LIMITS OF DEGENERATE PARABOLIC QUADRATIC RATIONAL MAPS

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Abstract. We investigate the closure in moduli space of the set of quadratic rational maps which possess a degenerate parabolic fixed point.

Contents

Introduction 1
1. Preliminaries 7
2. The cardinality of $\mathcal{A}_{p/q}$ 10
3. The rational map $\mathcal{R}_{p/q}$ 15
4. Parabolic renormalization 16
5. Horn maps for the family g_a 24
6. Hausdorff limit of $\mathcal{A}_{p/q}$ as $p/q \to 0$ 29
7. Finite type analytic maps 33
8. Order of growth of $\mathfrak{C}_{r/s}$ 35
9. Multizetas and multitangents 39
References 45

Introduction

The moduli space of Möbius conjugacy classes of quadratic rational maps is isomorphic to $\mathbb{C}^2$. This space is swept out by a family of lines, each consisting of the classes of maps possessing a fixed point of given multiplier; a generic point of moduli space lies on precisely three such lines [M1]. In this article, we are concerned with the geography of the parameter lines whose associated multiplier is a root of unity, and their limits when the multiplier tends to 1 in a suitably controlled fashion.

If the multiplier is $\omega_{p/q} = \mathrm{e}^{2\pi \mathrm{i} p/q}$ (the rational number p/q always assumed to be in least terms) then the corresponding fixed point is parabolic. The local dynamics is well understood. In particular, there exists a positive integer ν, the parabolic *degeneracy* with the following property: there are νq *attracting petals* and νq *repelling petals*, which alternate cyclically around the fixed point. Attracting petals are disjoint Jordan domain and two attracting petal boundaries touch at the fixed point and are otherwise disjoint. Similarly, repelling petals are disjoint Jordan domain and two attracting petal boundaries touch at the fixed point and are otherwise disjoint. Moreover, each repelling petal intersects two attracting petals, and each attracting petal intersects two repelling petals; see Figure 1. Each attracting petal

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is mapped into the attracting petal which is νp times further counterclockwise and the image of each repelling petal contains the repelling petal that is νp times further counterclockwise.

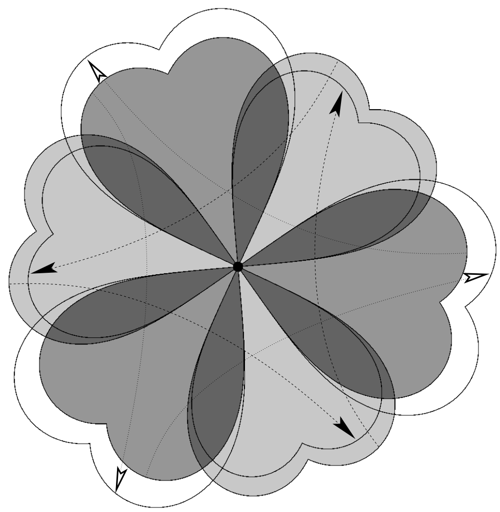


FIGURE 1. An example of petals for a multiplier $\omega_{1/3}$. The attracting petals are light grey and the repelling petals are dark grey. The fixed point is at the center of the flower.

There is an additional formal invariant, the *résidu itératif*, which takes values in $\mathbb{C}$ (for details, see Section 1). This invariant may be used to detect whether the fixed point is virtually attracting, indifferent or repelling (see [Ep4], [M2, Section 12] or [Bu]): this terminology will not be employed in our article.

The global dynamics is also well understood. Recall that a rational map g has a *Fatou set* $\mathcal{F}(g)$, on which the family of iterates $\{g^{\circ n}\}$ is normal, and a *Julia set* $\mathcal{J}(g) = \mathbb{P}^1 - \mathcal{F}(g)$ which is the closure of the set of repelling periodic points. A parabolic fixed point β has a *basin of attraction*

$$\{z \in \mathbb{P}^1 |\ g^{\circ n}(z) \xrightarrow[n\to+\infty]{\neq} \beta\}$$

which is a union of connected components of the Fatou set. The attracting petals of β are contained in the basin of attraction. The *immediate basin* of β is the union of the connected components of $\mathcal{F}(g)$ containing the attracting petals. Fatou [F] proved that the immediate basin of a parabolic point of degeneracy ν contains at least ν critical points. Since a quadratic rational map has 2 critical points, necessarily $\nu \leq 2$.

There are finitely many Möbius conjugacy classes of quadratic rational maps having a degenerate parabolic fixed point of given multiplier $\omega_{p/q}$. In this article, we prove that there are $q-2$ such classes (see Figure 3 and Theorem 0.1). Following [Sha, Section 2], it should be possible to give a combinatorial interpretation. There, it is shown that these invariants suffice to classify the corresponding hyperbolic maps - those with periodic critical points belonging to disjoint q-cycles whose immediate basins cluster at a fixed point. Moreover, it is shown that each

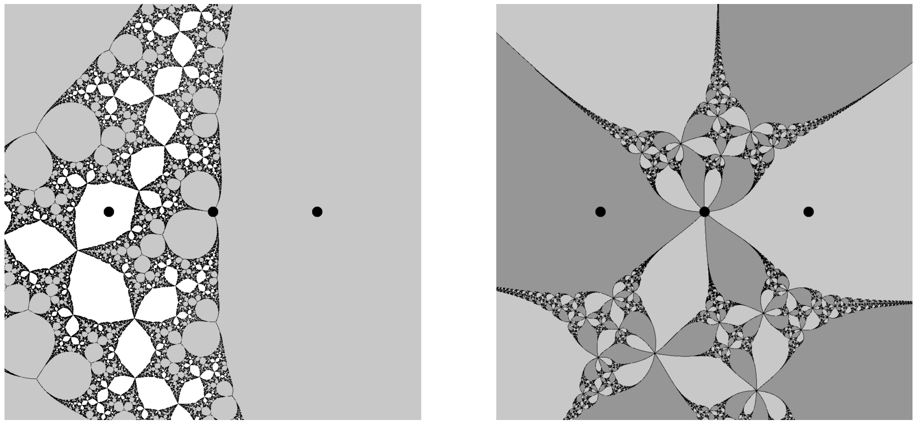


FIGURE 2. Left: an example with $p/q = 1/3$. The Julia set is black. The parabolic degeneracy is 1. There is one cycle of attracting petals (of period 3) whose basin is grey. There is a superattracting cycle of period 3 whose basin is white. Right: an example with $p/q = 1/4$. The parabolic degeneracy is 2. There are 2 cycles of attracting petals whose basins are light grey and dark grey. The points which are marked are the parabolic fixed point at 0 and the critical points at ± 1.

such example arises as a *mating* of an ordered pair of quadratic polynomials, and in precisely two ways. The parabolic case may follow from a pinching argument as in [HT].

For convenience, we work with the 2 parameter family of quadratic rational maps

$$g_{\rho,a}(z) = \frac{\rho z}{1 + az + z^2}, \qquad \text{with } a \in \mathbb{C} \text{ and } \rho \in \mathbb{C}^*.$$

Note that $g_{\rho,a}$ has critical points at ± 1 and that 0 is a fixed point of multiplier ρ. Conversely, every quadratic rational map with labeled critical points and a distinguished fixed point of multiplier $\rho \neq 0$ is Möbius conjugate to $g_{\rho,a}$ for some unique $a \in \mathbb{C}$; the involution $z \mapsto -z$, which interchanges the critical points ± 1, conjugates $g_{\rho,a}$ to $g_{\rho,-a}$. We write $g_a = g_{1,a}$, so that $g_{\rho,a} = \rho \cdot g_a$.

For a rational number p/q, consider the set

$$\mathcal{A}_{p/q} = \{a \in \mathbb{C} \mid \text{the map } g_{\omega_{p/q},a} \text{ has a degenerate parabolic fixed point at } 0\}.$$

Note that $\mathcal{A}_{p/q}$ is symmetric with respect to 0, and that $\mathcal{A}_{0/1} = \{0\}$; as we shall see, $0 \in \mathcal{A}_{p/q}$ precisely when q is odd.

Our main results are as follows.

Theorem 0.1. *The cardinality of $\mathcal{A}_{p/q}$ is $q - 2$, for any p/q with $q \geq 2$.*

We prove Theorem 0.1 in Section 2, by an argument which illustrates and applies transversality principles developed in [Ep5].

For parameters $a \notin \mathcal{A}_{p/q}$, the résidu itératif of $g_{\omega_{p/q},a}$ at 0 is given by a rational function $\mathcal{R}_{p/q} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, *e.g.* $\mathcal{R}_{0/1}(a) = 1/a^2$. In Section 3 we deduce the following:

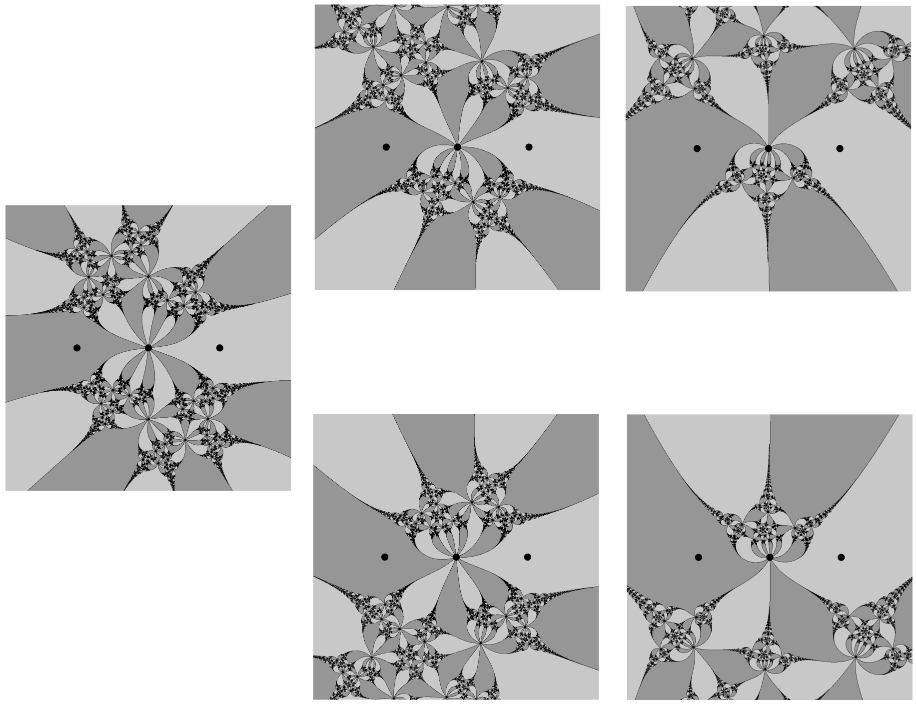


FIGURE 3. The Julia sets of the 5 quadratic rational maps having a degenerate parabolic fixed point with multiplier $\omega_{1/7}$ at 0 and critical points at ± 1.

Theorem 0.2. *If $q \geq 2$ then $\mathcal{R}_{p/q}$ is a rational function of degree $2q-2$. Moreover, every pole of $\mathcal{R}_{p/q}$ is double: there is a pole at ∞ and the finite poles are the points in $\mathcal{A}_{p/q}$.*

Theorem 0.3. *Let r/s be a rational number. Set $p_k = s$ and $q_k = ks + r$ so that*

$$\frac{p_k}{q_k} = \frac{1}{k + r/s}.$$

Then:

- *The sequence of sets $\mathcal{A}_{p_k/q_k}$ is uniformly bounded, and Hausdorff convergent as $k \to +\infty$. The Hausdorff limit consists of 0 together with a bounded set $\mathfrak{A}_{r/s}$ which is infinite and discrete in $\mathbb{C} - \{0\}$.*
- *There is a meromorphic function $\mathfrak{R}_{r/s} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ such that the sequence of rational functions $\left(\frac{p_k}{q_k}\right)^2 \mathcal{R}_{p_k/q_k}$ converges to $a \mapsto \mathfrak{R}_{r/s}(1/a^2)$ uniformly on every compact subset of $\mathbb{C} - \overline{\mathfrak{A}}_{r/s}$. Furthermore*

$$\mathfrak{A}_{r/s} = \left\{a \in \mathbb{C} - \{0\} \ : \ 1/a^2 \in \mathfrak{B}_{r/s}\right\},$$

  where $\mathfrak{B}_{r/s}$ is the set of poles of $\mathfrak{R}_{r/s}$.

For each p/q, the set $\mathcal{A}_{p/q}$ is contained in the bifurcation locus $\mathcal{L}_{p/q}$ of the family $a \mapsto g_{\omega_{p/q},a}$. This bifurcation locus may be defined as the closure of the set of parameters a such that $g_{\omega_{p/q},a}$ has a parabolic cycle with period greater than

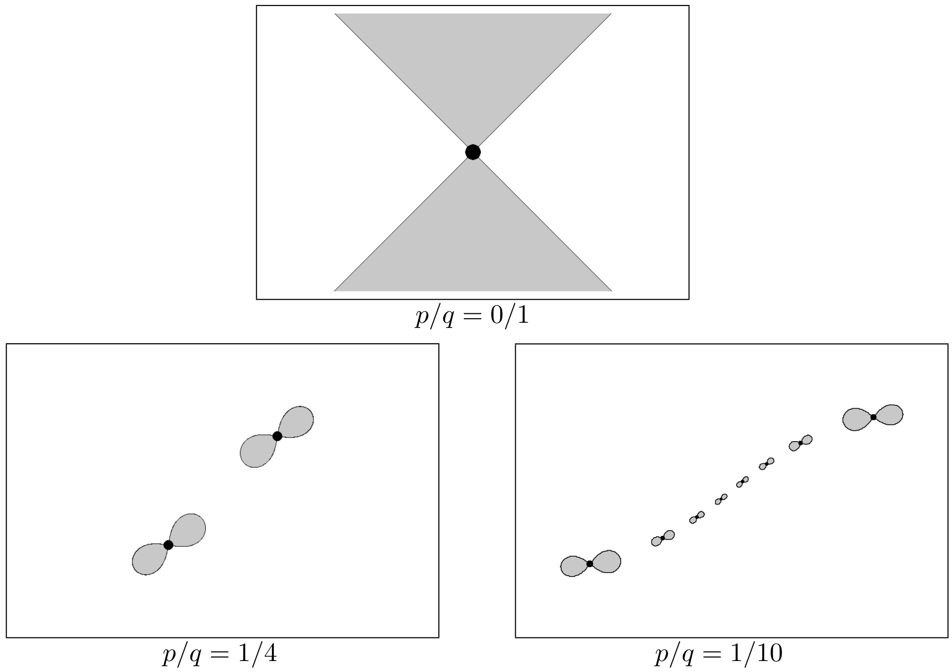


FIGURE 4. Points in $\mathcal{A}_{p/q}$ are represented by black dots (in fact the set where $|\mathcal{R}_{p/q}| > 100$). The connected components of the set $\mathcal{R}_{p/q}^{-1}\{z \in \mathbb{C} \;:\; \mathrm{Re}(z) < 0\}$ which are adherent to those points are grey. It follows from additional transversality results in [Ep5] that the restriction of $\mathcal{R}_{p/q}$ to each component is an isomorphism to the left half-plane. For each p/q there is a pair of unbounded components; these appear in the first frame, but they are out of range on the second and the third frames. Since each point of $\mathcal{A}_{p/q}$ is a double pole of $\mathcal{R}_{p/q}$, there are two components of the set attached to each point in $\mathcal{A}_{p/q}$; it is shown in [EpU] that these components are bounded.

1. Thus, by the Fatou-Shishikura Inequality [Ep4], the region where $\mathrm{Re}(\mathcal{R}_{p/q}) < 0$ does not intersect $\mathcal{L}_{p/q}$.

It is tempting to conjecture that for p_k/q_k tending to 0 as above, the sets $\mathcal{L}_{p_k/q_k}$ have a Hausdorff limit $\mathfrak{L}_{r/s}$. Figure 6 illustrates this phenomenon for $p_k/q_k = 1/k$. This Hausdorff limit $\mathfrak{L}_{r/s}$ should be the bifurcation locus of a family of *horn maps* defined in Section 5. However, this may require proving that the Julia sets of those horn maps do not carry invariant line fields, which seems a serious difficulty.

The proof of Theorem 0.3 is somewhat indirect: it would be interesting to consider the combinatorial interpretation of the sequences with given limit in $\mathbb{C} - \{0\}$. Our argument involves considerations of *parabolic renormalization*, discussed in Section 4, and applied in Section 5 to define a family of *horn maps* associated to the family $g_{\omega_{p/q},a}$. The set $\mathfrak{A}_{r/s}$ will correspond to those horn maps having a degenerate parabolic fixed point. Another key ingredient is that such horn maps are *finite type* analytic maps (see Section 7); they share certain dynamical properties with

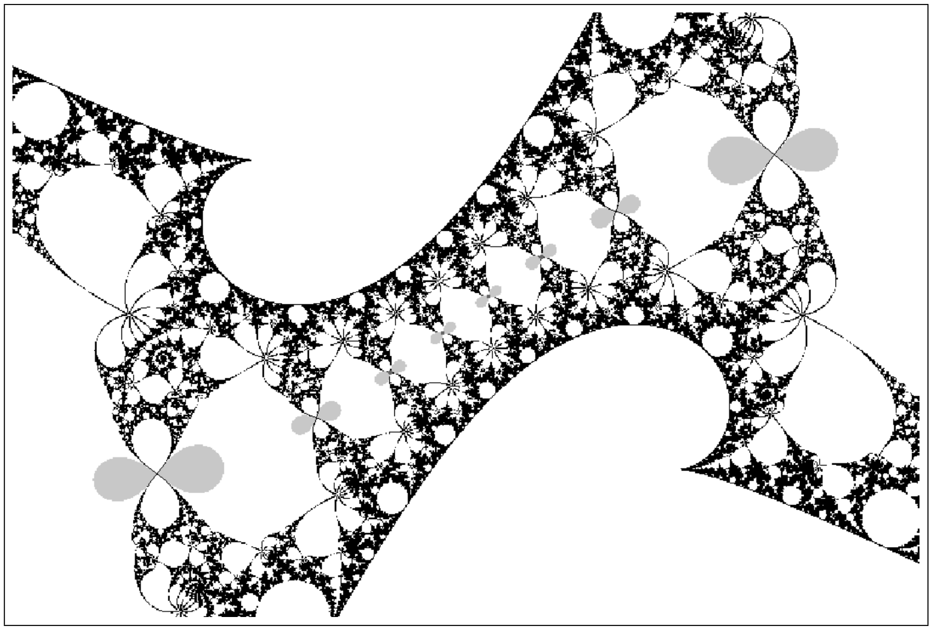


FIGURE 5. The set $\{\text{Re}(\mathcal{R}_{1/10}) < 0\}$ (light grey) and the bifurcation locus $\mathcal{L}_{1/10}$ (black).

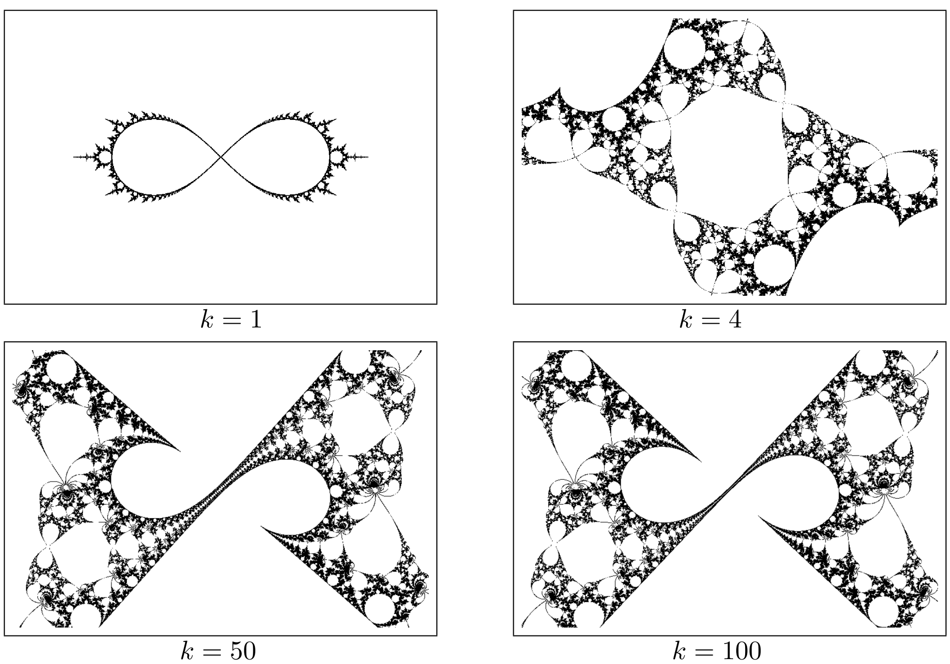


FIGURE 6. The bifurcation loci $\mathcal{L}_{1/k}$ for several values of k.

rational maps. We prove Theorem 0.3 in Section 6. The fact that $\mathfrak{A}_{r/s}$ is infinite rests on a growth estimate for a certain entire function that vanishes precisely at the points of $\mathfrak{A}_{r/s}$. This function admits an intriguing expansion which we discuss in Section 9.

1. Preliminaries

1.1. **Formal invariants.** Let $f : (\mathbb{C}, \beta) \to (\mathbb{C}, \beta)$ be an analytic map fixing $\beta \in \mathbb{C}$. Recall that the *multiplier* of f at β is the complex number $\rho_\beta(f) = f'(\beta)$. If f is not the identity near β, then the *topological multiplicity* and *holomorphic index* may be defined as the residues

$$\begin{aligned} \mathrm{mult}_\beta(f) &= \mathrm{res}_\beta\left(\frac{1-f'(z)}{z-f(z)}dz\right) \quad \text{and} \\ \mathrm{index}_\beta(f) &= \mathrm{res}_\beta\left(\frac{1}{z-f(z)}dz\right). \end{aligned}$$

The multiplicity $\mathrm{mult}_\beta(f)$ is known to topologists as the *Lefschetz fixed point index.* It is easy to verify that these quantities are invariant under holomorphic change of coordinates, and may thereby be sensibly defined for $\beta = \infty$ (for example, see [M2, Section 12]). The topological multiplicity is always a positive integer, but the holomorphic index may assume any complex value. It follows from the residue theorem that if $f : U \to \mathbb{C}$ is analytic on U, then

$$\begin{aligned} \sum_{\beta=f(\beta)\in V} \mathrm{mult}_\beta(f) &= \frac{1}{2\pi \mathrm{i}}\int_{\partial V} \frac{1-f'(z)}{z-f(z)}dz \quad \text{and} \\ \sum_{\beta=f(\beta)\in V} \mathrm{index}_\beta(f) &= \frac{1}{2\pi \mathrm{i}}\int_{\partial V} \frac{1}{z-f(z)}dz \end{aligned}$$

for any open set V compactly contained in U, so long as ∂V is a rectifiable curve containing no fixed points. These integrals evidently depend continuously on f. Moreover, it follows from these relations that

$$\begin{aligned} \sum_{\beta=f(\beta)\in \mathbb{P}^1} \mathrm{mult}_\beta(f) &= D+1 \quad \text{and} \\ \sum_{\beta=f(\beta)\in \mathbb{P}^1} \mathrm{index}_\beta(f) &= 1 \end{aligned}$$

for any degree D rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Note that $\mathrm{mult}_\beta(f) = 1$ precisely when $\rho_\beta(f) \neq 1$, and then

$$\mathrm{index}_\beta(f) = \frac{1}{1-\rho_\beta(f)}.$$

Moreover, if $|\rho_\beta(f)| \neq 1$ or $\rho_\beta(f) = 1$, then $\mathrm{mult}_\beta(f^{\circ n}) = \mathrm{mult}_\beta(f)$ for $n > 0$ (and if additionally $\rho_\beta(f) \neq 0$, then also for $n < 0$). When $\rho_\beta(f) = 1$, it is convenient to work with the *résidu itératif*

$$\mathrm{résit}_\beta(f) = \frac{1}{2}\mathrm{mult}_\beta(f) - \mathrm{index}_\beta(f)$$

since

$$\mathrm{résit}_\beta(f^{\circ n}) = \frac{1}{n}\mathrm{résit}_\beta(f)$$

for all $n \neq 0$ (see for example [BuEp, Introduction]). More generally, if the multiplier $\rho_\beta(f)$ is a primitive q-th root of unity $\omega_{p/q}$ but $f^{\circ q}$ is not the identity, then we say that β is a *parabolic* fixed point with *rotation number* p/q. In this situation, a straightforward calculation shows that $\text{mult}_\beta(f^{\circ q}) = \nu q + 1$ for some positive integer ν: we refer to ν as the *parabolic degeneracy* of f at β, and say that β is a *nondegenerate* parabolic fixed point if $\nu = 1$, a *degenerate* parabolic fixed point otherwise. The formal conjugacy class of such a fixed point is completely determined by the invariants p/q, ν and

$$\text{résit}_\beta(f) := q \cdot \text{résit}_\beta(f^{\circ q}).$$

For details, see [BuEp, Appendix]; see also [Ca] for the topological classification and [Éc], [Ma],[MR] or [V] for the analytic classification.

Now, if

$$f(z) = \omega_{p/q} \cdot \left(z + \sum_{k=2}^{\infty} c_k z^k \right)$$

then we may write

$$f^{\circ q}(z) = z + \sum_{k=2}^{\infty} \mathbf{C}_{p/q}^k (c_2, \ldots, c_k) z^k,$$

where $\mathbf{C}_{p/q}^k$ are polynomials in the variables $c_2, \ldots, c_k$. For example,

$$\mathbf{C}_{0/1}^k (c_2, \ldots, c_k) = c_k$$

for $k \geq 2$, while

$$\mathbf{C}_{1/2}^3 (c_2, c_3) = 2c_3 - 2c_2^2.$$

The polynomials $\mathbf{C}_{p/q}^2, \ldots, \mathbf{C}_{p/q}^q$ identically vanish, as noted above, and the fixed point 0 is parabolic with degeneracy ν if and only if for all $k \leq \nu q$,

$$\mathbf{C}_{p/q}^k (c_2, \ldots, c_k) = 0 \quad \text{and} \quad \mathbf{C}_{p/q}^{\nu q+1} (c_2, \ldots, c_{\nu q+1}) \neq 0.$$

We denote by $\mathbf{I}^m$ the rational functions such that $\text{index}_0(f) = \mathbf{I}^m(c_2, \ldots, c_{2m+1})$ whenever f fixes 0 with multiplicity $m \geq 1$, for example

$$\begin{aligned} \mathbf{I}^1(c_2, c_3) &= \frac{c_3}{c_2^2} \\ \mathbf{I}^2(c_2, c_3, c_4, c_5) &= \frac{3c_3^3 - 2c_3c_5 + 2c_4^2}{2c_3^3}. \end{aligned}$$

In addition, we set

$$\mathbf{R}_{p/q}^\nu (c_2, \ldots, c_{2\nu q+1}) = \frac{\nu q + 1}{2} - \mathbf{I}^{\nu q} \left(\mathbf{C}_{p/q}^2 (c_2), \ldots, \mathbf{C}_{p/q}^{2\nu q+1} (c_2, \ldots, c_{2\nu q+1}) \right)$$

so that $\text{résit}_0(f) = q \cdot \mathbf{R}_{p/q}^\nu (c_2, \ldots, c_{2\nu q+1})$ whenever the fixed point 0 has parabolic degeneracy ν. For example, $\mathbf{R}_{0/1}^\nu = \dfrac{\nu+1}{2} - \mathbf{I}^\nu$ for every $\nu \geq 1$, while

$$\mathbf{R}_{1/2}^1 (c_2, c_3, c_4, c_5) = \frac{11c_2^4 + 6c_2^3 - 25c_2^2c_3 + 12c_2c_4 - 4c_5}{4(c_2^2 - c_3)^2}.$$

1.2. **Variational formulae.** Here and henceforth, we will consider various holomorphic families $t \mapsto \gamma_t$ defined near 0 in $\mathbb{C}$. We will employ the notation

$$\gamma = \gamma_0 \quad \text{and} \quad \dot\gamma = \frac{d\gamma_t}{dt}\Big|_{t=0}.$$

Assume $t \mapsto f_t$ is a holomorphic family of germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Let us first assume that the multiplier $\rho_0(f)$ is not zero. Then, we may write $f_t = f \circ \chi_t$ with $\chi = \mathrm{id}$ and $\chi_t(0) = 0$. In that case, $\mathbf{v} = \dot\chi$ is a vector field, defined and holomorphic in a neighborhood of 0, vanishing at 0, and $\dot f = Df \circ \mathbf{v}$:

$$f_t = f + t \cdot Df \circ \mathbf{v} + \mathcal{O}(t^2).$$

In addition, an elementary induction yields that for all integers $N \geq 1$,

$$f_t^{\circ N} = f^{\circ N} + t \cdot Df^{\circ N} \circ \mathbf{v}_N + \mathcal{O}(t^2) \quad \text{with} \quad \mathbf{v}_N = \sum_{n=0}^{N-1} (f^{\circ n})^* \mathbf{v}.$$

In this situation we have the following:

Lemma 1.1. *Assume $t \mapsto f_t$ is a holomorphic family of germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that the multiplier of f_t at 0 is $\omega_{p/q}$. Then, $\mathbf{v}_q$ has a zero of order at least $q+1$ at 0. If $\mathbf{v}_q$ has a zero of order exactly $q+1$ at 0, then for $t \neq 0$ sufficiently close to 0, the parabolic degeneracy of f_t at 0 is 1.*

Proof. Since $f_t^{\circ q} - \mathrm{id}$ vanishes at 0 to order at least $q+1$ for all t, $\mathbf{v}_q$ also vanishes to order at least $q+1$ at 0:

$$\mathbf{v}_q = \left(cz^{q+1} + \mathcal{O}(z^{q+2})\right)\frac{d}{dz} \quad \text{with} \quad c \in \mathbb{C}.$$

As t and z tend to 0, we therefore have the expansion

$$f_t^{\circ q}(z) = f^{\circ q}(z) + tcz^{q+1} + \mathcal{O}(tz^{q+2}) + \mathcal{O}(t^2 z^{q+1}). \tag{1.1}$$

In particular, if $\mathbf{v}_q$ has a zero of order exactly $q+1$ at 0, i.e., if $c \neq 0$, then for $t \neq 0$ sufficiently close to 0, the multiplicity $m_0(f_t^{\circ q})$ is $q+1$ and the parabolic degeneracy of f_t at 0 is 1. □

Lemma 1.2. *Assume $t \mapsto f_t$ is a holomorphic family of germs $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that the multiplier of f_t at 0 is $\omega_{p/q}$. If $f^{\circ q} = \mathrm{id}$, and $\mathbf{v}_q$ does not identically vanish, then as $t \to 0$,*

$$\text{résit}_0(f_t) = \mathrm{res}_0\left(\frac{q}{\mathbf{v}_q}\right) \cdot \frac{1}{t} + \mathcal{O}(1).$$

Proof. If $f^{\circ q} = \mathrm{id}$, then:

$$f_t^{\circ q} = \mathrm{id} + t \cdot \mathbf{v}_q + \mathcal{O}(t^2).$$

and if $\mathbf{v}_q$ is not identically 0,

$$\mathrm{index}_0(f_t^{\circ q}) = \mathrm{res}_0\left(-\frac{1}{t \cdot \mathbf{v}_q + \mathcal{O}(t^2)}\right) = \mathrm{res}_0\left(-\frac{1}{t\mathbf{v}_q} + H(t, z)\right)$$

where H is a germ which is defined near $(t, z) = (0, 0)$, holomorphic with respect to t and meromorphic with respect to z. It follow that

$$\text{résit}_0(f_t) = \mathrm{res}_0\left(\frac{q}{\mathbf{v}_q}\right) \cdot \frac{1}{t} + h(t)$$

where h is a germ which is defined and holomorphic near $t = 0$. □

Lemma 1.3. *Assume $t \mapsto f_t$ is a holomorphic family of germs $(\mathbb{C},0) \to (\mathbb{C},0)$ such that the multiplier of f_t at 0 is $\omega_{p/q}$. If the parabolic degeneracy of f at 0 is 2 and if $\mathbf{v}_q$ has a zero of order exactly $q+1$ at 0 then the function $t \mapsto \text{résit}_0(f_t)$ has a double pole at $t=0$.*

Proof. Since the parabolic degeneracy of f at 0 is 2,

$$f^{\circ q}(z) = z + Cz^{2q+1} + \mathcal{O}(z^{2q+2}) \quad \text{with} \quad C \neq 0.$$

It follows that

$$f_t^{\circ q}(z) = z + c(t)z^{q+1} + Cz^{2q+1} + \mathcal{O}(tz^{q+2}) + \mathcal{O}(z^{2q+2})$$

where $c : (\mathbb{C},0) \to (\mathbb{C},0)$ is a holomorphic germ vanishing at $t=0$. According to Equation (1.1), we have

$$\mathbf{v}_q = \bigl(c'(0)z^{q+1} + \mathcal{O}(z^{q+2})\bigr)\frac{d}{dz}.$$

Since $\mathbf{v}_q$ has a zero of order exactly $q+1$ at $z=0$, the function c has a simple zero at $t=0$. As $z \to 0$,

$$\begin{aligned}\frac{1}{z - f_t^{\circ q}(z)} &= -\frac{1}{c(t)z^{q+1}}\left(1 - \frac{C}{c(t)}z^q + \mathcal{O}(tz) + \mathcal{O}(z^{q+1})\right)\\ &= -\frac{1}{c(t)z^{q+1}} + \frac{C}{[c(t)]^2 z} + \mathcal{O}\left(\frac{1}{z^q}\right) + \mathcal{O}\left(\frac{1}{t}\right).\end{aligned}$$

As a consequence, as $t \to 0$,

$$\text{index}_0(f_t^{\circ q}) = \text{res}_0\left(\frac{dz}{z - f_t^{\circ q}(z)}\right) = \frac{C}{[c(t)]^2} + +\mathcal{O}\left(\frac{1}{t}\right).$$

Thus, $\text{résit}_0(f_t) = -qC/[c(t)]^2 + \mathcal{O}(1/t)$ has a double pole at $t=0$. □

2. The cardinality of $\mathcal{A}_{p/q}$

Let us fix p and q coprime with $q \geq 2$ and set $\rho = \omega_{p/q}$. Our goal in this section is to prove that there are exactly $q-2$ distinct parameters $a \in \mathbb{C}$ such that $g_{\rho,a}$ has a degenerate parabolic fixed point at 0.

As mentioned in the Introduction, for z near 0 we have the expansion

$$g_{\rho,a}^{\circ q}(z) = z + \mathbf{C}_{p/q}(a)z^{q+1} + \mathcal{O}\left(z^{q+2}\right) \tag{2.1}$$

for some polynomial $\mathbf{C}_{p/q}$. The parameters $a \in \mathbb{C}$ for which $g_{\rho,a}$ has a degenerate parabolic fixed point at 0 are the roots of $\mathbf{C}_{p/q}$. We will first show that the degree of $\mathbf{C}_{p/q}$ is $q-2$. We will then prove that the roots of $\mathbf{C}_{p/q}$ are simple.

Since $z \mapsto -z$ conjugates $g_{\rho,a}$ to $g_{\rho,-a}$, the polynomial $\mathbf{C}_{p/q}$ satisfies

$$\mathbf{C}_{p/q}(-a) = (-1)^q\mathbf{C}_{p/q}(a).$$

Consequently, when q is odd, the polynomial $\mathbf{C}_{p/q}$ is odd. It vanishes at 0 which therefore belongs to $\mathcal{A}_{p/q}$. Another route to this result involves observing that $g_{\rho,0}$ is an odd map, and so, the number of attracting directions of $g_{\rho,0}$ is always even. Thus, when q is odd, the parabolic point at 0 is necessarily degenerate, whence the number of attracting directions is $2q$.

When q is even, the polynomial $\mathbf{C}_{p/q}$ is even. It cannot vanish at 0 since otherwise, this would be a double root of $\mathbf{C}_{p/q}$. Thus, when q is even, $0 \notin \mathcal{A}_{p/q}$.

2.1. **The degree of the polynomial $\mathbf{C}_{p/q}$ is $q-2$.** In order to determine the degree of $\mathbf{C}_{p/q}$, it is enough to study the asymptotics of $\mathbf{C}_{p/q}(a)$ as $a \to \infty$. Let us make the change of parameter $t = 1/a^2$ and the change of dynamical variable $y = az$, so that $g_{\rho,a}$ becomes

$$f_t(y) = a g_{\rho,a}(y/a) = \frac{\rho y}{1 + y + ty^2}.$$

On the one hand, near $y = 0$, we have the expansion

$$f_t^{\circ q}(y) = a g_{\rho,a}^{\circ q}(y/a) = y + \frac{\mathbf{C}_{p/q}(a)}{a^q} y^{q+1} + \mathcal{O}(y^{q+2}). \tag{2.2}$$

On the other hand, $f = f_0$ is a Möbius transformation of order q fixing 0 with derivative ρ and $y_0 = \rho - 1$ with derivative $1/\rho$. It is conjugate (via $y \mapsto y/(y-y_0)$) to the rotation of angle p/q and so, $f^{\circ q} = \mathrm{id}$. In addition,

$$f_t = f \circ \chi_t \quad \text{with} \quad \chi_t(y) = \frac{y}{1+ty^2}.$$

Thus, according to the previous section,

$$f_t^{\circ q} = \mathrm{id} + t\mathbf{v}_q + \mathcal{O}(t^2) \quad \text{with} \quad \mathbf{v}_q = \sum_{k=0}^{q-1} (f^{\circ k})^* \dot\chi.$$

Lemma 2.1. *The vector field $\mathbf{v}_q$ has a zero of order $q+1$ at the point 0.*

Proof. It is convenient to make a change of coordinates $y \mapsto x = y/(y - y_0)$ which fixes 0 and sends $y_0 \mapsto \infty$ and $\infty \mapsto 1$. In this new coordinate, f is conjugate to the rotation $x \mapsto \rho x$. We have

$$\dot\chi = -y^3 \frac{d}{dy} = -y_0^2 \frac{x^3}{1-x}\frac{d}{dx} = -y_0^2 \sum_{n\geq 3} x^n \frac{d}{dx}$$

and thus

$$(f^{\circ k})^* \dot\chi = -y_0^2 \sum_{n\geq 3} \rho^{k(n-1)} x^n \frac{d}{dx}.$$

Consequently,

$$\begin{aligned}
\mathbf{v}_q = \sum_{k=0}^{q-1} (f^{\circ k})^* \dot\chi &= -y_0^2 \sum_{k=0}^{q-1} \left(\sum_{n\geq 3} \rho^{k(n-1)} x^n \frac{d}{dx} \right) \\
&= -y_0^2 \sum_{n\geq 3} \left(\sum_{k=0}^{q-1} \rho^{k(n-1)} \right) x^n \frac{d}{dx} \\
&= -y_0^2 \sum_{m\geq 1} q x^{mq+1} \frac{d}{dx} = -q y_0^2 \frac{x^{q+1}}{(1-x^q)} \frac{d}{dx}.
\end{aligned}$$

□

Now since

$$\mathbf{v}_q = \left(c y^{q+1} + \mathcal{O}(y^{q+2})\right) \frac{d}{dy}$$

with $c \neq 0$, (1.1) may be rewritten as

$$f_t^{\circ q}(y) = y + \frac{c}{a^2} y^{q+1} + \mathcal{O}\left(\frac{y^{q+2}}{a^2}\right) + \mathcal{O}\left(\frac{y^{q+1}}{a^4}\right). \tag{2.3}$$

Comparing Equations (2.2) and (2.3), we see that as $a \to \infty$, we have

$$\frac{\mathbf{C}_{p/q}(a)}{a^q} \underset{a\to\infty}{=} \frac{c}{a^2} + \mathcal{O}\left(\frac{1}{a^4}\right)$$

where $c \neq 0$, whence $\mathbf{C}_{p/q}(a)$ is a polynomial of degree $q-2$.

2.2. **The polynomial $\mathbf{C}_{p/q}$ has simple roots.** This is a transversality result which may be addressed with the results of [Ep5]. We present a self-contained proof here.

Assume that $a_0 \in \mathbb{C}$ is a root of $\mathbf{C}_{p/q}$. For $t \in \mathbb{C}$, set

$$f_t = g_{\rho, a_0+t}.$$

Let $\mathbf{v}$ be the meromorphic vector field on $\mathbb{P}^1$ which satisfies

$$f_t = f + t \cdot Df \circ \mathbf{v} + \mathcal{O}(t^2).$$

Outside the critical point set $\mathcal{C}_f$, we have $\mathbf{v}(z) = D_z f^{-1}\big(\dot f(z)\big)$. Let $\zeta : (\mathbb{C},0) \to (\mathbb{C},0)$ be a local analytic coordinate tangent to the identity, in which the expression of f is

$$\zeta \circ f = \rho\zeta \cdot \big(1 + \mathcal{O}(\zeta^{2q})\big).$$

Finally, let

$$\mathbf{q} = \left(\frac{b_3}{z^3} + \cdots + \frac{b_{q+2}}{z^{q+2}}\right) dz^2$$

be the meromorphic quadratic differential on $\mathbb{P}^1$ having the same polar part of order ≤ -3 at 0 as $d\zeta^2/\zeta^{q+2}$ (a quadratic differential is a section of the complex line bundle given by $\otimes$-square of the cotangent bundle).

Lemma 2.2. *The polynomial $\mathbf{C}_{p/q}$ has a simple root at a_0 if and only* $\mathrm{res}_0(\mathbf{q}\cdot\mathbf{v}) = 0$.

Proof. Since $f_t = g_{\rho,a_0+t}$ and according to Equation (2.1) defining $\mathbf{C}_{p/q}$, we may write

$$f_t^{\circ q}(z) = z + \mathbf{C}_{p/q}(a_0+t) z^{q+1} + z^{q+2} G(t,z)$$

for some function G defined and analytic near $(0,0)$ in $\mathbb{C}^2$. In particular for $t=0$, we have that

$$f^{\circ q}(z) = z + z^{q+2} G(0,z).$$

Now

$$\begin{aligned} f_t^{\circ q}(z) &= z + \big(t\mathbf{C}'_{p/q}(a_0) + \mathcal{O}(t^2)\big) z^{q+1} + z^{q+2}\big(G(0,z) + \mathcal{O}(t)\big) \\ &= z + z^{q+2}G(0,z) + t\mathbf{C}'_{p/q}(a_0) z^{q+1} + \mathcal{O}(tz^{q+2}) + \mathcal{O}(t^2 z^{q+1}). \end{aligned}$$

Thus,

$$f_t^{\circ q}(z) = f^{\circ q}(z) + t\mathbf{C}'_{p/q}(a_0) z^{q+1} + \mathcal{O}(tz^{q+2}) + \mathcal{O}(t^2 z^{q+1}). \tag{2.4}$$

Since

$$\dot f(z) = -\frac{\rho z^2}{(1+a_0 z + z^2)^2}\frac{d}{dz}$$

and

$$f'(z) = -\frac{\rho\big(1 + a_0 z + z^2 - z(a_0+2z)\big)}{(1+a_0z+z^2)^2} = \frac{\rho(1-z^2)}{(1+a_0z+z^2)^2},$$

we have

$$\mathbf{v} = -\frac{\rho z^2}{\rho(1-z^2)}\frac{d}{dz} = \frac{z^2}{z^2-1}\frac{d}{dz}.$$

The vector field $\mathbf{v}$ is holomorphic outside $\mathcal{C}_f$, has simple poles at the critical points $c_1 = 1$ and $c_2 = -1$, and has a double zero at 0. According to Equation (1.1),

$$f_t^{\circ q}(z) = f^{\circ q}(z) + tcz^{q+1} + \mathcal{O}(tz^{q+2}) + \mathcal{O}(t^2 z^{q+1}) \tag{2.5}$$

with

$$\mathbf{v}_q = \sum_{k=0}^{q-1} (f^{\circ k})^* \mathbf{v} = \big(cz^{q+1} + \mathcal{O}(z^{q+2})\big)\frac{d}{dz}.$$

Comparing Equations (2.4) and (2.5) we deduce that $\mathbf{C}'_{p/q}(a_0) = c$.

Since $\zeta \circ f = \rho\zeta\big(1 + \mathcal{O}(\zeta^{2q})\big)$, for $j \geq 1$, we have that

$$f^*\left(\zeta^j \frac{d}{d\zeta}\right) = \rho^{j-1}\zeta^j\big(1 + \mathcal{O}(\zeta^{2q})\big)\frac{d}{d\zeta}.$$

Thus,

$$\mathbf{v} = \sum_{j\geq 2} c_j \zeta^j \frac{d}{d\zeta} \quad \Longrightarrow \quad \mathbf{v}_q = qc_{q+1}\zeta^{q+1}\big(1 + \mathcal{O}(\zeta^{2q})\big)\frac{d}{d\zeta}.$$

Since $\mathbf{v}$ has a double zero at $z = 0$,

$$\operatorname{res}_0(\mathbf{q}\cdot\mathbf{v}) = \operatorname{res}_0\left(\frac{d\zeta^2}{\zeta^{q+2}}\cdot\mathbf{v}\right) = c_{q+1} = \frac{1}{q}\mathbf{C}'_{p/q}(a_0).$$ □

A quadratic differential may be pushed forward by f as follows:

$$f_*\mathbf{q} = \sum_h h^*\mathbf{q} \quad \text{where the sum ranges over the inverse branches of } f.$$

Then, $f_*\mathbf{q}$ is a meromorphic quadratic differential. If $\mathbf{q}$ is holomorphic outside $A \subset \mathbb{P}^1$, then $f_*\mathbf{q}$ is holomorphic outside $f(A) \cup \mathcal{V}_f$, where $\mathcal{V}_f$ is the critical value set of f.

Lemma 2.3. *We have* $\operatorname{res}_0(\mathbf{q}\cdot\mathbf{v}) = 0$ *if and only if* $\mathbf{q} - f_*\mathbf{q} = 0$.

Proof. Note that $\mathbf{q}$ is holomorphic on $\mathbb{C} - \{0\}$ and has at most a simple pole at ∞. The product $\mathbf{q}\cdot\mathbf{v}$ is globally meromorphic on $\mathbb{P}^1$, with poles in $\{0\}\cup\mathcal{C}_f = \{0, c_1, c_2\}$ (there is no pole at ∞ since $\mathbf{q}$ has at most a simple pole at ∞ and $\mathbf{v}$ has a double zero at ∞). The sum of residues of $\mathbf{q}\cdot\mathbf{v}$ is 0 and so,

$$\operatorname{res}_0(\mathbf{q}\cdot\mathbf{v}) = -\operatorname{res}_{c_1}(\mathbf{q}\cdot\mathbf{v}) - \operatorname{res}_{c_2}(\mathbf{q}\cdot\mathbf{v}).$$

Let θ be a vector field, defined and holomorphic near $\mathcal{V}_f = \{v_1, v_2\}$ with $v_1 = f(c_1)$ and $v_2 = f(c_2)$, such that

$$\theta(v_1) = \dot{f}(c_1) = \frac{-\rho}{(2+a_0)^2}\frac{d}{dz} = \frac{-v_1^2}{\rho}\frac{d}{dz}$$

and

$$\theta(v_2) = \dot{f}(c_2) = \frac{-\rho}{(2-a_0)^2}\frac{d}{dz} = \frac{-v_2^2}{\rho}\frac{d}{dz}.$$

The vector field $f^*\theta$ is defined and meromorphic near $\mathcal{C}_f = \{c_1, c_2\}$. In addition, $f^*\theta - \mathbf{v}$ is holomorphic near $\mathcal{C}_f$ and vanishes at c_1 and c_2. As a consequence,

$$\operatorname{res}_{c_1}(\mathbf{q}\cdot\mathbf{v}) + \operatorname{res}_{c_2}(\mathbf{q}\cdot\mathbf{v}) = \operatorname{res}_{c_1}(\mathbf{q}\cdot f^*\theta) + \operatorname{res}_{c_2}(\mathbf{q}\cdot f^*\theta) = \operatorname{res}_{v_1}(f_*\mathbf{q}\cdot\theta) + \operatorname{res}_{v_2}(f_*\mathbf{q}\cdot\theta).$$

The poles of $f_*\mathbf{q}$ are contained in $\{0\}\cup\mathcal{V}_f = \{0, v_1, v_2\}$. Let $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the inverse branch of f fixing 0. Then,

$$\zeta \circ h = \rho^{-1}\zeta\big(1 + \mathcal{O}(\zeta^{2q})\big).$$

As a consequence

$$h^*\left(\frac{d\zeta^2}{\zeta^2}\right) = \frac{d\zeta^2}{\zeta^2}\cdot\big(1+\mathcal{O}(\zeta^{2q})\big) \quad\text{and}\quad h^*\left(\frac{d\zeta^2}{\zeta^{q+2}}\right) = \frac{d\zeta^2}{\zeta^{q+2}}\cdot\big(1+\mathcal{O}(\zeta^{2q})\big)$$

have respectively the same polar parts at 0 as $d\zeta^2/\zeta^2$ and $d\zeta^2/\zeta^{q+2}$. The inverse branch of f mapping 0 to ∞ contributes to at worst a simple pole for $f_*\mathbf{q}$. It follows that the quadratic differential $\mathbf{q}-f_*\mathbf{q}$ has at most a simple pole at 0. Therefore, $\mathbf{q}-f_*\mathbf{q}$ has at most 4 simple poles contained in $\{0,\infty,v_1,v_2\}$. As a consequence,

$$\mathbf{q}-f_*\mathbf{q} = \lambda\mathbf{q}_0 \quad\text{with}\quad \mathbf{q}_0 = \frac{dz^2}{z(z-v_1)(z-v_2)}. \tag{2.6}$$

Since $\mathbf{q}$ is holomorphic near $\mathcal{V}_f$,

$$\begin{aligned}\operatorname{res}_{v_1}(f_*\mathbf{q}\cdot\theta)+\operatorname{res}_{v_2}(f_*\mathbf{q}\cdot\theta) &= -\operatorname{res}_{v_1}\big((\mathbf{q}-f_*\mathbf{q})\cdot\theta\big)-\operatorname{res}_{v_2}\big((\mathbf{q}-f_*\mathbf{q})\cdot\theta\big)\\ &= -\operatorname{res}_{v_1}(\lambda\mathbf{q}_0\cdot\theta)-\operatorname{res}_{v_2}(\lambda\mathbf{q}_0\cdot\theta)\\ &= \frac{\lambda}{v_1(v_1-v_2)}\cdot\frac{v_1^2}{\rho}+\frac{\lambda}{v_2(v_2-v_1)}\cdot\frac{v_2^2}{\rho} = \frac{\lambda}{\rho}.\end{aligned}$$

All this shows that

$$\mathbf{q}-f_*\mathbf{q} = -\rho\cdot\operatorname{res}_0(\mathbf{q}\cdot\mathbf{v})\cdot\mathbf{q}_0.$$ □

It is therefore enough to show that $\mathbf{q}-f_*\mathbf{q}\neq 0$. This is given in a more general setting by [Ep4, Proposition 2]. For completeness, we include a proof in our case here. Note that in the following lemma, the strict inequality guarantees that $\mathbf{q}-f_*\mathbf{q}\neq 0$.

Lemma 2.4. *We have the inequality* $\displaystyle\int_{\mathbb{P}^1}|\mathbf{q}-f_*\mathbf{q}|>0$.

Proof. For $r>0$ sufficiently close to 0, let V_r be the Euclidean disk centered at 0 with radius r in the ζ-coordinate and let U_r^0 and U_r^∞ be the components of $U_r=f^{-1}(V_r)$ containing 0 and ∞ respectively.

We shall first prove that

$$\int_{V_r-U_r}|\mathbf{q}| \underset{r\to 0}{\longrightarrow} 0 \quad\text{and}\quad \int_{U_r-V_r}|\mathbf{q}| \underset{r\to 0}{\longrightarrow} 0.$$

Indeed, since $\zeta\circ f=\rho\zeta\cdot\big(1+o(\zeta^q)\big)$, the set V_r-U_r is contained in an annulus of the form $\big\{s<|\zeta|<r\big\}$ with $s=r\cdot\big(1+o(r^q)\big)$. Since q has the same polar part as $d\zeta^2/\zeta^{q+2}$, we have that

$$\int_{V_r-U_r}|q|\le\mathcal{O}\left(\int_s^r\frac{|d\zeta^2|}{|\zeta|^{q+2}}\right)=\mathcal{O}\left(\frac{1}{s^q}-\frac{1}{r^q}\right)\underset{r\to 0}{\longrightarrow}0.$$

Similarly, the set $U_r^0-V_r$ is contained in an annulus of the form $\big\{r<|\zeta|<s\big\}$ with $s=r\cdot\big(1+o(r^q)\big)$ and

$$\int_{U_r^0-V_r}|q|\le\mathcal{O}\left(\int_r^s\frac{|d\zeta^2|}{|\zeta|^{q+2}}\right)=\mathcal{O}\left(\frac{1}{r^q}-\frac{1}{s^q}\right)\underset{r\to 0}{\longrightarrow}0.$$

Finally, since $\mathbf{q}$ has at worst a simple pole at ∞, we have that $|\mathbf{q}|$ is integrable near ∞, so,

$$\int_{U_r^\infty-V_r}|\mathbf{q}|\underset{r\to 0}{\longrightarrow}0.$$

We now prove that the function

$$h : r \mapsto \int_{\mathbb{P}^1 - U_r} |\mathbf{q}| - \int_{\mathbb{P}^1 - V_r} |f_* \mathbf{q}|$$

is positive and decreasing; in particular, it has a positive limit as $r \to 0$. Indeed, thanks to the triangle inequality, if $s < r$, we have:

$$\int_{V_r - V_s} |f_* \mathbf{q}| = \int_{V_r - V_s} \left| \sum h^* \mathbf{q} \right| < \int_{V_r - V_s} \sum |h^* \mathbf{q}| = \int_{U_r - U_s} |\mathbf{q}|,$$

where the sum ranges over the two inverse branches of f. The inequality is indeed strict: otherwise the two inverse branches $h^* \mathbf{q}$ would be proportional, in which case $\mathbf{q}$ would also have a multiple pole at ∞, contrary to assumption.

It follows that

$$\begin{aligned}
\int_{\mathbb{P}^1} |\mathbf{q} - f_* \mathbf{q}| &\geq \lim_{r \to 0} \left(\int_{\mathbb{P}^1 - V_r} |\mathbf{q}| - \int_{\mathbb{P}^1 - V_r} |f_* \mathbf{q}| \right) \\
&= \lim_{r \to 0} \left(h(r) + \int_{\mathbb{P}^1 - V_r} |\mathbf{q}| - \int_{\mathbb{P}^1 - U_r} |\mathbf{q}| \right) \\
&= \lim_{r \to 0} \left(h(r) + \int_{U_r - V_r} |\mathbf{q}| - \int_{V_r - U_r} |\mathbf{q}| \right) = \lim_{r \to 0} h(r) > 0. \qquad \square
\end{aligned}$$

3. The rational map $\mathcal{R}_{p/q}$

Here, we prove Theorem 0.2. Recall that outside the finite set $\mathcal{A}_{p/q}$, the résidu itératif of $g_{\omega_{p/q}, a}$ is given by

$$\mathcal{R}_{p/q}(a) = \text{résit}_0(g_{\omega_{p/q}, a}) = q \cdot \left(\frac{q+1}{2} - \text{index}_0(g_{\omega_{p/q}, a}) \right),$$

where $\mathcal{R}_{p/q} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a rational function. For example, using the notation ω in place of $\omega_{p/q}$, we have

$$\begin{aligned}
\mathcal{R}_{0/1}(a) &= \frac{1}{a^2} \\
\mathcal{R}_{1/2}(a) &= \frac{a^2 + 2}{4} \\
\mathcal{R}_{\pm 1/3}(a) &= \frac{(1 + \omega) a^4 + (8 - 2\omega) a^2 - 3 - 6\omega}{3a^2} \\
\mathcal{R}_{\pm 1/4}(a) &= \frac{\omega a^6 + (15 - 2\omega) a^4 - (6 + 32\omega) a^2 - 6}{2(a^2 - \omega)^2}
\end{aligned}$$

by direct (computer-assisted) computation. Note that each $\mathcal{R}_{p/q}$ is an even function: indeed, the résidu itératif is a conjugacy invariant, and the maps associated to the parameters $\pm a$ are conjugate via the involution $z \mapsto -z$.

Since the cardinality of $\mathcal{A}_{p/q}$ is $q - 2$, Theorem 0.2 is a consequence of the following two assertions.

Proposition 3.1. *If $q \geq 2$ then the rational function $\mathcal{R}_{p/q}$ has a double pole at ∞.*

Proof. Let us use the notations of Section 2.1: $t = 1/a^2$, $f_t(y) = \rho y/(1 + y + ty^2)$ with $\rho = \omega_{p/q}$, $y_0 = \rho - 1$ and $x = y/(y - y_0)$. As proved in Section 2.1,

$$f_t^{\circ q} = \mathrm{id} + t\mathbf{v}_q + \mathcal{O}(t^2) \quad \text{with} \quad \mathbf{v}_q = -qy_0^2 \frac{x^{q+1}}{1 - x^q} \frac{d}{dx}.$$

Note that

$$\mathrm{res}_0 \left(\frac{1}{\mathbf{v}_q} \right) = -\frac{1}{qy_0^2} \mathrm{res}_0 \left(\frac{1 - x^q}{x^{q+1}} dx \right) = \frac{1}{qy_0^2}.$$

According to Lemma 1.2, as $t \to 0$,

$$\mathcal{R}_{p/q}(a) = \text{résit}_0(f_t) = \frac{1}{ty_0^2} + \mathcal{O}(1) = \frac{a^2}{(\rho - 1)^2} + \mathcal{O}(1). \qquad \square$$

Proposition 3.2. *The rational function $\mathcal{R}_{p/q}$ has a double pole at each point in $\mathcal{A}_{p/q}$.*

Proof. Let us use the notations of Section 2.2: $a_0 \in \mathcal{A}_{p/q}$ and $f_t = g_{\rho,a_0+t}$ with $\rho = \omega_{p/q}$. The parabolic degeneracy of $f = f_0$ at 0 is 2 and according to Section 2.2, $\mathbf{v}_q$ has a zero of order exactly $q+1$ at $z = 0$. According to Lemma 1.3, the function $t \mapsto \text{résit}_0(f_t)$ has a double pole at $t = 0$, whence $a \mapsto \text{résit}_0(g_{\rho,a}) = \text{résit}_0(f_{a-a_0})$ has a double pole at $a = a_0$. $\square$

4. Parabolic renormalization

In order to understand the behavior of the sets $\mathcal{A}_{p_k/q_k}$ when $p_k/q_k \to 0$ as in Theorem 0.3, we will use the theory of parabolic renormalization that we now recall. We will essentially follow the presentation of Shishikura [Shi2].

4.1. **Fatou coordinates.** Assume $g : z \mapsto z - az^2 + \mathcal{O}(z^3)$ with $a \in \mathbb{C} - \{0\}$ is defined and holomorphic near 0. In the coordinate $Z = 1/(az)$, the expression of g becomes

$$F(Z) = Z + 1 + \frac{b}{Z} + \mathcal{O}(1/Z^2) \quad \text{with} \quad b \in \mathbb{C}.$$

Assume R is chosen sufficiently large so that for all $Z \in \mathbb{C} - D(0, R)$

$$\big|F(Z) - Z - 1\big| < \frac{1}{4} \quad \text{and} \quad \big|F'(Z) - 1\big| < \frac{1}{4} \quad \text{when} \quad |Z| \geq R.$$

Set

$$\Omega^{\mathrm{att}} = \{Z \in \mathbb{C} \ : \ -3\pi/4 < \arg(Z - 2R) < 3\pi/4\}$$

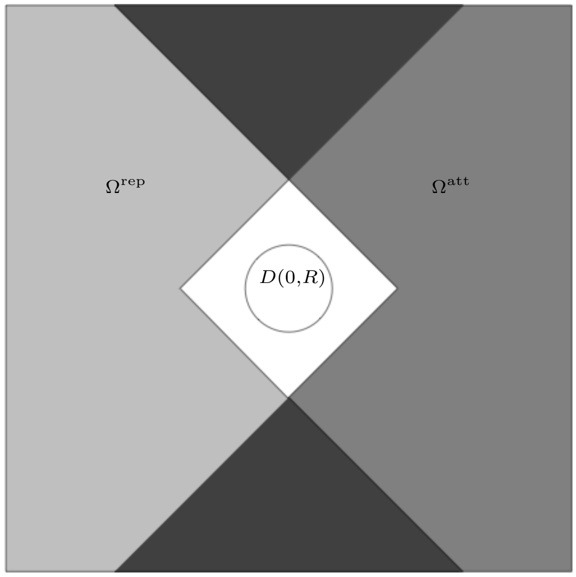
and

$$\Omega^{\mathrm{rep}} = \{Z \in \mathbb{C} \ : \ \pi/4 < \arg(Z + 2R) < 7\pi/4\}.$$

Then, there exist univalent maps $\Phi^{\mathrm{att}} : \Omega^{\mathrm{att}} \to \mathbb{C}$ and $\Phi^{\mathrm{rep}} : \Omega^{\mathrm{rep}} \to \mathbb{C}$ satisfying

$$\Phi^{\mathrm{att}}\big(F(Z)\big) = \Phi^{\mathrm{att}}(Z) + 1 \quad \text{and} \quad \Phi^{\mathrm{rep}}\big(F(Z)\big) = \Phi^{\mathrm{rep}}(Z) + 1.$$

Those *Fatou coordinates* $\Phi^{\mathrm{att/rep}}$ are unique up to additive constants.



4.2. **Estimates on the Fatou coordinates.**

Lemma 4.1 ([Shi2] Proposition 2.6.2)**.** *If $C \in (0,+\infty)$ and $D \in (0,+\infty)$ are large enough, then the following holds. Let $\Omega \subset \mathbb{C}$ be an open set, $\Phi : \Omega \to \mathbb{C}$ and $F : \Omega \to \mathbb{C}$ be holomorphic maps such that*

- *Φ is univalent on Ω,*
- *$\big|F(Z) - Z - 1\big| < 1/4$ for $Z \in \Omega$ and*
- *$\Phi \circ F(Z) = \Phi(Z) + 1$ when Z and $F(Z)$ are in Ω.*

If $\mathrm{dist}(Z, \mathbb{C} - \Omega) \geq D$, *then*

$$\left|\Phi'(Z) - \frac{1}{F(Z) - Z}\right| \leq C\left(\frac{1}{\big[\mathrm{dist}(Z, \mathbb{C} - \Omega)\big]^2} + \big|F'(Z) - 1\big|\right).$$

We can derive the following estimates on the Fatou coordinates, where log is the branch of logarithm defined on $\mathbb{C} - (-\infty, 0]$ with $\log(1) = 0$.

Proposition 4.2. *If $C \in (0,+\infty)$ and $D \in (0,+\infty)$ are large enough, then the following holds. Assume F is holomorphic on $\mathbb{C} - D(0,R)$ with*

$$\big|F(Z) - Z - 1\big| < 1/4 \quad \text{and} \quad \big|F'(Z) - 1\big| < 1/4.$$

Then, there is $C^{\rm att} \in \mathbb{C}$ such that, for all $Z \in \Omega^{\rm att}$ with $\mathrm{dist}(Z, \mathbb{C} - \Omega^{\rm att}) > D$,

$$\big|\Phi^{\rm att}(Z) - Z + b\log Z - C^{\rm att}\big| \leq C\frac{1 + R^2 + |b|^2}{\mathrm{dist}(Z, \mathbb{C} - \Omega^{\rm att})}$$

and there is $C^{\rm rep} \in \mathbb{C}$ such that for all $Z \in \Omega^{\rm rep}$ with $\mathrm{dist}(Z, \mathbb{C} - \Omega^{\rm rep}) > D$,

$$\big|\Phi^{\rm rep}(Z) - Z + b\log(-Z) - C^{\rm rep}\big| \leq C\frac{1 + R^2 + |b|^2}{\mathrm{dist}(Z, \mathbb{C} - \Omega^{\rm rep})}.$$

Proof. The functions $Z \mapsto Z^2 \cdot \big(F'(Z) - 1\big)$ and $Z \mapsto Z^2 \cdot \big(F(Z) - Z - 1 - b/Z\big)$ are holomorphic in $\mathbb{C} - D(0,R)$ and bounded as $Z \to \infty$. By the Maximum Modulus

Principle, we see that for all $Z \in \mathbb{C} - D(0,R)$,

$$\bigl|F'(Z) - 1\bigr| \le \frac{1}{4} \cdot \frac{R^2}{|Z|^2} \quad \text{and} \quad \left|F(Z) - Z - 1 - \frac{b}{Z}\right| \le \left(\frac{1}{4} + \frac{|b|}{R}\right) \cdot \frac{R^2}{|Z|^2}.$$

Set $v(Z) = F(Z) - Z - 1 - b/Z$. Then,

$$\begin{aligned}\left|\frac{1}{F(Z)-Z} - \left(1 - \frac{b}{Z}\right)\right| &= \left|\frac{v(Z)\cdot(-1+b/Z) + b^2/Z^2}{F(Z)-Z}\right| \\ &\le \frac{4}{3}\frac{\bigl(R^2/4 + R|b|\bigr)\cdot\bigl(1+|b|/R\bigr) + |b|^2}{|Z|^2} \\ &\le \frac{4}{3}\frac{2R^2 + 2R|b| + 2|b|^2}{|Z|^2} \\ &= 4\frac{R^2+|b|^2}{|Z|^2} - \frac{4}{3}\frac{(R-|b|)^2}{|Z|^2} \le 4\frac{R^2+|b|^2}{|Z|^2}.\end{aligned}$$

Let $\Phi^{\text{att}} : \Omega^{\text{att}} \to \mathbb{C}$ be a Fatou coordinate. Using the previous Lemma, we see that if $Z \in \Omega^{\text{att}}$ with $\text{dist}(Z, \mathbb{C} - \Omega^{\text{att}}) \ge D$, then

$$\left|(\Phi^{\text{att}})'(Z) - \left(1 - \frac{b}{Z}\right)\right| \le C \cdot \frac{1 + R^2 + |b|^2}{\bigl[\text{dist}(Z, \mathbb{C} - \Omega^{\text{att}})\bigr]^2}$$

for some universal constant C. For $t \in [0, +\infty)$, we have the inequality

$$\text{dist}(Z + t, \mathbb{C} - \Omega^{\text{att}}) \ge \text{dist}(Z, \mathbb{C} - \Omega^{\text{att}}) + t/\sqrt{2}.$$

Integration with respect to t yields the inequality

$$\begin{aligned}\bigl|\Phi^{\text{att}}(Z) - Z + b\log Z - C^{\text{att}}\bigr| &\le \int_0^{+\infty} \frac{C\cdot\bigl(1 + R^2 + |b|^2\bigr)}{\bigl[\text{dist}(Z, \mathbb{C} - \Omega^{\text{att}}) + t/\sqrt{2}\bigr]^2}\,dt \\ &= C\sqrt{2}\frac{1 + R^2 + |b|^2}{\text{dist}(Z, \mathbb{C} - \Omega^{\text{att}})}\end{aligned}$$

for an appropriate $C^{\text{att}} \in \mathbb{C}$. The estimate on Φ^{rep} is obtained similarly. □

In particular, we have the expansions

$$\Phi^{\text{att}}(Z) = Z - b\log Z + C^{\text{att}} + o(1) \quad \text{as} \quad \text{dist}(Z, \mathbb{C} - \Omega^{\text{att}}) \to +\infty$$

and

$$\Phi^{\text{rep}}(Z) = Z - b\log(-Z) + C^{\text{rep}} + o(1) \quad \text{as} \quad \text{dist}(Z, \mathbb{C} - \Omega^{\text{rep}}) \to +\infty.$$

4.3. **Unwrapping coordinates.** If $f(z) = \mathrm{e}^{2\pi\mathrm{i}\alpha_f} z + \mathcal{O}(z^2)$, with $\alpha_f \ne 0$, is sufficiently close to g, then f has a fixed point at 0 and a fixed point at $\sigma_f \ne 0$ with

$$\sigma_f = \frac{2\pi\mathrm{i}\alpha_f}{a}\bigl(1 + o(1)\bigr)$$

as f tends to g uniformly in some fixed neighborhood of 0 (note that $\alpha_f \to 0$ as $f \to g$).

Let us consider the universal covering $\tau_f : \mathbb{C} \to \mathbb{P}^1 - \{0, \sigma_f\}$ defined by

$$\tau_f(Z) = \frac{\sigma_f}{1 - \mathrm{e}^{-2\pi\mathrm{i}\alpha_f Z}}.$$

The covering transformation group is generated by the translation

$$T_f : Z \mapsto Z - 1/\alpha_f.$$

Moreover, as $f \to g$, the family of maps τ_f converges locally uniformly in $\mathbb{C}$ to $\tau : Z \mapsto 1/(aZ)$. Finally, for any $\varepsilon > 0$, there exists R_0 such that for all f sufficiently close to g, the image of

$$U_f = \mathbb{C} - \bigcup_{k\in\mathbb{Z}} T_f^k \overline{B}(0, R_0)$$

by τ_f is contained in $B(0, \varepsilon)$.

Therefore, if R_0 is sufficiently large, the map $f : B(0, \varepsilon) \to \mathbb{C}$ lifts via τ_f to a map $F_f : U_f \to \mathbb{C}$ such that

- $f \circ \tau_f = \tau_f \circ F_f$,
- $T_f \circ F_f = F_f \circ T_f$,
- $F_f(Z) = Z + 1 + o(1)$ as $\mathrm{Im}(\alpha Z) \to +\infty$ and
- for $Z \in U_f$,

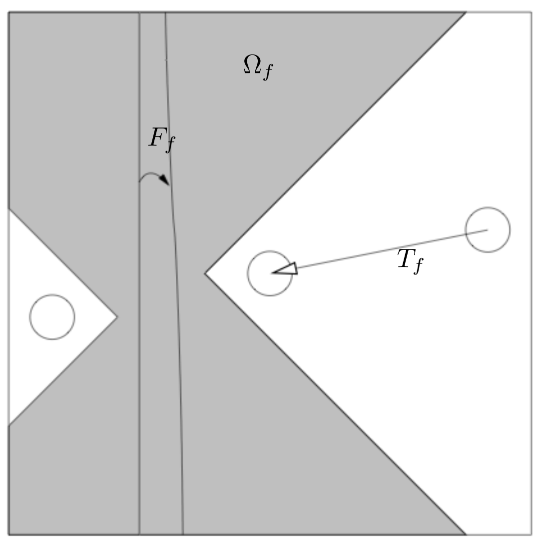
$$|F_f(Z) - Z - 1| < \frac{1}{4} \quad \text{and} \quad |F_f'(Z) - 1| < \frac{1}{4}.$$

As $f \to g$, $F_f \to F$ uniformly on every compact subset of $\mathbb{C} - \overline{B}(0, R_0)$.

4.4. **Perturbed Fatou coordinates.** Assume now that $f(z) = \mathrm{e}^{2\pi \mathrm{i} \alpha_f} z + \mathcal{O}(z^2)$ is a small perturbation of g with $\alpha_f \neq 0$ and $\left|\arg(\alpha_f)\right| < \pi/4$ (this forces $\mathrm{Re}(\alpha_f) > 0$). If f is sufficiently close to g, the set

$$\Omega_f = \Omega^{\mathrm{rep}} \cap T_f(\Omega^{\mathrm{att}})$$

is contained in U_f and contains a vertical strip of width ≥ 2.



Then, there exist a univalent map $\Phi_f : \Omega_f \to \mathbb{C}$ satisfying

$$\Phi_f\big(F_f(Z)\big) = \Phi_f(Z) + 1.$$

Since F_f commutes with the translation T_f, we have

$$F_f(Z) - Z - 1 = \mathcal{O}\left(e^{-2\pi \mathrm{Im}(\alpha_f Z)}\right) \quad \text{and} \quad F'(Z) - 1 = \mathcal{O}\left(e^{-2\pi \mathrm{Im}(\alpha_f Z)}\right)$$

as $\mathrm{Im}(\alpha_f Z)\to+\infty$. According to Lemma 4.1, as Z tends to ∞ within a sector of the form $\{Z\in\mathbb{C} \;:\; \theta_1<\arg(Z-iR)<\theta_2\}$ with $\pi/4<\theta_1<\theta_2<3\pi/4$, we have

$$\Phi_f'(Z)=1+\mathcal{O}\big(1/|Z|^2\big)$$

whence there is a constant $C_f\in\mathbb{C}$ such that

$$\Phi_f(Z)=Z+C_f+o(1).$$

The *perturbed Fatou coordinate* Φ_f is unique up to additive constant. Assume $Z_f\in\Omega_f$ depends analytically on f and tends to $Z_0\in\Omega^{\mathrm{rep}}$. Assume also that $\Phi_f(Z_f)$ depends analytically on f and tends to $\Phi^{\mathrm{rep}}(Z_0)$ as f tends to g. Then, Φ_f depends analytically on f and $\Phi_f\to\Phi^{\mathrm{rep}}$ uniformly on every compact subset of Ω^{rep} as $f\to g$.

4.5. **Horn maps.** Set

$$V=\Omega^{\mathrm{rep}}\cap\Omega^{\mathrm{att}}\quad\text{and}\quad W=\Phi^{\mathrm{rep}}(V).$$

The map $\mathfrak{H}=\Phi^{\mathrm{att}}\circ(\Phi^{\mathrm{rep}})^{-1}:W\to\mathbb{C}$ satisfies

$$\mathfrak{H}(Z+1)=\mathfrak{H}(Z)+1$$

whenever both sides are defined. Using this relation, we can extend $\mathfrak{H}$ to the region $\big\{Z\in\mathbb{C} \;:\; |\mathrm{Im}Z|>R'\big\}$ for some $R'>0$. We refer to $\mathfrak{H}$ as a *lifted horn map* associated to F. As $\mathrm{Im}(Z)\to+\infty$, we have $\mathfrak{H}(Z)=Z+C^{\mathrm{att}}-C^{\mathrm{rep}}-b\pi\mathrm{i}+o(1)$ and as $\mathrm{Im}(Z)\to-\infty$, we have $\mathfrak{H}(Z)=Z+C^{\mathrm{att}}-C^{\mathrm{rep}}+b\pi\mathrm{i}+o(1)$. The map $\mathfrak{H}$ projects via $\Pi:Z\mapsto\mathrm{e}^{2i\pi Z}$ to a map $\mathfrak{h}$ defined near 0 and ∞ in $\mathbb{C}^*$ with $\Pi\circ\mathfrak{H}=\mathfrak{h}\circ\Pi$. The map $\mathfrak{h}$ extends analytically to 0 and ∞ with

$$\mathfrak{h}(z)=\mathrm{e}^{2\pi^2 b}\cdot\mathrm{e}^{2\pi\mathrm{i}(C^{\mathrm{att}}-C^{\mathrm{rep}})}\cdot z\cdot\big(1+\mathcal{O}(z)\big)\quad\text{as}\quad z\to0$$

and

$$\mathfrak{h}(z)=e^{-2\pi^2 b}\cdot\mathrm{e}^{2\pi\mathrm{i}(C^{\mathrm{att}}-C^{\mathrm{rep}})}\cdot z\cdot\big(1+\mathcal{O}(1/z)\big)\quad\text{as}\quad z\to\infty.$$

This map $\mathfrak{h}$ is called a *horn map* associated to F.

4.6. **Near-parabolic renormalization.** Observe that $V=\Omega_f\cap T_f^{-1}(\Omega_f)$. Set $W_f=\Phi_f(V)$. The map $\Phi_f:V\to W_f$ conjugates $T_f:U\to\Omega_f$ to a map $\mathfrak{F}_f:W_f\to\mathbb{C}$ such that, as $\mathrm{Im}(Z)\to+\infty$,

$$\mathfrak{F}_f(Z)=Z-\frac{1}{\alpha_f}+o(1)\quad\text{and}\quad\mathfrak{F}_f(Z+1)=\mathfrak{F}_f(Z)+1.$$

Increasing R' if necessary and using the second relation, we may extend $\mathfrak{F}_f$ to the region $\big\{Z\in\mathbb{C} \;:\; |\mathrm{Im}Z|>R'\big\}$. The map $\mathfrak{F}_F$ projects via Π to a map $\mathfrak{f}_f$ defined near 0 and ∞ in $\mathbb{C}-\{0\}$ with $\Pi\circ\mathfrak{F}_f=\mathfrak{f}_f\circ\Pi$. The map $\mathfrak{f}_f$ extends analytically to 0 and ∞ with $\mathfrak{f}_f(z)=\mathrm{e}^{-2\pi\mathrm{i}/\alpha_f}z+\mathcal{O}(z^2)$ as $z\to0$.

The key result in the theory of parabolic renormalization is the following (see [Shi2] Prop. 3.2.3).

Proposition 4.3. *If $f\to g$ with $\big|\arg(\alpha_f)\big|<\pi/4$, then*

$$\mathfrak{F}_f+\frac{1}{\alpha_f}\to\mathfrak{H}+C^{\mathrm{rep}}-C^{\mathrm{att}}+b\pi\mathrm{i}$$

locally uniformly in $\big\{Z\in\mathbb{C} \;:\; |\mathrm{Im}Z|>R'\big\}$.

For $\rho \in \mathbb{C}$, we set

$$\mathfrak{g}_\rho = \rho \cdot \mathrm{e}^{-2\pi^2 b} \cdot \mathrm{e}^{2\pi\mathrm{i}(C^{\mathrm{rep}} - C^{\mathrm{att}})} \cdot \mathfrak{h}.$$

Note that $\mathfrak{g}_\rho$ fixes 0 with multiplier ρ. It follows from the previous proposition that if $f \to g$ with $\mathrm{Re}(\alpha_f) > 0$ and if $\mathrm{e}^{-2\pi\mathrm{i}/\alpha_f} \to \rho \in \mathbb{C}$, then $\mathfrak{f}_f \to \mathfrak{g}_\rho$ locally uniformly near 0.

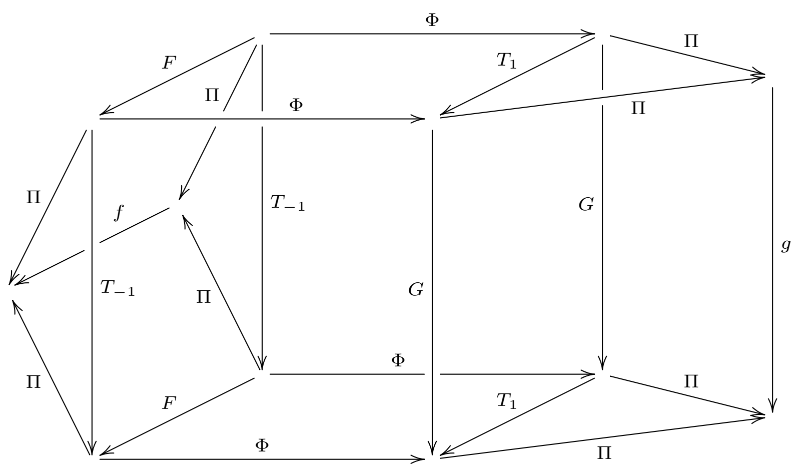
4.7. **Résidu itératif and renormalization.** It will be useful to understand how $\text{résit}_0(f)$ and $\text{résit}_0(\mathfrak{f}_f)$ are related when $\alpha_f = p/q$ is a rational number. In order to study this relation, we will work in a slightly more general context than the near-parabolic renormalization described above.

An *upper sector* is a sector contained in an upper half-plane. We shall say that Z tends to ∞ well inside a sector $V \subset \mathbb{C}$ if Z tends to ∞ and the distance from Z to $\mathbb{C} - V$ is bounded from below by $c|Z|$ for some $c > 0$ (Z remains in a subsector of V of smaller opening). We shall use the notation $Z \xrightarrow{\in V} \infty$.

Assume $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $\alpha \in \mathbb{C} - (-\infty, 0]$ are such that $f'(0) = \mathrm{e}^{2\pi\mathrm{i}\alpha}$. We shall say that a germ $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is an α-renormalization of f if:

- there are holomorphic maps F and G defined in upper half-planes, such that $\Pi \circ F = f \circ \Pi$ and $\Pi \circ G = g \circ \Pi$ with $\Pi(Z) = \mathrm{e}^{2\pi\mathrm{i}Z}$,
- $F(Z) = Z + \alpha + o(1)$ as $\mathrm{Im}(Z) \to +\infty$ and
- there is a univalent map $\Phi : V \to \mathbb{C}$ defined on an upper sector with image contained in a upper half-plane, such that $\Phi \circ F(Z) = \Phi(Z) + 1$ and $\Phi(Z - 1) = G \circ \Phi(Z)$ when both sides are defined.

The relation between those maps is summarized on the following commutative diagram:



Proposition 4.4. *Assume $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ fixes 0 with multiplier $\mathrm{e}^{2\pi\mathrm{i}p/q}$ and let $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a p/q-renormalization of f. Then,*

$$g'(0) = \mathrm{e}^{-2\pi\mathrm{i}q/p}, \quad \nu_0(g) = \nu_0(f) \quad \textit{and} \quad \text{résit}_0(g) = \left(\frac{p}{q}\right)^2 \text{résit}_0(f) + \frac{1}{2\pi\mathrm{i}} \cdot \frac{p}{q}.$$

Proof. We set $\alpha = p/q$ and we let F, G and $\Phi : V \to \mathbb{C}$ be as in the definition of α-renormalization.

As $\mathrm{Im}(Z) \to +\infty$, $F(Z) = Z + \alpha + \mathcal{O}(\mathrm{e}^{2\pi \mathrm{i} Z})$, whence $|F'(Z) - 1| = o(1/|Z|^2)$. It therefore follows from [Shi2] Prop. 2.6.2 that when $Z \overset{\in V}{\to} \infty$,

$$\Phi'(Z) = \frac{1}{F(Z) - Z} + \mathcal{O}\big(1/|Z|^2\big) \quad \text{and} \quad \Phi(Z) = \frac{Z}{\alpha} + C + o(1)$$

for some $C \in \mathbb{C}$. Consequently, as $Z \overset{\in V}{\to} \infty$,

$$G\big(\Phi(Z)\big) = \Phi(Z-1) = \frac{Z}{\alpha} - \frac{1}{\alpha} + C + o(1) = \Phi(Z) - \frac{1}{\alpha} + o(1).$$

It follows that $G(W) - W$ tends to $-1/\alpha$ as $\mathrm{Im}(W) \to +\infty$, whence $g'(0) = \mathrm{e}^{-2\pi \mathrm{i}/\alpha}$.

Consider the maps

$$F_q = F^{\circ q} - p \quad \text{and} \quad G_p = G^{\circ p} + q$$

which respectively lift $f^{\circ q}$ and $g^{\circ p}$ and satisfy

$$F_q(Z) = Z + o(1) \quad \text{and} \quad G_p(Z) = Z + o(1) \quad \text{as} \quad \mathrm{Im}(Z) \to +\infty.$$

Note that Φ conjugates F_q to G_p: $\Phi \circ F_q = G_p \circ \Phi$.

Assume $Z \overset{\in V}{\to} \infty$ and set $W = \Phi(Z)$. Then, $\mathrm{Im}(Z) \to +\infty$ and $\mathrm{Im}(W) \to +\infty$. We say that two quantities depending on Z (or W) are comparable if their ratio is bounded and bounded away from 0. We use the notation $\asymp$ to relate comparable quantities.

Set $\nu = \nu_0(f)$, so that $F_q(Z) - Z \asymp \mathrm{e}^{2\pi \mathrm{i} \nu q Z}$. Since $\Phi(Z) = \frac{q}{p} Z + C + o(1)$,

$$\begin{aligned} G_p(W) - W &= G_p\big(\Phi(Z)\big) - \Phi(Z) \\ &= \Phi\big(F_q(Z)\big) - \Phi(Z) \\ &\asymp F_q(Z) - Z \asymp \mathrm{e}^{2\pi \mathrm{i} \nu q Z} \sim \mathrm{e}^{2\pi \mathrm{i} \nu q \frac{p}{q}(W - C)} \asymp \mathrm{e}^{2\pi \mathrm{i} \nu p W}. \end{aligned}$$

Thus $\nu_0(g) = \nu$.

Before establishing the relation between $\text{résit}_0(f)$ and $\text{résit}_0(g)$, it will be convenient to introduce the forms

$$\eta_F = \frac{dZ}{Z - F_q(Z)} \quad \text{and} \quad \eta_G = \frac{dW}{W - G_p(W)}.$$

Those forms are invariant under translation by 1 and descends to $\mathbb{C}/\mathbb{Z}$. Denote by $\mathrm{res}_+(\eta_F)$ and $\mathrm{res}_+(\eta_G)$ their residue:

$$\mathrm{res}_+(\eta_F) = \frac{1}{2\pi \mathrm{i}} \int_A^{A+1} \eta_F \quad \text{and} \quad \mathrm{res}_+(\eta_G) = \frac{1}{2\pi \mathrm{i}} \int_A^{A+1} \eta_G$$

for $\mathrm{Im}(A)$ large enough. Note that

$$\begin{aligned} \Pi^* \left(\frac{dz}{z - f^{\circ q}(z)} \right) &= \frac{2\pi \mathrm{i} \mathrm{e}^{2\pi \mathrm{i} Z} dZ}{\mathrm{e}^{2\pi \mathrm{i} Z} - \mathrm{e}^{2\pi \mathrm{i} F_q(Z)}} \\ &= \frac{2\pi \mathrm{i} dZ}{1 - \mathrm{e}^{2\pi \mathrm{i}(F_q(Z) - Z)}} = \eta_F + \pi \mathrm{i} dZ + o(dZ). \end{aligned}$$

Thus,

$$\text{résit}_0(f^{\circ q}) = \frac{\nu q + 1}{2} - \mathrm{res}_0 \left(\frac{dz}{z - f^{\circ q}(z)} \right) = \frac{\nu q}{2} - \mathrm{res}_+(\eta_F).$$

Similarly,

$$\text{résit}_0(g^{\circ p}) = \frac{\nu p}{2} - \mathrm{res}_+(\eta_G).$$

The key estimate to relate the residue of η_F to that of η_G is the following.

Lemma 4.5. *As* $Z \overset{\in V}{\to} \infty$,

$$\Phi^*\eta_G = \eta_F + o(dZ).$$

Proof. Assume $Z \overset{\in V}{\to} \infty$. The maps

$$\phi_Z : u \mapsto \frac{\Phi(Z+u) - \Phi(Z)}{\Phi'(Z)}$$

are defined and univalent on disks whose radii tend to ∞, and they fix 0 with derivative 1. So, they lie in a compact family. Moreover, any limit value is a univalent entire map which fixes 0 with derivative 1, hence the identity. It follows that the maps

$$\psi_Z : u \mapsto \frac{\Phi(Z+u) - \Phi(Z) - \Phi'(Z)u}{u^2}$$

which are defined and holomorphic on disks whose radii tend to ∞, converge locally uniformly to 0. Since for $Z \in V$, the values $F_q(Z) - Z$ lie in a compact set, we see that

$$\psi_Z\big(F_q(Z) - Z\big) \to 0$$

whence

$$\frac{G_p\big(\Phi(Z)\big) - \Phi(Z)}{\Phi'(Z)} = \frac{\Phi\big(F_q(Z)\big) - \Phi(Z)}{\Phi'(Z)} = F_q(Z) - Z + o\big(F(Z) - Z\big)^2.$$

As a consequence,

$$\Phi^*\eta_G = \frac{\Phi'(Z)dZ}{\Phi(Z) - G_p\big(\Phi(Z)\big)} = \frac{dZ}{Z - F_q(Z)} + o(dZ). \qquad \square$$

We finally come to the relation between résit$_0(f)$ and résit$_0(g)$. A local isomorphism $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form $\psi(z) = \mathrm{e}^{2\pi \mathrm{i} C} z + o(z)$ lifts via Π to a map Ψ which is defined and univalent in a upper half-plane, and satisfies $\Psi(Z) = Z + C + o(1)$. As a consequence, if f_1 and f_2 are conjugate by ψ near 0, i.e., $f_1 \circ \psi = \psi \circ f_2$, and if g is an α-renormalization of f_1, then g is also an α-renormalization of f_2 (with Φ_1 replaced by $\Phi_2 = \Phi_1 \circ \Psi$).

Therefore, without loss of generality, we may work in preferred coordinates for f where the expression is of the form

$$f(z) = \mathrm{e}^{2\pi i p/q} z \left(1 + z^{\nu q} + \alpha z^{2\nu q}\right) + o(z^{2\nu q+1}) \quad \text{with} \quad \alpha = \frac{\nu q + 1}{2} - \text{résit}_0(f).$$

In that case,

$$F(Z) = Z + \frac{p}{q} + \frac{1}{2\pi \mathrm{i}} e^{2\pi \mathrm{i}\nu q Z}\big(1 + A\mathrm{e}^{2\pi \mathrm{i}\nu q Z} + o(\mathrm{e}^{2\pi \mathrm{i}\nu q Z})\big) \text{ with } A = \frac{\nu q}{2} - \text{résit}_0(f)$$

and

$$F_q(Z) = Z + \frac{q}{2\pi \mathrm{i}} e^{2\pi \mathrm{i}\nu q Z}\big(1 + qB\mathrm{e}^{2\pi \mathrm{i}\nu q Z} + o(\mathrm{e}^{2\pi \mathrm{i}\nu q Z})\big) \text{ with } B = \frac{\nu q}{2} - \text{résit}_0(f^{\circ q}).$$

According to the previous lemma

$$\Phi^*\eta_G = \eta_F + o(dZ) = -\frac{2\pi \mathrm{i} dZ}{q\mathrm{e}^{2\pi \mathrm{i}\nu q Z}} + 2\pi \mathrm{i} B dZ + o(dZ).$$

On the one hand, as $\mathrm{Im}(Z) \to +\infty$,

$$\begin{aligned}\int_Z^{F(Z)} \mathrm{e}^{-2\pi \mathrm{i}\nu q U} dU &= \frac{\mathrm{e}^{-2\pi\mathrm{i}\nu q Z} - \mathrm{e}^{-2\pi\mathrm{i}\nu q F(Z)}}{2\pi\mathrm{i}\nu q} \\ &= \mathrm{e}^{-2\pi\mathrm{i}\nu q Z}\frac{1-\mathrm{e}^{-2\pi\mathrm{i}\nu q (F(Z)-Z-p/q)}}{2\pi\mathrm{i}\nu q} \\ &\sim \mathrm{e}^{-2\pi\mathrm{i}\nu q Z}(F(Z)-Z-p/q) \to \frac{1}{2\pi\mathrm{i}}.\end{aligned}$$

In the second line, we use that $e^{2\pi \mathrm{i}\nu p} = 1$ and in the third line, we use that $F(Z) - Z - p/q \to 0$. On the other hand, as $\mathrm{Im}(Z) \to +\infty$,

$$\int_Z^{F(Z)} dU = F(U) - U \to \frac{p}{q}.$$

As a consequence,

$$\mathrm{res}_+(\eta_G) = \frac{1}{2\pi\mathrm{i}}\int_W^{W+1} \eta_G = \lim_{Z \overset{\in V}{\to} \infty} \frac{1}{2\pi\mathrm{i}}\int_Z^{F(Z)} \eta_F = -\frac{1}{2\pi\mathrm{i}q} + B\frac{p}{q}.$$

Thus,

$$\text{résit}_0(g^{\circ p}) = \frac{\nu p}{2} - \mathrm{res}_+(\eta_G) = \frac{\nu p}{2} + \frac{1}{2\pi\mathrm{i}q} - \frac{\nu p}{2} + \frac{p}{q}\text{résit}_0(f^{\circ q}),$$

thus

$$\text{résit}_0(g) = p\cdot\text{résit}_0(g^{\circ p}) = \frac{p^2}{q}\text{résit}_0(f^{\circ q}) + \frac{p}{2\pi\mathrm{i}q} = \frac{p^2}{q^2}\text{résit}_0(f) + \frac{p}{2\pi\mathrm{i}q}.$$

This completes the proof of Proposition 4.4. □

5. Horn maps for the family g_a

We shall now study the previous construction in the case of the family of maps

$$g_a = z \mapsto \frac{z}{1+az+z^2}.$$

We will fix the normalization of the Fatou coordinates and study the dependence of the Fatou coordinates and horn maps with respect to the parameter $b = 1/a^2$. Let us perform the change of dynamical variable $Z = 1/(az)$ in which f_a is conjugate to

$$F_b : Z \mapsto Z + 1 + b/Z.$$

We shall denote by $\mathcal{B}_b$ the parabolic basin of ∞ for F_b. For $b = 0$, F_b is the translation by 1 and by convention, we set

$$\mathcal{B}_0 = \mathbb{C} - \{0, -1, -2, -3, \ldots\}.$$

Lemma 5.1. *The set*

$$\mathcal{B} = \big\{(b, Z) \ : \ b \in \mathbb{C} \text{ and } Z \in \mathcal{B}_b\big\}$$

is an open subset of $\mathbb{C} \times \big(\mathbb{C} - \{0\}\big) \subset \mathbb{C} \times \mathbb{C}$.

Proof. The map $F : \mathbb{C} \times (\mathbb{C} - \{0\}) \to \mathbb{C} \times \mathbb{C}$ defined by

$$F(b, Z) = \big(b, F_b(Z)\big)$$

is continuous. If $|Z| > 4|b|$, then $|F(Z) - Z - 1| < 1/4$. It follows that

$$\mathcal{B}_b = \bigcup_{n\geq 0} F_b^{-n}(\mathcal{U}_b) \quad \text{with} \quad \mathcal{U}_b = \big\{Z \ : \ \mathrm{Re}(Z) > 4|b|\big\}.$$

Set $\mathcal{U} = \big\{(b, Z) \ : \ \mathrm{Re}(Z) > 4|b|\big\} \subset \mathbb{C} \times \mathbb{C}$. For all $n \geq 0$, the set $F^{-n}(\mathcal{U})$ is an open subset of $\mathbb{C} \times (\mathbb{C} - \{0\}) \subset \mathbb{C} \times \mathbb{C}$. As a consequence,

$$\mathcal{B} = \bigcup_{n\geq 0} F^{-n}(\mathcal{U})$$

is an open subset of $\mathbb{C} \times \big(\mathbb{C} - \{0\}\big) \subset \mathbb{C} \times \mathbb{C}$. □

We shall study the Fatou coordinates and horn maps associated to F_b. But first, let us recall the definition of the *generalized harmonic number function* which will be used in our normalization of Fatou coordinates.

5.1. **The generalized harmonic number function.**

Definition 5.2. *We will denote by* $\mathcal{H} : \mathbb{C} - \{-1, -2, -3, \ldots\} \to \mathbb{C}$ *the function*

$$\mathcal{H}(Z) = \sum_{0<j<+\infty} \left(\frac{1}{j} - \frac{1}{j+Z}\right).$$

Note that $\mathcal{H}$ is holomorphic since the series converges locally uniformly. Also note that $\mathcal{H}(0) = 0$ and $\mathcal{H}(k)$ is the k-th harmonic number for $k \geq 1$:

$$\mathcal{H}(k) = \sum_{0<j\leq k} \frac{1}{j}.$$

We shall use the following facts:

(1) For $Z \in \mathbb{C} - \mathbb{Z}$,

$$\frac{1}{Z} - \mathcal{H}(Z) + \mathcal{H}(-Z) = \frac{1}{Z} + \sum_{k>0} \left(\frac{1}{k+Z} + \frac{1}{-k+Z}\right) = \pi \cot(\pi Z).$$

(2) $\mathcal{H}(Z) = \Psi(Z+1) + \gamma$ where $\Psi = \Gamma'/\Gamma$ is the digamma function and γ is the Euler constant.

(3) As $\mathrm{Re}(Z) \to +\infty$,

$$\mathcal{H}(Z) = \log Z + \gamma + o(1).$$

(4) For any compact set $K \subset \mathbb{C} - \mathbb{Z}$, as $Z \in K$ and $n \to \pm\infty$,

$$\mathcal{H}(Z + n) = \mathcal{O}(\log |n|).$$

Fact (1) is a classical Reflection Formula: the sum is a 1-periodic meromorphic function which has simple poles with residue 1 at the integers, is holomorphic elsewhere, and tends to 0 as $\mathrm{Im}(Z)$ tends to $\pm\infty$; this function is unique by Liouville's Theorem. Fact (2) and Fact (3) may be derived from, for example, [WW, Sections 12.16 and 12.31]. Fact (4) with $n \to +\infty$ follows from Fact (3). Finally, to get Fact (4) with $n \to -\infty$, we apply Fact (1) and Fact (4) with $n \to +\infty$.

5.2. **Attracting Fatou coordinates.** For $n \geq 0$, let $\Phi^n : \mathcal{B} \to \mathbb{C}$ be the analytic map defined by
$$\Phi^n(b, Z) = F_b^{\circ n}(Z) - n - b\mathcal{H}(n).$$

Proposition 5.3. *The sequence* $(\Phi^n : \mathcal{B} \to \mathbb{C})$ *converges locally uniformly to an analytic map* $\Phi : \mathcal{B} \to \mathbb{C}$.

Proof. Let $\mathcal{K} \subset \mathcal{B}$ be a compact set and let $K \subset \mathbb{C}$ be the projection of $\mathcal{K}$ on the first coordinate. Let R be sufficiently large so that for all $b \in K$ and all $Z \in \mathbb{C} - D(0, R)$, we have
$$\bigl|F_b(Z) - Z - 1\bigr| = \bigl|b/Z\bigr| < 1/4 \quad \text{and} \quad \bigl|F_b'(Z) - 1\bigr| = \bigl|b/Z^2\bigr| < 1/4.$$
Let $\Phi_b^{\rm att} : \Omega^{\rm att} \to \mathbb{C}$ be the Fatou coordinates normalized by $C_b^{\rm att} = 0$. According to Section 4.2, there are constants C_K and D such that for all $b \in K$,
$$\bigl|\Phi_b^{\rm att}(Z) - Z + b \log Z\bigr| \leq \frac{C_K}{\operatorname{dist}(Z, \mathbb{C} - \Omega^{\rm att})}$$
as soon as $\operatorname{dist}(Z, \mathbb{C} - \Omega^{\rm att}) \geq D$.

Choose n_0 large enough so that for all $(b, Z) \in \mathcal{K}$, we have
$$\operatorname{dist}\bigl(F_b^{\circ n_0}(Z), \mathbb{C} - \Omega^{\rm att}\bigr) \geq D.$$
Then, for $n \geq n_0$, we have
$$\bigl|F_b^{\circ n}(Z)\bigr| > \operatorname{dist}\bigl(F_b^{\circ n}(Z), \mathbb{C} - \Omega^{\rm att}\bigr) \geq c \cdot (n - n_0) \quad \text{with} \quad c = \frac{1}{\sqrt{2}} - \frac{1}{4} > 0.$$
Since $\Phi_b^{\rm att}(Z) + n = \Phi_b^{\rm att}\bigl(F_b^{\circ n}(Z)\bigr)$, we have
$$\bigl|\Phi_b^{\rm att}(Z) - F_b^{\circ n}(Z) + n + b \log F^{\circ n}(Z)\bigr| \leq \frac{C_K}{c \cdot (n - n_0)}.$$
This shows that the sequence $F_b^{\circ n}(Z) - n - b \log F^{\circ n}(Z)$ converges uniformly on $\mathcal{K}$ to $\Phi_b^{\rm att}(Z)$. The convergence being uniform, the limit is analytic with respect to b and Z, in particular bounded on $\mathcal{K}$. It follows that the sequence $F_b^{\circ n}(Z)/n$ converges uniformly on $\mathcal{K}$ to 1, and the sequence Φ^n converges uniformly on $\mathcal{K}$ to
$$\Phi : (b, Z) \mapsto \Phi_b^{\rm att}(Z) - b\gamma,$$
where γ is the Euler constant. □

For $b \in \mathbb{C}$, the function $\Phi_b : \mathcal{B}_b \to \mathbb{C}$ defined by $\Phi_b(Z) = \Phi(b, Z)$ is an attracting Fatou coordinate (extended to the basin $\mathcal{B}_b$). It is normalized by the condition $C_b^{\rm att} = -b\gamma$. We shall explain later the advantage of this normalization. In particular,
$$\Phi_b(Z) = Z - b\mathcal{H}(Z) + o(1) \quad \text{as} \quad \operatorname{Re}(Z) \to +\infty.$$

5.3. **Repelling Fatou parametrizations.** Let us note that the map
$$(\mathbb{C} \times \mathbb{C}) - \bigl\{(0, 0)\bigr\} \ni (b, Z) \mapsto F_b(Z) \in \mathbb{P}^1$$
is analytic with a singularity at $(0, 0)$. It will be convenient to introduce the notation
$$\mathcal{D} = (\mathbb{C} \times \mathbb{C}) - \bigl\{(0, k) \ : \ k \geq 1 \text{ is an integer}\bigr\}$$
and to observe that for all $n \geq 0$, the map
$$\Psi^n : (b, Z) \mapsto F_b^{\circ n}\bigl(Z - n + b\mathcal{H}(n)\bigr)$$
is defined and analytic on $\mathcal{D}$ with values in $\mathbb{P}^1$.

Proposition 5.4. *The sequence* $(\Psi^n : \mathcal{D} \to \mathbb{P}^1)$ *converges locally uniformly to an analytic map* $\Psi : \mathcal{D} \to \mathbb{P}^1$.

Proof. Let $B \subset \mathbb{C}$ be an open set with compact closure. Let R be sufficiently large so that for all $b \in B$ and all $Z \in \mathbb{C} - D(0, R)$, we have

$$\bigl|F_b(Z) - Z - 1\bigr| = \bigl|b/Z\bigr| < 1/4 \quad \text{and} \quad \bigl|F_b'(Z) - 1\bigr| = \bigl|b/Z^2\bigr| < 1/4.$$

Let $\Phi_b^{\mathrm{rep}} : \Omega^{\mathrm{rep}} \to \mathbb{C}$ be the Fatou coordinates normalized by $C_b^{\mathrm{rep}} = 0$ and let $\Psi_b^{\mathrm{rep}} : \mathbb{C} \to \mathbb{C}$ be defined by

$$\Psi_b^{\mathrm{rep}}(Z) = F_b^{\circ n}\bigl((\Phi_b^{\mathrm{rep}})^{-1}(Z - n)\bigr)$$

where $n \geq 0$ is chosen sufficiently large so that $Z - n \in \Phi_b^{\mathrm{rep}}(\Omega^{\mathrm{rep}})$.

As in the previous Proposition, the sequence of maps

$$B \times \Omega^{\mathrm{rep}} \ni (b, Z) \mapsto F_b^{\circ(-n)}(Z) + n - b \log n \in \mathbb{C}$$

converges locally uniformly to the analytic map

$$B \times \Omega^{\mathrm{rep}} \ni (b, Z) \mapsto \Phi_b^{\mathrm{rep}}(Z) \in \mathbb{C}.$$

If R' is large enough, then the left half-plane $U := \{Z \ : \ \mathrm{Re}(Z) < R'\}$ is contained in $\Phi_b^{\mathrm{rep}}(\Omega^{\mathrm{rep}})$ for all $b \in B$. The map $Z \mapsto F_b^{\circ n}(Z - n + b \log n)$ is the inverse of $Z \mapsto F_b^{\circ(-n)}(Z) + n - b \log n$. The sequence of maps

$$B \times U \ni (b, Z) \mapsto F_b^{\circ n}(Z - n + b \log n) \in \Omega^{\mathrm{rep}}$$

is normal (the maps take their values in the fixed domain Ω^{rep}) and converges to

$$B \times U \ni (b, Z) \mapsto \Psi_b^{\mathrm{rep}}(Z) \in \Omega^{\mathrm{rep}}$$

locally uniformly.

Now, if $\mathcal{K} \subset \mathcal{D}$ is a compact set contained in $B \times \mathbb{C}$, then there is an integer n_0 such that $Z - n_0 + b\gamma \in U$ for all $(b, Z) \in \mathcal{K}$. As $n \to +\infty$

$$F_b^{\circ n}\bigl(Z - n + b\mathcal{H}(n)\bigr) = F_b^{\circ n_0} \circ F_b^{\circ(n-n_0)}\bigl(Z_n - (n - n_0) + b \log(n - n_0)\bigr)$$

where

$$Z_n = Z - n_0 + b\mathcal{H}(n) - b \log(n - n_0)$$

converges uniformly on $\mathcal{K}$ to $Z - n_0 + b\gamma \in U$. It follows that the sequence of maps $\Psi^n : \mathcal{K} \to \mathbb{P}^1$ converges uniformly to

$$(b, z) \mapsto F_b^{\circ n_0}\bigl(\Psi_b^{\mathrm{rep}}(Z + b\gamma - n_0)\bigr) = \Psi_b^{\mathrm{rep}}(Z + b\gamma). \qquad \square$$

For $b \in \mathbb{C}$, the function $\Psi_b : \mathbb{C} \to \widehat{\mathbb{C}}$ defined by $\Psi_b(Z) = \Psi(b, Z)$ is the inverse of a repelling Fatou coordinate (extended to $\mathbb{C}$). It is normalized by the condition $C_b^{\mathrm{rep}} = -b\gamma$. In particular,

$$\Psi_b(Z) = Z + b\mathcal{H}(-Z) + o(1) \quad \text{as} \quad \mathrm{Re}(Z) \to -\infty.$$

5.4. **Lifted horn maps.** Let us now consider the lifted horn maps

$$\mathfrak{H}_b = \Phi_b \circ \Psi_b : \mathfrak{D}_b \to \mathbb{C} \quad \text{with} \quad \mathfrak{D}_b := \Psi_b^{-1}(\mathcal{B}_b).$$

The domain of definition $\mathfrak{D}_b$ of $\mathfrak{H}_b$ is an open subset of $\mathbb{C}$ (usually with fractal boundary) which contains an upper half-plane and a lower half-plane. It is invariant under the translation by 1 and $\mathfrak{H}_b - \mathrm{id}$ is periodic of period 1. The set

$$\mathfrak{D} = \{(b, Z) \mid b \in \mathbb{C} \text{ and } Z \in \mathfrak{D}_b\}$$

is open and the map $\mathfrak{H} : (b, Z) \mapsto \mathfrak{H}_b(Z)$ is analytic.

Let us first study the behaviour of $\mathfrak{H}$ as b tends to 0. For $b = 0$, we have $\mathfrak{D}_0 = \mathbb{C} - \mathbb{Z}$ and $\mathfrak{H}_0 = \mathrm{id} : \mathfrak{D}_0 \to \mathfrak{D}_0$. As a consequence,

$$\dot{\mathfrak{H}} = \frac{d\mathfrak{H}_b}{db}\Big|_{b=0}$$

is a vector field, defined and holomorphic in $\mathbb{C} - \mathbb{Z}$. This vector field may be written

$$\dot{\mathfrak{H}} = \mathfrak{S}_1(Z) \frac{d}{dZ}$$

for some holomorphic map $\mathfrak{S}_1 : \mathbb{C} - \mathbb{Z} \to \mathbb{C}$. In addition, as $b \to 0$ and Z remains in a compact subset of $\mathbb{C} - \mathbb{Z}$, we have

$$\mathfrak{H}_b(Z) = Z + b \cdot \mathfrak{S}_1(Z) + \mathcal{O}(b^2)$$

where the constant of the $\mathcal{O}(b^2)$ only depends on the compact subset of $\mathbb{C} - \mathbb{Z}$.

Proposition 5.5. *We have* $\mathfrak{S}_1(Z) = \pi \cot(\pi Z)$.

Proof. Recall that $F(b, Z) = F_b(Z) = Z + 1 + b/Z$. For $n \geq 1$, set

$$F^n(b, Z) = F_b^{\circ n}(Z)$$

so that $F^n = F_b \circ F^{n-1}$. Then

$$\begin{aligned}\frac{\partial F^n}{\partial b}(0, Z) &= \frac{\partial F}{\partial b}\big(0, F_0^{\circ(n-1)}(Z)\big) + \frac{\partial F}{\partial Z}\big(0, F_0^{\circ(n-1)}(Z)\big) \cdot \frac{\partial F^{n-1}}{\partial b}(0, Z)\\ &= \frac{1}{Z+n-1} + \frac{\partial F^{n-1}}{\partial b}(0, Z) = \frac{1}{Z+n-1} + \cdots + \frac{1}{Z}.\end{aligned}$$

On the one hand, Φ is the limit of the sequence of maps

$$\Phi^n : (b, Z) \mapsto F_b^{\circ n}(Z) - n - b\mathcal{H}(n).$$

We have that

$$\frac{\partial \Phi^n}{\partial b}(0, Z) = \left(\sum_{k=0}^{n-1} \frac{1}{Z+k}\right) - \mathcal{H}(n),$$

and

$$\frac{\partial \Phi_b(Z)}{\partial b}\Big|_{b=0} = \frac{1}{Z} - \mathcal{H}(Z).$$

On the other hand, Ψ is the limit of the sequence of maps

$$\Psi^n : (b, Z) \mapsto F_b^{\circ n}\big(Z - n + b\mathcal{H}(n)\big).$$

We have that

$$\frac{\partial \Psi^n}{\partial b}(0, Z) = \left(\sum_{k=-n}^{-1} \frac{1}{Z+k}\right) + \mathcal{H}(n),$$

and

$$\frac{\partial \Psi_b(Z)}{db}\Big|_{b=0} = \mathcal{H}(-Z).$$

Now, for $Z \in \mathbb{C} - \mathbb{Z}$, we have that $\mathfrak{H}_b(Z) = \Phi_b \circ \Psi_b(Z)$ with $\Phi_0(Z) = Z$ and $\Psi_0(Z) = Z$. As a consequence,

$$\mathfrak{S}_1(Z) = \frac{\partial \Phi_b(Z)}{\partial b}\Big|_{b=0} + \frac{\partial \Psi_b(Z)}{\partial b}\Big|_{b=0} = \frac{1}{Z} - \mathcal{H}(Z) + \mathcal{H}(-Z) = \pi \cot(\pi Z). \qquad \square$$

Let us now study the behaviour of $\mathfrak{H}$ as $\mathrm{Im}(Z)$ tends to $\pm\infty$. Since $\mathfrak{H}_b - \mathrm{id}$ is a periodic function of period 1 whose domain of definition contains an upper half-plane and a lower half-plane, we may consider the Fourier expansions:

$$\mathfrak{H}_b(Z) - Z = \mathfrak{c}_0^+(b) + \sum_{k\geq 1} \mathfrak{c}_k(b) \mathrm{e}^{2\pi \mathrm{i} k Z} \quad \text{as } \mathrm{Im}(Z) \to +\infty$$

and

$$\mathfrak{H}_b(Z) - Z = \mathfrak{c}_0^-(b) + \sum_{k\leq -1} \mathfrak{c}_k(b) \mathrm{e}^{2\pi \mathrm{i} k Z} \quad \text{as } \mathrm{Im}(Z) \to -\infty,$$

the functions $\mathfrak{c}_0^+ : \mathbb{C} \to \mathbb{C}$, $\mathfrak{c}_0^- : \mathbb{C} \to \mathbb{C}$ and $\mathfrak{c}_k : \mathbb{C} \to \mathbb{C}$ being entire. The orientation-reversing involution $Z \mapsto \bar{Z}$ conjugates F_b to $F_{\bar{b}}$. It follows that $\mathfrak{H}_{\bar{b}}(Z) = \overline{\mathfrak{H}_b(\overline{Z})}$, whence

$$\mathfrak{c}_0^-(b) = \overline{\mathfrak{c}_0^+(\bar{b})} \quad \text{and} \quad \mathfrak{c}_{-k}(b) = \overline{\mathfrak{c}_k(\bar{b})}.$$

We shall therefore focus our study on the Fourier expansion as $\mathrm{Im}(Z) \to +\infty$.

Proposition 5.6. *We have* $\mathfrak{c}_0^+(b) = -\pi \mathrm{i} b$ *and* $\mathfrak{c}_k(b) = -2\pi \mathrm{i} b + \mathcal{O}(b^2)$ *for* $k \geq 1$.

Proof. According to Section 4.5,

$$\mathfrak{c}_0^+(b) = C_b^{\mathrm{att}} - C_b^{\mathrm{rep}} - b\pi\mathrm{i} = (-b\gamma) - (-b\gamma) - b\pi\mathrm{i} = -\pi\mathrm{i}b.$$

For $k \geq 1$, apply Proposition 5.5 to the Fourier expansions:

$$-\pi\mathrm{i} + \sum_{k\geq 1} \mathfrak{c}_k'(0) \mathrm{e}^{2\pi\mathrm{i}kZ} = \frac{\partial \mathfrak{H}(b,Z) - Z}{\partial b}\Big|_{b=0} = \pi\cot(\pi Z) = -\pi\mathrm{i} - 2\pi\mathrm{i}\sum_{k\geq 1} \mathrm{e}^{2\pi\mathrm{i}kZ}.$$

This shows that for $k \geq 1$, we have $\mathfrak{c}_k'(0) = -2\pi\mathrm{i}$, whence $\mathfrak{c}_k(b) = -2\pi\mathrm{i}b + \mathcal{O}(b^2)$. $\square$

In particular, for $k \geq 1$, the entire function $\mathfrak{c}_k$ is non-constant. Presumably, these functions are transcendental.

5.5. **Horn maps.** The family of horn maps $\mathfrak{h}_b : \mathfrak{U}_b \to \widehat{\mathbb{C}}$ is defined by

$$\mathfrak{h}_b \circ \Pi = \Pi \circ \mathfrak{H}_b,$$

with

$$\mathfrak{U}_b = \Pi(\mathfrak{D}_b) \cup \{0, \infty\}, \quad \Pi(Z) := \mathrm{e}^{2i\pi Z} \quad \text{and} \quad \widehat{\mathbb{C}} = \Pi(\mathbb{C}) \cup \{0,\infty\}.$$

The map $\mathfrak{h}_b$ fixes both 0 and ∞ with multiplier $\mathrm{e}^{2\pi^2 b}$.

6. Hausdorff limit of $\mathcal{A}_{p/q}$ as $p/q \to 0$

From now on, we assume that $r/s \in \mathbb{Q}$ is a rational number with r and s coprime and we set $p_k = s$ and $q_k = ks + r$ so that

$$\frac{p_k}{q_k} = \frac{1}{k + r/s}.$$

We wish to prove Theorem 0.3 which asserts that the sequence of sets $\mathcal{A}_{p_k/q_k}$ is uniformly bounded, and Hausdorff convergent as $k \to \infty$, the Hausdorff limit $\mathfrak{A}_{r/s}$ consisting of 0 together with a bounded set which is infinite and discrete in $\mathbb{C} - \{0\}$. For this purpose, consider

- the family of analytic maps
$$\mathfrak{g}_b = \omega_{r/s} \cdot \mathrm{e}^{-2\pi^2 b} \cdot \mathfrak{h}_b : \mathfrak{U}_b \to \widehat{\mathbb{C}}$$
which fix 0 with multiplier $\omega_{r/s}$,
- the entire function $\mathfrak{C}_{r/s} : \mathbb{C} \to \mathbb{C}$ defined by
$$\mathfrak{g}_b^{\circ s}(z) = z + \mathfrak{C}_{r/s}(b) z^{s+1} + \mathcal{O}(z^{s+2}).$$

Our strategy is the following.

(1) The set $\mathfrak{B}_{r/s}$ of parameters $b \in \mathbb{C}$ such that $\mathfrak{g}_b$ has parabolic degeneracy greater than 1 at 0 is the set of zeroes of the entire function $\mathfrak{C}_{r/s}$. We shall prove that $\mathfrak{C}_{r/s}$ has a simple zero at $b = 0$. As a consequence, $\mathfrak{C}_{r/s}$ does not identically vanish and $\mathfrak{B}_{r/s}$ is discrete in $\mathbb{C}$. We set
$$\mathfrak{A}_{r/s} = \big\{a \in \mathbb{C} - \{0\} \ : \ 1/a^2 \in \mathfrak{B}_{r/s}\big\}.$$
(2) We will prove that the sets $\mathcal{A}_{p_k/q_k}$ are uniformly bounded and that
$$\lim_{k\to+\infty} \mathcal{A}_{p_k/q_k} = \overline{\mathfrak{A}}_{r/s}.$$
The proof relies on the near-parabolic renormalization of Section 4.
(3) We will prove that $\mathfrak{B}_{r/s}$ is infinite, whence $\overline{\mathfrak{A}}_{r/s} = \mathfrak{A}_{r/s} \cup \{0\}$. The proof relies on careful control of the *order of growth* of the entire function $\mathfrak{C}_{r/s}$, which will be addressed in Section 8.
(4) Consider the meromorphic function $\mathfrak{R}_{r/s} : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ which is defined outside $\mathfrak{B}_{r/s}$ as the résidu itératif of $\mathfrak{g}_b$. We will prove that the sequence of rational functions $\left(\frac{p_k}{q_k}\right)^2 \mathcal{R}_{p_k/q_k}$ converges uniformly to $a \mapsto \mathfrak{R}_{r/s}\left(\frac{1}{a^2}\right)$ on every compact subset of $\mathbb{C} - \overline{\mathfrak{A}}_{r/s}$.
(5) We will prove that each point of $\mathfrak{B}_{r/s}$ is a pole of $\mathfrak{R}_{r/s}$. The proof relies on the fact that the parabolic degeneracy at 0 of the maps $\mathfrak{g}_b$ is at most 2. This fact will be proved in Section 7 using the notion of *finite type analytic maps*.

6.1. **The function $\mathfrak{C}_{r/s}$ has a simple zero at $b = 0$.** Setting $z = \Pi(Z) = \mathrm{e}^{2\pi \mathrm{i} Z}$, we have
$$\pi \cot(\pi Z) = \pi \mathrm{i} \frac{z+1}{z-1}.$$
Thus, according to Proposition 5.5, we have the following expansion as $b \to 0$:
$$\begin{aligned}\mathfrak{g}_b(z) &= \omega_{r/s} \cdot \exp\big(2\pi \mathrm{i} Z + 2\pi \mathrm{i} b \mathfrak{S}_1(Z) - 2\pi^2 b + \mathcal{O}(b^2)\big)\\ &= \omega_{r/s} z \cdot \big(1 + 2\pi \mathrm{i} b \mathfrak{S}_1(Z) - 2\pi^2 b + \mathcal{O}(b^2)\big)\\ &= \omega_{r/s} z \cdot \left(1 - b \cdot 2\pi^2 \frac{z+1}{z-1} - 2\pi^2 b + \mathcal{O}(b^2)\right)\\ &= \omega_{r/s} \cdot \left(z + b \cdot 4\pi^2 \frac{z^2}{1-z} + \mathcal{O}(b^2)\right).\end{aligned}$$
Note that $\mathfrak{g} = \mathfrak{g}_0$ is the rotation $z \mapsto \omega_{r/s} z$ and as $b \to 0$,
$$\mathfrak{g}_b = \mathfrak{g} + b \cdot D\mathfrak{g} \circ \mathbf{v} + \mathcal{O}(b^2) \quad \text{with} \quad \mathbf{v} = 4\pi^2 \frac{z^2}{1-z} \frac{d}{dz}.$$
According to Section 1.2, we have
$$\mathfrak{g}_b^{\circ s} = \mathrm{id} + b \cdot \mathbf{v}_s + \mathcal{O}(b^2)$$

with

$$\begin{aligned}\mathbf{v}_s = 4\pi^2 \sum_{n=0}^{s-1} (\mathfrak{g}^{\circ n})^* \mathbf{v} &= 4\pi^2 \sum_{n=0}^{s-1} \sum_{k=2}^{+\infty} \omega_{r/s}^{n(k-1)} z^k \frac{d}{dz} \\ &= 4\pi^2 \sum_{m=1}^{+\infty} s z^{ms+1} \frac{d}{dz} = 4\pi^2 s \frac{z^{s+1}}{1-z^s} \frac{d}{dz}.\end{aligned}$$

In particular, as b tends to 0,

$$\mathfrak{C}_{r/s}(b) = 4\pi^2 s b + \mathcal{O}(b^2).$$

So, $\mathfrak{C}_{r/s}$ has a simple zero at $b = 0$.

6.2. **Near-parabolic renormalization.** Make the change of variables $w = az$ and the change of parameters $b = 1/a^2$, so that g_a is conjugate to

$$f_b : w \mapsto \frac{w}{1 + w + bw^2}$$

and $\omega_{p_k/q_k} \cdot g_a$ is conjugate to

$$f_{k,b} = \omega_{p_k/q_k} \cdot f_b.$$

For all $b \in \mathbb{C}$, we have $f_b(w) = w - w^2 + \mathcal{O}(w^3)$ and so, we may apply the results of Section 4 to the maps f_b and their perturbations $f_{k,b}$.

In particular, if $B \subset \mathbb{C}$ is an open set with compact closure and k is large enough, we may associate to $f_{k,b}$

- a perturbed Fatou coordinate $\Phi_{k,b} : \Omega_{k,b} \to \mathbb{C}$ and
- a near-parabolic renormalization $\mathfrak{f}_{k,b} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which fixes 0 with multiplier $\omega_{r/s}$.

If Z_* has sufficiently negative real part and k is large enough, we may normalize the perturbed Fatou coordinates $\Phi_{k,b}$ by the condition

$$\Phi_{k,b}(Z_b) = Z_* \quad \text{with} \quad Z_b = \Psi_b(Z_*).$$

Since Z_b depends analytically on b, so do $\Phi_{k,b}$ and $\mathfrak{f}_{k,b}$ for k sufficiently large. Moreover, as $k \to +\infty$ and $B \ni b_k \to b$, we have $Z_{b_k} \to Z_b$ and so, $\mathfrak{f}_{k,b_k} \to \mathfrak{g}_b$ uniformly in some fixed neighborhood of 0.

Let $\mathfrak{s}_k : B \to \mathbb{C}$ be the holomorphic function defined by

$$\mathfrak{f}_{k,b}^{\circ s}(z) = z + \mathfrak{s}_k(b) \cdot z^{s+1} + \mathcal{O}(z^{s+2}).$$

Note that $\mathfrak{s}_k$ vanishes if and only if $\mathfrak{f}_{k,b}$ has a degenerate parabolic point at 0. According to Proposition 4.4 this is the case if and only if $\mathfrak{f}_{k,b}$ has a degenerate parabolic point at 0, whence $b = 1/a^2$ with $a \in \mathcal{A}_{p_k/q_k}$.

As $k \to \infty$ and $B \ni b_k \to b$, we have $\mathfrak{s}_k(b_k) \to \mathfrak{C}_{r/s}(b)$. It follows that the sequence of functions $\mathfrak{s}_k$ converges uniformly to $\mathfrak{C}_{r/s}$ in B. On the one hand, since $\mathfrak{C}_{r/s}$ has a simple zero at $b = 0$ and since $\mathfrak{s}_k(0) = 0$ (because $\mathfrak{f}_{k,0}$ is the rotation of angle r/s), there is a neighborhood $B' \subseteq B$ of $b = 0$ in $\mathbb{C}$, such that the only zero of $\mathfrak{s}_k$ in B' is $b = 0$. It follows that the sets $\mathcal{A}_{p_k/q_k}$ are uniformly bounded. On the other hand, in $B - \{0\}$ any zero of $\mathfrak{C}_{r/s}$ is accumulated by zeroes of $\mathfrak{s}_k$ and any accumulation point of a sequence of zeroes of s_k is a zero of $\mathfrak{C}_{r/s}$. It follows that

$$\lim_{k \to +\infty} \mathcal{A}_{p_k/q_k} = \overline{\mathfrak{A}}_{r/s}.$$

6.3. **The set $\mathfrak{B}_{r/s}$ is infinite.** The proof relies on careful control of the order of growth of the entire function $\mathfrak{C}_{r/s}$.

Definition 6.1. *The order of growth of an entire function $\mathfrak{c} : \mathbb{C} \to \mathbb{C}$ is the possibly infinite quantity*

$$\operatorname{ord}(\mathfrak{c}) = \limsup_{r\to\infty} \frac{\log\log \max\limits_{|z|=r} |\mathfrak{c}(z)|}{\log r}.$$

In other words, $\operatorname{ord}(\mathfrak{c})$ is the infimum of ρ such that

$$\mathfrak{c}(b) \in \mathcal{O}\big(\exp(|b|^\rho)\big) \quad \text{as } b \to \infty.$$

The following Lemma, at the center of the argument, will be proved in Section 7.

Lemma 6.2. *The order of growth of the entire function $\mathfrak{C}_{r/s}$ is exactly 1. Moreover, there is $c > 0$ such that*

$$\log\big|\mathfrak{C}_{r/s}(1/2 - \mathrm{i}t)\big| \geq c \cdot t \log t \quad \textit{as} \quad t \in \mathbb{R} \textit{ tends to } +\infty.$$

Now, if $\mathfrak{C}_{r/s}$ were vanishing only N times counting multiplicities, the Hadamard Factorization Theorem would imply that $\mathfrak{C}_{r/s}(b) = P(b)\mathrm{e}^{Q(b)}$ for a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree N and a polynomial $Q : \mathbb{C} \to \mathbb{C}$ of degree 1. It would follow that $\log\big|\mathfrak{C}_{r/s}(b)\big| = \mathcal{O}\big(|b|\big)$ as b tends to ∞. This would contradict Lemma 6.2.

6.4. **The meromorphic function $\mathfrak{R}_{r/s}$.** Let us consider the meromorphic function $\mathfrak{R}_{r/s} : \mathbb{C} \to \widehat{\mathbb{C}}$ which is defined outside $\mathfrak{B}_{r/s}$ as the résidu itératif of $\mathfrak{g}_b$.

According to Sections 4.6 and 4.7, we have that

$$\left(\frac{p_k}{q_k}\right)^2 \cdot \mathcal{R}_{p_k/q_k}(a) + \frac{1}{2\pi\mathrm{i}} \cdot \frac{p_k}{q_k} \underset{k\to\infty}{\longrightarrow} \mathfrak{R}_{r/s}\left(\frac{1}{a^2}\right),$$

the convergence being locally uniform in $\mathbb{C} - \overline{\mathfrak{A}}_{r/s}$.

6.5. **Poles of $\mathfrak{R}_{r/s}$.** The set of poles of $\mathfrak{R}_{r/s}$ is contained in the set of parameters b such that $\mathfrak{g}_b$ has a degenerate parabolic point at 0, i.e., the set $\mathfrak{B}_{r/s}$. We need to prove that each point of $\mathfrak{B}_{r/s}$ is a pole of $\mathfrak{R}_{r/s}$.

Proposition 6.3. *The meromorphic function $\mathfrak{R}_{r/s}$ has a simple pole at $b = 0$.*

Proof. According to Section 1.2, for $b \neq 0$ close to 0, we have

$$\mathfrak{R}_{r/s}(b) = \text{résit}_0(\mathfrak{g}_b) = \frac{s}{b}\text{res}_0\left(\frac{1}{\mathbf{v}_s}\right) + \mathcal{O}(1) = -\frac{1}{4\pi^2 b} + \mathcal{O}(1). \qquad \square$$

The proof of the next Proposition relies on the following Lemma which will be addressed in the next Section.

Lemma 6.4. *If $b \in \mathfrak{B}_{r/s} - \{0\}$, then the parabolic degeneracy of $\mathfrak{g}_b$ is 2.*

Proposition 6.5. *The meromorphic function $\mathfrak{R}_{r/s}$ has a pole at each $b \in \mathfrak{B}_{r/s} - \{0\}$.*

Proof. Fix $b \in \mathfrak{B}_{r/s} - \{0\}$ and set $\mathfrak{f}_t = \mathfrak{g}_{b+t}$. The parabolic degeneracy of $\mathfrak{f} = \mathfrak{f}_0$ at 0 is 2. Thus,

$$\mathfrak{f}^{\circ s}(z) = z + Cz^{2s+1} + \mathcal{O}(z^{2s+2}) \quad \text{with} \quad C \in \mathbb{C} - \{0\},$$

whence,

$$\mathfrak{f}_t^{\circ s}(z) = z + \mathfrak{c}(t)z^{s+1} + Cz^{2s+1} + \mathcal{O}(tz^{s+2}) + \mathcal{O}(z^{2s+2}) \quad \text{with} \quad \mathfrak{c}(0) = 0.$$

According to Section 1.2, as $t \to 0$,

$$\mathfrak{R}_{r/s}(b+t) = \frac{sC}{[\mathfrak{c}(t)]^2} + \mathcal{O}(t). \qquad \square$$

Remark. It is possible to generalize the transversality argument of Section 2.2 to show that $\mathfrak{c}$ has a simple root at b, and thus, that each point of $\mathfrak{B}_{r/s} - \{0\}$ is a double pole of $\mathfrak{R}_{r/s}$.

7. Finite type analytic maps

7.1. **Definition.** The notion of finite type analytic maps originates in [Ep1].

Let $f : W \to X$ be an analytic map of complex 1-manifolds, possibly disconnected. An open set $U \subseteq X$ is *evenly covered* by f if $f_{|V} : V \to U$ is a homeomorphism for each component V of $f^{-1}(U)$; we say that $x \in X$ is a *regular value* for f if some neighborhood $U \ni x$ is evenly covered, and a *singular value* for f otherwise. Note that the set $S(f)$ of singular values is closed. Recall that $w \in W$ is a *critical point* if the derivative of f at w vanishes, and then $f(w) \in X$ is a *critical value.* We say that $x \in X$ is an *asymptotic value* if f approaches x along some path tending to infinity relative to W. It follows from elementary covering space theory that the critical values together with the asymptotic values form a dense subset of $S(f)$. In particular, every isolated point of $S(f)$ is a critical or asymptotic value.

Definition 7.1. *An analytic map $f : W \to X$ of complex 1-manifolds is of* finite type *if*

- *f is nowhere locally constant,*
- *f has no isolated removable singularities,*
- *X is a finite union of compact Riemann surfaces, and*
- *$S(f)$ is finite.*

If X is connected, we define $\deg f$ as the finite or infinite number $\#f^{-1}(x)$ which is independent of $x \in X - S(f)$. Every nonconstant analytic map of compact Riemann surfaces is a finite type map of finite degree. The infinite degree case includes the exponential and trigonometric functions, the Weierstrass $\wp$-function, and the elliptic modular function. Further examples arise from the fact that the class of finite type analytic maps is closed under composition: to be precise, if $f_1 : W_1 \to X_1$ and $f_2 : W_2 \to X_2$ are finite type analytic maps, where $W_2 \subseteq X_1$ so that the composition $f_2 \circ f_1 : f_1^{-1}(W_2) \to X_2$ makes sense, then $f_2 \circ f_1$ is a finite type analytic map and

$$S(f_2 \circ f_1) \subseteq S(f_2) \cup f_2\big(S(f_1)\big).$$

When $f : W \to X$ is a finite type analytic map with $W \subseteq X$, we say that f is a finite type analytic map on X.

7.2. **Extended horn maps are of finite type.** The result we are interested in is the following.

Proposition 7.2. *For $b \in \mathbb{C} - \{0\}$, the horn map $\mathfrak{h}_b : \mathfrak{U}_b \to \widehat{\mathbb{C}}$ is a finite type analytic map on $\widehat{\mathbb{C}}$. There are either three or four singular values:*

- *0 and ∞ which are fixed asymptotic values, and*
- *one or two critical values which are the image by $\Pi \circ \Phi_b$ of the critical orbits of F_b contained in $\mathcal{B}_b$.*

In fact, this result is valid in greater generality. Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map, let $\Phi^{\rm att} : \Omega^{\rm att} \to \mathbb{C}$ be an attracting Fatou coordinate defined on some attracting petal of some fixed point of F with multiplier 1 and let $\Phi^{\rm rep} : \Omega^{\rm rep} \to \mathbb{C}$ be a repelling Fatou coordinate defined on some repelling petal of some fixed point of F with multiplier 1. Denote by $\mathcal{B}$ the open set of points whose orbit under F eventually enters the attracting petal $\Omega^{\rm att}$. Then, the attracting Fatou coordinate extends to an analytic map $\Phi : \mathcal{B} \to \mathbb{C}$ via

$$\Phi(z) = \Phi^{\rm att} \circ F^{\circ k}(z) - k$$

where $k \geq 0$ is chosen large enough so that $F^{\circ k}(z) \in \Omega^{\rm att}$. In addition, the inverse of the repelling Fatou coordinate extends to an analytic map $\Psi : \mathbb{C} \to \mathbb{P}^1$ via the relation

$$\Psi(z) = F^{\circ k} \circ (\Phi^{\rm rep})^{-1}(z-k)$$

where $k \geq 0$ is chosen large enough so that $z - k \in \Phi^{\rm rep}(\Omega^{\rm rep})$. Define

$$\mathfrak{H} = \Phi \circ \Psi : \mathfrak{D} \to \mathbb{C} \quad \text{with} \quad \mathfrak{D} = (\Psi)^{-1}(\mathcal{B}).$$

Finally set $\mathfrak{U} = \Pi(\mathfrak{D}) \cup \{0, \infty\}$ and let $\mathfrak{h} : \mathfrak{U} \to \widehat{\mathbb{C}}$ be defined by

$$\Pi \circ \mathfrak{H} = \mathfrak{h} \circ \Pi.$$

The general result from which Proposition 7.2 is a corollary is the following.

Proposition 7.3. *The map $\mathfrak{h} : \mathfrak{U} \to \widehat{\mathbb{C}}$ is a finite type analytic map on $\widehat{\mathbb{C}}$. The singular values are:*

- *0 and ∞, which are fixed asymptotic values of $\mathfrak{h}$, and*
- *the images by $\Pi \circ \Phi$ of the critical orbits of F contained in $\mathcal{B}$, which are critical values of $\mathfrak{h}$.*

Proof. The proof that $\mathfrak{h} : \mathfrak{U} \to \widehat{\mathbb{C}}$ is a finite type analytic map is essentially given in [BuEp, Section 2.5] , where the authors make use of the notion of *ramified covering*. An analytic mapping $f : X \to Y$, where X and Y are Riemann surfaces, is said to be a *ramified covering* if and only if for each point $y \in Y$, there exists a neighborhood V of y such that every connected component of $f^{-1}(V)$ is proper over V. A point $x \in X$ is said to be critical if the local degree of f at x is greater than 1. The image of a critical point is called a critical value. The proof of Proposition 4 in [BuEp] is not correct since the assertion "the composition of ramified coverings is a ramified covering" is false. We shall use Propositions 2 and 3 in [BuEp] and work with compositions of coverings instead.

Let $\mathcal{O}^+$ be the union of the critical value set of $F^{\circ n}$ for $n \geq 1$ and let $\mathcal{O}^-$ be the union of the critical point set of $F^{\circ n}$ for $n \geq 1$. The set $\mathcal{O}^+$ is customarily referred to as the postcritical set of F and $\mathcal{O}^-$ is the set of iterated preimages of the critical points of F. Let $\mathcal{GO}$ be the union of grand orbits of critical points of F, i.e., the set of points $z \in \mathbb{C} \cup \{\infty\}$ such that $F^{\circ n_1}(z) = F^{\circ n_2}(c)$ for some integers $n_1, n_2 \geq 1$ and some critical point c.

According to Proposition 2 in [BuEp], the map $\Phi : \mathcal{B} \to \mathbb{C}$ is a ramified covering. Its critical point set is $\mathcal{O}^- \cap \mathcal{B}$. The grand orbit of any point in $\mathcal{B}$ projects to a single point under $\Pi \circ \Phi$ and the preimage of a point in $\mathbb{C} - \{0\}$ by $\Pi \circ \Phi$ is a grand orbit. It follows that the set

$$S = \Pi \circ \Phi(\mathcal{GO}) \cup \{0, \infty\}$$

is finite (a rational map has finitely any critical points) and

$$\Pi \circ \Phi : \mathcal{B} - \mathcal{GO} \to \widehat{\mathbb{C}} - S$$

is a covering map.

According to Proposition 3 in [BuEp], the restriction $\Psi : \mathbb{C} - \Psi^{-1}(\mathcal{A}) \to \mathbb{P}^1 - \mathcal{A}$ is a ramified covering, where $\mathcal{A}$ is the accumulation set of the forward orbits of critical points of F. Its critical value set is the union of the forward orbits of the critical values of F. Since the forward orbit of a point in $\mathcal{B}$ only accumulates at a parabolic point in the boundary of $\mathcal{B}$, the restriction $\Psi : \mathfrak{D} \to \mathcal{B}$ is a ramified covering which restricts to a covering map above $\mathcal{B} - \mathcal{GO}$.

The composition of covering maps is a covering map. Therefore

$$\Pi \circ \Phi \circ \Psi : \mathfrak{D} - \Psi^{-1}(\mathcal{GO}) \to \widehat{\mathbb{C}} - S$$

is a covering map. It follows that $\Pi \circ \Phi \circ \Psi : \mathfrak{D} \to \widehat{\mathbb{C}}$ and $\mathfrak{h} : \mathfrak{U} \to \widehat{\mathbb{C}}$ are finite type maps, that 0 and ∞ are asymptotic values, and that points in $S - \{0, \infty\}$ are critical values. □

7.3. **Proof of Lemma 6.4.** Lemma 6.4 is an immediate corollary of the following more general result.

Lemma 7.4. *Let $f : W \to X$ be a finite type analytic map on X. Then, the parabolic degeneracy of a parabolic fixed point of f is either infinity or at most the cardinality of the set of singular values with infinite forward orbits.*

Proof. This follows from the Fatou-Shishikura inequality for finite type analytic maps. A proof may be given along the lines of [Ep4], generalizing the argument given there to finite type analytic maps. An alternative, which goes back to Fatou in the setting of rational maps, is to observe that each cycle of petals must attract the forward orbit of a singular value of f. Otherwise, there would be a component B of the immediate basin which would be fixed by some iterate $f^{\circ q}$ and would not contain any singular value of $f^{\circ q}$. The restriction $f^{\circ q} : B \to B$ would be a covering and the corresponding attracting Fatou coordinate would extend to a covering map $\Phi : B \to \mathbb{C}$. This would force B to be isomorphic to $\mathbb{C}$ and f to be conjugate to the translation by 1 in a whole neighborhood of the parabolic point. In particular, the parabolic degeneracy would not be finite. □

8. Order of growth of $\mathfrak{C}_{r/s}$

8.1. **A technical Lemma.** Given $b \in \mathbb{C} - \{0\}$, let $Y_b \in \mathbb{R}$ be the smallest real number such that $\mathfrak{H}_b$ is univalent in the upper half-plane $\{Z : \text{Im}(Z) > Y_b\}$. Since $\mathfrak{H}_b(Z) = \overline{\mathfrak{H}_{\bar{b}}(\overline{Z})}$, we have that $Y_{\bar{b}}$ is the smallest real number such that $\mathfrak{H}_b$ is univalent on the lower half-plane $\{Z : \text{Im}(Z) < -Y_{\bar{b}}\}$. Those half-planes are contained in $\mathfrak{D}_b$. They are separated by the Julia set, whence they are disjoint.

Lemma 8.1. *As $b \to \infty$,*

$$Y_b = -\text{Im}(b) \log |b| + \mathcal{O}(|b|).$$

Proof. For $b \in \mathbb{C} - \{0\}$, the upper half-plane $\{Z : \text{Im}(Z) > Y_b\}$ and the lower half-plane $\{Z : \text{Im}(Z) < -Y_{\bar{b}}\}$ are disjoint: $Y_b \geq -Y_{\bar{b}}$. Thus, it is enough to show that $Y_b \leq Y'_b$ with $Y'_b = -\text{Im}(b) \log |b| + \mathcal{O}(|b|)$.

Set $R_b = 4|b|$ and assume $|b| \geq 1$. Then, for all $Z \in \mathbb{C} - D(0, R_b)$,

$$\left|F_b(Z) - Z - 1\right| < 1/4 \quad \text{and} \quad \left|F'(Z) - 1\right| \leq 1/4.$$

Thus, Φ_b^{att} and Φ_b^{rep} are univalent on respectively

$$\Omega_b^{\mathrm{att}} = \{Z \in \mathbb{C} \ : \ -3\pi/4 < \arg(Z - 8|b|) < 3\pi/4\}$$

and

$$\Omega_b^{\mathrm{rep}} = \{Z \in \mathbb{C} \ : \ \pi/4 < \arg(Z + 8|b|) < 7\pi/4\}.$$

As a consequence, $\mathfrak{H}_b$ is defined and univalent on

$$\Phi_b^{\mathrm{rep}}(V_b) \quad \text{with} \quad V_b = \Omega_b^{\mathrm{rep}} \cap \Omega_b^{\mathrm{att}}.$$

Let $C \in (0, +\infty)$ and $D \in (0, +\infty)$ be sufficiently large so that Lemma 4.1 and Proposition 4.2 are valid. Given $\varepsilon > 0$, we can choose $M > 0$ large enough, so that for all Z in the subsector $V_b' \subset V_b$ defined by

$$V_b' = \{Z \in \mathbb{C} \ : \ \pi/4 < \arg(Z - \mathrm{i}M|b|) < 3\pi/4\},$$

we have

$$\mathrm{dist}(Z, \mathbb{C} - \Omega_b^{\mathrm{rep}}) > D, \qquad \left|\frac{1}{F(Z) - Z} - 1\right| < \varepsilon, \qquad C\frac{1 + R_b^2 + |b|^2}{\mathrm{dist}(Z, \mathbb{C} - \Omega_b^{\mathrm{rep}})} < |b|$$

and

$$C\left(\frac{1}{\left[\mathrm{dist}(Z, \mathbb{C} - \Omega_b^{\mathrm{rep}})\right]^2} + \left|F'(Z) - 1\right|\right) < \varepsilon.$$

If ε is sufficiently small, it follows from Lemma 4.1 that the argument of Φ' is less than $\pi/12$ on the boundary of V_b'. In that case, $\Phi_b^{\mathrm{rep}}(V_b')$ contains the sector

$$\{Z \in \mathbb{C} \ : \ \pi/3 < \arg(Z - W_b) < 2\pi/3\} \quad \text{with} \quad W_b = \Phi_b^{\mathrm{rep}}(\mathrm{i}M|b|)$$

and so, we can take $Y_b' = \mathrm{Im}(W_b) + \sqrt{3}/2$. Proposition 4.2 finally yields

$$W_b = b \log|b| + \mathcal{O}(|b|). \qquad \square$$

8.2. **The order of growth is at most** 1.

Proposition 8.2. *For each $k \geq 1$, the order of growth of the entire function $\mathfrak{c}_k$ is at most* 1*.*

Proof. The map $Z \mapsto \mathfrak{H}_b(\mathrm{i}Y_b + Z) - \mathrm{i}Y_b + b\pi\mathrm{i}$ is univalent in the upper half-plane. It projects via $\Pi : Z \mapsto \mathrm{e}^{2\pi\mathrm{i}Z}$ to a map which is univalent in the unit disk and fixes 0 with derivative 1. By compactness of this set of maps, for each k, the k-th Fourier coefficient of $Z \mapsto \mathfrak{H}_b(\mathrm{i}Y_b + Z) - \mathrm{i}Y_b + b\pi\mathrm{i}$ is bounded independently of b. Thus, for each $k \geq 1$, there is a constant C_k such that for all $b \in \mathbb{C}$

$$\left|\mathfrak{c}_k(b)\right| \leq C_k \mathrm{e}^{2\pi k Y_b}.$$

Therefore, as $b \to \infty$,

$$\log\left|\mathfrak{c}_k(b)\right| = \mathcal{O}(Y_b) = \mathcal{O}\big(|b| \log|b|\big),$$

which proves that the order of growth of $\mathfrak{c}_k$ is at most 1 as required. $\square$

Corollary 8.3. *The order of growth of the entire function $\mathfrak{C}_{r/s}$ is at most* 1*.*

Proof. We have that

$$\mathfrak{C}_{r/s}(b) = \mathfrak{P}_{r/s}\big(\mathfrak{c}_1(b), \ldots, \mathfrak{c}_s(b)\big)$$

with $\mathfrak{P}_{r/s}$ a polynomial in s variables and $\mathfrak{c}_1(b), \ldots, \mathfrak{c}_s(b)$ the Fourier coefficients of the lifted horn map $\mathfrak{H}_b$. In particular, the order of growth of the entire function $\mathfrak{C}_{r/s}$ is at most 1. □

8.3. **Structure of the horn maps.** We apply the results just obtained to estimate the growth of $\mathfrak{C}_{r/s}$. In certain cases, we have a better understanding of the horn map $\mathfrak{h}_b$. First, when $b = 1/4$, the critical points of the rational map F_b are $\pm 1/2$. The critical point $-1/2$ is mapped to 0 which is mapped to ∞. In that case,

- the Julia set is the segment $[-\infty, 0]$;
- the Fatou set $\widehat{\mathbb{C}} - [-\infty, 0]$ is the immediate basin of the parabolic point at infinity;
- the domain $\mathfrak{D}_{1/4}$ is $\mathbb{C} - \mathbb{R}$;
- the domain $\mathfrak{U}_{1/4}$ is the complement of the unit circle in $\widehat{\mathbb{C}}$.

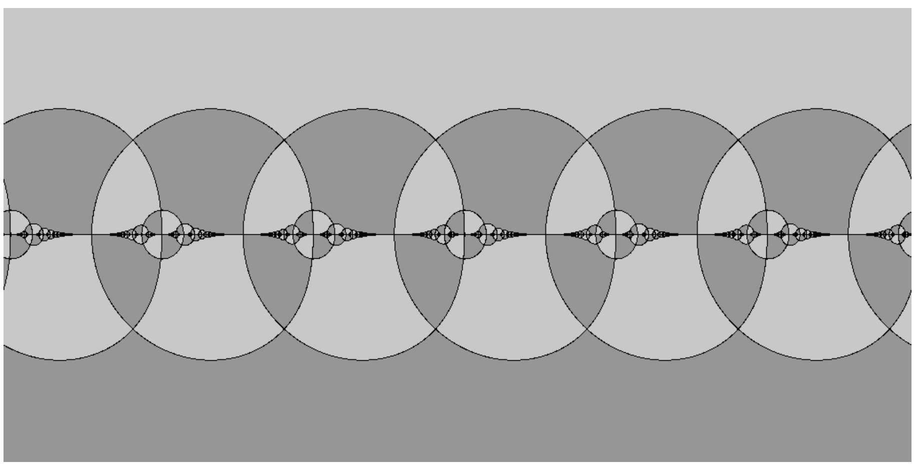


FIGURE 7. The domain $\mathfrak{D}_{1/4}$ is $\mathbb{C} - \mathbb{R}$. It is tiled according to the following rule: the lifted horn map $\mathfrak{H}_{1/4}$ maps each light grey region isomorphically to the upper half-plane and each dark grey region isomorphically to the lower half-plane.

Second, let us assume that $\mathrm{Re}(b) \geq 1/2$. The rational map F_b has a fixed point at $z = -b$ with multiplier $1 - 1/b$. If $\mathrm{Re}(b) \geq 1/2$, this fixed point is nonrepelling. As a consequence, the parabolic basin of ∞ contains only one critical point. The parabolic basin of ∞ is simply connected, so, there is an isomorphism φ_b between the parabolic basin of $F_{1/4}$ and the parabolic basin of F_b conjugating the two rational maps on their parabolic basins. It follows (see the proof of [BuEp, Theorem A] for details) that if we denote by $\mathfrak{U}_b^+$ the connected component of $\mathfrak{U}_b$ containing 0 and by $\mathfrak{U}_b^-$ the one containing ∞, then $\mathfrak{U}_b^+ \cap \mathfrak{U}_b^- = \emptyset$ and

$$\mathfrak{h}_{1/4} = \lambda_b \cdot \mathfrak{h}_b \circ \phi_b$$

for some complex number $\lambda_b \in \mathbb{C} - \{0\}$ and some isomorphism $\phi_b : \mathfrak{U}_{1/4} \to \mathfrak{U}_b$ fixing 0 and ∞.

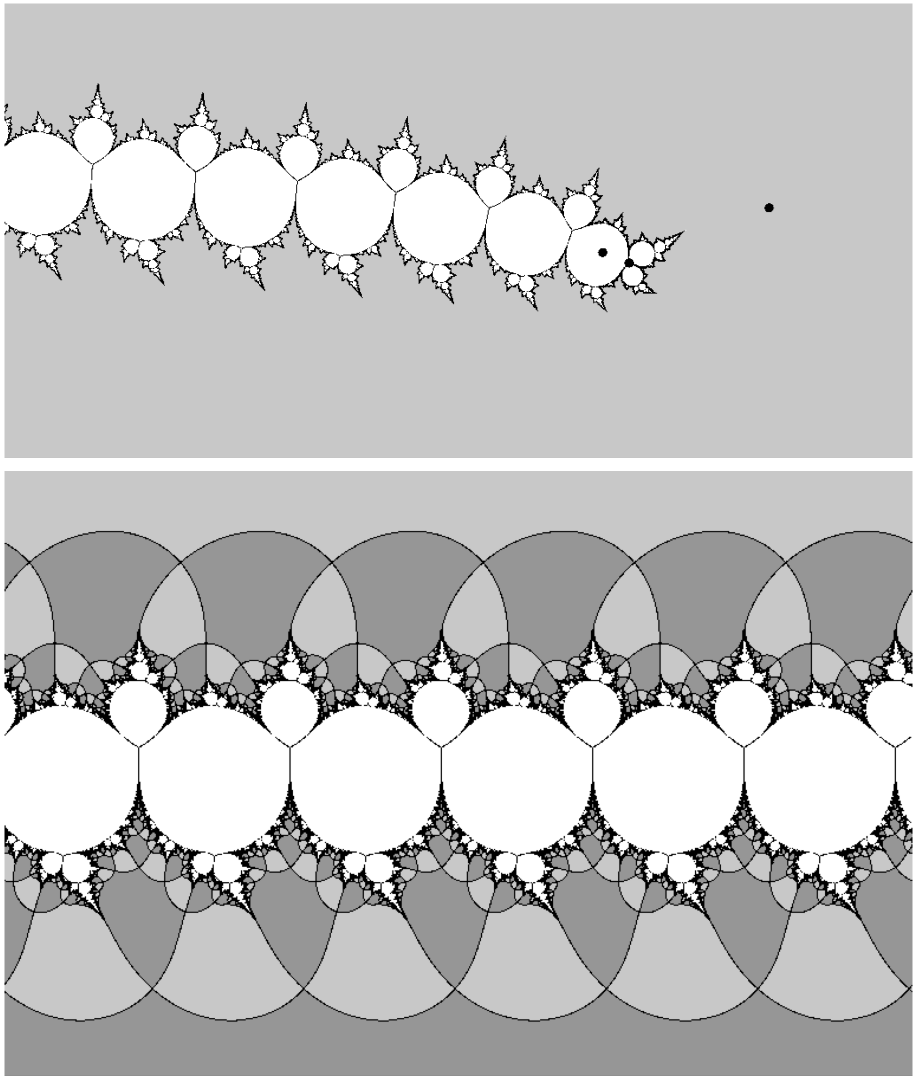


FIGURE 8. Top: the parabolic basin for $b = 1/2 + \mathrm{i}\sqrt{3}/6$. The marked points are the critical points $Z = \pm\sqrt{b}$ and the fixed point $Z = -b$ which has multiplier $\mathrm{e}^{2\pi\mathrm{i}/3}$. The parabolic basin contains only one critical point. Bottom: the domain of the lifted horn map $\mathfrak{H}_b$ is colored with two tones of grey. Each light grey region maps isomorphically to an upper half-plane and each dark grey region maps isomorphically to a lower half-plane.

8.4. **The order of growth is exactly** 1.

Lemma 8.4. *There exists $c > 0$ such that as $\mathbb{R} \ni t \to +\infty$,*

$$\log\left|\mathfrak{C}_{r/s}(1/2 - \mathrm{i}t)\right| \geq c \cdot t \log t.$$

Proof. Assume $b = 1/2 - \mathrm{i}t$ with $t \in \mathbb{R}$. Recall that $\mathfrak{D}_b$ contains the lower half-plane $\{Z \ :\ \mathrm{Im}(Z) < -Y_{\bar b}\}$. It follows that $\mathfrak{U}_b^-$ contains $\mathbb{C} - \overline{D}(0, r_b)$ with

$$\log r_b = 2\pi Y_{\bar b} = -2\pi t \log |t| + \mathcal{O}\big(|t|\big).$$

As mentioned in Section 7, $\mathfrak{U}_b^+ \cap \mathfrak{U}_b^- = \emptyset$, and so $\mathfrak{U}_b^+ \subset D(0, r_b)$. In addition,

$$\mathfrak{g}_{1/4} = \lambda_b \cdot \mathfrak{g}_b \circ \phi_b$$

for some complex number $\lambda_b \in \mathbb{C} - \{0\}$ and some isomorphism $\phi_b : \mathbb{D} \to \mathfrak{U}_b^+$ fixing the origin. Since $\mathfrak{g}_{1/4}'(0) = \mathfrak{g}_b'(0) = \omega_{r/s}$, we have $\lambda_b \cdot \phi_b'(0) = 1$, and since $\mathfrak{U}_b^+ \subset D(0, r_b)$, we have $|\lambda_b| = 1/\big|\phi_b'(0)\big| \geq 1/r_b$.

Consider the set $\mathfrak{G}$ of analytic maps of the form

$$\mathfrak{g} = \mathfrak{g}_{1/4} \circ \phi^{-1}$$

with $\phi : \mathbb{D} \to \mathbb{C}$ a univalent map fixing 0 with derivative 1. Note that any $\mathfrak{g} \in \mathfrak{G}$ is a finite type map which fixes 0 with multiplier $\omega_{r/s}$. Since there is only one singular orbit, the parabolic degeneracy is 1. Thus, for any $\mathfrak{g} \in \mathfrak{G}$, we have

$$\mathfrak{g}^{\circ s}(z) = z + c_{\mathfrak{g}} z^{s+1} + \mathcal{O}(z^{s+2}) \quad \text{with} \quad c_{\mathfrak{g}} \in \mathbb{C} - \{0\}.$$

By compactness of the family of normalized univalent maps ϕ, there is a uniform positive lower bound on $|c_{\mathfrak{g}}|$ for all $\mathfrak{g} \in \mathfrak{G}$. For $\mathfrak{g}(z) = \lambda_b \cdot \mathfrak{g}_b(z/\lambda_b)$, we have

$$\mathfrak{g}^{\circ s}(z) = \lambda_b \mathfrak{g}_b^{\circ s}(z/\lambda_b) = z + \frac{\mathfrak{C}_{r/s}(b)}{\lambda_b^s} z^{s+1} + \mathcal{O}(z^{s+2}).$$

Thus, there is a $c > 0$ such that for all $t \in \mathbb{R}$,

$$\big|\mathfrak{C}_{r/s}(1/2 - \mathrm{i}t)\big| \geq c|\lambda_b|^s.$$

It follows that as $t \to +\infty$,

$$\log\big|\mathfrak{C}_{r/s}(1/2 - \mathrm{i}t)\big| \geq s \log |\lambda_b| + \mathcal{O}(1) \geq -s \log r_b + \mathcal{O}(1) = 2\pi s t \log |t| + \mathcal{O}\big(|t|\big). \quad \square$$

9. Multizetas and multitangents

Our proof of Theorem 0.3 is now complete, but we shall push further the study of the dependence of the lifted horn map $\mathfrak{H}(b, Z)$ with respect to b: we will show that $\mathfrak{H}(b, Z)$ admits an intriguing expansion with respect to b. For further developments regarding results in this section, see [BoEc].

9.1. **Multizeta values.** For all integers $s \geq 2$, we denote by $\zeta(s)$ the Riemann ζ-function:

$$\zeta(s) := \sum_{k>0} \frac{1}{k^s}.$$

It is well-known that $\zeta(2) = \pi^2/6$. Moreover, $\zeta(2n)$ is a rational multiple of $\zeta(2)^n$. It is also known that $\zeta(3)$ is irrational (we will not use this result).

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} = 1.6449\ldots, \\
\zeta(3) &= 1.2020\ldots, \\
\zeta(4) &= \frac{2}{5}\zeta(2)\zeta(2) = 1.0823\ldots \\
\zeta(5) &= 1.0369\ldots.
\end{aligned}$$

For r-tuples $(s_1,\dots,s_r)$ of integers with $s_1\geq 2$, $s_2\geq 1$, $s_3\geq 1$, …, $s_r\geq 1$, set

$$\zeta(s_1,\dots,s_r)=\sum_{0<n_r<\ldots<n_2<n_1<\infty}\frac{1}{n_r^{s_r}}\cdots\frac{1}{n_2^{s_2}}\cdot\frac{1}{n_1^{s_1}}.$$

For example,

$$\zeta(2,1)=\sum_{0<j<k<\infty}\frac{1}{jk^2}.$$

Note that we have the multiplication rules:

$$\zeta(s_1)\zeta(s_2)=\zeta(s_1,s_2)+\zeta(s_1+s_2)+\zeta(s_2,s_1),$$

$$\zeta(s_1)\zeta(s_2,s_3)=\zeta(s_1,s_2,s_3)+\zeta(s_1+s_2,s_3)+\zeta(s_2,s_1,s_3)+\zeta(s_2,s_1+s_3)+\zeta(s_2,s_3,s_1),$$

and so on (the two sequences get shuffled and then possibly contracted). As an example, we have

$$\zeta(2)\zeta(2)=2\zeta(2,2)+\zeta(4),$$

and thus,

$$\zeta(2,2)=\frac{\pi^4}{120}.$$

There are other known rational relations between multizeta values. The most famous due to Euler is

$$\zeta(2,1)=\zeta(3).$$

We shall give a proof below.

9.2. **Tangents and multitangents.** We define

$$\mathrm{Cot}^{(1)}(Z)=\pi\cot(\pi Z)=\pi\mathrm{i}\frac{\mathrm{e}^{\pi\mathrm{i}Z}+\mathrm{e}^{-\pi\mathrm{i}Z}}{\mathrm{e}^{\pi\mathrm{i}Z}-\mathrm{e}^{-\pi\mathrm{i}Z}}$$

and for $n\geq 2$,

$$\mathrm{Cot}^{(n)}=\frac{(-1)^{n-1}}{(n-1)!}\cdot\frac{\partial^{n-1}\mathrm{Cot}^{(1)}}{\partial Z^{n-1}}=\sum_{k\in\mathbb{Z}}\frac{1}{(Z+k)^n}.$$

Let us now consider a series of the form

$$F(Z)=\sum_{-\infty<k_1<k_2<\ldots<k_r<\infty}\phi_1(Z+k_1)\phi_2(Z+k_2)\cdots\phi_r(Z+k_r),$$

where

$$\phi_j(Z)=\frac{\mathcal{H}^{n_j}(-Z)}{Z^{m_j}}\quad\text{with}\quad n_j\geq 0\text{ and }m_j\geq 2.$$

Since

$$\phi_j(Z+k)=\mathcal{O}\left(\frac{(\log|k|)^{n_j}}{|k|^{m_j}}\right)$$

the series is convergent. Shifting all the indices by 1, we see that F is periodic of period 1. It is holomorphic outside $\mathbb{Z}$ and has poles of order $\leq\max m_j$ on $\mathbb{Z}$. Moreover, $F(Z)$ tends to 0 as $\mathrm{Im}(Z)$ tends to $\pm\infty$.

Integrating F on a rectangle R with vertices at $\pm 1/2\pm iM$, we see that the residue of F at $Z=0$ vanishes. Indeed, the integral of F on this rectangle is equal to the integral on the horizontal sides since F is 1-periodic. As M tends to ∞, this integral tends to 0.

Since F is 1-periodic, since $F(Z)\to 0$ as $\mathrm{Im}(Z)\to\pm\infty$ and since the residue of F at 0 vanishes, F can be expressed as a linear combination of the tangents $\mathrm{Cot}^{(n)}$

with $2 \le n \le \max m_j$. In order to determine the coefficient of $\mathrm{Cot}^{(n)}$, it is sufficient to compute the coefficient of $1/Z^n$ in the polar part of F at $Z = 0$.

As an example, let us consider the series

$$F(Z) = \sum_{k\in\mathbb{Z}} \frac{\mathcal{H}^2(-Z-k)}{(Z+k)^2}.$$

Note that $\mathcal{H}^2(-Z-k)/(Z+k)^2$ has a priori a pole at $Z = 0$ for $k \ge 0$. In fact, for $k = 0$ there is no pole, since $\mathcal{H}(0) = 0$. Now, for $k > 0$,

$$\mathcal{H}(-Z-k) = \frac{1}{Z} + \mathcal{O}(1)$$

and so,

$$\frac{\mathcal{H}^2(-Z-k)}{(Z+k)^2} = \frac{1}{k^2Z^2} + \mathcal{O}(1/Z).$$

It follows that the coefficient of $1/Z^2$ in the polar part of F at $Z = 0$ is

$$\sum_{k>0} \frac{1}{k^2} = \zeta(2).$$

So, $F = \zeta(2)\mathrm{Cot}^{(2)}$.

In fact,

$$\frac{\mathcal{H}^2(-Z-k)}{(Z+k)^2} = \frac{1}{k^2Z^2} + \frac{2}{Z}\left(-\frac{1}{k^3} + \sum_{0<j<k} \frac{1}{jk^2}\right) + \mathcal{O}(1).$$

We know that the residue of F at $Z = 0$ vanishes. This shows that

$$\zeta(2,1) = \sum_{0<j<k<\infty} \frac{1}{jk^2} = \sum_{k>0} \frac{1}{k^3} = \zeta(3).$$

9.3. Expansion of $\mathfrak{H}$ with respect to b.

Proposition 9.1. *As b tends to 0 and Z ranges in a compact subset of $\mathbb{C} - \mathbb{Z}$, we have the expansion:*

$$\mathfrak{H}(b, Z) = Z + b\cdot\mathfrak{S}_1(Z) + b^2\cdot\mathfrak{S}_2(Z) + b^3\cdot\mathfrak{S}_3(Z) + b^4\cdot\mathfrak{S}_4(Z) + \mathcal{O}(b^5)$$

with

$$\mathfrak{S}_1 = \mathrm{Cot}^{(1)}, \quad \mathfrak{S}_2 = 0, \quad \mathfrak{S}_3 = 3\zeta(3)\mathrm{Cot}^{(2)},$$

and

$$\mathfrak{S}_4 = -\zeta(4)\mathrm{Cot}^{(3)} + 10\zeta(5)\mathrm{Cot}^{(2)}.$$

Remark. We obtain $\mathfrak{S}_2 = 0$ thanks to our normalization of the Fatou coordinates via the harmonic numbers $\mathcal{H}(n)$ rather than $\log n$. If we had used the standard normalization, the Euler constant γ would appear in several places in the expansion.

A further computer-assisted calculation (see [BoEc]) gives

$$\mathfrak{S}_5 = \zeta(5)\mathrm{Cot}^{(4)} + \left(9\zeta(3)^2 - \frac{244}{105}\zeta(2)^3\right)\mathrm{Cot}^{(3)} + \frac{77}{2}\zeta(7)\mathrm{Cot}^{(2)}$$

and

$$\mathfrak{S}_6 = -\zeta(6)\mathrm{Cot}^{(5)} + \left(-\frac{18}{5}\zeta(2)^2\zeta(3) + 10\zeta(7) + 3\zeta(2)\zeta(5)\right)\mathrm{Cot}^{(4)} + \left(3\zeta(6,2) + 54\zeta(3)\zeta(5) - \frac{878}{105}\zeta(2)^4\right)\mathrm{Cot}^{(3)} + 151\zeta(9)\mathrm{Cot}^{(2)}.$$

Proof. Assume K is a compact subset of $\mathbb{C} - \mathbb{Z}$. If $\varepsilon > 0$ is sufficiently small, then, as $n \longrightarrow +\infty$,

$$(b, Z) \mapsto F_b^{\circ 2n+1}\big(Z - n + b\mathcal{H}(n - Z)\big) - b\mathcal{H}(n + Z) - (n + 1)$$

converges uniformly to $(b, Z) \mapsto \mathfrak{H}_b(Z)$ on $B(0, \varepsilon) \times K$. Fix an integer n and for $k \geq -n$, define

$$G_{n,k}(b, Z) = F_b^{\circ n+k}\big(Z - n + b\mathcal{H}(n - Z)\big).$$

In particular,

$$G_{n,-n}(b, Z) = Z - n + b\mathcal{H}(n - Z) \quad \text{and} \quad G_{n,n+1}(b, Z) = F_b^{\circ 2n+1}(Z - n + b\mathcal{H}(n - Z)).$$

The lifted horn maps $\mathfrak{H}(b, Z)$ are obtained as the limit of

$$G_{n,n+1}(b, Z) - b\mathcal{H}(n + Z) - (n + 1).$$

In order to determine the expansion of $\mathfrak{H}(b, Z)$ with respect to b at $b = 0$, we will consider the expansion of $G_{n,k}$ with respect to b at $b = 0$:

$$G_{n,k}(b, Z) = (Z + k) + bG_{n,k}^{(1)} + b^2 G_{n,k}^{(2)} + \ldots$$

Note that $G_{n,-n}^{(1)} = \mathcal{H}(n - Z)$ and for $j \geq 2$, $G_{n,-n}^{(j)} = 0$. Since

$$G_{n,k+1}(b, Z) = G_{n,k}(b, Z) + 1 + \frac{b}{G_{n,k}(b, Z)},$$

we have a recursive relation for the coefficients $G_{n,k}^{(j)}$. In particular, we have, for $k \geq -n$,

$$\begin{aligned}
G_{n,k+1}^{(1)} &= G_{n,k}^{(1)} + \frac{1}{Z + k}, \\
G_{n,k+1}^{(2)} &= G_{n,k}^{(2)} - \frac{G_{n,k}^{(1)}}{(Z + k)^2}, \\
G_{n,k+1}^{(3)} &= G_{n,k}^{(3)} - \frac{G_{n,k}^{(2)}}{(Z + k)^2} + \frac{[G_{n,k}^{(1)}]^2}{(Z + k)^3} \text{ and} \\
G_{n,k+1}^{(4)} &= G_{n,k}^{(4)} - \frac{G_{n,k}^{(3)}}{(Z + k)^2} + 2\frac{G_{n,k}^{(1)}G_{n,k}^{(2)}}{(Z + k)^3} - \frac{[G_{n,k}^{(1)}]^3}{(Z + k)^4}.
\end{aligned}$$

Therefore

$$G_{n,k}^{(1)}(Z) = \mathcal{H}(n - Z) + \sum_{j=-n}^{k-1} \frac{1}{Z + j} = \mathcal{H}(-k - Z).$$

Thus,

$$G_{n,n+1}^{(1)}(Z) = \mathcal{H}(-n - 1 - Z)$$

and since

$$\mathcal{H}(-n - 1 - Z) - \mathcal{H}(n + Z) = \pi\cot(\pi Z) = \mathrm{Cot}^{(1)},$$

we have

$$\boxed{\mathfrak{S}_1 = \mathrm{Cot}^{(1)}.}$$

Next, we have

$$G^{(2)}_{n,k}(Z) = -\sum_{j=-n}^{k-1} \frac{G^{(1)}_{n,j}(Z)}{(Z+j)^2} = -\sum_{j=-n}^{k-1} \frac{\mathcal{H}(-j-Z)}{(Z+j)^2}$$

and

$$G^{(2)}_{n,n+1} = -\sum_{j=-n}^{n} \frac{\mathcal{H}(-j-Z)}{(Z+j)^2}.$$

The only term which could contribute to the polar part of order 2 at $Z = 0$ would be $\mathcal{H}(-Z)/Z^2 = \mathcal{O}(1/Z)$ (for $j = 0$). So there, is no pole of order 2 and

$$\boxed{\mathfrak{S}_2 = 0.}$$

Now,

$$\begin{aligned} G^{(3)}_{n,k}(Z) &= -\sum_{j=-n}^{k-1} \frac{G^{(2)}_{n,j}(Z)}{(Z+j)^2} + \frac{[G^{(1)}_{n,j}]^2(Z)}{(Z+j)^3} \\ &= \sum_{-n\le i<j<k} \frac{\mathcal{H}(-i-Z)}{(Z+i)^2(Z+j)^2} + \sum_{-n\le j<k} \frac{\mathcal{H}^2(-j-Z)}{(Z+j)^3}. \end{aligned}$$

So,

$$\mathfrak{S}_3 = \sum_{-\infty<i<j<\infty} \frac{\mathcal{H}(-i-Z)}{(Z+i)^2(Z+j)^2} + \sum_{-\infty<j<\infty} \frac{\mathcal{H}^2(-j-Z)}{(Z+j)^3}.$$

For the first sum, the polar part of order ≥ 2 only comes from $j = 0$. Indeed, the poles of $\mathcal{H}$ are simple poles and $\mathcal{H}(0) = 0$, so for $i = 0$ and $j > 0$, we have

$$\frac{\mathcal{H}(-Z)}{Z^2(Z+j)^2} = \mathcal{O}(1/Z).$$

Now, for $j = 0$, we have

$$\frac{\mathcal{H}(-i-Z)}{(Z+i)^2 Z^2} = \frac{\mathcal{H}(-i)}{i^2 Z^2} + \mathcal{O}(1/Z) = \frac{1}{Z^2}\left(\sum_{0<k<-i} \frac{1}{ki^2} - \frac{1}{i^3}\right) + \mathcal{O}(1/Z).$$

For the second sum, the polar part of order ≥ 2 only comes from $j > 0$:

$$\frac{\mathcal{H}^2(-j-Z)}{(Z+j)^3} = \frac{1}{j^3 Z^2} + \mathcal{O}(1/Z).$$

So, $\mathfrak{S}_3$ has a pole of order 2 at $Z = 0$ and the coefficient is

$$\sum_{0<k<-i<\infty} \frac{1}{ki^2} + \sum_{0<-i<\infty} \frac{1}{i^3} + \sum_{0<j<\infty} \frac{1}{j^3} = \zeta(2,1) + \zeta(3) + \zeta(3) = 3\zeta(3).$$

Therefore,

$$\boxed{\mathfrak{S}_3(Z) = 3\zeta(3)\mathrm{Cot}^{(2)}.}$$

We finally come to the computation of $\mathfrak{S}_4$. We have

$$\begin{aligned} \mathfrak{S}_4 &= -\sum_{-\infty<i<j<k<\infty} \frac{\mathcal{H}(-i-Z)}{(Z+i)^2(Z+j)^2(Z+k)^2} - \sum_{-\infty<j<k<\infty} \frac{\mathcal{H}^2(-j-Z)}{(Z+j)^3(Z+k)^2} \\ &\quad +2\sum_{-\infty<j<k<\infty} \frac{\mathcal{H}(-j-Z)\mathcal{H}(-k-Z)}{(Z+j)^2(Z+k)^3} - \sum_{-\infty<k<\infty} \frac{\mathcal{H}^3(-k-Z)}{(Z+k)^4}. \end{aligned}$$

This function has a pole of order at most 3 at $Z = 0$. Indeed, $\mathcal{H}^3(-Z)/Z^4$ only has a simple pole at $Z = 0$. The only term which contributes to the polar part of order 3 is the last term. This term only has poles when $k > 0$ and in that case, we have

$$-\frac{\mathcal{H}^3(-k-Z)}{(Z+k)^4} = -\frac{1}{Z^3}\left(1 + 3Z\sum_{0<j<k}\frac{1}{j}\right)\left(\frac{1}{k^4} - 4\frac{Z}{k^5}\right) + \mathcal{O}(1/Z).$$

Therefore, the contribution of the last term is

$$-\zeta(4)\mathrm{Cot}^{(3)} + (-3\zeta(4,1) + 4\zeta(5))\mathrm{Cot}^{(2)}.$$

The other terms only contribute to the polar part of order 2, i.e., to the coefficient of $\mathrm{Cot}^{(2)}$. We will now determine the contribution of each term to the coefficient of $\mathrm{Cot}^{(2)}$.

First term. For $j = 0$, the contribution is

$$\begin{aligned} -\sum_{i<0<k}\frac{\mathcal{H}(-i)}{i^2k^2} &= -\left(\sum_{0<n\le -i}\frac{1}{n\cdot(-i)^2}\right)\left(\sum_{k>0}\frac{1}{k^2}\right) \\ &= -\zeta(2)\left(\sum_{0<n<i}\frac{1}{ni^2} + \sum_{i>0}\frac{1}{i^3}\right) \\ &= -\zeta(2)(\zeta(2,1)+\zeta(3)). \end{aligned}$$

For $k = 0$, the contribution is

$$\begin{aligned} -\sum_{i<j<0}\frac{\mathcal{H}(-i)}{i^2j^2} &= -\sum_{0<-j<-i}\left(\sum_{n\le -i}\frac{1}{n\cdot(-i)^2\cdot(-j)^2}\right) \\ &= -\sum_{0<j<i}\left(\frac{1}{i^3j^2} + \sum_{n<i}\frac{1}{ni^2j^2}\right) \\ &= -\zeta(3,2) - \sum_{n<j<i}\frac{1}{nj^2i^2} - \sum_{j(=n)<i}\frac{1}{j^3i^2} - \sum_{j<n<i}\frac{1}{j^2ni^2} \\ &= -\zeta(3,2) - \zeta(2,2,1) - \zeta(2,3) - \zeta(2,1,2). \end{aligned}$$

Second term. For $k = 0$, the contribution is

$$\begin{aligned} -\sum_{j<0}\frac{\mathcal{H}^2(-j)}{j^3} &= \sum_{0<-j}\left(\sum_{0<i\le -j}\frac{1}{i}\right)^2\frac{1}{(-j)^3} \\ &= \sum_{0<j}\left(\sum_{0<i_1\le j}\frac{1}{i_1}\right)\left(\sum_{0<i_2\le j}\frac{1}{i_2}\right)\frac{1}{j^3} \\ &= \sum_{0<i_1<i_2<j}\frac{1}{i_1i_2j^3} + \sum_{0<i_1=i_2<j}\frac{1}{i_1^2j^3} + \sum_{0<i_2<i_1<j}\frac{1}{i_2i_1j^3} \\ &\quad + \sum_{0<i_2<i_1=j}\frac{1}{i_2j^4} + \sum_{0<i_1<i_2=j}\frac{1}{i_1j^4} + \sum_{0<i_1=i_2=j}\frac{1}{j^5} \\ &= 2\zeta(3,1,1) + \zeta(3,2) + 2\zeta(4,1) + \zeta(5). \end{aligned}$$

For $j > 0$, the contribution is

$$-\sum_{0<j<k} \frac{1}{j^3k^2} = -\zeta(2,3).$$

<u>Third term.</u> For $k = 0$, the contribution is

$$-2\sum_{j<0} \frac{\mathcal{H}(-j)\cdot(-\mathcal{H}'(0))}{j^2} = 2\zeta(2)(\zeta(2,1)+\zeta(3)).$$

For $j = 0$ and $k > 0$, the contribution is

$$-2\sum_{k>0} \frac{-\mathcal{H}'(0)}{k^3} = 2\zeta(2)\zeta(3).$$

Finally, for $0 < j < k$, the contribution is

$$-2\sum_{0<j<k} \frac{1}{j^2k^3} = -2\zeta(3,2).$$

Collecting the terms, and using the $\mathbb{Q}$-relations between multizeta values, we deduce that the coefficient of $\mathrm{Cot}^{(2)}$ is $10\zeta(5)$:

$$\boxed{\mathfrak{S}_4 = -\zeta(4)\mathrm{Cot}^{(3)} + 10\zeta(5)\mathrm{Cot}^{(2)}.}$$

This completes the proof of Proposition 9.1. □

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