{f^*\rho_{Y'}}(x) = 1.$$

Lemma 3. *Let X be a hyperbolic Riemann surface, X' be an open subset of X and assume that $x_0 \in X \setminus X'$ is an isolated point of $X \setminus X'$. Then, in any analytic chart, if we note $\rho_{X'} = \rho_{X'}(x)|dx|$, we have*

$$\rho_{X'}(x) \underset{x \to x_0}{\sim} \frac{1}{|x - x_0| \log \dfrac{1}{|x - x_0|}}.$$

Note that the formula is independent of the chosen chart.

Proof. We will work in the local coordinates given by the universal covering $\pi_X : (\mathbb{D}, 0) \to (X, x_0)$. We set $U = \pi_X^{-1}(X')$. Let $x = \pi_X(w)$ Then, we have $\rho_{X'} = \rho_U(w)|dw|$. We claim that

$$\rho_U(w) \underset{w \to 0}{\sim} \frac{1}{|w| \log \dfrac{1}{|w|}}.$$

Indeed, we may find $\varepsilon > 0$ such that $\mathbb{D}^*_\varepsilon \subset U \subset \mathbb{D}^*$. Then,

$$\rho_{\mathbb{D}^*}(w) = \frac{1}{|w|\log\dfrac{1}{|w|}} \le \rho_U(w) \le \rho_{\mathbb{D}^*_\varepsilon}(w) = \frac{1}{|w|\log\dfrac{\varepsilon}{|w|}}.$$

Then, since $\rho_{X'}(x) = \rho_{X'}(\pi_X(w)) = \rho_U(w)/|\pi'_X(w)|$, and $|x-x_0| \sim |\pi'_X(0)|\cdot|w|$ when $w \to 0$, the result follows. □

Now, let us choose analytic charts for X near x_0 and for Y near y_0, and note $r = |x - x_0|$. Then, as $x \to x_0$, we have

$$|f(x) - f(x_0)| \sim Ar^d \quad \text{and} \quad |f'(x)| \sim Adr^{d-1},$$

where d is the local degree of f at x_0 and $A > 0$. Thus, as $x \to x_0$, we have

$$\frac{\rho_{X'}}{|dx|}(x) \sim \frac{1}{r\log\dfrac{1}{r}},$$

and

$$\frac{f^*\rho_{Y'}}{|dx|}(x) = |f'(x)|\rho_Y(f(x)) \sim \frac{Adr^{d-1}}{Ar^d\log\dfrac{1}{Ar^d}} \sim \frac{1}{r\log\dfrac{1}{r}}.$$

The claim follows.

Step 3. As we have just seen, we may extend σ continuously to $X \setminus X'$ by setting $\sigma = f^*\rho_Y$ there, and since $f^*\rho_{Y'} \le \rho_{X'}$, we see that $\sigma \ge f^*\rho_Y$. If σ does not vanish at x_0, i.e., if x_0 is not a critical point of f, then $f^*\rho_Y$ is C^2 in a neighborhood of x_0, has curvature -1 and coincides with σ at x_0. Thus, condition (ii) in the definition of ultrahyperbolic metrics is satisfied: $f^*\rho_Y$ is a "supporting metric" at x_0 and we have proved that σ is ultrahyperbolic. Thus, if Y' is obtained by removing one point from Y, the relative Schwarz's lemma is proved.

By induction, the lemma is therefore proved when Y' is obtained by removing finitely many points from Y. In order to prove the lemma for an arbitrary open subset $Y' \subset Y$, we may choose a dense countable set $\{y_n,\ n \ge 0\} \subset Y \setminus Y'$, define $Y_n = Y \setminus \{y_k,\ k \le n\}$ and set $X_n = f^{-1}(Y_n)$. Then, for all $n \ge 0$,

$$\frac{f^*\rho_Y}{\rho_X} \le \frac{f^*\rho_{Y_n}}{\rho_{X_n}} \le 1. \tag{1}$$

Lemma 4. *Assume $(U_n)_{n\ge 0}$ is a decreasing sequence of hyperbolic Riemann surfaces. Let U be the interior of $\bigcap_{n\ge 0} U_n$. As $n \to +\infty$, the Poincaré metrics ρ_{U_n} converge uniformly on every compact subset of U to the Poincaré metric ρ_U.*

Proof. Let a be an arbitrary point in U and let U_a be the connected component of U that contains a. Let $\phi_n : (\mathbb{D}, 0) \to (U_n, a)$ and $\phi : (\mathbb{D}, 0) \to (U_a, a)$ be the universal coverings which have real and positive derivatives at 0 (for some chart around a in U_a). We will show that the maps ϕ_n converge uniformly to ϕ on every compact subset of U_a. The lemma follows easily.

The maps ϕ_n all take their values in U_0 which is hyperbolic. So, they form a normal family. Let $\psi : (\mathbb{D}, 0) \to (U_a, a)$ be a limit value. For all $n \ge 0$, the map ϕ takes its values in U_n, and thus, $\phi'(0) \le \phi'_n(0)$. Similarly, the map ψ takes its values in U_a and thus, $\psi'(0) \le \phi'(0)$. Since ψ is a limit value of the sequence ϕ_n, we have $\psi'(0) = \phi'(0)$ and by the classical Schwarz lemma, $\psi = \phi$. □

As a consequence, as $n \to +\infty$, the Poincaré metrics ρ_{X_n} and ρ_{Y_n} converge uniformly on every compact subset of X' and Y' to the Poincaré metrics $\rho_{X'}$ and $\rho_{Y'}$. Passing to the limit in inequality (1) gives the required result:

$$\frac{f^*\rho_Y}{\rho_X} \le \frac{f^*\rho_{Y'}}{\rho_{X'}} \le 1.$$

■

7.2. **Proof of proposition 10.** Let us recall the problem. We assume $U, V \subset \mathbb{C}$ are hyperbolic domains containing 0, we assume $\chi : (U, 0) \to (V, 0)$ is holomorphic, and we assume that $\chi(S)$ avoids 0 (in which case S also avoids 0). We wish to conclude that

$$\frac{\mathrm{rad}(V \setminus \chi(S))}{\mathrm{rad}(V)} \le \frac{\mathrm{rad}(U \setminus S)}{\mathrm{rad}(U)}.$$

The conformal radius $\mathrm{rad}(U)$ is related to the coefficient of the Poincaré metric $\rho_U(0)$ as follows:

$$\mathrm{rad}(U) = \frac{2}{\rho_U(0)}.$$

We will apply the relative Schwarz's lemma with $X = U$, $Y = V$, $f = \chi$, $Y' = V \setminus \chi(S)$ and $X' = \chi^{-1}(Y')$. We have

$$\frac{f^*\rho_Y}{\rho_X} \le \frac{f^*\rho_{Y'}}{\rho_{X'}},$$

which may be rewritten as

$$\frac{\rho_{X'}}{\rho_X} \le \frac{f^*\rho_{Y'}}{f^*\rho_Y} = \frac{\rho_{Y'}}{\rho_Y} \circ f.$$

Evaluating this inequality at 0, and using the relation between the conformal radius and the coefficient of the Poincaré metric, we get

$$\frac{\mathrm{rad}(V \setminus \chi(S))}{\mathrm{rad}(V)} = \frac{\mathrm{rad}(Y')}{\mathrm{rad}(Y)} \le \frac{\mathrm{rad}(X')}{\mathrm{rad}(X)} = \frac{\mathrm{rad}(U \setminus \chi^{-1}(\chi(S)))}{\mathrm{rad}(U)}.$$

The result follows since $U \setminus \chi^{-1}(\chi(S)) \subset U \setminus S$, and so,

$$\mathrm{rad}(U \setminus \chi^{-1}(\chi(S))) \le \mathrm{rad}(U \setminus S).$$

8. Holomorphic motions.

To prove proposition 8, we must now take into account the fact that for $i \ge 2$, $\chi_{p_{n_i}/q_{n_i}}$ does not take its values in V_{i-1} but rather that $\chi_{p_{n_i}/q_{n_i}}(\delta)$ belongs to $V_{i-1}(\alpha(\delta))$ with $\alpha(\delta) = p_{n_i}/q_{n_i} + \delta^{q_{n_i}}$. The sets $V_{i-1}(\alpha)$ move holomorphically with respect to $\alpha \in D_i$ and when δ ranges in U_i, $\alpha(\delta)$ remains in B_i which is well inside D_i (the ratio of the radii is q_{n_i} and $q_{n_i} \ge 2$ as $\alpha \in]0, 1/2[$).

To begin with, let us work in quite a general, but normalized, setting under the following assumptions. We assume that V_λ are hyperbolic subdomains of $\mathbb{C}$ which contain 0 and move holomorphically with respect to $\lambda \in \mathbb{D}$. By Slodkowski's theorem, we can assume that the holomorphic motion is a holomorphic motion of the whole complex plane. We set

$$\mathcal{V} = \big\{(\lambda, z) \bigm| \lambda \in \mathbb{D} \text{ and } z \in V_\lambda\big\}.$$

The maps $p_1 : \mathcal{V} \to \mathbb{D}$ and $p_2 : \mathcal{V} \to \mathbb{C}$ are the projections to the first and the second coordinates.

Proposition 13. *There exists a family of simply connected open sets $\widetilde{V}_\lambda$ and of universal coverings $\pi_\lambda : \widetilde{V}_\lambda \to V_\lambda$ such that $\widetilde{V_0} = \mathbb{D}$, the set*

$$\widetilde{\mathcal{V}} = \left\{(\lambda, z) \in \mathbb{D} \times \mathbb{C} \,\middle|\, z \in \widetilde{V}_\lambda\right\}$$

is open, and $\Pi : (\lambda, z) \in \widetilde{\mathcal{V}} \mapsto \pi_\lambda(z)$ is analytic.
For all $\lambda \in \mathbb{D}$,

$$\widetilde{V}_\lambda \subset B(0,\rho) \text{ with } \log\rho = \frac{2\log 4}{1 + |\lambda|^{-1}}.$$

Proof. We want to construct universal coverings $\pi_\lambda : \widetilde{V}_\lambda \to V_\lambda$ such that π_λ depend holomorphically on λ. For this purpose, we use Bers's embedding.

By hypothesis, the set V_0 is hyperbolic, i.e. its analytic universal coverings are isomorphic to $\mathbb{D}$. Let $\pi_0 : \mathbb{D} \to V_0$ be a universal covering mapping 0 to 0. Let $h_\lambda : V_0 \to V_\lambda$ be the quasiconformal homeomorphism provided by the holomorphic motion. Let μ_λ be the Beltrami form on V_0 defined by $\mu_\lambda = \overline{\partial} h_\lambda / \partial h_\lambda$. Finally, let $\widetilde{\mu}_\lambda$ be the Beltrami form defined on $\mathbb{C}$ by $\widetilde{\mu}_\lambda = \pi_0^* \mu_\lambda$ on $\mathbb{D}$ and $\widetilde{\mu}_\lambda = 0$ on $\mathbb{C} \setminus \mathbb{D}$.

There exist quasiconformal homeomorphisms $\widetilde{h}_\lambda : \mathbb{C} \to \mathbb{C}$ such that $\widetilde{\mu}_\lambda = \overline{\partial}\widetilde{h}_\lambda / \partial \widetilde{h}_\lambda$. Those homeomorphisms are univalent outside $\mathbb{D}$. We can normalize them by the conditions $\widetilde{h}_\lambda(0) = 0$ and $\widetilde{h}_\lambda(z) = z + \mathcal{O}(1)$ as $z \to \infty$. Then $\widetilde{h}_\lambda$ is uniquely defined.

Now, set $\widetilde{V}_\lambda = \widetilde{h}_\lambda(\mathbb{D})$ and define

$$\widetilde{\mathcal{V}} = \left\{(\lambda, z) \in \mathbb{D} \times \mathbb{C} \,\middle|\, z \in \widetilde{V}_\lambda\right\}$$

The map $(\lambda, z) \mapsto (\lambda, \widetilde{h}_\lambda(z))$ is a homeomorphism from $\mathbb{D} \times \mathbb{C}$ to $\mathbb{D} \times \mathbb{C}$ and $\widetilde{\mathcal{V}}$ is the image of $\mathbb{D} \times \mathbb{D}$. Thus, $\widetilde{\mathcal{V}}$ is open. The maps $\pi_\lambda : \widetilde{V}_\lambda \to V_\lambda$ defined by $\pi_\lambda = h_\lambda \circ \pi_0 \circ \widetilde{h}_\lambda^{-1}$ are universal coverings. The computation to prove that $(\lambda, z) \mapsto \pi_\lambda(z)$ is analytic is becoming well known, but since we know no reference for this, we include the proof here : indeed, for every fixed λ, the null Beltrami differential is mapped by $\widetilde{h}_\lambda^{-1}$ to $\widetilde{\mu}_\lambda$, which is mapped by π_0 to μ_λ, and then by h_λ to 0. Thus each π_λ is a holomorphic function. Then, $\widetilde{\mu}_\lambda$ depends holomorphically on λ, and thus, $\partial\widetilde{h}_\lambda / \partial\overline{\lambda} = 0$.

The maps π_λ and π_0 are analytic (and thus C^∞) whereas the homeomorphisms $\widetilde{h}_\lambda$ and h_λ are only quasiconformal. Approximating $\widetilde{h}_\lambda$ and h_λ by C^∞ functions, it is fairly easy to see that we can use the chain rule to compute the derivative of the expression $\pi_\lambda \circ \widetilde{h}_\lambda = h_\lambda \circ \pi_0$ with respect to $\overline{\lambda}$ in the sense of distributions, we get

$$\frac{\partial \pi_\lambda}{\partial \overline{\lambda}}\Big|_{h_\lambda(z)} + \frac{\partial \pi_\lambda}{\partial z}\Big|_{h_\lambda(z)} \cdot \frac{\partial \widetilde{h}_\lambda}{\partial \overline{\lambda}}\Big|_z + \frac{\partial \pi_\lambda}{\partial \overline{z}}\Big|_{h_\lambda(z)} \cdot \overline{\frac{\partial \widetilde{h}_\lambda}{\partial \lambda}}\Big|_z = \frac{\partial h_\lambda}{\partial \overline{\lambda}}\Big|_{\pi_0(z)}.$$

Since $\partial\widetilde{h}_\lambda / \partial\overline{\lambda} = 0$, $\partial\pi_\lambda / \partial\overline{z} = 0$ and $\partial h_\lambda / \partial\overline{\lambda} = 0$, the previous expression simplifies to

$$\frac{\partial \pi_\lambda}{\partial \overline{\lambda}}\Big|_{h_\lambda(z)} = 0.$$

So, by Weyl's lemma, π_λ depends analytically on λ.

We can now estimate the conformal radius of the sets $\widetilde{V}_\lambda$. For this purpose, note that by the area theorem, since $\tilde{h}_\lambda$ is univalent outside $\mathbb{D}$ and normalized to be tangent to the identity at ∞, the set $\widetilde{V}_\lambda$ is contained in the disk $B(0,4)$. The boundary moves holomorphically in $B(0,4) \setminus \{0\}$. For $\lambda = 0$ the boundary is the unit circle. It follows from Schwarz's lemma that the hyperbolic distance in $B(0,4)\setminus\{0\}$ between the boundary of $\widetilde{V}_\lambda$ and S^1 is less than or equal to the hyperbolic distance in $\mathbb{D}$ between λ and 0. This and an elementary computation yield

$$\widetilde{V}_\lambda \subset B(0,\rho_2) \quad \text{with} \quad \log\rho_2 = \frac{\log 16}{1+|\lambda|^{-1}}.$$

■

Let us recall that $\alpha \in D_i = B(p_{n_i}/q_{n_i}, 1/q_{n_i}^2)$. Let $r = 1/q_{n_i}^2$. The real number α_0 belongs to $B_i = B(p_{n_i}/q_{n_i}, 1/q_{n_i}^3)$ thus $\dfrac{\alpha_0 - p_{n_i}/q_{n_i}}{r}$ belongs to $\mathbb{D}$. Let us apply the previous proposition to our problem with

$$\lambda = \lambda(\alpha) = \zeta\Big(\frac{\alpha - p_{n_i}/q_{n_i}}{r}\Big)$$

where ζ is any automorphism of $\mathbb{D}$ that maps $\dfrac{\alpha_0 - p_{n_i}/q_{n_i}}{r}$ to 0, and with $V_\lambda = V_{i-1}(\alpha)$. Let

$$\phi(\delta) = \big(\lambda\circ\alpha(\delta), \chi(\delta)\big).$$

where $\alpha(\delta) = \frac{p_{n_i}}{q_{n_i}} + \delta^q_{n_i}$ and $\chi = \chi_{p_{n_i}/q_{n_i}}$. In section 5, we proved that $|\alpha_0 - p_{n_i}/q_{n_i}| \le 1/2q_{n_i}^3$. Thus

$$\lambda(B_i) \subset B\Big(0, \frac{3}{2q_{n_i}}\Big).$$

This proves proposition 11.

Now, set

$$\widetilde{\mathcal{V}} = \{(\lambda, z) \mid \lambda\in\mathbb{D} \text{ and } z \in \widetilde{V}_\lambda\}.$$

We keep the notation p_1 and p_2 for the projections on the first and the second coordinates. We can lift the map $\phi : U_i \to \mathcal{V}$ to a map $\tilde\phi : U_i \to \widetilde{\mathcal{V}}$ such that for all $\delta \in U_i$, we have

- $p_1\circ\phi(\delta) = p_1\circ\tilde\phi(\delta) = \lambda$,
- $p_2\circ\phi(\delta) = \pi_\lambda\circ p_2\circ\tilde\phi(\delta)$ and
- $p_2\circ\tilde\phi(0) = 0$.

We then define $\widehat{\phi} = p_2\circ\tilde\phi$.

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Appendix A. Arithmetic conventions

By convention,

$$[a_0, a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

The n-th approximant of an irrational number $\alpha = [a_0, a_1, \dots] \in \mathbb{R}\setminus\mathbb{Q}$ is the number

$$p_n/q_n = [a_0, \dots, a_n] = a_0 + \cfrac{1}{\ddots + \cfrac{1}{a_n}},$$

where q_n is a positive integer, and the fraction p_n/q_n is in it's lowest terms.

We always have $q_0 = 1$, and if we set $q_{-1} = 0$ and $q_{-2} = 1$, then the following recurrence relation holds for all $n \in \mathbb{N}$:

$$q_n = a_n q_{n-1} + q_{n-2}.$$

Thus $q_1 = a_1$, $q_2 = a_2 a_1 + 1$, ..., and q_n never depends on a_0.

For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the Bruno sum is defined by

$$B(\alpha) = \sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n} \in \,]0, +\infty].$$

Lemma 5. *For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $B(\alpha + 1) = B(\alpha)$ and $B(1 - \alpha) = B(\alpha)$.*

Proof. The first comes from $\alpha + 1 = [a_0 + 1, a_1, a_2, \ldots]$. For the second, we may assume that $\alpha \in]1/2, 1[$. This is equivalent to $a_0 = 0$ and $a_1 = 1$. Thus $q_1 = 1$ and $q_2 = a_2 + 1$. It is easy to check that $1 - \alpha = [0, a_2 + 1, a_3, a_4, \ldots]$. Thus, if we note p'_n/q'_n the approximants of $1 - \alpha$, then $q'_0 = 1 = q_1$, $q'_1 = a_2 + 1 = q_2$, and one then checks by induction that $q'_n = q_{n+1}$ for all $n \in \mathbb{N}$. Thus $B(\alpha) = B(1 - \alpha) + \log(q_1)/q_0$, and $\log(q_1)/q_0 = \log(1)/1 = 0$. ■

At some point, we defined the Fibonacci numbers F_n, by $F_0 = 1$, $F_1 = 2$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$. The reader should note that the indexing may be different than what is usually found in the litterature. It is designed for the situation when $\alpha \in]0, 1/2[$. Then, for all $n \in \mathbb{N}$, $q_n \geq F_n$, as can be proved by induction.

In this article, we make use a few times of the following fact, that we state here (the proof is left as an exercise to the reader)

Lemma 6. *For all $\lambda > 81/64$, the sequence*

$$\log(\lambda n^2)/n,$$

defined for $n \geq 2$, is decreasing. As a corollary, if $\alpha \in]0, 1/2[$ is irrationnal, then for all $n \geq 1$,

$$\frac{\log(\lambda q_n^2)}{q_n} \leq \frac{\log(\lambda F_n^2)}{F_n}.$$

We also include here, for reference, the following computations:

$$\sum_{n=1}^{+\infty} \frac{\log F_n}{F_n} = 1.96\ldots \qquad \sum_{n=1}^{+\infty} \frac{1}{F_n} = 1.35\ldots$$

where the rounding is to the lower.

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